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SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS AND THEIR CHARACTERISTIC SURFACES.[1]

By Tracy Yerkes Thomas and Edwin Warren Titt.

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§ 1. Introduction.

It is the object of this paper to treat in a general and systematic manner the existence theorems for systems of partial differential equations of first order (Part I), and to study their characteristic surfaces (Part II). Our work includes in particular a detailed treatment of the systems of invariantive type and is so developed as to bring into relationship the methods used in a series of previous papers by one of us[2] and a most interesting paper by Cartan[3] on this same subject.

[1] Received, June 1, 1932.

[2] T. Y. Thomas, *Determination of affine and metric spaces by their differential invariants.* Math. Ann., 101 (1929), pp. 713–728. *The existence theorems in the problem of the determination of affine and metric spaces by their differential invariants.* Amer. Jour. Math.,

The work in Part II is an extension of the treatment of characteristic surfaces in a series of notes in the *Proceedings of the National Academy of Sciences*[4] and is believed to be the first adequate general treatment of this problem. The method used in proving the existence theorems is patterned after Riquier's[5] theory of orthonomic systems. This has necessitated the development of a theory of sets of monomials which will apply to a system of differential equations, part of which hold only over a subspace; this theory contains as a special case the theory presented by Janet.[6]

Throughout the paper illustrative examples have been given in fine print. In particular we have given for the first time a detailed study of the characteristic surfaces and the associated existence theorems for Einstein's gravitational equations in free space.[7]

PART I.

GENERAL EXISTENCE THEOREMS.

§ 2. Regular Systems of Differential Equations.

Consider a system of L partial differential equations, linear and of the first order in w dependent variables $v_1, \cdots, v_w$ and n independent variables $x^1, \cdots, x^n$, namely

$$\sum_{k=1}^{w} \sum_{\alpha=1}^{n} a_{ik}^{\alpha} \frac{\partial v_k}{\partial x^{\alpha}} + c_i = 0 \qquad (i = 1, \cdots, L). \tag{2.1}$$

The coefficients a_{ik}^{α} and c_i are functions of x^{α} and v_k. It is assumed also that the left members of (2.1) are linearly independent in the derivatives of the dependent variables v_k.

Let us suppose that there are $L_1 \leqq L$ equations (2.1) which are independent in the derivatives $\partial v_k / \partial x^1$ and, as the integer L_1 is conceivably

52 (1930), pp. 225–250. *Invariantive systems of partial differential equations,* Ann. of Math. (2), 31 (1930), pp. 687–713. *Space structure as a boundary value problem.* Ann. of Math. (2), 31 (1930), pp. 714–726. In order to shorten the work references will be made to the above papers frequently. The designation M., Jour., Ann. (1), Ann. (2) respectively will be used when reference is made to the above papers.

[3] Élie Cartan, *Sur la théorie des systèmes en involution et ses applications à la Relativité.* Bull. Sc. Math., 59 (1931), pp. 88–118.

[4] T. Y. Thomas, *On the Unified Field Theory,* Proc. Nat. Acad. Sciences; 16 (1930), Notes I and II, pp. 761–766, 830–835; 17 (1931), Notes III–VI, pp. 48–58, 111–119, 199–210, 325–329. These will be referred to as Proc. Note I, etc.

[5] Riquier, *Les systèmes d'équations aux dérivées partielles,* (1910).

[6] Maurice Janet, *Sur les systèmes d'équations aux dérivées partielles,* Journ. de Math. (8), 3 (1920), pp. 65–144.

[7] The differential equations defining the 3-dimensional characteristic surfaces for Einstein's gravitational equations have been given by T. Levi-Civita, *Caratteristiche e bicaratteristiche delle equazioni gravitazionali di Einstein,* Rend. Accad. Lincei, (6), 11 (1930), pp. 1–11; ibid. pp. 113–121.

dependent on the coördinate system (x), let us suppose that coördinates x^α have been selected for which L_1 will have its maximum value. We can then divide our equations into two sets: a set S_1 consisting of L_1 equations which can be solved for L_1 of the derivatives $\partial v_k/\partial x^1$ and a set S_2 from which these derivatives can be eliminated. Now suppose that the set of equations S_2 is independent in L_2 of the derivatives $\partial v_k/\partial x^2$ and in fact that coördinates x^α are selected so that L_2 has its maximum possible value, under the restriction that the above integer L_1 is unchanged. This makes it possible to divide the set S_2 into two sets: S_2^* and S_3^* such that S_2^*, consisting of L_2 equations, can be solved for L_2 of the derivatives $\partial v_k/\partial x^2$ and such that these latter derivatives can be eliminated entirely from the set S_3^*. Proceeding in this way we arrive at a coördinate system (which is obviously one of an infinity of such coördinate systems) for which our system of equations (2.1) can be put into the form

$$\sum_{k=1}^{w}\sum_{\alpha=1}^{n} b_{ik\beta}^{\alpha}\frac{\partial v_k}{\partial x^\alpha} + c_{i\beta} = 0, \qquad \begin{pmatrix}\beta = 1, \cdots, n\\ i = 1, \cdots, L_\beta\end{pmatrix}, \tag{2.2}$$

where $b_{ik\beta}^{\alpha} = 0$ if $\alpha < \beta$. A system of coördinates x^α with respect to which (2.1) can be put into the form (2.2) in which the integers L_β are characterized by the above mentioned property, is said to be *non-singular;* otherwise the coördinate system is said to be *singular.*

If (2.1) is written in the form

$$\sum_{k=1}^{w}\sum_{\alpha=1}^{n} b_{ik\beta}^{\alpha}\frac{\partial v_k}{\partial x^\alpha} + c_{i\beta} = 0, \qquad \begin{pmatrix}\beta = 1, \cdots, n\\ i = 1, \cdots, J_\beta\end{pmatrix},$$

where $b_{ik\beta}^{\alpha} = 0$ if $\alpha < \beta$, with respect to a singular coördinate system. then $J_1 < L_1$ or if $J_i = L_i$ for $i = 1, \cdots, r$ then $J_{r+1} < L_{r+1}$. Obviously the inequality $r \leqq n-1$ is here satisfied.

Now assume a non-singular choice of independent variables x^α and make the transformation

$$\bar{x}^\sigma = x^\sigma + m_\tau^\sigma x^\tau, \tag{2.3}$$

where the m_τ^σ are constants. If the dependent variables v_k transform as *scalars* the law of transformation of their derivatives is given by

$$\frac{\partial v_k}{\partial x^\alpha} = \frac{\partial v_k}{\partial \bar{x}^\alpha} + m_\alpha^\sigma \frac{\partial v_k}{\partial \bar{x}^\sigma},$$

and hence equations (2.2) become

$$\sum_{k=1}^{w}\sum_{\alpha=1}^{n} (b_{ik\beta}^{\alpha} + m_\sigma^\alpha b_{ik\beta}^{\sigma})\frac{\partial v_k}{\partial \bar{x}^\alpha} + c_{i\beta} = 0. \tag{2.4}$$

For a fixed value α belonging to the set $1, \cdots, n-1$ assume that $m_\tau^\sigma = 0$ except when $\sigma = \alpha$. Then if the constants m_τ^α are sufficiently small equations (2.4) for $\beta = \alpha$ must be linearly independent with respect to the derivatives $\partial v_k / \partial \bar{x}^\alpha$. For the above choice of the constants m_τ^σ the coefficients of the derivatives $\partial v_k / \partial \bar{x}^\gamma$ for $\gamma = 1, \cdots, \alpha-1$ in the equations (2.4) will be equal to the coefficients of the corresponding derivatives $\partial v_k / \partial x^\gamma$ in the system (2.2). Hence the set of forms,

$$\sum_{k=1}^{w} m_\sigma^\alpha \, b_{ik\beta}^\sigma \frac{\partial v_k}{\partial \bar{x}^\alpha} \qquad (\beta > \alpha, \ \alpha \text{ not summed}),$$

will be linearly dependent on the forms

$$\sum_{k=1}^{w} (b_{ik\alpha}^\alpha + m_\sigma^\alpha \, b_{ik\alpha}^\sigma) \frac{\partial v_k}{\partial \bar{x}^\alpha} \qquad (\alpha \text{ not summed}),$$

since otherwise the original choice of variables x^α would be singular contrary to hypothesis. Or

$$m_\sigma^\alpha \, b_{ik\beta}^\sigma = \sum_{j=1}^{L_\alpha} \lambda_{ij\beta}^{*\alpha} (b_{jk\alpha}^\alpha + m_\sigma^\alpha \, b_{jk\alpha}^\sigma) \qquad (\beta > \alpha \text{ and } \alpha \text{ not summed}). \tag{2.5}$$

Now take $m_\beta^\alpha = m$, and $m_\tau^\sigma = 0$ otherwise. Then (2.5) gives

$$m \, b_{ik\beta}^\beta = \sum_{j=1}^{L_\alpha} \lambda_{ij\beta}^{*\alpha} (b_{jk\alpha}^\alpha + m \, b_{jk\alpha}^\beta) \quad (\beta > \alpha, \ \alpha \text{ and } \beta \text{ not summed}).$$

Hence, if we let m approach zero, we have

$$b_{ik\beta}^\beta = \sum_{j=1}^{L_\alpha} \lambda_{ij\beta}^\alpha \, b_{jk\alpha}^\alpha, \qquad \begin{pmatrix} \beta > \alpha, \ i = 1, \cdots, L_\beta \\ \alpha, \beta \text{ not summed} \end{pmatrix}, \tag{2.6}$$

where

$$\lambda_{ij\beta}^\alpha = \lim_{m \to 0} \left[\frac{\lambda_{ij\beta}^{*\alpha}}{m} \right].$$

More generally, take $m_\gamma^\alpha = m$ for a single index $\gamma \geqq \beta$ and $m_\tau^\sigma = 0$ otherwise. Then (2.5) gives

$$m \, b_{ik\beta}^\gamma = \sum_{j=1}^{L_\alpha} \lambda_{ij\beta}^{*\alpha} (b_{jk\alpha}^\alpha + m \, b_{jk\alpha}^\gamma),$$

and this becomes

$$b_{ik\beta}^\gamma = \sum_{j=1}^{L_\alpha} \lambda_{ij\beta}^{\alpha\gamma} \, b_{jk\alpha}^\alpha \qquad (\gamma \geqq \beta > \alpha), \tag{2.7}$$

when m is allowed to approach zero.

It is clear that

$$L_1 \geqq L_2 \geqq L_3 \cdots \geqq L_n,$$

otherwise a transformation of the independent variables x^α, producing merely a permutation of the indices of these variables, would show that

the original choice of the variables x^α was singular, contrary to hypothesis.

Now suppose that equations (2.2) for $\beta = 1$ can be solved for the derivatives

$$\frac{\partial v_1}{\partial x^1}, \quad \frac{\partial v_2}{\partial x^1}, \quad \cdots, \quad \frac{\partial v_{L_1}}{\partial x^1},$$

or in other words the matrix of the quantities b^1_{jk1}, where $j, k = 1, \cdots, L_1$, is non-singular. Put $\alpha = 1$, $\beta = 2$ in (2.6) and consider the matrix of the quantities λ^1_{ij2} appearing in these equations. If this matrix is of rank R, then $L_2 \leqq R$; this follows from a theorem in Algebra[8] and the fact that the matrix of the quantities b^2_{ik2} is of rank L_2. Hence $R = L_2$ since R can obviously not be greater than L_2. It then follows from a second theorem in Algebra[9] and (2.6) that the matrix of the quantities b^2_{ik2} for $i = 1, \cdots, L_2$ and $k = 1, \cdots, L_1$ is of rank L_2. Hence equations (2.2) for $\beta = 2$ can be solved for the derivatives

$$\frac{\partial v_1}{\partial x^2}, \quad \cdots, \quad \frac{\partial v_{L_2}}{\partial x^2}$$

after a suitable choice of the indices of the dependent variables v_k has been made. By a continuation of this process it is evident that, if we change the notation for the independent variables v_k in accordance with the following scheme:

$$\begin{aligned}
v_{i0} &\sim v_{L_1+1}, \cdots, v_w,\\
v_{i1} &\sim v_{L_2+1}, \cdots, v_{L_1},\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
v_{in-1} &\sim v_{L_n+1}, \cdots, v_{L_{n-1}},\\
v_{in} &\sim v_1, \quad \cdots, v_{L_n},
\end{aligned}$$

equations (2.2) can be written

$$\frac{\partial v_{ik}}{\partial x^\alpha} = \sum (x, v) \frac{\partial v_{pq}}{\partial x^\beta} + \star, \qquad \begin{pmatrix} k = 1, \cdots, n \\ i = 1, \cdots, w_k \\ \alpha = 1, \cdots, k \end{pmatrix}, \tag{2.8}$$

where

$$\begin{aligned}
w_0 &= w - L_1,\\
w_1 &= L_1 - L_2,\\
w_2 &= L_2 - L_3,\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
w_{n-1} &= L_{n-1} - L_n,\\
w_n &= L_n
\end{aligned} \tag{2.9}$$

[8] See, for example, Dickson: *Modern Algebraic Theories*, p. 51.

[9] Dickson, *loc. cit.*, p. 51.

and $\beta \geqq \alpha$, $\beta > q$. The coefficients (x, v) in (2.8) depend upon the quantities x and v_{rs}, and the $\ast$ denotes terms containing no derivatives of the v_{rs}; in the sequel the $\ast$ will be used to denote terms of lower order than those written down explicitly. A system of the form (2.8) will be said to be *regular*.[10]

Addition of corresponding members of (2.9) gives

$$w = w_0 + w_1 + \cdots + w_n.$$

§ 3. Extension to Tensor Differential Equations.

The results of § 2 can be extended to systems of equations the left members of which are linear in the first derivatives of the components of a tensor T. Thus consider

$$D_{ip\cdots q}^{r\cdots s\alpha} \frac{\partial T_{r\cdots s}^{p\cdots q}}{\partial x^\alpha} + \ast = 0, \qquad (i = 1, \cdots, L), \tag{3.1}$$

where the coefficients D are functions of the independent variables x^α and the unknowns T; the same is true of the $\ast$ terms.

With respect to a non-singular coördinate system (defined as in § 2) equations (3.1) can be written

$$D_{ip\cdots q\beta}^{r\cdots s\alpha} \frac{\partial T_{r\cdots s}^{p\cdots q}}{\partial x^\alpha} + \ast = 0, \qquad \begin{pmatrix} \beta = 1, \cdots, n \\ i = 1, \cdots, L_\beta \end{pmatrix}, \tag{3.2}$$

where $D = 0$ if $\alpha < \beta$. Under the transformation (2.3) the derivatives of the components of the tensor T transform according to the equations

$$\frac{\partial T_{r\cdots s}^{p\cdots q}}{\partial x^\alpha} = \frac{\partial \bar{T}_{c\cdots d}^{a\cdots b}}{\partial \bar{x}^\sigma} (\delta_\alpha^\sigma + m_\alpha^\sigma) \frac{\partial \bar{x}^c}{\partial x^r} \cdots \frac{\partial \bar{x}^d}{\partial x^s} \frac{\partial x^p}{\partial \bar{x}^a} \cdots \frac{\partial x^q}{\partial \bar{x}^b}.$$

In the coördinate system $(\bar{x})$ equations (3.2) therefore take the form

$$(\bar{D}_{ia\cdots b\beta}^{c\cdots d\alpha} + m_\sigma^\alpha \bar{D}_{ia\cdots b\beta}^{c\cdots d\sigma}) \frac{\partial \bar{T}_{c\cdots d}^{a\cdots b}}{\partial \bar{x}^\alpha} + \ast = 0 \tag{3.3}$$

with $\bar{D}_{ia\cdots b\beta}^{c\cdots d\alpha} = 0$ if $\alpha < \beta$. Letting α be a particular number of the set $1, \cdots, n-1$ and assuming that $m_\sigma^\tau = 0$ if $\alpha \neq \tau$, we obtain by an argument analogous to that employed in § 2 that

$$m_\sigma^\alpha \bar{D}_{ia\cdots b\beta}^{c\cdots d\sigma} = \sum_{j=1}^{L_\alpha} \lambda_{ij\beta}^{\ast\alpha} (\bar{D}_{ja\cdots b\alpha}^{c\cdots d\alpha} + m_\sigma^\alpha \bar{D}_{ja\cdots b\alpha}^{c\cdots d\sigma}) \quad (\beta > \alpha, \ \alpha \text{ not summed}) \tag{3.4}$$

[10] Without the restriction $\beta \geqq \alpha$, the system (2.8) is called *regular and immediate* by Méray and Riquier *Sur la convergence des développements des intégrales ordinaires*, Ann. Ec. Norm. Sup., (3), 7, (1890), p. 44.

as a result of the assumption that the original coördinate system (x) is non-singular. Then putting $m_\gamma^\alpha = m$ for a single $\gamma \geqq \beta$ and $m_\tau^\sigma = 0$ otherwise, we obtain, on allowing m to approach zero, that

$$D_{ip\cdots qB}^{r\cdots s\gamma} = \sum \lambda_{ijB}^{\alpha\gamma} D_{jp\cdots q\alpha}^{r\cdots s\alpha} \qquad (\gamma \geqq \beta > \alpha). \tag{3.5}$$

If all the components $T_{r\cdots s}^{p\cdots q}$ are independent they can be represented by v_k and the system (3.2) together with (3.5) can be written

$$\sum_{k=1}^{w} b_{ik\beta}^{\alpha} \frac{\partial v_k}{\partial x^\alpha} + c_{i\beta} = 0 \qquad \begin{pmatrix} \beta = 1, \cdots, n \\ i = 1, \cdots, L_\beta \end{pmatrix}, \tag{3.6}$$

$$b_{ik\beta}^{\gamma} = \sum_j \lambda_{ij\beta}^{\alpha\gamma} b_{jk\alpha}^{\alpha} \qquad (\gamma \geqq \beta > \alpha), \tag{3.7}$$

where $b_{ik\beta}^{\alpha} = 0$ if $\alpha < \beta$. Suppose, however, that the components $T_{r\cdots s}^{p\cdots q}$ satisfy linear relations of the form

$$\sum T_{m\cdots n}^{k\cdots l} = 0, \tag{3.8}$$

where the indices $k \cdots l$, $m \cdots n$ are obtainable from $p \cdots q$, $r \cdots s$ by permutations. If, in this case, v_k is used to denote the independent components $T_{r\cdots s}^{p\cdots q}$ when account is taken of (3.8), equations (3.6) and (3.7) will likewise apply. On the basis of the discussion in § 2 the system (3.2) can then be replaced by a system of equations in the regular form (2.8).

§ 4. Application to Invariantive Systems. Affine and Metric Cases.

Consider a system of partial, differential equations of the form

$$T_{\gamma\cdots\delta}^{\mu\cdots\nu}\left(\Gamma_{\alpha\beta}^{i};\ \frac{\partial \Gamma_{\alpha\beta}^{i}}{\partial x^\gamma};\ \frac{\partial^2 \Gamma_{\alpha\beta}^{i}}{\partial x^\gamma\, \partial x^\vartheta}\right) = 0, \tag{4.1}$$

where the T's are the components of an absolute or relative tensor invariant of weight W and the $\Gamma_{\alpha\beta}^{i}$ are components of affine connection. The equations (4.1) will be assumed linear in the second derivatives of the $\Gamma_{\alpha\beta}^{i}$ and as indicated, to depend on the $\Gamma_{\alpha\beta}^{i}$ and their first derivatives. Along with this system consider

$$T_{\gamma\cdots\delta}^{\mu\cdots\nu}\left(g_{\alpha\beta};\ \frac{\partial g_{\alpha\beta}}{\partial x^\gamma};\ \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma\, \partial x^\vartheta};\ \frac{\partial^3 g_{\alpha\beta}}{\partial x^\gamma\, \partial x^\vartheta\, \partial x^\varepsilon}\right) = 0, \tag{4.2}$$

where the T's are now tensor invariants linear in the third derivatives of the components $g_{\alpha\beta}$ of the fundamental metric tensor and depending on the components $g_{\alpha\beta}$ themselves together with their first and second derivatives. In particular (4.1) or (4.2) may consist of a single equation or of several

equations in which the left members are scalar invariants.[11] The system (4.1) is completely equivalent to one of the form

$$\sum T_{\alpha i}^{\beta\gamma\delta\varepsilon}(A)\frac{\partial A_{\beta\gamma\delta}^{\alpha}}{\partial x^{\varepsilon}} + \ast = 0, \tag{4.3}$$

where the left members are again components of a tensor or scalar invariant. The system is linear in first derivatives of the components $A_{\beta\gamma\delta}^{\alpha}$ of the first normal tensor and the coefficients are functions of $A_{\beta\gamma\delta}^{\alpha}$. Similarly (4.2) can be replaced by a system of the form

$$\sum T_{i}^{\alpha\beta\gamma\delta\varepsilon}(g_{\alpha\beta};\ g_{\alpha\beta,\gamma\delta})\frac{\partial g_{\alpha\beta,\gamma\delta}}{\partial x^{\varepsilon}} + \ast = 0, \tag{4.4}$$

linear in the first derivatives of the components $g_{\alpha\beta,\gamma\delta}$ of the first metric normal tensor with coefficients which are functions of $g_{\alpha\beta,\gamma\delta}$ and the components $g_{\alpha\beta}$ of the fundamental metric tensor. In the system (4.3) the $\ast$ terms depend on the components $A_{\beta\gamma\delta}^{\alpha}$ and $\Gamma_{\beta\gamma}^{\alpha}$; in (4.4) the $\ast$ terms depend on $g_{\alpha\beta}$, $g_{\alpha\beta,\gamma\delta}$ and $\Gamma_{\beta\gamma}^{\alpha}$ where the components $\Gamma_{\beta\gamma}^{\alpha}$ denote the Christoffel symbols for this latter case. In addition to the conditions (4.3) on the derivatives of the components $A_{\beta\gamma\delta}^{\alpha}$ there exists a system of identical relations

$$\frac{\partial A_{ik}}{\partial x^{\alpha}} = \sum \frac{\partial A_{pq}}{\partial x^{r}} + \ast \qquad \begin{pmatrix} k = 1, \cdots, n-2 \\ i = 1, \cdots, A_k \\ \alpha = 1, \cdots, k \\ q < r \geqq \alpha \end{pmatrix}, \tag{4.5}$$

in which the summation $\sum$ denotes a linear form in the derivatives $\partial A_{\beta\gamma\delta}^{\alpha}/\partial x^{\varepsilon}$ with constant coefficients. Equations (4.5) express the conditions that the quantities $A_{\beta\gamma\delta}^{\alpha}$ should be components of a normal tensor.[12]

We suppose (4.3) to consist of N independent equations; certain of

[11] T. Y. Thomas and A. D. Michal, *Differential invariants of affinely connected manifolds;* Ann. of Math., (2), 28 (1927), p. 196; also ibid., (2), 28 (1927), p. 631.

[12] In the former treatment, loc. cit. Jour. p. 246, the inequality $q \leqq k$ is established and used in place of the inequality $r \geqq \alpha$ appearing in (4.5). To establish this latter inequality we proceed as follows: The system (6.2) Jour. can be written

$$\text{(a)}\quad \frac{\partial A_{jk\alpha}^{i}}{\partial x^{\beta}} = \frac{\partial A_{jk\beta}^{i}}{\partial x^{\alpha}} + \frac{2}{3}\frac{\partial A_{j\beta\alpha}^{i}}{\partial x^{k}} + \frac{1}{3}\frac{\partial A_{\alpha\beta j}^{i}}{\partial x^{k}} + \frac{1}{3}\frac{\partial A_{\beta k\alpha}^{i}}{\partial x^{j}} - \frac{1}{3}\frac{\partial A_{\alpha k\beta}^{i}}{\partial x^{j}} + \ast,$$

where $\alpha = \mu+1$; $k > \alpha > \beta$; $j \leqq k$. The component $A_{jk\beta}^{i}$ for particular values of the indices j, k, α, β belongs to the group $G_{\beta-1}$; since $\alpha \geqq \beta+1$ the first derivative on the right of this particular equation (a) is not the left member of any equation (a). In the previous paper it is shown that $A_{j\beta\alpha}^{i}$ and $A_{\alpha\beta j}^{i}$ contribute components to the groups

these equations may, however, be dependent on the remaining equations of the system (4.3) in consequence of the identities (4.5). These dependent equations which we will assume to be M in number thus possess the property that they are satisfied in consequence of (4.5) and the remaining $N-M$ equations of the system (4.3). If the M dependent equations are excluded, the remainder together with the system of identities (4.5) can be put in the form

$$\frac{\partial A^*_{ik}}{\partial x^\alpha} = \sum (A^*) \frac{\partial A^*_{pq}}{\partial x^r} + \star \qquad \begin{pmatrix} i = 1, \cdots, A^*_k \\ k = 1, \cdots, n \\ \alpha = 1, \cdots, k \\ q < r \geqq \alpha \end{pmatrix}, \tag{4.6}$$

on the basis of the theory of § 3. The notation A^*_{ik} is used in place of A_{ik} to denote a different grouping of the independent components $A^\alpha_{\beta\gamma\delta}$ from that used in (4.5). Together with the system (4.6) we consider the set of equations[13]

$$\frac{\partial \gamma_{lm}}{\partial x^k} = \frac{\partial \gamma_{pq}}{\partial x^r} + \sum A^*_{pq} + \sum \gamma_{uv}\gamma_{pq} \qquad \begin{pmatrix} m = 1, \cdots, n-1 \\ l = 1, \cdots, \gamma_m \\ k = 1, \cdots, m \\ q < r > k \end{pmatrix} \tag{4.7}$$

which define the components A^*_{ik} in terms of the independent components γ_{lm} of the affine connection Γ. Since the inequalities $q < r \geqq \alpha$ and $q < r > k$ are satisfied by the indices of the derivatives in (4.6) and (4.7) respectively, the system composed of (4.6) and (4.7) is regular.

G_μ, $G_{\beta-1}$, G_{j-1} where $j \leqq \mu$. Since k exceeds each of the numbers $\mu, \beta-1$, and $j-1$ the derivatives with respect to x^k which appear on the right of any particular equation (a) cannot appear on the left of any equation (a). From the form of the pth and the qth equations, Jour. p. 246, it is seen that the elimination of the derivative $\partial A^i_{\beta k\alpha}/\partial x^j$ where $j \leqq \mu$ from the right of a particular equation (a) cannot bring in any derivatives for which $r < \alpha$ in (4.5). In order to eliminate the last derivative we must use an equation of the form

$$\frac{\partial A^i_{\alpha k\beta}}{\partial x^j} = \cdots, \tag{b}$$

where $j < \beta$. To eliminate the derivative on the left of the particular equation (b) from the right of the corresponding equation (a) we multiply (b) by a factor and add to (a). From the form of (b) it is seen this could not bring any derivatives into the right of (a) for which $r < \alpha$ in (4.5).

A similar discussion can be made for the system Jour. (6.7). The details will be omitted when we come to the analogous discussion in the treatment of the system (4.4).

In case $n = 2$ equations (4.5) and Jour. (6.7), i. e. the following equations (4.8), are satisfied identically, *loc. cit.* Jour., p. 235.

[13] *loc. cit.* Ann. (2), p. 715.

A corresponding discussion can be made for the system (4.4). In place of the set of identities (4.5) we now have a system of the form

$$\frac{\partial B_{lm}}{\partial x^k} = \sum \frac{\partial B_{pq}}{\partial x^r} + \star \qquad \begin{pmatrix} m = 1, \cdots, n-2 \\ l = 1, \cdots, B_m \\ k = 1, \cdots, m \\ q < r \geqq k \end{pmatrix} \tag{4.8}$$

which expresses the condition that the quantities B_{lm} constitute the components of the second extension of a fundamental metric tensor. The remarks made in connection with the independence of the equations (4.3) and (4.5) apply equally well to the systems (4.4) and (4.8). The $N - M$ independent equations of the system (4.4) together with the equations (4.8) can be put in the form

$$\frac{\partial B^*_{lm}}{\partial x^\alpha} = \sum (g, B^*) \frac{\partial B^*_{pq}}{\partial x^r} + \star \qquad \begin{pmatrix} l = 1, \cdots, B^*_m \\ m = 1, \cdots, n \\ \alpha = 1, \cdots, m \\ q < r \geqq \alpha \end{pmatrix}, \tag{4.9}$$

notation B^*_{lm} being used to denote a regrouping of the components B_{lm}. Equations (4.7) likewise pertain to the present discussion provided that the $\sum A^*_{pq}$ in the right members of these equations be replaced by the corresponding term $\sum B^*_{pq}$ in the components B^*_{pq} of the second extension of the fundamental metric tensor, i. e.

$$\frac{\partial \gamma_{lm}}{\partial x^k} = \frac{\partial \gamma_{pq}}{\partial x^r} + \sum (g) B^*_{pq} + \sum \gamma_{uv} \gamma_{pq} \qquad \begin{pmatrix} m = 1, \cdots, n-1 \\ l = 1, \cdots, \gamma_m \\ k = 1, \cdots, m \\ q < r > k \end{pmatrix}, \tag{4.10}$$

where the coefficients of the quantities B^*_{pq} are rational functions of the components $g_{\alpha\beta}$. We must now add

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = g_{\sigma\beta} \Gamma^\sigma_{\alpha\gamma} + g_{\alpha\sigma} \Gamma^\sigma_{\beta\gamma} \tag{4.11}$$

as the expression of the conditions that the $\Gamma^\alpha_{\beta\gamma}$ be Christoffel symbols with respect to the components $g_{\alpha\beta}$ of some fundamental metric tensor. Then, obviously, the equations (4.9), (4.10) and (4.11) comprise a regular system.

§ 5. Application to Invariantive Systems. Vector Case.

There is a certain analogy between the local coördinates which can be introduced into a space of distant parallelism[14] and the normal coördinates of an affinely connected space: Differentiation of the components of a tensor or set of scalar invariants and evaluation at the origin of a system of local coördinates gives rise to a set of absolute scalar invariants. This

[14] *loc. cit.* Proc. Note I.

was treated in a paper by one of us[14] with particular reference to the case of four dimensions but the theory can be extended immediately to the n dimensional case. Now consider a system of equations of the form

$$T^{\mu\nu\cdots\varepsilon}_{\gamma\delta\cdots\eta}\left(h^i_\alpha;\ \frac{\partial h^i_\alpha}{\partial x^\beta};\ \frac{\partial^2 h^i_\alpha}{\partial x^\beta \partial x^\gamma}\right) = 0, \tag{5.1}$$

where the left members are components of a tensor invariant or a set of scalar invariants linear in the second derivatives of the fundamental vectors h^i_α and depending also on first derivatives and the vectors themselves. By a replacement theorem analogous to that for normal coördinates the system (5.1) is seen to be equivalent to one of the form

$$\sum (h^i_\alpha;\ h^i_{j,k})\frac{\partial h^i_{j,k}}{\partial x^\alpha} + \star = 0, \tag{5.2}$$

where the summation denotes a linear expression in the derivatives of the invariants $h^i_{j,k}$ with coefficients which are functions of the components $h^i_{j,k}$ and h^i_α; the $\star$ terms depend on the $h^i_{j,k}$. In addition there is the set of equations

$$h^i_{j,k,l}+h^i_{k,l,j}+h^i_{l,j,k} = 2\,[h^i_{m,j}\,h^m_{k,l}+h^i_{m,l}\,h^m_{j,k}+h^i_{m,k}\,h^m_{l,j}] \tag{5.3}$$

which expresses the condition that the set of quantities $h^i_{j,k}$ be the invariants arising from a set of fundamental vectors h^i_α. As in § 4 we will suppose that the dependent equations of the system composed of (5.2) and (5.3) have been excluded and the unknowns $h^i_{j,k}$ have been replaced by an independent set K_{lm}.[15] By the theory of § 2 the combined system (5.2) and (5.3) can be put into the form

$$\frac{\partial K^*_{lm}}{\partial x^\alpha} = \sum (h^i_\alpha,\ K^*_{lm})\frac{\partial K^*_{pq}}{\partial x^r} + \star \qquad \begin{pmatrix} m = 1, \cdots, n \\ l = 1, \cdots, K^*_m \\ \alpha = 1, \cdots, m \\ q < r \geqq \alpha \end{pmatrix} \tag{5.4}$$

where K^*_{lm} denotes a regrouping of the independent components K_{lm}. The combination of (5.4) and the equations

$$\frac{\partial h^i_\alpha}{\partial x^\beta} = \frac{\partial h^i_\beta}{\partial x^\alpha} + 2\,h^i_{j,k}\,h^j_\alpha\,h^k_\beta \qquad \begin{pmatrix} \alpha = 2, \cdots, n \\ \beta = 1, \cdots, \alpha-1 \\ i = 1, \cdots, n \end{pmatrix} \tag{5.5}$$

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[15] The rule for separating the independent components $h^i_{j,k}$ into groups, which is given in Proc. Note III for the four dimensional case, can be extended immediately to the case of n dimensions. Rule: *The group G_m $(m = 0, 1, \cdots, n-2)$ for the components $h^i_{j,k}$ is composed of all components that can be formed from $h^i_{j,k}$ by taking $k = m+1$ and $i, j = 1, \cdots, n$ subject to the condition $j > m+1$.* There are $K_m = n^2 - n\,m - n$ components K_{lm} in each group G_m.

which define the set of invariants $h^i_{j,k}$ in terms of the components of the fundamental vectors h^i_α obviously constitute a regular system.

The following example illustrates the above theory by showing how the differential equations for an unrestricted space of distant parallelism can be put into the regular form. In four dimensions the set of invariants $h^i_{3,2}$; $h^i_{4,2}$; $h^i_{4,3}$; $h^i_{2,1}$; $h^i_{3,1}$; $h^i_{4,1}$ are independent, i. e. not connected by a linear relation; when the system of identities (5.3) is referred to these invariants and their derivatives, the matrix of the coefficients of the derivatives is that exhibited in the adjacent table in which

$$\alpha = h^1_1, \quad \beta = h^1_2, \quad \gamma = h^1_3, \quad \delta = h^1_4, \quad a = h^2_1, \quad b = h^2_2$$

j, k, l	$\frac{\partial h^i_{3,2}}{\partial x^1}$	$\frac{\partial h^i_{4,2}}{\partial x^1}$	$\frac{\partial h^i_{4,3}}{\partial x^1}$	$\frac{\partial h^i_{2,1}}{\partial x^1}$	$\frac{\partial h^i_{3,1}}{\partial x^1}$	$\frac{\partial h^i_{4,1}}{\partial x^1}$	$\frac{\partial h^i_{4,3}}{\partial x^2}$
1, 2, 3	$-\alpha$			$-\gamma$	β		
1, 2, 4		$-\alpha$		$-\delta$		β	
1, 3, 4			$-\alpha$		$-\delta$	γ	$-a$
2, 3, 4	$-\delta$	γ	$-\beta$				$-b$

in the contravariant components h^α_i. Rows in this table correspond to equation (5.3) for which the indices j, k, l have the values indicated at the left. Each element in the table is the coefficient of the derivative at the top of the column in which the element appears. The matrix formed by the elements of the first six columns of the table is of rank three. If α and $\alpha b - \beta a$ do not vanish the system (5.3) can be put into the form

$$5\,(a) \qquad \frac{\partial K_{lm}}{\partial x^\alpha} = \sum R(h^i_\alpha)\frac{\partial K_{pq}}{\partial x^r} + \sum R(h^i_\alpha) K^2 \qquad \begin{pmatrix} m = 1, 2 \\ l = 1, \cdots, K_m \\ \alpha = 1, \cdots, m \end{pmatrix}$$

where $R(h^i_\alpha)$ denotes a rational function of the quantities h^i_α and where K_{11} is composed of $h^i_{3,2}$ and $h^i_{4,2}$ and the group K_{12} is composed of $h^i_{4,3}$. [See, Proc. Note III (2.3).] The system composed of (5a) and (5.5) is regular since we have solved (5.3) for the maximum number of derivatives with respect to x^1, namely 12.

It will be proved in § 8 that this latter system is completely integrable and hence by a proper assignment of arbitrary functions in accordance with the general existence theorems of § 6, this system of equations will completely determine the quantities $h^i_{j,k}$ as the scalar invariants arising from a fundamental set of vectors h^i_α.

§ 6. General Existence Theorems.

We shall now impose on the system (2.1) the following two fundamental restrictions.

CONDITION I. *The coefficients a^α_{ik} and c_i in* (2.1) *are analytic functions in the neighborhood of some set of values $x^i = q^i$ and $v_k = (v_k)_q$ of their arguments.*

CONDITION II. *The regular system* (2.8) *is completely integrable.*

The first condition carries with it the consequence that the coefficients (x, v) and the $\star$ terms in (2.8) are analytic functions in the neighborhood of some set of values p^i and $(v_{ik})_p$, these being values in the neighborhood

of q^i and $(v_k)_q$. A system of equations will be said to be completely integrable if the integrability conditions resulting from one differentiation of the system are satisfied identically in the parametric derivatives.[16] It follows from the work of Meray and Riquier[17] that if Conditions I and II are satisfied, the system (2.1) has a unique solution, given by a set of convergent power series expansions, corresponding to the arbitrary assignment of analytic data predicted by the form of the left members of the system (2.8).

EXISTENCE THEOREM. *Suppose that* (2.1) *is a system such that Conditions* I *and* II *are satisfied. Let*

$$\varphi_{ik}(x^{k+1}, \cdots, x^n), \qquad (i = 1, \cdots, w_k),$$

where $k \neq n$ *be an arbitrary function of the variables* $x^{k+1}, \cdots, x^n$ *analytic in the neighborhood of* $x^i = p^i$ *of their arguments, such that* $\varphi_{ik}(p) = (v_{ik})_p$ *for all values of the indices for which the* φ_{ik} *are defined. Then there exists one and only one solution,* $v_{ik}(x)$, *of* (2.1), *each function* $v_{ik}(x)$ *being analytic in the neighborhood of the set of values* $x^i = p^i$, (1) *such that* $v_{in}(p) = (v_{in})_p$ *for* $i = 1, \cdots, w_n$ *and* (2) *such that*

$$v_{i0}(x^1, \cdots, x^n) = \varphi_{i0}(x^1, \cdots, x^n) \quad (i = 1, \cdots, w_0)$$

$$v_{ik}(x^{k+1}, \cdots, x^n) = \varphi_{ik}(x^{k+1}, \cdots, x^n) \quad \left[\begin{matrix} i = 1, \cdots, w_k \\ x^1 = p^1, \cdots, x^k = p^k \end{matrix}\right].$$

With regard to the system (4.3) we shall set up the conditions

CONDITION $\mathrm{I_A}$. *The coefficients* $T(A)$ *and the* ✱ *terms in* (4.3) *are analytic functions in the neighborhood of some set of values* $(A_{lm})_q$ *of their arguments.*

CONDITION $\mathrm{II_A}$. *The regular system composed of* (4.6) *and* (4.7) *is completely integrable.*

Condition I carries with it the consequence that the coefficients and the ✱ terms in (4.6) will be analytic functions in the neighborhood of some set of values $(A^*_{lm})_p$, this being some set of values in the neighborhood of the set of values $(A_{lm})_q$.

EXISTENCE THEOREM. *Suppose that* (4.3) *is such that the conditions* $\mathrm{I_A}$ *and* $\mathrm{II_A}$ *are satisfied. Let*

$$\varphi_{ik}(x^{k+1}, \cdots, x^n), \qquad \left[\begin{matrix} 0 \leqq k \leqq n-1 \\ i = 1, \cdots, A^*_k \end{matrix}\right],$$

where the indices i, k *have the same range of values as the indices of the components* A^*_{ik} $(k \neq n)$, *and*

[16] The usual definition of a principle derivative is understood here, namely one which can be obtained by differentiation of a left member of the system, or one of the left members themselves. All other derivatives are parametric.

[17] Meray and Riquier, *loc. cit.*, (1890).

$$\psi_{ik}(x^{k+1}, \cdots, x^n), \qquad \begin{bmatrix} k = 0, \cdots, n-1 \\ i = 1, \cdots, \gamma_k \end{bmatrix}$$

denote functions of the variables $x^{k+1}, \cdots, x^n$ *analytic in the neighborhood of the set of values* $x^i = p^i$ *of their arguments, such that* $\varphi_{ik}(p) = (A^*_{ik})_p$ *for all values of the indices* i *and* k *for which the* φ_{ik} *are defined. Then there exists one and only one affine connection with components* $\Gamma^\alpha_{\beta\gamma} (= \Gamma^\alpha_{\gamma\beta})$ *in a system of coördinates* x^α, *each function* $\Gamma^\alpha_{\beta\gamma}(x)$ *being analytic in the neighborhood of the set of values* $x^i = p^i$, *which constitutes a set of integrals of the system of equations* (4.3) *and which is* (1) *such that* $A^*_{in}(p) = (A^*_{in})_p$ *and* (2) *such that*

$$A^*_{i0} = \varphi_{i0}(x^1, \cdots, x^n) \quad [i = 1, \cdots, A^*_0] \qquad \gamma_{l0} = \psi_{l0}(x^1, \cdots, x^n) \quad [l = 1, \cdots, \gamma_0]$$

$$A^*_{ik} = \varphi_{ik}(x^{k+1}, \cdots, x^n) \quad \begin{bmatrix} k = 1, \cdots, n-1 \\ i = 1, \cdots, A^*_k \\ x^1 = p^1, \cdots, x^k = p^k \end{bmatrix} \qquad \gamma_{lm} = \psi_{lm}(x^{m+1}, \cdots, x^n) \quad \begin{bmatrix} m = 1, \cdots, n-1 \\ l = 1, \cdots, \gamma_m \\ x^1 = p^1, \cdots, x^m = p^m \end{bmatrix}.$$

The corresponding conditions and existence theorems for the system (4.4) can be stated immediately.

CONDITION I_G. *The coefficients* $T(g, B)$ *and the* $\star$ *terms in* (4.4) *are analytic functions in the neighborhood of some set of values* $(B_{lm})_q$ *and* $(g_{\alpha\beta})_q = (g_{\beta\alpha})_q$ *of their arguments, the determinant* $|(g_{\alpha\beta})_q|$ *being different from zero.*

CONDITION II_G. *The regular system of equations* (4.9), (4.10) *and* (4.11) *is completely integrable.*

As a consequence of Condition I_G the coefficients and the $\star$ terms in (4.9) will be analytic functions in the neighborhood of some set of values $(g_{\alpha\beta})_p$ and $(B^*_{lm})_p$ lying in the neighborhood of the values $(g_{\alpha\beta})_q$ and $(B_{lm})_q$ respectively, such that $(g_{\alpha\beta})_p = (g_{\beta\alpha})_p$ and also such that the determinant $|(g_{\alpha\beta})_p|$ does not vanish.

EXISTENCE THEOREM. *Suppose that* (4.4) *is such that Conditions* I_G *and* II_G *are satisfied. Let*

$$\varphi_{ik}(x^{k+1}, \cdots, x^n), \qquad \begin{bmatrix} 0 \leqq k \leqq n-1 \\ i = 1, \cdots, B^*_k \end{bmatrix},$$

where the indices i, k *have the same ranges of values as the indices of the components* B^*_{ik} $(k \neq n)$ *and*

$$\psi_{ik}(x^{k+1}, \cdots, x^n) \qquad \begin{bmatrix} k = 0, \cdots, n-1 \\ i = 1, \cdots, \gamma_k \end{bmatrix}$$

denote functions of the variables $x^{k+1}, \cdots, x^n$ *analytic in the neighborhood of the set of values* $x^i = p^i$ *of their arguments, such that* $\varphi_{ik}(p) = (B^*_{ik})_p$ *for all values of the indices for which the* φ_{ik} *are defined. Then there exists one, and only one, fundamental metric tensor with components* $g_{\alpha\beta}\,(= g_{\beta\alpha})$ *in a system of coördinates* x^α *each function* $g_{\alpha\beta}(x)$ *being analytic in the neighborhood of the point* $x^\alpha = p^\alpha$ *which constitutes a set of integrals of the system* (4.4) *and which* (1) *is such that* $g_{\alpha\beta}(p) = (g_{\alpha\beta})_p$ *and* $B^*_{in}(p) = (B^*_{in})_p$ *and* (2) *such that*

$$B^*_{i0} = \varphi_{i0}(x^1, \cdots, x^n) \quad [i = 1, \cdots, B^*_0] \qquad \gamma_{l0} = \psi_{l0}(x^1, \cdots, x^n) \quad [l = 1, \cdots, \gamma_0]$$

$$B^*_{ik} = \varphi_{ik}(x^{k+1}, \cdots, x^n) \quad \begin{bmatrix} k = 1, \cdots, n-1 \\ i = 1, \cdots, B^*_k \\ x^1 = p^1, \cdots, x^k = p^k \end{bmatrix} \qquad \gamma_{lm} = \psi_{lm}(x^{m+1}, \cdots, x^n) \quad \begin{bmatrix} m = 1, \cdots, n-1 \\ i = 1, \cdots, \gamma_m \\ x^1 = p^1, \cdots, x^m = p^m \end{bmatrix}.$$

On the basis of the discussion in § 5, the corresponding existence theorem for spaces of distant parallelism can be stated.

CONDITION $\mathrm{I_H}$. *The coefficients* $(h^i_\alpha,\ h^i_{j,k})$ *and the* ✱ *terms in the system* (5.2) *are analytic functions in neighborhood of some set values* $(h^i_{j,k})_q\,[= -(h^i_{k,j})_q]$ *and* $(h^i_\alpha)_q$ *of their arguments, such that the determinant* $|(h^i_\alpha)_q|$ *does not vanish.*

CONDITION $\mathrm{II_H}$. *The regular system composed of* (5.4) *and* (5.5) *is completely integrable.*

Condition $\mathrm{I_H}$ implies that the coefficients $(h^i_\alpha,\ K^*_{lm})$ and the ✱ terms in the system (5.4) are analytic functions in the neighborhood some set of values $(h^i_\alpha)_p$ and $(K^*_{lm})_p$ in the neighborhood of the values $(h^i_\alpha)_q$ and $(h^i_{j,k})_q$ such that the determinant $|(h^i_\alpha)_p|$ does not vanish.

EXISTENCE THEOREM. *Suppose that* (5.2) *is such that conditions* $\mathrm{I_H}$ *and* $\mathrm{II_H}$ *are satisfied. Let*

$$\varphi_{lm}(x^{m+1}, \cdots, x^n), \qquad \begin{bmatrix} 0 \leqq m \leqq n-1 \\ l = 1, \cdots, K^*_m \end{bmatrix},$$

where l, m *have the same range of values as the indices of the components* K^*_{lm} $(m \neq n)$, *and*

$$\psi_{i\alpha}(x^\alpha, \cdots, x^n) \qquad \begin{bmatrix} \alpha = 1, \cdots, n \\ i = 1, \cdots, n \end{bmatrix}$$

denote functions of the variables $x^{m+1}, \cdots, x^n$ *or* $x^\alpha, \cdots, x^n$ *as the case may be, analytic in the neighborhood of the set of values* $x^\alpha = p^\alpha$ *of their arguments such that* $\varphi_{lm}(p) = (K^*_{lm})_p$ *and* $\psi_{i\alpha}(p) = (h^i_\alpha)_p$ *for all values of the indices for which the functions* φ_{lm} *and* $\psi_{i\alpha}$ *are defined. Then there exists one and only one set of fundamental vectors with components* h^i_α *in*

a system of coördinates x^α *each function* $h^i_\alpha(x)$ *being analytic in the neighborhood of the set of values* $x^\alpha = p^\alpha$ *which constitutes a set of integrals of the system* (5.2) *and which is* (1) *such that* $K^*_{ln}(p) = (K^*_{ln})_p$ *and* (2) *such that*

$h^i_1 = \psi_{i1}(x^1, \cdots, x^n)$ $[i = 1, \cdots, n]$	$K^*_{l0} = \varphi_{l0}(x^1, \cdots, x^n)$ $[l = 1, \cdots, K^*_0]$
$h^i_\alpha = \psi_{i\alpha}(x^\alpha, \cdots, x^n)$ $\left[\begin{array}{c} i = 1, \cdots, n \\ \alpha = 2, \cdots, n \\ x^1 = p^1, \cdots, x^{\alpha-1} = p^{\alpha-1} \end{array}\right]$	$K^*_{lm} = \varphi_{lm}(x^{m+1}, \cdots, x^n)$ $\left[\begin{array}{c} l = 1, \cdots, K^*_m \\ m = 1, \cdots, n-1 \\ x^1 = p^1, \cdots, x^m = p^m \end{array}\right].$

§ 7. Functional Systems.

Consider a system of partial differential equations of the form

$$T^{\mu\cdots\nu}_{\varepsilon\cdots\vartheta}\left(\Gamma^\alpha_{\beta\gamma}; \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\vartheta}\right) = 0, \tag{7.1}$$

where the T's are components of a tensor invariant of the type discussed in § 4, depending on the components $\Gamma^\alpha_{\beta\gamma}$ and their first derivatives. Along with this consider the invariantive system

$$T^{\mu\cdots\nu}_{\varepsilon\cdots\vartheta}\left(g_{\alpha\beta}; \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}; \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\vartheta}\right) = 0, \tag{7.2}$$

where the components T depend on the fundamental metric tensor $g_{\alpha\beta}$ and their first and second derivatives. By the replacement theorem equations (7.1) can be written in terms of the independent components A_{lm} of the affine normal tensor. Hence (7.1) takes the form

$$T(A_{lm}) = 0. \tag{7.3}$$

Assume that the equations (7.3) are independent and N in number; dependent equations are to be discarded from the system.

CONDITION $\mathrm{III}^*_\mathrm{A}$. *Equations* (7.3) *are such that* (1) *there exists a solution* $(A_{lm})_p$ *of these equations and* (2) *the components* $T(A_{lm})$ *of these equations are analytic functions in the neighborhood of the set of values* $(A_{lm})_p$.

By covariant differentiation of (7.3) we obtain a system of the type considered in § 4, namely

$$\frac{\partial T}{\partial A_{lm}} \frac{\partial A_{lm}}{\partial x^\alpha} = \bigstar, \tag{7.4}$$

where the $\bigstar$ indicates a bilinear form in the components T and $\Gamma^\alpha_{\beta\gamma}$. Applying the results of § 4, the system (7.4) together with the identities (4.5) can be replaced by equations of the form

$$\frac{\partial A^*_{ik}}{\partial x^\alpha} = \sum (A^*) \frac{\partial A^*_{pq}}{\partial x^r} + \ast, \qquad \begin{pmatrix} i = 1, \cdots, A^*_k \\ k = 1, \cdots, n \\ \alpha = 1, \cdots, k \\ q < r \geqq \alpha \end{pmatrix}. \tag{7.5}$$

where the A^*_{lm} represent independent components A^i_{jkl}. Corresponding to the previous Condition I_A we now impose the following

CONDITION I^*_A. *The coefficients* (A^*) *and the* $\ast$ *terms in* (7.5) *are analytic functions in the neighborhood of the set of values* $(A_{lm})_p$.

Assuming that (7.5) and (4.7) are completely integrable, these equations can be integrated in accordance with the general existence theorem stated in § 6. In order to see that the functions A^*_{lm} which are formed from the integrals $\Gamma^\alpha_{\beta\gamma}(x)$ of the system (7.5) by the process of calculating the components of the first normal tensor, satisfy (7.1) or the equivalent system (7.3) we make use of Condition $\mathrm{III}^*_\mathrm{A}$. The functions $A^*_{lm}(x)$ are analytic in the neighborhood of $x^i = p^i$ and hence, by Condition $\mathrm{III}^*_\mathrm{A}$, the components $T(A)$ are analytic, as functions of the x^i, in the neighborhood of the same point. Thus the components T admit power series expansions about the point $x^i = p^i$, the constant terms of which vanish by Condition $\mathrm{III}^*_\mathrm{A}$. On account of equations (7.4) the remainder of the coefficients in these expansions must vanish likewise, and hence the components T are identically zero as functions of the coördinates x^i.

In an analogous manner the system (7.2) can be replaced by equations of the form

$$T(g_{\alpha\beta};\ B_{lm}) = 0, \tag{7.6}$$

where the B_{lm} are the independent components of the first metric normal tensor $g_{\alpha\beta,\gamma\delta}$.

CONDITION $\mathrm{III}^*_\mathrm{G}$. *There exists a solution* $(g_{\alpha\beta})_p = (g_{\beta\alpha})_p$ *and* $(B_{lm})_p$ *of equations* (7.6) *such that* (1) *the determinant* $|(g_{\alpha\beta})_p|$ *does not vanish and* (2) *the components* T *are analytic functions in the neighborhood of the values* $(g_{\alpha\beta})_p$ *and* $(B_{lm})_p$.

By covariant differentiation of (7.6) we obtain

$$\frac{\partial T}{\partial B_{lm}} \frac{\partial B_{lm}}{\partial x^\alpha} = \ast, \tag{7.7}$$

where the $\ast$ denotes a bilinear form in the components T and the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$, and a linear form in the first derivatives of the components $g_{\alpha\beta}$ with coefficients which are functions of the $g_{\alpha\beta}$ and B_{lm}. Following the method of § 4 the system (7.7) is combined with the identities (4.8) to give a system of the form (4.9).

CONDITION I_G^*. *The coefficients* (g, B^*) *and the* $\star$ *terms in* (4.9) *are analytic functions in the neighborhood of the set of values* $(g_{\alpha\beta})_p$ *and* $(B_{lm})_p$ *of Condition* III_G^*.

Furthermore it can be shown that the integrals $g_{\alpha\beta}(x)$ of the system (4.9) actually satisfy equations (7.6) in case Conditions I_G^* and III_G^* are satisfied. On the basis of the results of § 5 an analogous treatment can be given for the invariantive system

$$T_{\varepsilon\ldots\delta}^{\mu\ldots\nu}\left(h_\alpha^i;\ \frac{\partial h_\alpha^i}{\partial x^\beta}\right) = 0.$$

As an example illustrating the theory consider Einstein's gravitational equations for free space

(7 a) $$R_{\alpha\beta} = 0$$

which express the fact that the contracted curvature tensor is equal to zero. In a previous paper by T. Y. Thomas, *On the Existence of Integrals of Einstein's Gravitational Equations for free space and their Extension to n-variables*, Proc. Nat. Acad. Sci., 15, (1929), p. 906, the problem of the existence of integrals for the system of equations (7 a) has been treated when the components of the fundamental metric tensor $g_{\alpha\beta}$ have the initial values $\pm\delta_\beta^\alpha$. In the present section we shall consider the problem in four dimensions without restricting the initial values of the components $g_{\alpha\beta}$; the coördinates x^i are to be arbitrary. The work of putting the system (7 a) in regular form will be carried out in detail.

The system (7 a) is completely equivalent to one of the form

(7 b) $$g^{\gamma\delta}\, g_{\alpha\beta,\,\gamma\delta} = 0$$

which corresponds to the system (7.6), and by covariant differentiation, we obtain

(7 c) $$g^{\gamma\delta}\, g_{\alpha\beta,\,\gamma\delta,\,\varepsilon} = 0.$$

The system of identities which furnish the conditions that the $g_{\alpha\beta,\,\gamma\delta}$ be the components of the first metric normal tensor formed from some fundamental tensor with components $g_{\alpha\beta}$ are of the form

(7 d) $$g_{\alpha\beta,\,\gamma\delta,\,\varepsilon} = \frac{1}{2}\left[g_{\varepsilon\beta,\,\gamma\delta,\,\alpha} + g_{\alpha\varepsilon,\,\gamma\delta,\,\beta} + g_{\alpha\beta,\,\varepsilon\delta,\,\gamma} + g_{\alpha\beta,\,\gamma\varepsilon,\,\delta}\right].$$

We shall write out in detail the solved form of these equations since it will be needed in the following work.

Let us adopt the following more convenient notation

$$\begin{array}{lllll} \alpha = g^{11}, & \beta = g^{22}, & \gamma = g^{33}, & \delta = g^{44}, & \\ a = g^{12}, & b = g^{13}, & c = g^{14}, & d = g^{23}, & e = g^{24}, \qquad f = g^{34}, \end{array}$$

with the understanding that $g^{\alpha\beta}$ is symmetric in α and β; also

$$\begin{array}{lll} u_1 = g_{11;22}, & v_1 = g_{22;33}, & w_1 = g_{13;24}, \\ u_2 = g_{11;23}, & v_2 = g_{12;33}, & w_2 = g_{14;33}, \\ u_3 = g_{11;24}, & v_3 = g_{13;22}, & w_3 = g_{13;44}, \\ u_4 = g_{11;33}, & v_4 = g_{14;22}, & w_4 = g_{24;33}, \\ u_5 = g_{11;34}, & v_5 = g_{12;44}, & w_5 = g_{23;44}, \\ u_6 = g_{11;44}, & v_6 = g_{12;34}, & w_6 = g_{33;44}, \\ & v_7 = g_{22;34}, & \\ & v_8 = g_{22;44}, & \end{array}$$

where the components u^α constitute the group G_0, v_α constitute G_1 and w_α constitute G_2 according to the rule for grouping independent components given in a former paper *loc. cit.*, Jour. p. 248. Then the system (7 d) takes the form

$$(1)\qquad \frac{\partial v_1}{\partial x^1} = \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3} + \ast,$$

$$(2)\qquad \frac{\partial v_2}{\partial x^1} = \frac{\partial u_4}{\partial x^2} - \frac{\partial u_3}{\partial x^3} + \ast,$$

$$(3)\qquad \frac{\partial v_3}{\partial x^1} = \frac{\partial u_1}{\partial x^3} - \frac{\partial u_2}{\partial x^2} + \ast,$$

$$(4)\qquad \frac{\partial v_4}{\partial x^1} = \frac{\partial u_1}{\partial x^4} - \frac{\partial u_3}{\partial x^2} + \ast,$$

$$(5)\qquad \frac{\partial v_5}{\partial x^1} = \frac{\partial u_6}{\partial x^2} - \frac{\partial u_3}{\partial x^4} + \ast,$$

$$(6)\qquad \frac{\partial v_6}{\partial x^1} = \frac{\partial u_5}{\partial x^2} - \frac{1}{2}\frac{\partial u_3}{\partial x^3} - \frac{1}{2}\frac{\partial u_2}{\partial x^4} + \ast,$$

$$(7)\qquad \frac{\partial v_7}{\partial x^1} = \frac{\partial v_6}{\partial x^2} + \frac{1}{2}\frac{\partial v_4}{\partial x^3} + \frac{1}{2}\frac{\partial v_3}{\partial x^4} + \ast,$$

$$(8)\qquad \frac{\partial v_8}{\partial x^1} = \frac{\partial v_5}{\partial x^2} + \frac{\partial v_4}{\partial x^4} + \ast,$$

$$(9)\qquad \frac{\partial w_1}{\partial x^1} = \frac{\partial u_3}{\partial x^3} - \frac{1}{2}\frac{\partial u_5}{\partial x^2} - \frac{1}{2}\frac{\partial u_2}{\partial x^4} + \ast,$$

$$(10)\qquad \frac{\partial w_2}{\partial x^1} = \frac{\partial u_4}{\partial x^4} - \frac{\partial u_5}{\partial x^3} + \ast,$$

$$(11)\qquad \frac{\partial w_3}{\partial x^1} = \frac{\partial u_6}{\partial x^3} - \frac{\partial u_5}{\partial x^4} + \ast,$$

$$(12)\qquad \frac{\partial w_4}{\partial x^1} = \frac{\partial v_2}{\partial x^4} + \frac{2}{3}\frac{\partial w_1}{\partial x^3} - \frac{2}{3}\frac{\partial v_6}{\partial x^3} + \ast,$$

$$(13)\qquad \frac{\partial w_5}{\partial x^1} = \frac{\partial v_5}{\partial x^3} - \frac{2}{3}\frac{\partial w_1}{\partial x^4} - \frac{4}{3}\frac{\partial v_6}{\partial x^4} + \ast,$$

$$(14)\qquad \frac{\partial w_6}{\partial x^1} = \frac{\partial w_3}{\partial x^3} - \frac{\partial w_2}{\partial x^4} + \ast,$$

$$(15)\qquad \frac{\partial w_1}{\partial x^2} = -\frac{3}{4}\frac{\partial v_4}{\partial x^3} + \frac{3}{4}\frac{\partial v_3}{\partial x^4} - \frac{1}{2}\frac{\partial v_6}{\partial x^2} + \ast,$$

$$(16)\qquad \frac{\partial w_2}{\partial x^2} = \frac{\partial v_3}{\partial x^4} - \frac{4}{3}\frac{\partial v_6}{\partial x^3} - \frac{2}{3}\frac{\partial w_1}{\partial x^3} + \ast,$$

$$(17)\qquad \frac{\partial w_3}{\partial x^2} = \frac{\partial v_5}{\partial x^3} + \frac{2}{3}\frac{\partial w_1}{\partial x^4} - \frac{2}{3}\frac{\partial v_6}{\partial x^4} + \ast,$$

$$(18)\qquad \frac{\partial w_4}{\partial x^2} = \frac{\partial v_1}{\partial x^4} - \frac{\partial v_7}{\partial x^3} + \ast,$$

$$(19)\qquad \frac{\partial w_5}{\partial x^2} = \frac{\partial v_8}{\partial x^3} - \frac{\partial v_7}{\partial x^4} + \ast,$$

$$(20)\qquad \frac{\partial w_6}{\partial x^2} = \frac{\partial w_5}{\partial x^3} + \frac{\partial w_4}{\partial x^4} + \ast.$$

It is to be noticed that the ranges of indices and the inequalities given for (4.8) hold for the above system of equations.

ε	X_{111}	X_{121}	X_{131}	X_{141}	X_{221}	X_{231}	X_{241}	X_{331}	X_{341}	X_{441}
1	α				$-\beta$	$-2d$	$-2e$	$-\gamma$	$-2f$	$-\delta$
2		2α			$2a$	$2b$	$2c$			
3			2α			$2a$		$2b$	$2c$	
4				2α			$2a$		$2b$	$2c$

Table I.

There are 40 equations in the system (7 c), but not all of these equations are independent. In fact, if we multiply through by $g^{\alpha\beta}\, g^{\gamma\delta}$ and sum on the indices $\alpha, \beta, \gamma, \delta$ we find

$$2 g^{\alpha\beta} X_{\varepsilon\alpha\beta} = g^{\alpha\beta} X_{\alpha\beta\varepsilon}, \tag{7e}$$

where $X_{\alpha\beta\varepsilon}$ has been used to denote the left member of (7 c). In Table I the matrix of the coefficients of the terms containing the quantities $X_{\alpha\beta 1}$ in (7 e) is exhibited; provided g^{11} is not zero we see that the system (7 e) can be solved for the quantities $X_{1\beta 1}$. Hence the four equations

$$X_{1\beta 1} = 0$$

will be omitted from the system (7 c).

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	α, β	u_1	u_2	u_3	u_4	u_5	u_6	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	w_1	w_2	w_3	w_4	w_5	w_6
1	2, 2	α						γ		$2b$	$2c$			$2f$	δ						
2	2, 3		α					$-d$	$-b$	$-a$			$-2c$	$-e$		$-2c$			$-f$	δ	
3	2, 4			α							$-a$	$-c$		$-d$	$-e$	$2b$			γ	$-f$	
4	3, 3				α			β	$2a$								$2c$		$2e$		δ
5	3, 4					α							$2a$	β			$-b$	$-c$	$-d$	$-e$	$-f$
6	4, 4						α					$2a$			β			$2b$		$2d$	γ
7	1, 1	β	$2d$	$2e$	γ	$2f$	δ														
8	1, 2	$-a$	$-b$	$-c$				γ	$-d$	$-e$	δ	$2f$									
9	1, 3		$-a$		$-b$	$-c$		$-d$	β							$2e$	$-f$	δ			
10	1, 4			$-a$		$-b$	$-c$				β	$-e$	$-2d$			$-2d$	γ	$-f$			

Table II.

If we write equations (7 c) in the form

$$g^{\gamma\delta} \frac{\partial g_{\alpha\beta,\gamma\delta}}{\partial x^{\varepsilon}} = \star$$

for a fixed value of ε the matrix of the coefficients of the derivatives $\partial g_{\alpha\beta,\gamma\delta}/\partial x^{\varepsilon}$ is exhibited in Table II; in this table the indices α, β give the row, and any element is the coefficient of the derivative of the unknown at the top of the column in which it appears When $\varepsilon = 1$ only the first six rows are to be considered, and this set of equations can be solved for the derivatives $\partial u_{\beta}/\partial x^1$ provided α is not zero. When $\varepsilon = 2, 3, 4$ it will be possible to solve the corresponding sets of equations taken from (7 c) for the derivatives $\partial u_{\sigma}/\partial x^{\varepsilon}$ and $\partial v_{\tau}/\partial x^{\varepsilon}$ where $\sigma = 1, \cdots, 6$ and $\tau = 1, 2, 3, 4$ provided the determinant

formed from the first ten columns of Table II does not vanish. Expanding this determinant in terms of fourth order minors formed from rows 3, 5, 6, 10 we see that it can be factored into $\alpha^2(\alpha\beta - a^2)$ and a sixth order determinant indicated in Table III, no regard being paid to algebraic sign.

When use is made of Laplace's expansion in terms of third order minors it is found that the determinant of Table III can be written

	$-\alpha$	$-\beta$			$2a$
$-\alpha$		$-\gamma$		$2b$	
$-\beta$	$-\gamma$		$2d$		
		d	α	$-a$	$-b$
	b		$-a$	β	$-d$
a			$-b$	$-d$	γ

Table III.

$$-2\begin{vmatrix} \alpha & a & b \\ a & \beta & d \\ b & d & \gamma \end{vmatrix}^2.$$

Hence if the quantity

$$2(g^{11})^2 \begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{vmatrix} \begin{vmatrix} g^{11} & g^{12} & g^{18} \\ g^{21} & g^{22} & g^{28} \\ g^{31} & g^{32} & g^{33} \end{vmatrix}^2 \tag{21}$$

does not vanish the system of equations (7c), excluding the dependent equations $X_{1\beta 1} = 0$ can be written in the form

$$\frac{\partial u_\alpha}{\partial x^1} = \sum(g)\frac{\partial v_\beta}{\partial x^1} + \sum(g)\frac{\partial w_\gamma}{\partial x^1} + \star, \tag{7f}$$

$$\left.\begin{matrix} \dfrac{\partial u_\alpha}{\partial x^2} \\ \dfrac{\partial v_\delta}{\partial x^2} \end{matrix}\right\} = \sum(g)\frac{\partial v_\varepsilon}{\partial x^2} + \sum(g)\frac{\partial w_\gamma}{\partial x^2} + \star, \tag{7g}$$

$$\left.\begin{matrix} \dfrac{\partial u_\alpha}{\partial x^3} \\ \dfrac{\partial v_\delta}{\partial x^3} \end{matrix}\right\} = \sum(g)\frac{\partial v_\varepsilon}{\partial x^3} + \sum(g)\frac{\partial w_\gamma}{\partial x^3} + \star, \tag{7h}$$

$$\left.\begin{matrix} \dfrac{\partial u_\alpha}{\partial x^4} \\ \dfrac{\partial v_\delta}{\partial x^4} \end{matrix}\right\} = \sum(g)\frac{\partial v_\varepsilon}{\partial x^4} + \sum(g)\frac{\partial w_\gamma}{\partial x^4} + \star, \tag{7i}$$

where $\alpha, \gamma = 1, \cdots, 6$; $\beta = 1, \cdots, 8$; $\delta = 1, \cdots, 4$; $\varepsilon = 5, \cdots, 8$. Equations (1) to (14) can be used to eliminate their left members from the right hand members of (7f) and from the resulting equations eliminate the left members of (7g), (7h), (7i). Thus (7f) is replaced by a system (F) having no derivatives of $g_{\alpha\beta,\gamma\delta}$ with respect to x^1 on the right and also no left members of the systems (7g), (7h), (7i). Equations (15) to (20) can be used to eliminate their left members from the right members of (F) and (7g); then use equations (7h) and (7i) to eliminate their left members from the resulting equations. Thus (F) and (7g) are replaced by systems (φ) and (G) respectively such that the system composed of (φ), (G), (7h), (7i) is solved for its left members and contains no left members of equations (1) to (20) on the right. Hence the systems (φ), (G), (7h), (7i) can be used to eliminate their left members from the right hand sides of equations (1) to (20) and we have the

combined system (7c) and (7d) solved for as many derivatives as there are independent equations in the system. Furthermore adopting the notation

$$B_{i4}^* \sim u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6,\ v_1,\ v_2,\ v_3,\ v_4;$$
$$B_{i2}^* \sim w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6;$$
$$B_{i1}^* \sim v_5,\ v_6,\ v_7,\ v_8$$

the combined system (7c) and (7d) can be replaced by one of the form

$$\frac{\partial B_{lm}^*}{\partial x^\alpha} = \sum (g) \frac{\partial B_{pq}^*}{\partial x^r} + \star, \qquad \begin{pmatrix} m = 1, 2, 4 \\ \alpha = 1, \cdots, m \\ l = 1, \cdots, B_m^* \\ r > q \end{pmatrix}, \tag{7j}$$

where $B_1^* = 4$, $B_2^* = 6$, $B_4^* = 10$. It is also evident from the method used in eliminating the left members from the right hand sides of the two systems that the inequality $r \geqq \alpha$ is satisfied and hence the system composed of (4.10), (4.11) and (7j) is regular.

§ 8. Identities. Conditions for Complete Integrability.

A system of partial differential equations of the form (2.1) can be replaced by a system of the form

$$X_{ikj} \equiv \frac{\partial v_{ik}}{\partial x^j} - F_{ikj}^{pqr} \frac{\partial v_{pq}}{\partial x^r} + \star = 0, \qquad \begin{pmatrix} k = 1, \cdots, n \\ i = 1, \cdots, w_k \\ j = 1, \cdots, k \end{pmatrix} \tag{8.1}$$

provided the coördinate system is non-singular. The index r is subject to the conditions $q < r \geqq j$. It must therefore be possible to put equations (2.1) in the form

$$Z_\alpha \equiv b_\alpha^{ikj} \frac{\partial v_{ik}}{\partial x^j} - b_\alpha^{ikj} F_{ikj}^{pqr} \frac{\partial v_{pq}}{\partial x^r} + \star = 0, \quad (\alpha = 1, \cdots, L), \tag{8.2}$$

where $L = \sum k w_k$ and the matrix of the quantities $b_\alpha^{ikj}(x, v)$ is non-singular, i. e. the Z_α are identically equal to the left members of (2.1). If the system (8.1) is completely integrable the set of equations

$$\frac{\partial X_{ikj}}{\partial x^l} - \frac{\partial X_{ikl}}{\partial x^j} = 0 \qquad \begin{pmatrix} l, j \leqq k \\ l < j \end{pmatrix} \tag{8.3}$$

must be satisfied identically when the left members are expressed in terms of the parametric derivatives determined by (8.1), i. e. there must exist

$$A = \sum_{k=2}^{n} w_k C(k, 2)$$

equations[18] of the type

$$\frac{\partial X_{ikj}}{\partial x^l} - \frac{\partial X_{ikl}}{\partial x^j} + M_{ikjl}^{IKJB} \frac{\partial X_{IKJ}}{\partial x^B} = 0, \tag{8.4}$$

where I, J, K have the same range as i, j, k and $B > K$, these equations being satisfied identically in consequence of (8.1). The quantities M are functions of x^α and v_k.

[18] $C(k, 2)$ is used to denote the number of combinations without repetitions of k things taken 2 at a time.

Conversely suppose that there are A independent equations of the form

$$A_\sigma^{\alpha d} \frac{\partial Z_\alpha}{\partial x^d} = 0 \qquad \begin{pmatrix} \sigma = 1, \cdots, A \\ \alpha = 1, \cdots, L \\ d = 1, \cdots, n \end{pmatrix} \tag{8.5}$$

which are satisfied identically in consequence of $Z_\alpha = 0$; the rank of the matrix of the quantities $A(x, v)$ therefore has its maximum value A. If we make use of the relations

$$Z_\alpha = b_\alpha^{ikj} X_{ikj},$$

the equations (8.5) can be written

$$A_\sigma^{\alpha d} b_\alpha^{ikj} \delta_d^c \frac{\partial X_{ikj}}{\partial x^c} + \sum X_{ikj} = 0. \tag{8.6}$$

The indices c, d run from 1 to n and the system (8.6) possesses the property that it is satisfied in consequence of $Z_\alpha = 0$, i. e. $X_{ikj} = 0$. Since the matrix of the quantities $b_\alpha^{ikj} \delta_d^c$ is non-singular the equations in (8.6) are independent in the derivatives $\partial X_{ikj} / \partial x^l$. If the equations (8.6) are to be satisfied identically in consequence of $X_{ikj} = 0$ the coefficients of the second derivatives of the unknowns v_{ik} must vanish identically. The system (8.6) can be written

$$A_\sigma^{\alpha d} b_\alpha^{ikj} \delta_d^c \frac{\partial^2 v_{ik}}{\partial x^j \partial x^c} - A_\sigma^{\alpha d} b_\alpha^{ikj} F_{ikj}^{pqr} \delta_d^c \frac{\partial^2 v_{pq}}{\partial x^r \partial x^c} + \star = 0. \tag{8.7}$$

Since $r > q$ and $j \leqq k$, it is evident from the form of (8.7) that

$$A_\sigma^{\alpha d} b_\alpha^{ikj} \delta_d^l + A_\sigma^{\alpha d} b_\alpha^{ikl} \delta_d^j \equiv 0, \tag{8.8}$$

where $l \leqq j \leqq k$. Consequently the equations (8.6) can be written

$$A_\sigma^{\alpha d} b_\alpha^{ikj} \delta_d^l \left(\frac{\partial X_{ikj}}{\partial x^l} - \frac{\partial X_{ikl}}{\partial x^j} \right) + A_\sigma^{\alpha d} b_\alpha^{ikj} \delta_d^\beta \frac{\partial X_{ikj}}{\partial x^\beta} + \sum X = 0, \tag{8.9}$$

where $l < j \leqq k$ and $\beta > k$. Suppose that the matrix of quantities $A_\sigma^{\alpha d} b_\alpha^{ikj} \delta_d^l$ is singular; it follows that (8.9) can be replaced by a system which contains at least one equation of the form

$$N^{ikj\beta} \frac{\partial X_{ikj}}{\partial x^\beta} + \sum X = 0 \qquad (\beta > k). \tag{8.10}$$

By differentiation of the system (8.1) we obtain

$$\frac{\partial X_{ikj}}{\partial x^\beta} \equiv \frac{\partial^2 v_{ik}}{\partial x^j \partial x^\beta} - F_{ikj}^{pqr} \frac{\partial^2 v_{pq}}{\partial x^r \partial x^\beta} + \star = 0 \qquad \begin{pmatrix} q < r \geqq j \\ \beta > k \geqq j \end{pmatrix}. \tag{8.11}$$

By the method used in a former paper,[19] equations (8.11) can be replaced by a system of the form

[19] *loc. cit.*, Jour. p. 246.

$$(8.12) \qquad Y_{ikj\beta} \equiv \frac{\partial^2 v_{ik}}{\partial x^\beta \partial x^j} - E^{pqrs}_{ikj\beta} \frac{\partial^2 v_{pq}}{\partial x^r \partial x^s} + \star = 0 \qquad \begin{pmatrix} r, s > q \\ \beta > k \geqq j \end{pmatrix},$$

where the indices i, k, j, β have the same range as in (8.11). It is to be noticed that throughout this process the equations obtained by differentiation of (8.1) with respect to x^l where $l \leqq k$ are not needed. Furthermore relations of the form

$$\frac{\partial X_{ikj}}{\partial x^\beta} = D^{IKJB}_{ikj\beta} Y_{IKJB}$$

must be satisfied where the matrix of the quantities D is non-singular. Hence (8.10) can be written

$$(8.13) \qquad N^{ikj\beta} D^{IKJB}_{ikj\beta} Y_{IKJB} + \sum X = 0.$$

However from the form of (8.12) it is clear that the equation (8.13) is not identically zero in consequence of $X = 0$ unless the coefficient $N^{ikj\beta} D^{IKJB}_{ikj\beta}$ is identically zero. This implies that the quantities N are all zero and hence that the equations (8.6) are not all independent. Therefore the matrix of the quantities $A^{\alpha d}_{\sigma} b^{ikj}_{\alpha} \delta^l_d$ in (8.8) must have a non-vanishing determinant from which it follows that (8.3) are satisfied in consequence of $X_{ikj} = 0$ and $\partial X_{ikj} / \partial x^\beta$ $(\beta > k)$, i. e. the system (8.1) is completely integrable.

Hence a necessary and sufficient condition for complete integrability of (8.1) is that there exist independent identities of the type (8.5) equal in number to the number of integrability conditions of (8.1). That there cannot exist more than A independent identities of the type (8.5) follows from the form of (8.9). The existence of A', $(A' > A)$, independent identities of the type (8.5) would lead to A' independent identities (8.6). Since there are only A distinct left members of (8.3), it follows that (8.9), which is equivalent to (8.6), could be replaced by a system containing at least one equation of the form (8.10). The above argument shows that the quantities N must be identically zero, i. e. equations (8.6) or in other words (8.5) would not be independent.

As an application of the theory of this section consider the system of equations 5 (a) together with a system of the type (5.5), i. e.

$$8\,(a) \qquad \frac{\partial h^i_\alpha}{\partial x^\beta} - \frac{\partial h^i_\beta}{\partial x^\alpha} - 2 h^i_{j,k} h^j_\alpha h^k_\beta = 0, \qquad \begin{pmatrix} \alpha = 2, 3, 4 \\ \beta = 1, \cdots, \alpha - 1 \\ i = 1, 2, 3, 4 \end{pmatrix}.$$

The above system 8 (a) can be solved for

$$\frac{\partial h^i_2}{\partial x^1}, \frac{\partial h^i_2}{\partial x^1}, \frac{\partial h^i_3}{\partial x^2}, \frac{\partial h^i_4}{\partial x^1}, \frac{\partial h^i_4}{\partial x^2}, \frac{\partial h^i_4}{\partial x^3}$$

from which it follows that there are 16 integrability conditions arising from 8 (a) alone. By differentiating the equations 8 (a) with respect to x^γ and permuting the indices α, β, γ cyclically and adding the three equations thus obtained, we arrive at the system (5.3), i. e.

$$8\,(b) \qquad h^i_{j,k,l} + h^i_{k,l,j} + h^i_{l,j,k} - 2\,[h^i_{m,j}\,h^m_{k,l} + h^i_{m,l}\,h^m_{j,k} + h^i_{m,k}\,h^m_{l,j}] = 0,$$

where the indices j, k, l have the sets of values $1,2,3;\ 1,2,4;\ 1,3,4;\ 2,3,4$. Equations 8 (b) form a set of 16 equations of the type (8.5) which are satisfied identically in consequence of 5 (a); in fact 5 (a) is but a different form of 8 (b). From 5 (a) we see that there will be four more integrability conditions or 20 in all for the combined system 5 (a) and 8 (a). Denote the left members of 8 (b) by X^i_{jkl} and consider the four equations

$$8\,(c) \qquad X^i_{128,4} + X^i_{142,3} + X^i_{134,2} + X^i_{324,1} = 0,$$

where the comma denotes absolute differentiation. It can be shown that equations 8 (c) are satisfied identically in consequence of 8 (a) and 5 (a). Hence the system of identities 8 (c) and 8 (b) numbering 20 in all, insures the complete integrability of 8 (a) and 5 (a).

When the system of field equations

$$\Delta_k\, h^i_{j,k} = 0,$$

[see Proc. Note III], is combined with the system 5 (a), a regular system (3.3) Proc. Note III is obtained [that (3.3) is regular follows from the fact that all derivatives with respect to x^1 appear on the left]. The 8 integrability conditions arising from (3.3) Proc. Note III taken together with the 16 conditions arising from 8 (a) make 24 in all. The corresponding set of 24 independent identities involving derivatives of the left members of the system is made up of equations 8 (c), 8 (b), and the equations (3.4) Proc. Note II, the latter stating that the divergence of the left members of the field equations vanishes identically.

§ 9. A Sufficient Condition for Complete Integrability.

To find a set of identities (8.5) which will insure complete integrability requires considerable work in the case of certain systems of differential equations. This difficulty can be met in some measure at least by use of a sufficient condition for complete integrability given by one of us in a former treatment;[20] this condition for complete integrability can be applied with facility to a large class of invariantive systems and in particular to the type of invariantive system which serves as field equations in the theory of relativity. In this section we shall consider this method of establishing complete integrability from the standpoint of the present paper.

In § 7 it was noted that a certain number M of the equations (7.4) may be linearly dependent on the remainder of (7.4) in consequence of the equations of the system (4.5). Let us express this dependence as follows

$$(9.1) \qquad \sum(A)\frac{\partial T}{\partial x^\alpha} + \star = 0 \qquad (M \text{ equations}).$$

From the theory of § 3 it follows that a coördinate system can be found in which (9.1) can be put in regular form, i. e. can be written

[20] *loc. cit.* Ann. (1), p. 690 et. seq.

$$\frac{\partial T_{ik}}{\partial x^\alpha} = \sum(A)\frac{\partial T}{\partial x} + \star, \qquad \begin{pmatrix} k &= 1, \cdots, n \\ \alpha &= 1, \cdots, k \\ i &= 1, \cdots, M_k \end{pmatrix}, \tag{9.2}$$

where $M = \sum k M_k$. From the fact that (9.2) is regular it follows that all derivatives of order r of the left members of (9.2) vanish in consequence of the vanishing of the components T, all derivatives of components T of orders 1 to r inclusive, and the remainder of the derivatives of the components T of order $r+1$.[21] Now set up the following condition with reference to the numbers A_i^* which arise in the system (7.5).[22]

CONDITION IV_A^*. *The numbers A_i^* are such that*

$$\begin{aligned} A_0^* &= A_0 - N + \sum_1^n M_\alpha, \\ A_i^* &= A_i - M_i \qquad (i = 1, \cdots, n-1), \\ M_{n-1} &= 0, \qquad M_n = 0.^{23} \end{aligned}$$

Since $\sum_0^n A_i^*$ is equal to $\sum_0^{n-2} A_i$, it follows that $A_n^* = N$ when use is made of Condition IV_A^*. A comparison with the set of conditions given previously[24] and the related discussion, shows that if Conditions I_A^*, III_A^*, IV_A^* are satisfied, the combined system (7.5) and (4.7) will be completely integrable.

Conditions similar to IV_A^* can be applied to invariantive systems of the metric and vector types.

As an illustration of the use of this method the reader is referred to a proof that the system of equations (7j) together with (4.10) and (4.11) is completely integrable, *loc. cit.* Ann. (1), p. 713; also Ann. (2) p. 725. This enables us to state the following

EXISTENCE THEOREM. *Let*

$$\varphi_{lm}(x^{m+1}, \cdots, x^4) \qquad \begin{bmatrix} m &= 1, 2 \\ l &= 1, \cdots, B_m^* \end{bmatrix}$$

and

$$\psi_{lm}(x^{m+1}, \cdots, x^4) \qquad \begin{bmatrix} m &= 0, 1, 2, 3 \\ l &= 1, \cdots, \gamma_m \end{bmatrix}$$

[21] Cf. *loc. cit.*, Jour. p. 246.

[22] It is immaterial whether or not the coördinate system (x) to which the system (9.2) is referred is identical with the coördinate system (x) to which (4.6) is referred.

[23] It may be noted that $M_{n-1} = 0$ follows from $A_{n-1}^* = A_{n-1} - M_{n-1}$ since $A_{n-1} = 0$ and the A_i^* and M_i are zero or positive.

[24] *loc. cit.*, Ann. (1), p. 690. If Conditions I_A^*, III_A^*, IV_A^* are satisfied, it follows that Conditions I–V in Ann. (1) are likewise satisfied; Condition I, Ann. (1) is equivalent to Condition III_A^*; Condition II, Ann. (1) follows from the fact that (9.2) is regular; Condition III and IV, Ann. (1) follow from Condition IV_A^*; Condition V, Ann. (1) is equivalent to Condition I_A^*.

denote functions of the variables $x^{m+1}, \cdots, x^4$ *analytic in the neighborhood of* $x^i = p^i$ *such that* $\varphi_{lm}(p) = (B^*_{lm})_p$ *for* $m = 1, 2$. *The constants* $(g_{\alpha\beta})_p = (g_{\beta\alpha})_p$ *and* $(B^*_{lm})_p$ *where* $m = 1, 2, 4$ *are to be chosen* (1) *such that they satisfy equations* (7 b), (2) *such that the determinant* $|(g_{\alpha\beta})_p|$ *does not vanish, and* (3) *such that quantity* (21) *does not vanish. Then there exists one and only one fundamental metric tensor with components* $g_{\alpha\beta}\,(= g_{\beta\alpha})$ *each function* $g_{\alpha\beta}(x)$ *being analytic in the neighborhood of* $x^i = p^i$, *which constitutes a set of integrals of the system* (7 b) *and which is* (1) *such that* $g_{\alpha\beta}(p) = (g_{\alpha\beta})_p$ *and* $B^*_{l4}(p) = (B^*_{l4})_p$, *where* $l = 1, \cdots, 10$ *and* (2) *such that*

$$\gamma_{l0} = \psi_{l0}(x^1, \cdots, x^4) \quad [l = 1, 2, 3, 4] \qquad \left|\begin{array}{l}\gamma_{lm} = \psi_{lm}(x^{m+1}, \cdots, x^4) \\ \left[\begin{array}{l} m = 1, 2, 3 \\ l = 1, \cdots, \gamma_m \\ x^1 = p^1, \cdots, x^m = p^m \end{array}\right]\end{array}\right| \qquad \left|\begin{array}{l} B^*_{lm} = \varphi_{lm}(x^{m+1}, \cdots, x^4) \\ \left[\begin{array}{l} m = 1, 2 \\ l = 1, \cdots, B^*_m \\ x^1 = p^1, \cdots, x^m = p^m \end{array}\right]\end{array}\right|.$$

§ 10. Existence Theorems in Normal Coördinates.

It is not possible to choose a priori some of the components $\Gamma^\alpha_{\beta\gamma}$ arbitrarily as functions of a part of the variables independently of the arbitrary functions corresponding to the components A^*_{lm} and at the same time to characterize the coördinates as normal coördinates y^i. However the discussion of this problem will utilize many former results. Let it suffice to say that in constructing the power series expansions for the components A^*_{lm} by the use of equations (4.6) that the initial conditions $\Gamma^\alpha_{\beta\gamma} = 0$ at $y^i = 0$ are to be imposed, also that the derivatives $\partial\,\Gamma^i_{jk}/\partial\,x^l$ at $y^i = 0$ be equal to $A^i_{jkl}(0)$ and that the higher derivatives are to be determined by equations of the type $M\,(2.10)$.[25] If the system composed of (4.6) and (4.7) is completely integrable the power series expansions for the A^*_{lm} and for the components $\Gamma^i_{\alpha\beta}$ will be determined uniquely for a suitable assignment of arbitrary data. The Γ series

$$\Gamma^i_{\alpha\beta} = (A^i_{\alpha\beta\gamma})_0\, y^\gamma + \cdots \tag{10.1}$$

will be shown to converge in § 11.

Existence Theorem. *Let* (4.6) *be a regular system which satisfies Conditions* $\mathrm{I_A}$ *and* $\mathrm{II_A}$. *Also let*

$$\varphi_{ik}(y^{k+1}, \cdots, y^n) \qquad \left[\begin{array}{l} 0 \leqq k \leqq n-1 \\ i = 1, \cdots, A^*_k \end{array}\right],$$

where the indices i, k *have the same range of values as the indices of the components* $A^*_{ik}\ (k \neq n)$, *denote a function of the variables* $y^{k+1}, \cdots, y^n$ *analytic in the neighborhood of* $y^i = 0$ *such that* $\varphi_{ik}(0) = (A^*_{ik})_0$ *for all*

[25] Cf. Jour. p. 246.

values of the indices for which the φ_{ik} *are defined. Then there exists one, and only one, set of components* $\Gamma^i_{\alpha\beta}\,(=\Gamma^i_{\beta\alpha})$ *of affine connection given by the convergent power series* (10.1), *which constitute a set of integrals of the system* (4.6) *and which is* (1) *such that* $A^*_{in}(0) = (A^*_{in})_0$ *where* $i = 1, \cdots, A^*_n$ *and* (2) *such that*

$$A^*_{i0} = \varphi_{i0}(y^1, \cdots, y^n) \quad [i = 1, \cdots, A^*_0] \qquad A^*_{ik} = \varphi_{ik}(y^{k+1}, \cdots, y^n) \quad \begin{bmatrix} k = 1, \cdots, n-1 \\ i = 1, \cdots, A^*_k \\ y^1 = 0, \cdots, y^k = 0 \end{bmatrix}.$$

Similarly for the metric case equations (4.9) are to be used in calculating the coefficients of the power series expansions of the components B^*_{lm}. Throughout this process the initial values $g_{\alpha\beta}(0) = g_{\beta\alpha}(0) = (g_{\alpha\beta})_0$ and $\partial g_{\alpha\beta}/\partial y^\gamma = 0$ at $y^i = 0$ are to be imposed; also let $\partial^2 g_{\alpha\beta}/\partial y^\gamma\, \partial y^\vartheta$ at $y^i = 0$ be equal to $(g_{\alpha\beta,\gamma\vartheta})_0$ and determine the higher derivatives by equations of the type M (2.16). If the system composed of equations (4.9), (4.7), and (4.10) is completely integrable a unique power series

$$g_{\alpha\beta} = g_{\alpha\beta}(0) + \frac{1}{2!}\, g_{\alpha\beta,\gamma\vartheta}(0)\, y^\gamma\, y^\vartheta + \cdots \tag{10.2}$$

is determined and this will be shown to converge in § 11.

Existence Theorem. *Let* (4.9) *be a regular system which satisfies Conditions* I_G *and* II_G. *Also let*

$$\varphi_{ik}(y^{k+1}, \cdots, y^n) \qquad \begin{bmatrix} 0 \leqq k \leqq n-1 \\ i = 1, \cdots, B^*_k \end{bmatrix},$$

where the indices i, k *have the same range of values as the indices of the components* $B^*_{ik} (k \neq n)$, *denote a function of the variables* $y^{k+1}, \cdots, y^n$ *analytic in the neighborhood of* $y^i = 0$ *such that* $\varphi_{ik}(0) = (B^*_{ik})_0$ *for all values of the indices for which the* φ_{ik} *are defined. Then there exists one, and only one set of components* $g_{\alpha\beta}\,(= g_{\beta\alpha})$ *of a fundamental metric tensor, given by the convergent power series* (10.2) *which constitutes a set of integrals of the system of equations* (4.9) *and which is* (1) *such that* $B^*_{in}(0) = (B^*_{in})_0$ *where* $i = 1, \cdots, B^*_n$ *and* (2) *such that*

$$B^*_{i0} = \varphi_{i0}(y^1, \cdots, y^n) \quad [i = 1, \cdots, B^*_0] \qquad B^*_{ik} = \varphi_{ik}(y^{k+1}, \cdots, y^n) \quad \begin{bmatrix} k = 1, \cdots, n-1 \\ i = 1, \cdots, B^*_k \\ y^1 = 0, \cdots, y^k = 0 \end{bmatrix}.$$

For spaces of distant parallelism a corresponding existence theorem can be stated where the integrals $h^i_\alpha(x)$ will constitute a set of fundamental vectors in a system of local coördinates z^i. In calculating the coefficients of

the power series expansions for the invariants $h^i_{j,k}$ with the aid of equations (5.4) the limited conditions $h^i_{|j|} = \delta^i_j$ at $z^i = 0$ and $\partial h^i_{|j|}/\partial z^k = (h^i_{j,k})_0$ at $z^i = 0$ are to be imposed; the higher derivatives of the components $h^i_{|j|}$ are to be determined by equations of the type Proc. Note II (4.2). If the system of equations composed of (5.4) and (5.5) is completely integrable, the power series expansions for the invariants $h^i_{j,k}$ and also the $h^i_{|j|}$ series of the form

$$h^i_{|j|} = \delta^i_j + h^i_{j,k}(0)\, z^k + \cdots \tag{10.3}$$

will be unique; the convergence of this latter series will be proved in § 11.

EXISTENCE THEOREM. *Let* (5.4) *be a regular system which satisfies Conditions* $\mathrm{I_H}$ *and* $\mathrm{II_H}$. *Also let*

$$\varphi_{lm}(z^{m+1}, \cdots, z^n) \qquad \begin{bmatrix} m = 0, \cdots, n-1 \\ l = 1, \cdots, K^*_m \end{bmatrix},$$

where the indices l, m have the same range of values as the indices of the components K^*_{lm} $(m \neq n)$, *denote a function of the variables* $z^{m+1}, \cdots, z^n$ *analytic in the neighborhood of* $z^i = 0$, *such that* $\varphi_{lm}(0) = (K^*_{lm})_0$ *for all values of the indices for which the* φ_{lm} *are defined. Then there exists one, and only one, set of components* $h^i_{|j|}$ *of the fundamental vectors given by the convergent power series* (10.3) *which constitute a set of integrals of the system* (5.4) *and which is* (1) *such that* $K^*_{ln}(0) = (K^*_{ln})_0$ *where* $i = 1, \cdots, K^*_n$ *and* (2) *such that*

$$K^*_{l0} = \varphi_{l0}(z^1, \cdots, z^n) \quad [i = 1, \cdots, K^*_0] \qquad \left| \quad K^*_{lm} = \varphi_{lm}(z^{m+1}, \cdots, z^n) \quad \begin{bmatrix} m = 1, \cdots, n-1 \\ i = 1, \cdots, K^*_m \\ z^1 = 0, \cdots, z^m = 0 \end{bmatrix}.\right.$$

The extension to functional systems can be made in accordance with the discussion in § 7.

§ 11. Convergence Proofs.

The equations (4.6) of the present theory have exactly the same form as the system of equations (2.7) Ann. (I) except that in (4.6) there may be a group of unknowns having derivatives with respect to $(n-1)$ of the y's appearing in the left members of the system, which was not the case in the former treatment. It is easily seen however that this greater generality will necessitate no change in the convergence proof of the Γ series. Hence the reader is referred to the former treatment for a proof of the convergence of the series (10.1). Similar remarks apply to convergence of the expansions (10.2) but the convergence of the series (10.3) remains to be demonstrated.

Consider the system of partial differential equations.

$$\frac{b_m}{a_h}\frac{\partial \mathfrak{K}^*_{lm}}{\partial z^h} = DF(z) + \frac{\mu\,\Theta(\mathfrak{K}^*, \mathfrak{H})}{1-\varepsilon\,\Theta(\mathfrak{K}^*, \mathfrak{H})}, \tag{11.1a}$$

$$\frac{1}{a_k}\frac{\partial \mathfrak{H}^i_j}{\partial z^k} = W\sum b_q\,\mathfrak{K}^*_{pq}\,\mathfrak{H}^c_d\,\mathfrak{H}^g_f \tag{11.1b}$$

in which a_h, b_m and μ are positive constants for which more exact values will be fixed later. The positive constants D and W are to be chosen so that

$$D \geqq \frac{b_m}{a_h}, \quad W \geqq \frac{2}{a\,b}, \tag{11.2}$$

where a and b are the least of the a_h and b_m respectively. The indices p, q, l, m in (11.1) assume all possible values as indices of the independent components K^*_{lm} and all other indices have values from 1 to n. All summations in (11.1) represent the sum of all terms obtainable from the representative term by giving different values to the indices involved. The quantity ε in (11.1a) is a positive constant less than unity and is to be considered as the sum of α positive constants $\nu_1+\cdots+\nu_\alpha$. The integer α in the sum $\nu_1+\cdots+\nu_\alpha$ for a particular equation (11.1a) in which the values of the indices in its left member are equal to the corresponding indices in the left member of some equation (5.4), is to be equal to the number of different derivatives $\partial K^*_{pq}/\partial x^r$ in the right member of the said equation of the system (5.4). The integer α for an equation (11.1a) which does not correspond to an equation of (5.4) can be taken to have the value unity. The function $F(z)$ in (11.1a) is defined by the equation

$$F(z) = \frac{\Omega}{1-\sum a_k z^k},$$

where the positive constants Ω and a_k are to be chosen so that the expression

$$\frac{\Omega}{1-a_{h+1}z^{h+1}-\cdots-a_n z^n} \tag{11.3}$$

is dominant for each derivative $\partial \varphi_{lm}/\partial z^h\,(h>m)$ of the K^*_m functions $\varphi_{lm}\,(l=1,\cdots,K^*_m)$. Finally

$$\Theta(\mathfrak{K}^*, \mathfrak{H}) = \left\{1-\sum b_q\,[\mathfrak{K}^*_{pq}-(\mathfrak{K}^*_{pq})_0]-\sum\frac{1}{\varrho}\,[\mathfrak{H}^i_j-(\mathfrak{H}^i_j)_0]\right\}^{-1},$$

where $\sum$ denotes a summation over all possible values of the indices p, q, i, j. The positive constant ϱ is to be chosen later. Equations (11.1) constitute a completely integrable system of total differential equations.

Hence according to the well known theorem for the existence of solutions of systems of total differential equations, there exists a unique solution $\mathfrak{K}^*_{lm}(z)$, $\mathfrak{H}^i_j(z)$ of the system (11.1) such that these integrals assume an arbitrary set of initial values $(\mathfrak{K}^*_{lm})_0$ and $(\mathfrak{H}^i_j)_0$. We shall choose the initial values so that the inequalities

$$(\mathfrak{H}^i_j)_0 \geqq \delta^i_j \tag{11.4}$$

and

$$(\mathfrak{K}^*_{lm})_0 \geqq |(K^*_{lm})_0| \tag{11.5}$$

are satisfied, where $(\mathfrak{H}^i_j)_0 = \mathfrak{H}^i_j(0)$ and $(\mathfrak{K}^*_{lm})_0 = \mathfrak{K}^*_{lm}(0)$. All derivatives at $z^i = 0$ of the integrals $\mathfrak{K}^*_{lm}$ and $\mathfrak{H}^i_j$ are then positive since any derivative of one of the sets of terms in the right member of (11.1) evaluated at $z^i = 0$ is a polynomial composed entirely of positive terms.[26] In consequence of the dominant property of the function $F(z)$ and the first inequality of (11.2) it follows from (11.1a) that

$$\mathfrak{K}^*_{lm/h_1 \cdots h_s}(0) \geqq \left| \frac{\partial^s \varphi_{lm}}{\partial z^{h_1} \cdots \partial z^{h_s}}(0) \right|, \tag{11.6}$$

where $h_1, \cdots, h_s > m$ and the notation $\mathfrak{K}^*_{lm/h_1 \cdots h_s}$ denotes the ordinary derivative of the function $\mathfrak{K}^*_{lm}$.

Since the right members of the equations (11.1a) are all equal, the equations

$$\frac{b_m}{a_h} \frac{\partial \mathfrak{K}^*_{lm}}{\partial z^h} = \frac{b_q}{a_r} \frac{\partial \mathfrak{K}^*_{pq}}{\partial z^r}$$

are satisfied for all values of the indices involved. It is therefore possible to write those equations of (11.1a) which have left members with indices corresponding to the indices of some left member of (5.4) in the form

$$\begin{aligned} \frac{\partial \mathfrak{K}^*_{lm}}{\partial z^h} = {} & \sum \nu \frac{a_h}{a_r} \frac{b_q}{b_m} \Theta(\mathfrak{K}^*, \mathfrak{H}) \frac{\partial \mathfrak{K}^*_{pq}}{\partial z^r} \\ & + DF(z) \frac{a_h}{b_m} [1 - \varepsilon \Theta(\mathfrak{K}^*, \mathfrak{H})] + \mu \frac{a_h}{b_m} \Theta(\mathfrak{K}^*, \mathfrak{H}), \end{aligned} \tag{11.7}$$

where the quantity ν assumes values ν_1 to ν_α so that the first summation in these equations denotes a sum of α terms in which the derivatives are taken to correspond to derivatives in the first summation in the right member of the corresponding equation (5.4). The reader is referred to the former discussion for the proof of the following[27]

[26] Meray and Riquier, *loc. cit.* p. 48.

[27] *loc. cit.* Ann. (1), p. 698.

LEMMA. *Given any two positive constants P and Q it is possible to assign values, each of which is greater than P to the constants $a_1, \cdots, a_n$ and $b_0, \cdots, b_n$ such that each of the coefficients*

$$\nu \frac{a_h}{a_r} \frac{b_q}{b_m} \tag{11.8}$$

of the derivatives in the right member of (11.7) *will be greater than Q.*

Choose the positive constants P, Q and r so that the expression

$$Q\{1 - P(z^{h+1} + \cdots + z^n)\}^{-1}$$

dominates each derivative $\partial \varphi_{lm} / \partial z^h$ $(h > m)$ of the K_m^* functions φ_{lm} $(l = 1, \cdots, K_m^*)$ and so that

$$Q\{1 - \sum P[\mathfrak{K}_{pq}^* - (\mathfrak{K}_{pq}^*)_0] - \sum \frac{1}{r}[\mathfrak{H}_j^i - (\mathfrak{H}_j^i)_0]\}^{-1}$$

dominates each of the coefficients (h_a^i, K_{lm}^*) in (5.4) and also the terms in (5.4) containing no derivatives. Then choose the positive constants $a_1, \cdots, a_n$ and $b_0, \cdots, b_n$ so that each is greater than P and so that the coefficients (11.8) are each greater than Q; also choose the positive constant $\varrho \leqq r$. Then the coefficients of the derivatives in the right members of the equations (11.7) will dominate the corresponding coefficients in equations (5.4). If the positive constant μ is chosen such that $\mu \geqq BQ/a$ then the quantities $\mu \frac{ah}{b_m} \Theta(\mathfrak{K}^*, \mathfrak{H})$ will dominate the terms in the right members of (5.4) containing no derivatives; the quantity a is used to denote the least of the a's and B the greatest of b's. Hence it follows from (11.6) and (11.7) that

$$\mathfrak{K}_{lm/h_1}^*(0) \geqq |K_{lm/h_1}^*(0)|, \tag{11.9}$$

where $h_1 \leqq m$. If the indices i, j, k in (11.1b) determine an independent component $h_{j,k}^i$, then we have from (11.2), (11.5), and the fact that all components $\mathfrak{K}_{pq}^*$ appear in the right member of each equation (11.1b) that

$$\mathfrak{H}_{j/k}^i(0) \geqq |(h_{j,k}^i)_0|. \tag{11.10}$$

The inequality holds also for values of the indices i, j, k which determine a dependent component $h_{j,k}^i$ since, in that case, $h_{j,k}^i$ is merely the negative of an independent component. In order to extend (11.10) to higher derivatives we shall assume the inequalities

$$\mathfrak{H}_{j/k_1 \cdots k_s}^i(0) \geqq |h_{j,k_1 \cdots k_s}^i(0)| \tag{11.11}$$

for $s<r$ $(r \geqq 2)$. In calculating derivatives of second or higher order of the components K^*_{lm} from equations (5.4), we use a process of differentiating equations (5.4) and eliminating derivatives from the right members of the resulting equations, which also appear on the left of equations (5.4) or the equations obtained by differentiating (5.4). Due to the form of the left members of (5.4) this elimination merely involves the substitution for a right hand derivative of its equivalent from some other equation[28] and hence it is easily seen by comparing the two systems of equations, one arising from (5.4) and the other from (11.7), that the inequalities

$$\mathfrak{K}^*_{lm/h_1 \cdots h_s}(0) \geqq |K^*_{lm/h_1 \cdots h_s}(0)| \tag{11.12}$$

hold where $s = 1, \cdots, r$; $k_1 \leqq m$; and h_i, for $i>1$, is arbitrary. Combining (11.12) and (11.6) it follows that (11.12) is satisfied for all values of the indices involved. Now differentiate equations (5.5) $(r-1)$ times and evaluate at the point $z^i = 0$; we obtain a system of the form

$$h^i_{j,k_1 \cdots k_r} - h^i_{k_1, jk_2 \cdots k_r} = 2\, h^i_{j,k_1,k_2 \cdots k_r} + \star .$$

If we compare these equations with the equations which can be obtained by differentiating (11.1b) it is seen, due to the inequalities (11.2), (11.4), (11.5), (11.11) and (11.12), that

$$\mathfrak{H}^i_{j,k_1 \cdots k_r}(0) \geqq |h^i_{j,k_1 \cdots k_r}(0) - h^i_{k_1, jk_2 \cdots k_r}(0)| \tag{11.13}$$

for all values of the indices involved, or interchanging indices

$$\mathfrak{H}^i_{j,k_1 \cdots k_r}(0) \geqq |h^i_{j,k_1 \cdots k_r}(0) - h^i_{\mu,\eta_1 \cdots \eta_r}(0)| \tag{11.14}$$

where $\mu\eta_1 \cdots \eta_r$ represents any permutation of the indices $jk_1 \cdots k_r$. Add together the $(r+1)$ inequalities obtained from (11.14) by allowing the indices $\mu\eta_1 \cdots \eta_r$ to assume the cyclical permutations of $jk_1, \cdots, k_r$. The inequalities

$$(r+1)\,\mathfrak{H}^i_{j/k_1 \cdots k_r}(0) \geqq |(r+1)\,h^i_{j,k_1 \cdots k_r}(0) - S[h^i_{j,k_1 \cdots k_r}(0)]| \tag{11.15}$$

result and hence (11.11) holds for $s = r$ since the summation S in the right member of (11.15) vanishes.[29] This recurrence process enables one to say that (11.11) and (11.12) are satisfied for $s = 1, 2, \cdots$. Hence the power series expansions for the functions $\mathfrak{H}^i_j$ dominate the corresponding h^i_j series with the result that the latter converge.

[28] *loc. cit.* Jour. (6.5).

[29] *loc. cit.* Proc. Note I (4.9).

PART II.

THEORY OF CHARACTERISTIC SURFACES.

§ 12. Definitions of Characteristic Surfaces.

The point of departure in our study of characteristic surfaces will be the Existence Theorem in § 6 for the system (2.1). The regular form (2.8), which is equivalent to (2.1), shows that if the quantities φ_{i0} and $\partial \varphi_{i0}/\partial x^1$ are assigned over the surface $x^1 = p^1$ and the remaining functions φ_{ik} are assigned as indicated in the statement of the existence theorem, the solution $v_{ik}(x)$ is uniquely determined over $x^1 = p^1$. In a similar manner if the functions

$$\varphi_{ik}, \frac{\partial \varphi_{ik}}{\partial x^{k+1}}, \cdots, \frac{\partial \varphi_{ik}}{\partial x^{\alpha}} \qquad (k = 0, \cdots, \alpha - 1) \tag{12.1}$$

are assigned over the $n - \alpha$ dimensional surface $x^1 = p^1, \cdots, x^\alpha = p^\alpha$ and the remaining functions φ_{ik} are assigned as indicated in the statement of the existence theorem, the solution $v_{ik}(x)$ is uniquely determined over $x^1 = p^1, \cdots, x^\alpha = p^\alpha$. These remarks apply equally well to a system of the invariantive type.

Let $v_k(x)$ denote a solution of (2.1) and consider the system of equations

$$\sum_{\alpha=1}^{n} \sum_{k=1}^{w} a_{ik}^{*\alpha}(x) \frac{\partial v_k}{\partial x^\alpha} + c_i^*(x) = 0 \qquad (i = 1, \cdots, L), \tag{12.2}$$

where $a_{ik}^{*\alpha}$ and c_i^* are functions of the variables $x^1, \cdots, x^n$ obtained by substituting the integral $v_k(x)$ into the quantities $a_{ik}^\alpha(x, v)$ and $c_i(x, v)$ respectively. *A surface C_{n-1} having $x^1 = 0$ as its equation will be called an $n-1$ dimensional characteristic surface for an integral $v_k(x)$ of the system* (2.1) *if it is impossible to find a coördinate system $(\overline{x})$ defined by the transformation*

$$x^1 = \overline{x}^1, \qquad x^i = f^i(\overline{x}^1, \cdots, \overline{x}^n) \qquad (i = 2, \cdots, n) \tag{12.3}$$

such that (12.2) *can be solved for L_1 derivatives of the set $\partial v^k/\partial \overline{x}^1$ at a point P on the surface C_{n-1}.* If the coefficients a_{ik}^α are functions of x^σ alone, the characteristic surfaces C_{n-1} are determined independently of the integrals v_k. An analogous definition and remark applies to the tensor equations (3.1); the same is true of the invariantive systems composed (a) of the equations (4.3), (4.5) and (4.7) for the affine case, (b) of the equations (4.4), (4.8), (4.10) and (4.11) for the metric case, and (c) of the equations (5.2), (5.3) and (5.5) for the vector case. It is to be noted, however, that under case (a) the equations (4.3) and (4.5), under case (b) the equations (4.4)

and (4.8), and under case (c) the equations (5.2) and (5.3) are alone of significance in the determination of the characteristic surfaces.[30]

It follows that if the data is assigned over a characteristic surface C_{n-1} of the system (2.1) or (3.1) the general existence theorem will fail to apply.

If the equation of a characteristic surface C_{n-1} of the system (2.1) has the general form $\Phi(x^1, \cdots, x^n) = 0$ we can, by a transformation of coördinates[31]

$$\bar{x}^1 = \Phi(x^1, \cdots, x^n), \quad \bar{x}^i = x^i \qquad (i = 2, \cdots, n) \tag{12.4}$$

reduce the equation of C_{n-1} to the form $\bar{x}^1 = 0$ with respect to the $(\bar{x})$ coördinate system. In this latter system of coördinates $\bar{x}^\alpha$ the equations (2.1) become

$$\sum_{\beta=1}^{n} \sum_{k=1}^{w} \bar{a}_{ik}^{\beta} \frac{\partial v_k}{\partial \bar{x}^\beta} + \bar{c}_i = 0, \qquad (i = 1, \cdots, L), \tag{12.5}$$

where

$$\bar{a}_{ik}^{\beta} = a_{ik}^{\alpha} \frac{\partial \bar{x}^\beta}{\partial x^\alpha}. \tag{12.6}$$

The fact that $\bar{x}^1 = 0$ is a characteristic surface implies that

$$\bar{W}_1 = 0, \cdots, \bar{W}_r = 0 \tag{12.7}$$

over $\bar{x}^1 = 0$, where the quantities $\bar{W}_i$ are the determinants of order L_1 in the matrix

$$\left\| \begin{matrix} \bar{a}_{11}^{*1} & \cdots & \bar{a}_{1w}^{*1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \bar{a}_{L1}^{*1} & \cdots & \bar{a}_{Lw}^{*1} \end{matrix} \right\|, \tag{12.8}$$

in which the $\bar{a}_{ik}^{*\alpha}$ are obtained from the quantities $a_{ik}^{*\alpha}$ by the transformation (12.6). The property (12.7) will persist under coördinate transformations. In fact we have

$$\bar{a}_{ik}^{1} = a_{ik}^{\alpha} \Phi_\alpha \tag{12.9}$$

[30] Several definitions of a characteristic surface of a system of equations have appeared in the literature. Cf. Cartan, *loc. cit.*, (3); also N. M. Günther, *On the theory of characteristics of systems of equations with partial derivatives.* St. Petersburg (1913). The above definition has the advantage that it applies to invariantive systems as well as systems in which the unknowns are scalars.

[31] The statement that equations (12.4) define a transformation of coördinates implies that the jacobian determinant of these equations does not vanish identically, i. e. that $\partial \Phi / \partial x^1$ is not identically zero. If such were the case we could, by a relettering of the variables (x), cause the derivative $\partial \Phi / \partial x^1$ to be different from zero; then the system (2.1) or (3.1) could be referred to the new set of variables.

since the a^{α}_{ik} transform as the components of a contravariant vector in the index α and the derivatives Φ_{α} of the function Φ are the components of a covariant vector. Hence the corresponding terms in (12.8) and the transformed matrix

$$\left\| \begin{matrix} a^{*\alpha}_{11}\Phi_{\alpha} & \cdots & a^{*\alpha}_{1w}\Phi_{\alpha} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a^{*\alpha}_{L1}\Phi_{\alpha} & \cdots & a^{*\alpha}_{Lw}\Phi_{\alpha} \end{matrix} \right\| \tag{12.10}$$

are equal at any point P in space; the determinants $W_i\,(i = 1, \cdots, r)$ in the transformed matrix (12.10) therefore vanish over $\Phi = 0$.

Conversely suppose we have a surface $\Phi = 0$ such that $W_1 = 0, \cdots, W_r = 0$ over $\Phi = 0$. Make the transformation (12.4). Then because $\overline{W}_1 = 0, \cdots, \overline{W}_r = 0$ over $\overline{x}^1 = 0$, it is impossible to solve the system (12.5) for L_1 derivatives $\partial v_k/\partial \overline{x}^1$. Under transformations of the type (12.3) namely

$$\overline{x}^1 = \tilde{x}^1, \quad \overline{x}^i = f^i(\tilde{x}^1, \cdots, \tilde{x}^n) \qquad (i = 2, \cdots, n) \tag{12.11}$$

we have from (12.6) that $\overline{a}^1_{ik} = \tilde{a}^1_{ik}$; hence no coördinate system $(\tilde{x})$ can be found in which the system (2.1) can be solved for L_1 derivatives $\partial v_k/\partial \tilde{x}^1$ at a point P on $\tilde{x}^1 = 0$, i. e. $\Phi = 0$ is an $(n-1)$-dimensional characteristic surface C_{n-1}. In other words *a necessary and sufficient condition for the surface $\Phi = 0$ to be an $(n-1)$-dimensional characteristic surface C_{n-1} for an integral $v_k(x)$ of the system* (2.1) *is that all the determinants $W_i\,(i = 1, \cdots, r)$ of order L_1 that can be formed from the matrix* (12.10) *vanish over $\Phi = 0$.*

The above discussion concerning the system (2.1) and its characteristic surface C_{n-1} is likewise applicable to the equations (5.2) and (5.3) of the vector invariantive system composed of (5.2), (5.3) and (5.5) and in fact a corresponding necessary and sufficient condition for the surface $\Phi = 0$ to be a characteristic surface C_{n-1} of the vector invariantive system (5.2), (5.3) and (5.5) can be stated. It is obvious that the quantities W_i whose vanishing constitutes the condition for the surface $\Phi = 0$ to be a characteristic surface C_{n-1} of the system (2.1) are differential parameters since in fact each of the elements of the matrix (12.10) is a differential parameter; an analogous remark applies to the vector invariantive system.[32]

[32] The observation of this fact is facilitated by noting that (5.2) has the particular form

$$\sum (h^p_{q,r})\, h^{\alpha}_i \frac{\partial h^i_{j,k}}{\partial x^{\alpha}} + \star = 0$$

in which the coefficients $(h^p_{q,r})\, h^{\alpha}_i$ are linear and homogeneous in the components h^{α}_i and the index α is to be summed as indicated; hence the coefficients $(h^p_{q,r})\, h^{\alpha}_i$ have the contravariant vector transformation in the index α and so correspond exactly to the coefficients a^{α}_{ik} in (2.1).

A surface $C_{n-\alpha}$ having $x^1 = 0, \cdots, x^\alpha = 0$ as its equations will be called an $(n-\alpha)$-dimensional characteristic surface of type β where $\beta = 1, \cdots, \alpha$ for an integral v_k of the system (2.1) *if it is impossible to find a coördinate system $(\bar{x})$ defined by the transformation*

(12.12) $x^i = \bar{x}^i \ (i = 1, \cdots, \alpha); \quad x^j = f^j(\bar{x}^1, \cdots, \bar{x}^n) \ (j = \alpha+1, \cdots, n)$

such that (12.2) *can be solved for*

$$L_1 \quad \text{derivatives:} \quad \frac{\partial v_k}{\partial \bar{x}^1}$$

$$\cdots\cdots\cdots$$

$$L_{\beta-1} \quad \text{derivatives:} \quad \frac{\partial v_k}{\partial \bar{x}^{\beta-1}}$$

$$L_\beta \quad \text{derivatives:} \quad \frac{\partial v_k}{\partial \bar{x}^\beta}$$

but possible to find a coördinate system $(\bar{x})$ defined by the above transformation such that (12.2) *can be solved for*

$$L_1 \text{ derivatives:} \quad \frac{\partial v_k}{\partial \bar{x}^1}$$

$$\cdots\cdots\cdots$$

$$L_{\beta-1} \text{ derivatives:} \quad \frac{\partial v_k}{\partial \bar{x}^{\beta-1}}$$

at a point P on the surface $C_{n-\alpha}$.

The characteristic surfaces $C_{n-\alpha}$ of the tensor equations (3.1) can be defined in a similar manner by replacing the above derivatives by the derivatives

$$\frac{\partial \bar{T}}{\partial \bar{x}^1}, \cdots, \frac{\partial \bar{T}}{\partial \bar{x}^\beta}$$

and the equations (12.2) by the corresponding system determined by (3.1). When the data is assigned over an $(n-\alpha)$-dimensional characteristic surface $C_{n-\alpha}$ the general existence theorem for the system (2.1) or (3.1) will fail to apply. Analogous definitions and remarks are to be understood to apply to the affine, metric and vector invariantive systems.

Now consider the matrix

$$(12.13) \qquad \left\| \begin{pmatrix} a^{*1}_{11} & \cdots & a^{*1}_{1w} \\ \cdots & \cdots & \cdots \\ a^{*1}_{L1} & \cdots & a^{*1}_{Lw} \end{pmatrix} \cdots \begin{pmatrix} a^{*\alpha}_{11} & \cdots & a^{*\alpha}_{1w} \\ \cdots & \cdots & \cdots \\ a^{*\alpha}_{L1} & \cdots & a^{*\alpha}_{Lw} \end{pmatrix} \right\|$$

which for brevity will be denoted by $\|a^{*\alpha}_{ik}\|$. We observe that the matrix $\|a^{*\alpha}_{ik}\|$ is composed of the α matrices in the parentheses in (12.13); the

first of these matrices in parenthesis will be referred to as the *first sub-matrix* of (12.13), the second as the *second sub-matrix*, etc. Let us denote by Ξ_β any determinant of (12.13) of order

$$\Lambda_\beta = \sum_{i=1}^{\beta} L_i,$$

where $\beta = 1, \cdots, \alpha$ that can be formed by selecting L_1 columns of the first sub-matrix, L_2 columns of the second sub-matrix, $\cdots$, L_β columns of the βth sub-matrix of (12.13); particular determinants Ξ_β will be denoted by Ξ_{β_1}, Ξ_{β_2}, etc. The condition that the equations $x^1 = 0, \cdots, x^\alpha = 0$ shall define an $(n - \alpha)$-dimensional characteristic surface $C_{n-\alpha}$ of type β for the system (2.1) is that there exists a determinant $\Xi_{\beta-1}$ which does not vanish but that all determinants Ξ_β shall vanish over this surface. More generally a *necessary and sufficient condition that* $\Phi^i = 0$ *where* $i = 1, \cdots, \alpha$ *should define an* $(n - \alpha)$*-dimensional characteristic surface* $C_{n-\alpha}$ *of type* β *is that there exists a determinant* $\Xi_{\beta-1}$ *in the matrix* $\| a_{ik}^{*\sigma} \Phi_\sigma^\alpha \|$ *which does not vanish where* Φ_ν^μ *denotes the partial derivatives of* Φ^μ $(\mu = 1, \cdots, \alpha)$ *while all determinants* Ξ_β *in this matrix vanish over the surface* $\Phi^i = 0$. This extends the previous condition for $n - 1$ dimensional characteristic surfaces; an analogous condition can evidently be stated for the vector invariantive system composed of (5.2), (5.3), and (5.5).

Let us now consider an affine invariantive system (4.3), (4.5) and (4.7) or a metric invariantive system (4.4), (4.8), (4.10) and (4.11); let us in fact represent either the equations (4.3) and (4.5) or the equations (4.4) and (4.8) by writing

$$\sum_{a=1}^{n} \sum_{b=1}^{Q} I_{bc}^a \frac{\partial J_b}{\partial x^a} + \star = 0, \tag{12.14}$$

where the coefficients I are functions of the components $A_{\beta\gamma\delta}^\alpha$ in the affine case and functions of the components $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma\delta}$ in the metric case; the quantities J_b where $b = 1, \cdots, Q$ are independent $A_{\beta\gamma\delta}^\alpha$ or $g_{\alpha\beta,\gamma\delta}$ in the affine or metric cases respectively.[33]

If $x^1 = 0, \cdots, x^\alpha = 0$ is an $n - \alpha$ dimensional characteristic surface $C_{n-\alpha}$ of type β where $\beta = 1, \cdots, \alpha$ of an affine or metric invariantive system, this implies that the determinants Ξ_β of order Λ_β in the matrix

$$\left\| \begin{pmatrix} I_{11}^1 & \cdots & I_{Q1}^1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ I_{1W}^1 & \cdots & I_{QW}^1 \end{pmatrix} \cdots \begin{pmatrix} I_{11}^\alpha & \cdots & I_{Q1}^\alpha \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ I_{1W}^\alpha & \cdots & I_{QW}^\alpha \end{pmatrix} \right\| \tag{12.15}$$

[33] For the values of Q in the affine and metric case, see *loc. cit.*, (11) p. 202 and p. 660.

in which W represents the number of equations (12.14), vanish over the surface $x^1 = 0, \cdots, x^\alpha = 0$ for an integral of the system (12.14) but that there exists a determinant $\Xi_{\beta-1}$ which does not vanish over this surface. Now let $U_1, \cdots, U_m$ denote a set of irreducible factors of the determinants Ξ_β in (12.15) such that the vanishing of the $U_1, \cdots, U_m$ implies the vanishing of all determinants Ξ_β. Make an arbitrary transformation of coördinates: $x^i = \Phi^i(\bar{x})$ and replace each component appearing in the U_k by its value in accordance with the tensor law of transformation. Thus the components $A^\alpha_{\beta\gamma\delta}$, $g_{\alpha\beta}$, $g_{\alpha\beta,\gamma\delta}$ will be replaced by expressions which are linear and homogeneous in $\bar{A}^\alpha_{\beta\gamma\delta}$, $\bar{g}_{\alpha\beta}$ and $\bar{g}_{\alpha\beta,\gamma\delta}$ respectively and rational in the derivatives $\partial \Phi^i / \partial \bar{x}^k$ of the coördinate transformation.[34] Hence we shall have

$$\text{Affine case: } U_k(A^\alpha_{\beta\gamma\delta}) = \Psi_k\left(\bar{A}^\alpha_{\beta\gamma\delta};\ \frac{\partial \Phi^1}{\partial \bar{x}^1}, \cdots, \frac{\partial \Phi^n}{\partial \bar{x}^n}\right),$$

$$\text{Metric case: } U_k(g_{\alpha\beta};\ g_{\alpha\beta,\gamma\delta}) = \Psi_k\left(\bar{g}_{\alpha\beta};\ \bar{g}_{\alpha\beta,\gamma\delta};\ \frac{\partial \Phi^1}{\partial \bar{x}^1}, \cdots, \frac{\partial \Phi^n}{\partial \bar{x}^n}\right).$$

It is obvious from their method of formations that the functions Ψ_k are differential parameters. If the U_k vanish over the surface $x^1 = 0, \cdots, x^\alpha = 0$, the parameters Ψ_k will vanish over the surface $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ and conversely. Hence, *a necessary and sufficient condition that the equations* $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ *define an* $n-\alpha$ *dimensional characteristic surface of type* β *for an integral* $\Gamma^\alpha_{\beta\gamma}$ *or* $g_{\alpha\beta}$ *of an affine or metric invariantive system respectively, is that* (1) *there exists a differential parameter* Ψ *corresponding to a determinant* $\Xi_{\beta-1}$ *in the matrix* (12.15) *which does not vanish over* $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ *and* (2) *that all the differential parameters* Ψ_k *vanish over the above surface.* It should be observed that in stating the above condition we suppose that the jacobian determinant of the functions $\Phi^1, \cdots, \Phi^n$ occurring in the differential parameters Ψ_k, does not vanish over the surface $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$; also it can be supposed that the functions Φ^i, etc. with which we are concerned in the above necessary and sufficient condition are referred to coördinates (x) instead of the coördinates $(\bar{x})$ as originally considered.

§ 13. Differential Equations of the Characteristic Surfaces.

The conditions obtained by equating to zero the differential parameters Ξ_β formed from the matrix $\| a^{*\sigma}_{ik} \Phi^\alpha_\sigma \|$, namely

[34] The denominators of these rational expressions in the derivatives $\partial \Phi^i / \partial \bar{x}^k$ will be the jacobian determinant of the above coördinate transformation and hence will not vanish identically.

$$\Xi_{\beta_1} = 0, \cdots, \Xi_{\beta_m} = 0 \tag{13.1}$$

over $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ constitute a system of first order partial differential equations for the determination of the α independent functions Φ^j. When the functions Φ^j are so determined, the equations $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ define an $n-\alpha$ dimensional characteristic surface $C_{n-\alpha}$ of type β for the integrals $v_k(x)$ of the system (2.1) provided that there exists an expression $\Xi_{\beta-1}$ which does not vanish over the above surface. Analogous remarks apply to the case of the vector invariantive system composed of (5.2), (5.3), and (5.5). Similarly the equations of type

$$\Psi_1 = 0, \cdots, \Psi_m = 0 \tag{13.2}$$

over $\Phi^1 = 0, \cdots, \Phi^\alpha = 0$ determine the $n-\alpha$ dimensional characteristic surfaces $C_{n-\alpha}$ of type β of the affine and metric invariantive systems; in case the above equations (13.2) do not possess the required solution, the surfaces $C_{n-\alpha}$ will fail to exist.

The above characteristic surfaces $C_{n-\alpha}$ will have a unique determination in space if and only if the coefficients a^α_{ik} in (2.1) or the coefficients I^a_{bc} in (12.13) are independent of the integrals of the corresponding systems.

If the functions $\Phi^j(x)$ satisfy the partial differential equations (13.1) over the entire x-space, then the equations

$$\Phi^1 = c^1, \cdots, \Phi^\alpha = c^\alpha$$

in which the c's are arbitrary constants within suitable limits, define a family of characteristic surfaces $C_{n-\alpha}$ of type β. A similar remark can be made for the equations (13.2).

§ 14. Sets of Monomials.

The theory of monomials here presented contains as a special case the theory given by Janet[6] in an exposition of Riquier's work. Only properties of monomials will be considered which are essential in the discussion of existence theorems yet to be treated.

By a monomial is meant the product $x_1^{a_1} \cdots x_n^{a_n}$ in which the exponents $a_1, \cdots, a_n$ are positive integers or zero.[35] As the basis of the following discussion we shall consider a finite set (M) of monomials; the monomials of (M) will be supposed to be distinct. Any particular monomial of the set (M) will be denoted by $\bar{M}$, $\bar{\bar{M}}$, etc. and an analogous notation will be used for the monomials of other sets which will be introduced in the following discussion.

[35] Throughout this section and also throughout §§ 15–18 the notation $x_1, \cdots, x_n$ involving *subscripts* will be used for the coördinates.

The above set (M) will be divided into two mutually exclusive sets (L) and (K). The following definitions apply to the sets (L) and (K) respectively.

THE SET (L)

(a) *Multiple.* The monomial $\overline{L}: x_1^{a_1} \cdots x_n^{a_n}$ is a multiple of $\overline{\overline{L}}: x_1^{b_1} \cdots x_n^{b_n}$ if none of the differences

$$(a_1 - b_1), \cdots, (a_n - b_n) \tag{14.1}$$

is negative.

(b) *Multiplier.* The variable x_1 is a multiplier of $\overline{L}$ if the degree of x_1 in $\overline{L}$ is equal to the maximum of the degrees of x_1 in monomials of (M). Similarly $x_i\,(i = 2, \cdots, n)$ is a multiplier of

$$\overline{L}: \quad x_1^{\sigma_1} \cdots x_{i-1}^{\sigma_{i-1}} \; x_i^{\sigma_i} \cdots x_n^{\sigma_n}$$

if σ_i is equal to the maximum of all degrees of the variable x_i in all monomials of (M) having degrees in $x_1, \cdots, x_{i-1}$ equal respectively to the particular values $\sigma_1, \cdots, \sigma_{i-1}$ in $\overline{L}$.

(c) *Non-multiplier.* If x_i is not a multiplier of a monomial $\overline{L}$ it will be called a non-multiplier of $\overline{L}$.

THE SET (K)

Let α have any one of the values $0, 1, \cdots, n-1$; when the value of α has been selected it will be held fixed throughout the discussion of this section.

(a) *Multiple with respect to α.* The monomial $\overline{K}: x_1^{a_1} \cdots x_n^{a_n}$ is a multiple with respect to α of $\overline{\overline{K}}: x_1^{b_1} \cdots x_n^{b_n}$ if the first α of the differences (14.1) are zero and none of the remaining is negative.

(b) *Multiplier with respect to α.* The variable $x_i\,(i = \alpha+1, \cdots, n;\ \alpha \geqq 1)$ is a multiplier with respect to α of

$$\overline{K}: \quad x_1^{\tau_1} \cdots x_{i-1}^{\tau_{i-1}} \; x_i^{\tau_i} \cdots x_n^{\tau_n}$$

if τ_i is equal to the maximum of all degrees of the variable x_i in all monomials of (M) having degrees in $x_1, \cdots, x_{i-1}$ equal respectively to the particular values $\tau_1, \cdots, \tau_{i-1}$ in $\overline{K}$. If $\alpha = 0$, the multipliers of $\overline{K}$ are determined as though this monomial were a monomial $\overline{L}$.

(c) *Non-multiplier with respect to α.* If $x_i\,(i = \alpha+1, \cdots, n)$ is not a multiplier of a monomial $\overline{K}$ it will be called a non-multiplier of $\overline{K}$.

(d) *Variables $x_1, \cdots, x_\alpha$.* The variables $x_1, \cdots, x_\alpha$ will be neither multipliers nor non-multipliers with respect to α of any monomial $\overline{K}$.

For brevity in the following discussion the above phrase "with respect to α" will be omitted since it is to be understood that the value of α, when once selected is to be held fixed throughout this section. It should

be noted that if $\alpha = 0$, the above definitions regarding the set (K) become identical with the corresponding definitions for the set (L).

The following example will serve to illustrate the above definitions. Let $\alpha = 2$.

	Monomials	Multipliers	Non-multipliers
	$x^2\ y^3\ z^2$	$y\ z$	$x\ \cdot\ \cdot$
(L)	$x^2\ y^2\ z^2$	$\cdot\ z$	$x\ y\ \cdot$
	$x^2\ y\ z^2$	$\cdot\ z$	$x\ y\ \cdot$
	$x\ y\ z^2$	$y\ z$	$x\ \cdot\ \cdot$
	x^3	$\cdot\ z$	$\cdot\ \cdot\ \cdot$
(K)	$x\ y$	$\cdot\ \cdot$	$\cdot\ \cdot\ z$

We shall now define $n+1$ different sets of monomials $(N_1), (N_2), \cdots, (N_n)$, (N_k) which together will comprise a set (N) of monomials. *The set* (N) *will be said to be complementary to the set* (M). Corresponding to the previous notation we will denote a monomial of (N) by $\bar{N}$, $\bar{\bar{N}}$, etc., a monomial of (N_1) will be denoted by $\bar{N}_1$, $\bar{\bar{N}}_1$, etc. etc.

THE SET (N)

(a) *The set* (N_1) is composed of all monomials x_1^{β} where β is a positive integer or zero which is less than the maximum of the exponents of x_1 in (M) and which does not appear among the exponents of x_1 in (M).

(b) *The set* (N_i) where $i = 2, \cdots, n$ is composed of all monomials that can be formed from

$$x_1^{\sigma_1} \cdots x_{i-1}^{\sigma_{i-1}} x_i^{\beta}$$

by taking $\sigma_1, \cdots, \sigma_{i-1}$ as any system of exponents of $x_1, \cdots, x_{i-1}$ respectively of a monomial of (M) and β as a positive integer or zero such that (1) β is less than the maximum exponent of x_i of monomials of (M) of the form

$$x_1^{\sigma_1} \cdots x_{i-1}^{\sigma_{i-1}} \cdots$$

and (2) β does not appear among the exponents of x_i in these monomials of (M).

(c) *The set* (N_k). Consider a monomial $\bar{K}$: $x_1^{a_1} \cdots x_n^{a_n}$ which would have at least one of the variables $x_1, \cdots, x_\alpha$ as a multiplier if all of the monomials of (M) were assumed to belong to the set (L). Let $x_{\nu_1}, \cdots, x_{\nu_r}$ where $\nu_i \leqq \alpha$ and $\alpha = 1, \cdots, n-1$ be the hypothetical multipliers of this monomial $\bar{K}$ and form the r $(\leqq \alpha)$ monomials

$$x_1^{a_1} \cdots x_{\nu_i}^{a_{\nu_i}+1} \cdots x_n^{a_n} \qquad (i = 1, \cdots, r).$$

The totality of these latter monomials constitutes the set (N_k). For $\alpha = 0$, the set (N_k) is not defined.

(d) *Multipliers of* $\bar{N}_1$. The variables $x_2, \cdots, x_n$ are multipliers for any monomial $\bar{N}_1$.

(e) *Multipliers of* $\bar{N}_i$ *where* $i = 2, \cdots, n$. The multipliers of a monomial $\bar{N}_i$ having the quantities $\sigma_1 \cdots \sigma_{i-1}$ as exponents of the variables $x_1, \cdots, x_{i-1}$ respectively are (1) the variables $x_{i+1}, \cdots, x_n$ and (2) those variables of the set $x_1, \cdots, x_{i-1}$ which would be multipliers of a monomial of (M) of the form

$$x_1^{\sigma_1} \cdots x_{i-1}^{\sigma_{i-1}} \cdots$$

if all the monomials of (M) were considered to belong to the set (L).

(f) *Multipliers of* $\bar{N}_k$. The multipliers of any monomial

$$\bar{N}_k: \quad x_1^{a_1} \cdots x_{\nu_i}^{a_{\nu_i}+1} \cdots x_n^{a_n}$$

are the multipliers of the corresponding monomial $\bar{K}$: $x_1^{a_1} \cdots x_n^{a_n}$ when all monomials of (M) are considered to belong to the set (L) with the exception of $x_{\nu_1}, \cdots, x_{\nu_{i-1}}$.

Let us find the complementary set to the set (M) treated in the above example.

	Complementary sets	Multipliers
(N_1)	1	$\cdot \quad y \quad z$
(N_2)	x	$\cdot \quad \cdot \quad z$
	x^2	$\cdot \quad \cdot \quad z$
(N_3)	$x \quad y \quad z$	$\cdot \quad y \quad \cdot$
	$x^2 \quad y^3 \quad z$	$\cdot \quad y \quad \cdot$
	$x^2 \quad y^3$	$\cdot \quad y \quad \cdot$
	$x^2 \quad y^2 \quad z$	$\cdot \quad \cdot \quad \cdot$
	$x^2 \quad y^2$	$\cdot \quad \cdot \quad \cdot$
	$x^2 \quad y \quad z$	$\cdot \quad \cdot \quad \cdot$
	$x^2 \quad y$	$\cdot \quad \cdot \quad \cdot$
(N_K)	x^4	$x \quad y \quad z$
	$x^3 \quad y$	$\cdot \quad y \quad z$
	$x \quad y^2$	$\cdot \quad y \quad \cdot$

The class $(\bar{\mathfrak{M}})$ *of a monomial* $\bar{M}$ *is the set of all monomials which can be obtained by forming the product of* $\bar{M}$ *and an arbitrary monomial in its multipliers.* It is to be observed that the class $(\bar{\mathfrak{M}})$ as so defined, contains the monomial $\bar{M}$ in case $\bar{M}$ possesses a multiplier and that in this case the class $(\bar{\mathfrak{M}})$ is composed of infinitely many monomials. *If* $\bar{M}$ *possesses no multipliers it will be assumed that* $(\bar{\mathfrak{M}})$ *consists of the single monomial* $\bar{M}$. Analogous definitions and remarks apply to the classes $(\bar{\mathfrak{N}}_i)$ where $i = 1, \cdots, n$ and $(\bar{\mathfrak{N}}_k)$ corresponding to a monomial $\bar{N}_i$ and $\bar{N}_k$ respectively.

We will say that a monomial of the class $(\bar{\mathfrak{M}})$, $(\bar{\mathfrak{N}}_i)$ or $(\bar{\mathfrak{N}}_k)$ arises from the monomial $\bar{M}$, $\bar{N}_i$ or $\bar{N}_k$ as basis. To designate a particular monomial of the class $(\bar{\mathfrak{M}})$ we will use the symbol $\bar{\mathfrak{M}}$, etc. The set of all classes $(\bar{\mathfrak{M}})$ will be denoted by $(\mathfrak{M})$; similarly the set of all classes $(\bar{\mathfrak{N}}_i)$ and $(\bar{\mathfrak{N}}_k)$ will be denoted by $(\mathfrak{N})$.

THEOREM I. *An arbitrary monomial P in $x_1, \cdots, x_n$ belongs to one and only one class $(\bar{\mathfrak{M}})$ or $(\bar{\mathfrak{N}})$.*

In proving this theorem we shall assume $\alpha \neq 0$; this is always permissable since by a change of notation a set (M) for which $\alpha = 0$ can be replaced by a set $(\tilde{M})$ for which the set $(\tilde{K})$ is vacuous. In case $n = 1$ the theorem follows immediately: all monomials in (M) then belong to the set (L) and any monomial P, i. e. x_1^a either belongs to (M) or (N) or is a multiple of x_1^b where $b(\geqq 0)$ is the highest power of x_1 in (M). Hence we shall assume the theorem for $n-1$ variables and prove it for n variables $(n \geqq 2)$. Let $e_1 < e_2 < \cdots < e_m$ be the exponents of x_1 in (M) and denote by $\lambda(\geqq 1)$ the degree of x_1 in P. The induction proof will be divided into the following cases.

Case 1. *λ is less than e_m and different from e_i $(i < m)$.* For this case P cannot belong to a class $(\bar{\mathfrak{L}})$ since x_1 is a multiplier only for those monomials of (L) for which the degree in x_1 is e_m; P cannot belong to a class $(\bar{\mathfrak{K}})$ since x_1 is not a multiplier of any monomial $\bar{K}$; P cannot belong to a class $(\bar{\mathfrak{N}}_k)$ since x_1 is a multiplier only for those monomials of the set (N_K) for which the degree in x_1 is e_m; likewise P cannot belong to a class $(\bar{\mathfrak{N}}_i)$ where $i = 2, \cdots, n$. Finally it is evident that P belongs to one and only one class $(\bar{\mathfrak{N}}_1)$.

Case 2. *λ is equal to e_h for $h < m$.* Here P can obviously arise only from some monomial $\bar{M}^\lambda$ or $\bar{N}^\lambda$ in (M) or (N) respectively for which x_1 has the exponent λ. Dividing a monomial $\bar{M}^\lambda$ or $\bar{N}^\lambda$ by x_1^λ let us denote the resulting monomial in the variables $x_2, \cdots, x_n$ by $\bar{M}^{*\lambda}$ or $\bar{N}^{*\lambda}$ respectively. The monomials $\bar{L}^{*\lambda}$ and $\bar{K}^{*\lambda}$ belong to the sets $(L^{*\lambda})$ and $(K^{*\lambda})$ of $(M^{*\lambda})$ for which $\alpha^* = \alpha - 1$ will be assumed. If $\alpha^* = 0$, $(M^{*\lambda})$ will be considered as a set $(\tilde{M})$ for which $(\tilde{K})$ is vacuous in accordance with the previous procedure; however the designations $\bar{L}^{*\lambda}$ and $\bar{K}^{*\lambda}$ will continue to be employed as indicative of the origin of these monomials from monomials $\bar{L}^\lambda$ and $\bar{K}^\lambda$ respectively. Then observe that the monomials $\bar{N}^{*\lambda}$ are complementary to the monomials of the set $(M^{*\lambda})$; also that the multipliers of a monomial $\bar{M}^{*\lambda}$ in the set $(M^{*\lambda})$ are the same as the multipliers of the corresponding monomial $\bar{M}^\lambda$ in (M); likewise that the multipliers of a monomial $\bar{N}^{*\lambda}$ in the set $(N^{*\lambda})$ are the same as the multipliers of the corresponding monomial $\bar{N}^\lambda$ in the set (N).

Now the monomial P/x_1^λ in the variables $x_2, \cdots, x_n$ arises from one and only one monomial $\bar{M}^{*\lambda}$ or $\bar{N}^{*\lambda}$ as basis since the above theorem is assumed for $n-1$ variables; hence P arises from the corresponding monomial $\bar{M}^\lambda$ or $\bar{N}^\lambda$ as basis, i. e. P belongs to the class $(\bar{\mathfrak{M}}^\lambda)$ or $(\bar{\mathfrak{N}}^\lambda)$. If P belongs to a class $(\bar{\mathfrak{M}}^\lambda)$ or $(\bar{\mathfrak{N}}^\lambda)$ the monomial P/x_1^λ belongs to the corresponding class $(\bar{\mathfrak{M}}^{*\lambda})$ or $(\bar{\mathfrak{N}}^{*\lambda})$ respectively; hence if P belonged to more than one class $(\bar{\mathfrak{M}}^\lambda)$ or $(\bar{\mathfrak{N}}^\lambda)$ the monomial P/x_1^λ would belong to more than one class $(\bar{\mathfrak{M}}^{*\lambda})$ or $(\bar{\mathfrak{N}}^{*\lambda})$ contrary to hypothesis.

Case 3. *λ is equal to or greater than e_m.* In this case the monomial P can arise only from a monomial $\bar{M}$ or $\bar{N}$ in which the exponent of x_1 is e_m or e_m+1. Just as in Case 2 let us consider the sets (M^{*e_m}) and (N^{*e_m}) obtained by dividing the monomials $\bar{M}^{e_m}$ and $\bar{N}^{e_m}$ by $x_1^{e_m}$ respectively. Then observe that the multipliers of the monomial $\bar{M}^{*e_m}$ in the set (M^{*e_m}) are the same as those for the corresponding monomial $\bar{M}^{e_m}$ in (M) with the exception of x_1 if $\bar{M}^{e_m}$ is a monomial $\bar{L}^{e_m}$; for the monomial $\bar{K}^{e_m}$ the variable x_1 does not occur as a multiplier. Also the monomials $\bar{N}^{*e_m}$ are complementary to the monomials of the set (M^{*e_m}); the multipliers of a monomial $\bar{N}^{*e_m}$ in the set (N^{*e_m}) are the same as those for the corresponding $\bar{N}^{e_m}$ in the set (N) with the exception of x_1. Now the monomial P/x_1^λ arises from one and only one monomial $\bar{M}^{*e_m}$ or $\bar{N}^{*e_m}$ as basis since the theorem is assumed for $n-1$ variables. If P/x_1^λ arises from a monomial $\bar{L}^{*e_m}$ or $\bar{N}_i^{*e_m}$, the monomial P can arise from the corresponding monomial $\bar{L}^{e_m}$ or $\bar{N}_i^{e_m}$. If P/x_1^λ arises from a monomial $\bar{K}^{*e_m}$ the monomial P arises from the corresponding $\bar{K}^{e_m}$ if $\lambda = e_m$ and from the monomial $\bar{N}_K^{e_m+1}$ as basis if $\lambda > e_m$; if P/x_1^λ arises from a monomial $\bar{N}_K^{*e_m}$, the monomial P arises from the corresponding monomial $\bar{N}_K^{e_m}$ if $\lambda = e_m$ and from the monomial $\bar{N}_K^{e_m+1}$ as basis if $\lambda > e_m$. Conversely suppose P belongs to any two of the classes $(\bar{\mathfrak{L}}^{e_m})$, $(\bar{\mathfrak{N}}_i^{e_m})$, $(\bar{\mathfrak{K}}^{e_m})$, $(\bar{\mathfrak{N}}_K^{e_m})$, $(\bar{\mathfrak{N}}_K^{e_m+1})$ with the exception of two classes $(\bar{\mathfrak{K}}^{e_m})$, $(\bar{\mathfrak{N}}_K^{e_m})$, $(\bar{\mathfrak{N}}_K^{e_m+1})$ of the monomials $\bar{K}^{e_m}$, $\bar{N}_K^{e_m}$, and $\bar{N}_K^{e_m+1}$ respectively where $\bar{N}_K^{e_m}$ and $\bar{N}_K^{e_m+1}$ correspond to the monomial $\bar{K}^{e_m}$, then P/x_1^λ belongs to two classes $(\bar{\mathfrak{M}}^{*e_m})$ and $(\bar{\mathfrak{N}}^{*e_m})$, contrary to hypothesis. It is easily seen that P cannot belong to two of the excluded classes $(\bar{\mathfrak{K}}^{e_m})$, $(\bar{\mathfrak{N}}_K^{e_m})$ and $(\bar{\mathfrak{N}}_K^{e_m+1})$.

The set (M) is said to be complete if any monomial P which is a multiple of at least one monomial of the set (M) belongs to the class $(\mathfrak{M})$. It follows from Theorem I that the multiples of a complete set (M) are separated into a finite number of mutually exclusive classes, namely the classes $(\bar{\mathfrak{M}})$; monomials which are not multiples of any $\bar{M}$ are likewise separated into a finite number of mutually exclusive classes, i. e. the classes $(\bar{\mathfrak{N}})$.

The set (M) *is said to be strongly complete if any monomial* P *which is a multiple of at least one monomial of the set* (L) *belongs to the class* $(\mathfrak{L})$ *and any monomial* Q *which is a multiple of at least one monomial of the set* (K) *belongs to the class* $(\mathfrak{M})$.[36] A set which is strongly complete is also complete; the definition of strong completeness is introduced since Theorem II below gives a convenient test for strong completeness while we do not have such a test for completeness.

THEOREM II. *A necessary and sufficient condition that a set* (M) *be strongly complete is that the product of any monomial* $\bar{L}$ *by one of its non-multipliers belongs to the class* $(\mathfrak{L})$ *and the product of any* $\bar{K}$ *by one of its non-multipliers belongs to the class* $(\mathfrak{M})$.

The necessity of the condition in the theorem follows immediately. The product of any monomial $\bar{L}$ by one of its non-multipliers is a multiple of $\bar{L}$ and hence belongs to the class $(\mathfrak{L})$. Likewise the product of any monomial $\bar{K}$ by one of its non-multipliers is a multiple of $\bar{K}$ and hence belongs to the class $(\mathfrak{M})$.

Let us prove the sufficiency condition for the case $n = 1$; this implies $\alpha = 0$. Suppose that $e_1 < e_2 < \cdots < e_m$ are the exponents of x_1 in the set (L) and $d_1 < d_2 < \cdots < d_q$ are the exponents of x_1 in the set (K). Then the class $(\mathfrak{M})$ consists of the monomials

$$x_1^{d_1}, \cdots, x_1^{d_q};\ x_1^{e_1}, \cdots, x_1^{e_m};\ x_1^{a+1}, x_1^{a+2}, \cdots \tag{14.2}$$

where a is the maximum exponent in (M). But the monomials

$$x_1^{e_1+1}, \cdots, x_1^{e_{m-1}+1}$$

belong to the class $(\mathfrak{L})$ by hypothesis. Hence $e_1+1 = e_2, \cdots, e_{m-1}+1 = e_m$ and the exponents $e_1, \cdots, e_m$ are consecutive integers. Suppose $e_m \neq a$, then x_1 is a non-multiplier for $x_1^{e_m}$ and so $x_1^{e_m+1}$ belongs to the class $(\mathfrak{L})$ by hypothesis. However upon examining (14.2) we find that if $e_m \neq a$ the monomial $x_1^{e_m+1}$ could only belong to the class $(\mathfrak{K})$ or $(\mathfrak{N}_1)$ and we have a contradiction. Hence $e_m = a$. By hypothesis $x_1^{d_q+1}$ then belongs to a class $(\bar{\mathfrak{M}})$ so that $d_q+1 = e_1$ and $d_1, \cdots, d_q$ are also consecutive integers. A set of monomials of this type is easily seen to be strongly complete.

In order to prove the sufficiency condition for $n > 1$ we assume that every product of an $\bar{L}$ by any one of its non-multipliers belongs to the class $(\mathfrak{L})$ and that every product of a $\bar{K}$ by any one of its non-multipliers belongs to the class $(\mathfrak{M})$; then we show that (1) the product of every monomial $\bar{L}$ by an arbitrary monomial in $x_1, \cdots, x_n$ belongs to the

[36] When $\alpha = 0$, this definition introduces a distinction between the sets (L) and (K).

class ($\mathfrak{L}$) and (2) the product of every monomial $\bar{K}$ by an arbitrary monomial in $x_{\alpha+1}, \cdots, x_n$ belongs to the class ($\mathfrak{M}$). Let us assume the theorem for $n-1$ variables and prove it for n variables.

Again denote by $\bar{M}^{*\lambda}$ the quotient of $\bar{M}^{\lambda}$ by x_1^{λ}. The monomials $\bar{L}^{*\lambda}$ and $\bar{K}^{*\lambda}$ will belong to the sets $(L^{*\lambda})$ and $(K^{*\lambda})$ with $\alpha^* = \alpha - 1$ for $\alpha \geqq 1$ and $\alpha^* = 0$ for $\alpha = 0$. Then those variables $x_2, \cdots, x_n$ which are multipliers or non-multipliers for $\bar{M}^{\lambda}$ in the set (M) are precisely the multipliers or non-multipliers respectively for $\bar{M}^{*\lambda}$ in $(M^{*\lambda})$. Hence the product of any monomial $\bar{\mathfrak{L}}^{*\lambda}$ of the class arising from $\bar{L}^{*\lambda}$ in the set $(M^{*\lambda})$ and x_1^{λ} belongs to the class $(\bar{\mathfrak{L}}^{\lambda})$. Similarly the product of any monomial $\bar{\mathfrak{R}}^{*\lambda}$ of the class arising from $\bar{K}^{*\lambda}$ in the set $(M^{*\lambda})$ and x_1^{λ} belongs to the class $(\bar{\mathfrak{R}}^{\lambda})$. By assumption the product of every $\bar{L}^{\lambda}$ by one of its non-multipliers selected from $x_2, \cdots, x_n$ belongs to the class ($\mathfrak{L}$); this product $x_\beta \bar{L}^{\lambda}$ is of degree λ in x_1 and hence must belong to a class $(\bar{\bar{\mathfrak{L}}}^{\lambda})$, i. e. the class arising from a monomial $\bar{\bar{L}}^{\lambda}$ in (M). Hence

$$x_1^{\lambda} x_\beta \bar{L}^{*\lambda} = x_1^{\lambda} p \bar{\bar{L}}^{*\lambda},$$

where p is a monomial in multipliers of $\bar{\bar{L}}^{*\lambda}$ in the set $(M^{*\lambda})$. That is (1) the product of a monomial $\bar{L}^{*\lambda}$ by a non-multiplier of $\bar{L}^{*\lambda}$ in the set $(M^{*\lambda})$ belongs to the class $(\bar{\bar{\mathfrak{L}}}^{*\lambda})$ of a monomial $\bar{\bar{L}}^{*\lambda}$ in $(M^{*\lambda})$. Similarly the product of $\bar{K}^{\lambda}$ by one of its non-multipliers x_β belongs by assumption to the class $(\bar{\bar{\mathfrak{M}}}^{\lambda})$. Hence

$$x_1^{\lambda} x_\beta \bar{K}^{*\lambda} = x_1^{\lambda} p \bar{\bar{M}}^{*\lambda},$$

where p is a monomial in multipliers of $\bar{\bar{M}}^{*\lambda}$ in the set $(M^{*\lambda})$. That is (2) the product of a monomial $\bar{K}^{*\lambda}$ by a non-multiplier of $\bar{K}^{*\lambda}$ in the set $(M^{*\lambda})$ belongs to the class $(\bar{\bar{\mathfrak{M}}}^{*\lambda})$ of a monomial $\bar{\bar{M}}^{*\lambda}$ in $(M^{*\lambda})$. By assumption for $n-1$ variables it follows therefore that the set $(M^{*\lambda})$ is complete on account of statements (1) and (2). Hence we have the strongly product of a monomial $\bar{L}^{*\lambda}$ by an arbitrary monomial in $x_2, \cdots, x_n$ belongs to the class $(\mathfrak{L}^{*\lambda})$; that the product of a monomial $\bar{K}^{*\lambda}$, with $\alpha = 0$, by an arbitrary monomial $x_2, \cdots, x_n$ belongs to the class $(\mathfrak{M}^{*\lambda})$; that the product of a monomial $\bar{K}^{*\lambda}$ with $\alpha \geqq 1$ by an arbitrary monomial in $x_{\alpha+1}, \cdots, x_n$ belongs also to the class $(\mathfrak{M}^{*\lambda})$. Thus we have proved that for $\alpha \geqq 1$ a multiple of a monomial $\bar{K}$ belongs to the class ($\mathfrak{M}$), that the product of a monomial $\bar{K}$, with $\alpha = 0$, by an arbitrary monomial in $x_2, \cdots, x_n$ belongs to the class ($\mathfrak{M}$), and that the product of a monomial $\bar{L}$ by an arbitrary monomial in $x_2, \cdots, x_n$ belongs to the class ($\mathfrak{L}$). It remains to prove that the products of a monomial $\bar{L}$ and a monomial $\bar{K}$, with

$\alpha = 0$, by an arbitrary monomial of degree $\varrho\,(\geqq 1)$ in x_1 belong to the class $(\mathfrak{L})$ and the class $(\mathfrak{M})$ respectively.

Now assume that the product of a monomial $\bar{L}$ by an arbitrary monomial of degree $\varrho\,(\geqq 0)$ in x_1 belongs to the class $(\mathfrak{L})$ and let us seek to prove the property for $\varrho+1$. By this hypothesis the product of a monomial $\bar{L}$ by an arbitrary monomial of degree $\varrho+1$ in x_1 can be considered as the product of x_1 and a particular monomial $\bar{\bar{\mathfrak{L}}}$. If the degree of $\bar{\bar{\mathfrak{L}}}$ is less than e_m the monomial $\bar{\bar{\mathfrak{L}}}$ belongs to a class arising from a monomial $\bar{\bar{L}}$ for which x_1 is not a multiplier; $x_1\,\bar{\bar{L}}$ is therefore at most of degree e_m in x_1. Hence $x_1\,\bar{\bar{\mathfrak{L}}}$ is the product of $x_1\,\bar{L}$ and a monomial in $x_2, \cdots, x_n$. But $x_1\,\bar{\bar{L}}$ belongs to the class $(\mathfrak{L})$ by hypothesis since x_1 is a non-multiplier of $\bar{\bar{L}}$. Hence $x_1\,\bar{\bar{L}}$ is the product of a monomial $\bar{\bar{L}}$ and a monomial in $x_2, \cdots, x_n$. By the above paragraph the monomial $x_1\,\bar{\bar{\mathfrak{L}}}$ therefore belongs to the class $(\mathfrak{L})$. If the degree of $\bar{\bar{\mathfrak{L}}}$ is equal to or greater than e_m the monomial $\bar{\bar{\mathfrak{L}}}$ belongs to the class $(\bar{\bar{\mathfrak{L}}})$ arising from a monomial $\bar{\bar{L}}$ for which x_1 is a multiplier and hence $x_1\,\bar{\bar{\mathfrak{L}}}$ belongs to the same class $(\bar{\bar{\mathfrak{L}}})$. In other words we have proved that the product of a monomial $\bar{L}$ by an arbitrary monomial belongs to a class $(\bar{\bar{\mathfrak{L}}})$. The argument used here can be applied without modification to show that the product of a monomial $\bar{K}$, with $\alpha = 0$, by an arbitrary monomial belongs to the class $(\mathfrak{M})$ and the theorem is proved.

Upon examination of the set (M) in the above example, one finds that the monomial $x^3 y^3 z^2$ does not belong to any class $(\bar{\bar{\mathfrak{M}}})$ and hence the set (M) is not complete.

The following example exhibits a set (M) with $\alpha = 1$ which is complete but not strongly complete.

	Monomials	Multipliers	Non-multipliers
(L)	$x^2\ y^2\ z$	$x\ y\ z$	$\cdot\ \cdot$
	$x^2\ y\ z$	$x\ \cdot\ z$	$\cdot\ y$
	$x\ y\ z$	$\cdot\ \cdot\ z$	$x\ y$
(K)	$x\ y^2$	$\cdot\ y\ z$	$\cdot\ \cdot$

It is evident by inspection that the above set (M) is complete. The requirement that the product of $\bar{L}$ by a non-multiplier belong to $(\mathfrak{L})$ would mean that the monomial $x\,y^2\,z$ would belong to the class $(\mathfrak{L})$; however the monomial $x\,y^2\,z$ belongs to the class $(\mathfrak{R})$.

The set (M) will be said to be normal if no monomial $\bar{K}$ when considered as an $\bar{L}$, is a multiple of any monomial $\bar{\bar{L}}$ in the set (L). In the applications of the theory of monomials we shall have occasion to deal with normal sets (M).

We are now going to give a procedure for obtaining a strongly complete set (M_ν) from an arbitrary normal set (M_1) such that the multiples of the monomials of (M_ν) and (M_1) are identical. Those products of a monomial $\bar{L}_1$ by one of its non-multipliers which do not belong to the

class $(\mathfrak{L}_1)$ will be added to the set (L_1) to form the set (M_2). On account of the fact that (M_1) is normal the set (M_2) will be composed of distinct monomials, etc. We must show that after a finite number of such operations A we will arrive at a set (M_μ) after which no new monomials will be added by this process. Then form the product of a monomial $\bar{K}_\mu$ with one of its non-multipliers and if this product does not belong to the class $(\mathfrak{M}_\mu)$ add the monomial to (K_μ) to form $(M_{\mu+1})$. Let this operation be denoted by B.

Consider the numbers $\sigma_1, \cdots, \sigma_n$ equal respectively to the maximum exponents of the corresponding variables $x_1, \cdots, x_n$ in the set (M_1). The operation A cannot add a monomial P: $x_1^{\tau_1} \cdots x_n^{\tau_n}$ with $\tau_\beta > \sigma_\beta$ for any value of $\beta\ (= 1, \cdots, n)$. For suppose such were the case; let x_γ be the variable $x_1, \cdots, x_n$ for which $\tau_\gamma = \sigma_\gamma + 1$; then we have

$$P \equiv x_\gamma \bar{L}_1 \equiv x_1^{\tau_1} \cdots x_{\gamma-1}^{\tau_{\gamma-1}} x_\gamma^{\sigma_{\gamma+1}} \cdots,$$

where x_γ is a non-multiplier of $\bar{L}_1$. But x_γ is not a non-multiplier of $\bar{L}_1$ since the exponent of x_γ in $\bar{L}_1$ is equal to the maximum of the exponents of x_γ in (M_1). Since there are only a finite number of monomials P with $\tau_i \leqq \sigma_i\ (i = 1, \cdots, n)$ we must after a finite number of operations A arrive at a set (M_μ) after which the operations A will cease to add new monomials. Similar remarks apply to the operation B.

Now consider the set (M_μ) formed from (M_1) by operations A, the set (M_μ) being such that no new monomials can be added by further operations A. It is evident that the set (M_μ) is normal. Let us show that the property of normality is likewise preserved when we make an operation B upon the set (M_μ). Suppose that the monomial

$$\bar{K}_\mu: \quad x_1^{a_1} \cdots x_\sigma^{a_\sigma - 1} \cdots x_n^{a_n}$$

possesses the non-multiplier x_σ and that the monomial

$$\bar{K}_{\mu+1}: \quad x_1^{a_1} \cdots x_\sigma^{a_\sigma} \cdots x_n^{a_n}$$

is added to the set (M_μ) by the operation B. Let us assume that $\bar{K}_{\mu+1}$ is a multiple of a monomial $\bar{L}_\mu$ when $\bar{K}_{\mu+1}$ is considered as belonging to the set (L_μ), i. e. that the set $(M_{\mu+1})$ is not normal. Hence we have

$$\bar{L}_\mu \equiv x_1^{b_1} \cdots x_n^{b_n} \qquad (b_j \leqq a_j).$$

If $b_1 < a_1$ it follows that x_1 is a non-multiplier for $\bar{L}_\mu$. Since operation A can add no new monomials to the set (L_μ) the monomial $x_1 \bar{L}_\mu$ must belong to the class $(\bar{\bar{\mathfrak{L}}}_\mu)$ with the basis

$$\bar{\bar{L}}_\mu: \quad x_1^{b_1+1} x_2^{c_2} \cdots x_n^{c_n} \qquad \begin{pmatrix} b_1 + 1 \leqq a_1 \\ c_j \leqq b_j \leqq a_j \\ j = 2, \cdots, n \end{pmatrix}.$$

If $b_1+1<a_1$, we repeat the process until we have finally that the monomial $\bar{K}_{\mu+1}$ is a multiple of

$$\tilde{L}_\mu: \quad x_1^{a_1} x_2^{d_2} \cdots x_n^{d_n} \qquad (d_j \leqq a_j;\ j=2, \cdots, n).$$

Then repeat the process for the variables $x_2, \cdots, x_\sigma$ each time comparing the monomial in question of the set (L_μ) with monomial $\bar{K}_\mu$ to determine whether or not the variable $x_2, \cdots, x_\sigma$ in question is a multiplier. By repetition we have finally that the monomial $\bar{K}_{\mu+1}$ is multiple of

$$\tilde{\tilde{L}}_\mu: \quad x_1^{a_1} \cdots x_\sigma^{a_\sigma} x_{\sigma+1}^{e_{\sigma+1}} \cdots x_n^{e_n} \quad \begin{pmatrix} e_j \leqq a_j \\ j=\sigma+1, \cdots, n \end{pmatrix}.$$

Let x_τ be the first of the variables $x_{\sigma+1}, \cdots, x_n$ such that (1) $e_\tau < a_\tau$ and (2) x_τ is a non-multiplier for $\tilde{\tilde{L}}_\mu$. Then $x_\tau \tilde{\tilde{L}}_\mu$ belongs to the class $(\hat{\Omega}_\mu)$ with basis

$$\hat{L}_\mu: \quad x_1^{a_1} \cdots x_\sigma^{a_\sigma} x_{\sigma+1}^{e_{\sigma+1}} \cdots x_\tau^{e_\tau+1} x_{\tau+1}^{s_{\tau+1}} \cdots x_n^{s_n} \qquad (s_i \leqq a_i;\ i=\tau+1, \cdots, n).$$

A continuation of this process will show that $\bar{K}_{\mu+1}$ belongs to the class of the monomial $L_\mu^*: x_1^{a_1} \cdots x_\sigma^{a_\sigma} \cdots$, where each of the variables x_ν of the set $x_{\sigma+1}, \cdots, x_n$ is either a multiplier for L_μ^* or else appears in L_μ^* with the exponent a_ν. Thus we have a contradiction and $\bar{K}_{\mu+1}$ is not added by operation B.

If a monomial $\bar{\bar{K}}_{\mu+1}$ is added by operation B then repeat operation A on the set $(M_{\mu+1})$ until no new monomials are added by further operations; then repeat operation B once, etc., until finally we arrive at a strongly complete set (M_ν). Since operations A and B add only multiples of monomials $\bar{L}$ and $\bar{K}$ to the sets (L) and (K) respectively in the process of forming (M_ν), it is evident that the multiples of the monomials of (M_1) and (M_ν) are identical.

The set (M) treated in the above example is normal. If one adds the monomials $x^3y^3z^2$, $x^3y^2z^2$, x^3yz^2 to the set (L_1) to form the set (L_2) and then adds the monomial xyz to the set (K_2) to form the set (K_3) it is seen that the resulting set satisfies the condition of Theorem II and hence (M_3) is complete.

	Monomials			Multipliers			Non-Multipliers		
	x^3	y^3	z^2	x	y	z	·	·	·
	x^3	y^2	z^2	x	·	z	·	y	·
	x^3	y	z^2	x	·	z	·	y	·
(L_2)	x^2	y^3	z^2	·	y	z	x	·	·
	x^2	y^2	z^2	·	·	z	x	y	·
	x^2	y	z^2	·	·	z	x	y	·
	x	y	z^2	·	y	z	x	·	·
	x^3			·	·	z	·	·	·
(K_3)	x	y	z	·	·	·	·	·	z
	x	y		·	·	·	·	·	z

A monomial $x_1^{a_1} \cdots x_n^{a_n}$ is said to be of *higher or lower rank* than a different monomial $x_1^{b_1} \cdots x_n^{b_n}$ according as the first of the differences (14.1) which is not zero is positive or negative.

THEOREM III. *The product of any monomial of a complete set by one of its non-multipliers is equal to the product of a monomial of the complete set of higher rank, by multipliers of the latter.*

By definition the product of a monomial $\bar{M}$ of a complete set (M) by one of its non-multipliers is the product of another monomial $\bar{\bar{M}}$ of the set (M) by certain multipliers of the latter. We must show that $\bar{\bar{M}}$ is of higher rank than $\bar{M}$. Consider a monomial $\bar{M}$: $x_1^{a_1} \cdots x_n^{a_n}$ and let x_1 be the non-multiplier in question. In order that $x_1 \bar{M}$ be a multiple of $\bar{\bar{M}}$: $x_1^{b_1} \cdots x_n^{b_n}$ we must have $a_1 + 1 \geqq b_1$. If $\bar{\bar{M}} = \bar{\bar{K}}$, with $\alpha \geqq 1$, x_1 would not be a multiplier for $\bar{\bar{K}}$; the degree of $\bar{\bar{K}}$ in x_1 in that case could not be raised to $a_1 + 1$ by multiplication by multipliers of $\bar{\bar{K}}$. Hence $a_1 + 1 = b_1$. Similarly if $\bar{\bar{M}} = \bar{\bar{L}}$ or if $\bar{\bar{M}} = \bar{\bar{K}}$, with $\alpha = 0$, and if b_1 were less than $a_1 + 1$ then x_1 would not be a multiplier of $\bar{\bar{L}}$ or $\bar{\bar{K}}$ since x_1 is a multiplier only for monomials of the set (M) which contain the highest power of x_1. If x_σ is the non-multiplier of $\bar{M}$ it follows in exactly the same manner that

$$a_1 \geqq b_1, \; a_2 \geqq b_2, \; \cdots, \; a_\sigma + 1 \geqq b_\sigma.$$

By repetition of the above argument

$$a_1 = b_1, \; \cdots, \; a_\sigma + 1 = b_\sigma;$$

the monomial $\bar{\bar{M}}$ is therefore of higher rank than $\bar{M}$.

§ 15. General Existence Theorem.

For the purpose of ordering a system of partial differential equations of the form

$$\frac{\partial^{a_1 + \cdots + a_n} u}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}} = F\left(x; \; u; \; \frac{\partial u}{\partial x}; \; \cdots\right) \tag{15.1}$$

we shall assign to each independent variable and to each unknown s successive cotes which as is well known are represented by zero, positive and negative integers.[37] The qth cote of an arbitrary derivative of u_i is then obtained by adding the qth cote of the unknown u_i and the qth cotes of all variables of differentiation, distinct or not. If the cotes of a derivative z, namely $C_1, \cdots, C_s$ are not all equal to the cotes of a derivative z^*, namely $C_1^*, \cdots, C_s^*$ then z will be said to precede or follow z^*, or to be of lower or higher rank than z^*, according as the first of the differences

$$C_1 - C_1^*, \; \cdots, \; C_s - C_s^*$$

[37] Cotes were introduced and used systematically by Riquier, *loc. cit.*, (5), p. 201.

which is not zero, is negative or positive. The order which is thereby established among the derivatives appearing in (15.1) by the first s cotes is therefore not effected by assigning additional cotes. If the order of all derivatives of the unknowns $u_1, \cdots, u_r$ with respect to variables $x_1, \cdots, x_n$ is not uniquely established by the assignment of the above s cotes ordering the derivatives appearing in (15.1), we can accomplish this by the assignment of additional cotes. For example, if we assign the $n+1$ additional cotes

	x_1	x_2	$\cdots$	x_n	u_1	u_2	$\cdots$	u_r
$s+1$	0	0	$\cdots$	0	1	2	$\cdots$	r
$s+2$	1	0	$\cdots$	0	0	0	$\cdots$	0
$s+3$	0	1	$\cdots$	0	0	0	$\cdots$	0
.	.	.	$\cdots$	.	.	.	$\cdots$	.
.	.	.	$\cdots$	.	.	.	$\cdots$	.
.	.	.	$\cdots$	.	.	.	$\cdots$	.
$s+n+1$	0	0	$\cdots$	1	0	0	$\cdots$	0

all derivatives are uniquely ordered. Thus, if two derivatives $D_i u_\sigma$ and $D_j u_\tau$ have exactly the same set of $s+n+1$ cotes, the equality of the $(s+1)$st cotes implies that the functions u_σ and u_τ are identical and from the equality of the remaining cotes the derivatives are identical.

Let us say that the totality of derivatives of the unknowns $u_1, \cdots, u_r$ of first cote C constitutes the class C as a matter of convenient terminology.

It will be assumed that the above system (15.1) satisfies the following conditions:

(A) The equations (15.1), the unknown functions $u_1, \cdots, u_r$ and the independent variables $x_1, \cdots, x_n$ are finite in number.

(B) The equations (15.1) are composed of two systems of equations, (1) the system R which holds throughout the region D: $|x_i| < c_i$ where the c's are constants and (2) the system S which holds throughout the subspace $\mathfrak{D}$: $x^1 = 0, \cdots, x^\alpha = 0$ of the region D.

(C) The equations R are solved for certain derivatives of the unknowns u and the functions F which constitute the right members of these equations depend on the independent variables $x_1, \cdots, x_n$, the unknowns $u_1, \cdots, u_r$ and the derivatives of the unknowns.[38]

(D) The equations S are solved for certain derivatives which are not left members of R and the functions F which constitute the right members of these equations depend on the independent variables $x_{\alpha+1}, \cdots, x_n$ and the derivatives of the unknowns which are not left members of R or S.

(E) The functions F which constitute the right members of the system (15.1) are analytic functions in the neighborhood of the values

[38] The term *derivative* is to be interpreted as including the unknowns u themselves.

$$x = 0, \quad u = (u)_0; \qquad \frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x}\right)_0; \cdots \tag{15.2}$$

of their arguments.

(*F*) Cotes, s in number, are assigned to the unknowns $u_1, \cdots, u_r$ and the independent variables $x_1, \cdots, x_n$ such that (1) the first cote of each variable (x) is unity and (2) the derivatives of the functions (u) are uniquely ordered.

(*G*) The order established by the above assignment of cotes is such that any derivative z in the right member of an equation of the system R except those for which the derivative $\partial F/\partial z$ vanishes throughout the subspace $\mathfrak{D}$, precedes the derivative in the left member of the equation; also any derivative in the right member of an equation of the system S, precedes the corresponding left member.

(*H*) In any equation of the system R the class of the derivatives in the right member does not exceed the class of the derivative in the left member of the equation.

It is to be noted that if a derivative z precedes a derivative z^*, that $\partial z/\partial x^\sigma$ will precede $\partial z^*/\partial x^\sigma$; likewise z will precede z^* if z is of lower class than z^*. All derivatives of a particular unknown u_i in the class C will moreover be of the same order.

Let us now associate with the derivatives of the unknown u_i which appear in the left members of the system (15.1), the set (M_i) of monomials $x_1^{a_1} \cdots x_n^{a_n}$ corresponding to these derivatives; the set (L_i) of (M_i) will be composed of those monomials which correspond to derivatives of u_i in the left members of the system R and the set (K_i) of (M_i) will be composed of monomials corresponding to derivatives of u_i in the left members of the equations of the system S.

(I) The set (M_i) of monomials is strongly complete.

A system of equations (15.1) *will be said to be normal if the above conditions* $A, \cdots, I$ *are satisfied.*

By differentiation of an equation of a normal system (15.1) we obtain an equation which satisfies conditions G and H. Let z^* denote the derivative in the left member of an equation R or S and let z be a derivative in the right member of the same equation such that z precedes the derivative z^*. By differentiation of this equation we obtain

$$\frac{\partial z^*}{\partial x^i} = \frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x^i} + \cdots \equiv F' \tag{15.3}$$

where the derivatives z and $\partial z/\partial x^i$ in the right member precede the derivative in the left member. Suppose now that z follows z^* for an

equation of the system R; this implies that $\partial F/\partial z = 0$ throughout the subspace $\mathfrak{D}$. Hence

$$\frac{\partial F'}{\partial z'} = \frac{\partial F}{\partial z} = 0 \quad \text{throughout } \mathfrak{D},$$

where the derivative $\partial z/\partial x^i$ has been denoted by z'. In other words the equation (15.3) satisfies properties G and H. Hence G and H will be satisfied by the system of equations which result from a normal system (15.1) by the addition of a finite number of equations obtained from (15.1) by partial differentiation.

Differentiating the equations of the system (15.1) with respect to multipliers (x) of the corresponding monomials, each principle derivative of an unknown $u_1, \cdots, u_r$ will be obtained as the left member of one and only one of the resulting equations; this follows from the characteristic property of a complete system of monomials. Equations so formed by differentiation of an equation of the system R are to be added to R and those formed by differentiation of an equation of S are to be added to S. Thus we arrive at a system of equations consisting (1) of the system R and all equations which were added to R and (2) of the system S and all equations which were added to the system S; we shall refer to these as the systems $\tilde{R}$ and $\tilde{S}$ respectively. It is evident that a derivative z cannot appear both in the left and right member of an equation of $\tilde{R}$ or $\tilde{S}$. Now consider the system E composed of all equations of $\tilde{R}$ and $\tilde{S}$ whose left members belong to classes C of serial number not exceeding N; we shall refer to a particular equation of the system E by E_1, E_2, etc. Let z^* be the derivative of lowest rank which appears in the right member of an equation E_1 of the system $\tilde{R}$ and in the left member of an equation E_2 of $\tilde{R}$; using the equation E_2 we eliminate the derivative z^* from the right member of E_1. Similarly if the derivative z^* appears in the right member of an equation E_3 of the system $\tilde{S}$ and also in the left member of an equation E_4 of $\tilde{R}$ or $\tilde{S}$, the equation E_4 is to be used to eliminate z^* from the right member of E_3. By repetition of this process we finally reach a system $\mathfrak{E}$ equivalent to the system E and such that $\mathfrak{E}$ involves only parametric derivatives in its right members; more precisely if we denote by $\mathfrak{E}_R$ and $\mathfrak{E}_S$ the equations of $\mathfrak{E}$ corresponding to $\tilde{R}$ and $\tilde{S}$ respectively, then the right members of $\mathfrak{E}_R$ depend only on parametric derivatives of the system R and the right members of $\mathfrak{E}_S$ depend only on parametric derivatives of the system (15.1). The fact that we can limit our considerations to the system E in the process of forming the system $\mathfrak{E}$ is a consequence of the above result that properties G and H are preserved under differentiation. It is easily seen that conditions analogous to $A, \cdots, H$ are satisfied by the system $\mathfrak{E}$.

We are looking for power series of the type

$$u_i = \sum_{a=0}^{\infty} \frac{B_{a_1 \cdots a_n}}{a_1! \cdots a_n!} x_1^{a_1} \cdots x_n^{a_n} \tag{15.4}$$

which will satisfy (15.1). Let us first, however, consider the problem of determining a solution $u_1, \cdots, u_r$ of the system (15.1) such that each equation of (15.1) is satisfied when we equate to zero the non-multipliers (x) of the corresponding monomial. For brevity the system (15.1) so modified will be referred to as the system U^*. It is evident that the equations of the above system $\mathfrak{E}$, evaluated at $x_\beta = 0$, can be arranged in the increasing order of the principle derivatives in their left members and such that any equation will involve in its right member only parametric derivatives of the system (15.1). Allowing the number N to increase without limit we therefore see that for a given assignment of the initial values of the parametric derivatives, the power series (15.4), i. e. the formal solution $u_1, \cdots, u_r$ of the system U^* will be uniquely determined. But all parametric derivatives of (15.1) at $x_\beta = 0$ are uniquely determined if the derivatives of each unknown u_i corresponding to monomials of the complementary set (N_i), are assigned as arbitrary functions of the multipliers (x) of the associated monomials, these functions being analytic in the neighborhood of the values $x_\beta = 0$ of their arguments. The functions thus associated with an unknown u_i will be called (following Riquier) the initial determination of the unknown u_i; the sum of the initial determinations of an unknown u_i will be denoted by φ_i. Hence the formal power series solution (15.4) of the system U^* is uniquely determined by the assignment of the initial determinations; the convergence of these series will be proved in § 17.[39]

Let us now adopt the notation

$$\overline{Z}_i \equiv D_i u - F = 0 \tag{15.5}$$

for an equation of the system (15.1); here $D_i u$ denotes the derivative in the left member of the equation (15.1) in question. According as the expression $\overline{Z}_i$ corresponds to an equation of the system R or the system S, it will be denoted by $\overline{R}_i$ or $\overline{S}_i$ respectively. We shall speak of the derivative $D_i u$ as the *first term* of the corresponding expression $\overline{Z}_i$. For convenience in terminology we will speak of the cotes, the rank and the class of the first term of an expression $\overline{Z}_i$ as the cotes, the rank, and the class respectively of the expression $\overline{Z}_i$ itself; an analogous terminology

[39] While the condition of strong completeness is imposed by condition I it may be observed that the less restrictive condition of completeness of the sets (M_i) is sufficient for the determination of these power series.

will be applied to an expression obtained from $\overline{Z}_i$ by differentiation. Likewise the multipliers and non-multipliers (x) of the monomial corresponding to the first term of an expression $\overline{Z}_i$ will be referred to as the multipliers and non-multipliers (x) of the expression $\overline{Z}_i$ itself. We shall also denote by $\overline{\mathfrak{Z}}_i$ either the expression $\overline{Z}_i$ or an expression resulting by differentiation of $\overline{Z}_i$ with respect to its multipliers; more particularly $\overline{\mathfrak{Z}}_i$ will be denoted by $\overline{\mathfrak{R}}_i$ or $\overline{\mathfrak{S}}_i$ according as $\overline{\mathfrak{Z}}_i$ was obtained from an expression $\overline{R}_i$ or $\overline{S}_i$ respectively.

The derivative of the first term of an expression $\overline{R}_1$ with respect to a non-multiplier x_b of $\overline{R}_1$ is identical with the first term of some expression $\overline{\mathfrak{R}}_2$ by condition I. Furthermore the difference

$$\frac{d\overline{R}_1}{dx_b} - \overline{\mathfrak{R}}_2 \equiv \Phi\,(Du,\, x), \tag{15.6}$$

where the derivative of $\overline{R}_1$ denotes the result of differentiating the expression $\overline{R}_1$ considered as a function of the independent variables (x) alone, is such that (1) the derivative $\partial\Phi/\partial(Du)$ vanishes throughout the subspace $\mathfrak{D}$ for any derivative Du in Φ which is of higher rank than the first term of $d\overline{R}_1/dx_b$ and (2) the class of any derivative Du in Φ is at most equal to that of the first term of $d\overline{R}_1/dx_b$. Similarly, if x_c is a non-multiplier of $\overline{S}_1$, the expression

$$\frac{d\overline{S}_1}{dx_c} - \overline{\mathfrak{Z}}_2 \equiv \Psi(Du, x), \tag{15.7}$$

where the first terms of $d\overline{S}_1/dx_c$ and $\overline{\mathfrak{Z}}_2$ are identical, is such that any derivative Du in Ψ precedes the first term of the expression $d\overline{S}_1/dx_c$.

Now use the equations $\mathfrak{E}_R$ to eliminate all principle derivatives of the system R from the function Φ in (15.6); also use the combined system $\mathfrak{E}$ to eliminate all principle derivatives of (15.1) from the function Ψ in (15.7). Suppose that

$$\Phi(Du, x) \to \Phi^*(Du, x), \qquad \Psi(Du, x) \to \Psi^*(Du, x)$$

as the result of this elimination. *The normal system* (15.1) *will be said to be completely integrable if all functions* Φ^* *and* Ψ^* *which can be constructed from the equations of the system* (15.1), *vanish identically.*

In the following section it will be shown that a completely integrable normal system (15.1) is satisfied by the power series solution (15.4) of the system U^*.

EXISTENCE THEOREM. *A completely integrable normal system* (15.1) *is satisfied by a unique set of analytic functions* $u_1, \cdots, u_r$ *determined by the assignment of the initial determinations predicted by the form of the left members of the system.*

§ 16. Equivalence of the Solutions of a Completely Integrable Normal System and its Associated System U^*.

We now consider the problem of showing that the power series solution (15.4) of the system U^* satisfies the normal system (15.1) provided that this latter system is completely integrable.

Corresponding to an equation $\overline{\mathfrak{R}}_i = 0$ or $\overline{\mathfrak{S}}_j = 0$ we have identities of the form

$$(16.1)\quad (a)\quad D_i u \equiv \overline{\mathfrak{R}}_i + \mathfrak{F}(Du, x), \qquad (b)\quad D_j u \equiv \overline{\mathfrak{S}}_j + \mathfrak{F}(Du, x)$$

derivable by differentiation of (15.5), where the functions $\mathfrak{F}$ in (16.1 a) and (16.1 b) have properties regarding the derivatives Du analogous to those stated in connection with equations (15.6) and (15.7) respectively. Hence equations of the type (16.1 a) can be used to eliminate principle derivatives Du of the system R which occur in the functions $\mathfrak{F}$ in these equations; the infinite system so obtained as a result of this elimination furnishes equations which can be used in turn to eliminate principle derivatives Du of the system R from the functions Φ in equations of the type (15.6). The equation (15.6) thereby becomes

$$(16.2)\qquad \frac{d\overline{R}_1}{dx_b} - \overline{\mathfrak{R}}_2 \equiv \tilde{\Phi}(\overline{\mathfrak{R}}, Du, x),$$

where (a) the derivatives Du in $\tilde{\Phi}$ are parametric for the system R, (b) the class of an expression $\overline{\mathfrak{R}}$ in $\tilde{\Phi}$ is at most equal to the class of $d\overline{R}_1/dx_b$ and (c) any expression $\overline{\mathfrak{R}}$ in $\tilde{\Phi}$ which does not precede $d\overline{R}_1/dx_b$ is such that the derivative $\partial\tilde{\Phi}/\partial\overline{\mathfrak{R}}$ vanishes throughout the subspace $\mathfrak{D}$. Similarly from (15.7) we have

$$(16.3)\qquad \frac{d\overline{S}_1}{dx_c} - \overline{\mathfrak{Z}}_2 \equiv \tilde{\Psi}(\overline{\mathfrak{R}}, \overline{\mathfrak{S}}, Du, x),$$

where (d) the derivatives Du in $\tilde{\Psi}$ are parametric for the system (15.1) and (e) any expression $\overline{\mathfrak{R}}$ or $\overline{\mathfrak{S}}$ in $\tilde{\Psi}$ precedes $d\overline{S}_1/dx_c$.

If (15.1) is completely integrable the functions $\tilde{\Phi}$ and $\tilde{\Psi}$ must be such that they vanish in consequence of the vanishing of the expressions $\overline{\mathfrak{R}}$ and $\overline{\mathfrak{S}}$ appearing in these functions; this follows as a result of the way in which the above functions Φ^* and Ψ^* were obtained.

Now consider the system of equations composed of all equations of the type (16.2) and (16.3) which can be formed from (15.1) and into these equations substitute the solution $u_1, \cdots, u_r$ of the system U^*. The resulting system

$$(16.4)\qquad \frac{\partial\overline{Z}_i^*}{\partial x_j} = D_k\overline{Z}_l^* + \Pi(D\overline{Z}^*, x)$$

will be regarded as a system of equations for the determination of the unknowns $\bar{Z}_i^*$ where $\bar{Z}_i^*$ denotes the result of substituting the solution $u_1, \cdots, u_r$ of the system U^* in the expression $\bar{Z}_i$; it will furthermore be understood that the variables $x_1 = \cdots = x_\alpha = 0$ in an expression $\bar{S}_i^*$. In any equation (16.4) the first, second and third terms indicated result respectively from the first, second and third terms of the corresponding equation (16.2) or (16.3). We shall show in the following that (16.4) is a normal system, i.e. that these equations satisfy the above conditions $A, \cdots, I$ provided that we consider those equations (16.4) which contain a derivative $\partial \bar{R}_i^* / \partial x_j$ in their left member as the system R and the remaining equations (16.4) as the system S of condition B.

The monomials of the set (M_i) corresponding to the derivatives of an unknown $\bar{R}_i^*$ which appear in the left members of (16.4) all belong to the set (L_i). It can be shown that any set (M) of monomials of this type in which each monomial consists of a single variable x_β, is strongly complete.[40] Since each monomial of the set (M_i) for an unknown $\bar{S}_i^*$ belongs to the set (K_i) and furthermore since each of these monomials consists of one of the variables $x_{\alpha+1}, \cdots, x_n$ it follows that the set (M_i) can be considered as a set (M) in the $n-\alpha$ variables $x_{\alpha+1}, \cdots, x_n$ where all monomials of (M) belong to the set (L); hence (M_i) is strongly complete by the above argument. The condition I is therefore satisfied by the system (16.4).

Let us assign the cotes for the system (16.4) in the following manner:

(α) The variables x_β will be given the same set of s cotes as in the normal system (15.1).

(β) The unknowns $\bar{Z}_i^*$ will be given the same set of s cotes as $\bar{Z}_i$ in the system (15.1).

(γ) The variables x_β will each be given an $(s+1)$st cote of zero.

(δ) Suppose that the expressions $\bar{Z}_i$ which have derivatives of the same unknown u_j as their first terms, are denoted by $\bar{Z}_1^{(j)}, \bar{Z}_2^{(j)}, \cdots$ when arranged in the order of decreasing rank of the monomials corresponding to their first terms. The unknown $\bar{Z}_i^{*(j)}$ will be given an $(s+1)$st cote equal to the integer i.

The first s cotes of the derivative $D_k \bar{Z}_l^*$ in the right member of an equation (16.4) will be identical with the first s cotes of the derivative in the left member of this equation; however the $(s+1)$st cote of $D^k \bar{Z}_l^*$ will be less than the $(s+1)$st cote of the derivative in the left member in consequence of the above conditions (γ) and (δ) and Theorem III of § 14. Hence $D_k \bar{Z}_l^*$ precedes the derivative in the left member of (16.4). Also

[40] cf. Janet, *loc. cit.*, (6), p. 85.

the derivatives $D\bar{Z}^*$ in the function Π of (16.4) either precede the derivative in the left member of this equation or are such that the derivative $\partial \Pi / \partial (D\bar{Z}^*)$ vanishes over the subspace $\mathfrak{D}$; these results follow in consequence of the above statements (c) and (e) made in connection with equations (16.2) and (16.3) respectively. Hence the system (16.4) satisfies condition G.

Consider those equations (16.4) which involve a derivative of an $\bar{R}_i^*$ in their left members; any derivative of an unknown $\bar{Z}_i^*$ appearing in the right member of one of these equations will have class at most equal to the class of the corresponding left member in consequence of statement (b) under equation (16.2). Condition H is therefore satisfied by (16.4).

To see that equations (16.4) are actually solved for the derivatives $\partial \bar{Z}_i^* / \partial x_j$ appearing in the left members of these equations we have merely to note that x_j is a non-multiplier of the monomial corresponding to the first term of $\bar{Z}_i$ while derivatives in the right members of these equations involve only differentiations with respect to multipliers (x) of the corresponding monomials; hence no derivative occurring in the right member of an equation (16.4) can appear on the left of any equation (16.4) and conversely. Hence conditions C and D are satisfied by (16.4).

If the $s+1$ sets of cotes given by conditions $\alpha, \beta, \gamma, \delta$ do not uniquely order all derivatives of $\bar{Z}_i^*$, the ordering can be made unique by the assignment of additional cotes in accordance with the discussion in § 15. Condition F can therefore be considered to be satisfied by (16.4). The remaining conditions A, B, E are obviously satisfied by (16.4) on account of the method of formation of these equations.

Consider the system V^* obtained from (16.4) by equating to zero in each equation the non-multipliers (x) of the monomial corresponding to the left member of the equation. Hence by the theory of § 15 there exist a power series solution $\bar{Z}_i^*$ of the system V^* which is uniquely determined by the initial determinations in the sense of § 15. In the functions $\bar{S}_i^*$ of the solution $\bar{Z}_i^*$ so obtained let us put $x_1 = \cdots = x_a = 0$. It then follows from the form of (16.4) that the solution $\bar{Z}_i^*$ as so modified also satisfies the system (16.4) and hence is to be regarded as the solution of (16.4) as above considered; hereafter by the solution $\bar{Z}_i^*$ of the system (16.4) this latter solution will be understood.

Now consider the set (M_i) of monomials corresponding to an unknown $\bar{R}_i^*$; all monomials $x_p, \cdots, x_q$ of this set (M_i) then belong to the set (L_i). The set complementary to (M_i) consists of one monomial $\bar{N}_q : 1$ with multipliers consisting (1) of the variables $x_{q+1}, \cdots, x_n$ and (2) of those variables $x_1, \cdots, x_{q-1}$ which do not belong to the set $x_p, \cdots, x_{q-1}$. Hence the initial determination of each $\bar{R}_i^*$ will be a function θ of the multipliers (x) of the corresponding expression $\bar{R}_i$ in the normal system

(15.1). Similarly the initial determination of each function $\overline{S}_i^*$ will be a function ψ of the multipliers (x) of the corresponding expression $\overline{S}_i$ in (15.1).

We wish to prove that the expressions $\overline{Z}_i$ reduce identically to zero when the unknowns $u_1, \cdots, u_r$ in $\overline{Z}_i$ are replaced by the solution $u_1, \cdots, u_r$ of the system U^*. To prove this we observe (1) that when the non-multipliers (x) of $\overline{R}_i$ are equated to zero, the expression $\overline{R}_i^* \equiv 0$ and when the variables $x_1, \cdots, x_\alpha$ together with the non-multipliers (x) of $\overline{S}_i$ are equated to zero, the expression $\overline{S}_i^* \equiv 0$, (2) when the non-multipliers (x) of $\overline{Z}_i$ are equated to zero, the function $\overline{Z}_i^*$ reduces to its initial determination and (3) that the analytic functions $\overline{Z}_i^* \equiv 0$ satisfy (16.4) and the initial determinations of this solution of (16.4) vanish identically. This proves the above statement and shows that the solution $u_1, \cdots, u_r$ of the system U^* satisfies (15.1).

§ 17. Convergence Proof.

Denote by b and B the minimum and maximum values of the classes of the derivatives appearing in the left members of the equations of the system U^*. Call the totality of equations of the system U^* whose left members are in the class b, the system T_b. The system T_{b+1} will be composed of all equations with principle derivatives of the class $b+1$ in their left members; if such an equation is lacking in the system U^* it can be obtained by differentiating a suitable equation of U^*. Continuing in this way we obtain the sequence of systems

$$T_b,\ T_{b+1},\ \cdots,\ T_B,\ T_{B+1},\ \cdots.$$

The equations of this sequence fall into two component systems, namely (1) the system composed of R^* and those equations obtainable by differentiation of the system R^* and (2) the system S^* and those equations resulting by differentiation of S^*; the first of these component systems will be referred to as *the extended system* R^* and second as *the extended system* S^*. It is evident that any equation of the system T_{B+1} is linear in derivatives of class $B+1$. By the method of elimination used in forming the system $\mathfrak{E}$ in § 15, the systems $T_b, \cdots, T_{B+1}$ can be replaced by equivalent systems $T_b^*, \cdots, T_{B+1}^*$ which evidently determine uniquely the principle derivatives of (15.1) at $x_\beta = 0$ in classes b to $B+1$ inclusive when the initial determinations are assigned.

Consider the transformation

$$u_i = \overline{u}_i + \varphi_i + \sum \frac{C_{a_1 \cdots a_n}}{a_1! \cdots a_n!} x_1^{a_1} \cdots x_n^{a_n} \tag{17.1}$$

of the unknowns $u_1, \cdots, u_r$ in which the summation $\sum$ is limited to those $C_{a_1 \cdots a_n}$ which give at $x_\beta = 0$ the values of those principle derivatives of the unknown u_i in classes b to B inclusive. Substituting (17.1) into the system (15.1), this latter system goes over into an analogous system involving the corresponding derivatives of $\bar{u}_1, \cdots, \bar{u}_r$ in its left members. On account of the form of (17.1) it is easily seen (1) that the principle derivatives of the unknowns $\bar{u}_1, \cdots, \bar{u}_r$ in classes b to B inclusive must be equal to zero at $x_\beta = 0$ and (2) that the initial determinations of the unknowns $\bar{u}_1, \cdots, \bar{u}_r$ vanish identically. It can therefore be assumed without loss of generality that the system (15.1) satisfies the following condition.

(J) The principle derivatives of $u_1, \cdots, u_r$ in classes b to B inclusive vanish at $x_\beta = 0$ and the initial determinations vanish identically.

Let us now write the system T^*_{B+1} in the form

$$D_i u_\sigma = \sum p_{i\sigma j\mu} D_j u_\mu + q_{i\sigma}; \tag{17.2}$$

we observe that this system has the following properties

(a) The derivatives $D_i u_\sigma$ in the left members of (17.2) are all in the class $B+1$.

(b) All principle derivatives of the system U^* in classes C where $C > B+1$ are determined once and only once by differentiation of the left members of (17.2).

(c) The right member of an equation (17.2) is linear in derivatives $D_j u_\mu$ of class $B+1$; the coefficients p and the q terms may contain derivatives of class not exceeding B.

(d) Any derivative $D_j u_\mu$ in the right member of an equation of R^* of (17.2) which follows the derivative $D_i u_\sigma$ in the left member of the equation, has a coefficient p which vanishes throughout the domain $\mathfrak{D}$.

(e) The coefficients p and the q terms in the right member of any equation (17.2) are analytic functions in the neighborhood of the values

$$x = 0, \quad Du = 0, \cdots \tag{17.3}$$

of their arguments.

We shall show that when the principle derivatives of $u_1, \cdots, u_r$ and the initial determinations are assigned for the normal system (15.1) in accordance with condition J, that the unique power series determinations of the $u_1, \cdots, u_r$ by the system U^*, converge. It will then follow that the series for the $u_1, \cdots, u_r$ as determined by the system U^* without the condition J, will converge, since these latter series for $u_1, \cdots, u_r$ are related to the above convergent series by an equation of the form (17.1).

The above convergence proof will depend on two lemmas which we will now proceed to state. Let us associate with the unknowns $u_1, \cdots, u_r$ the

positive constants $\zeta_1, \cdots, \zeta_r$ and with the variables $x_1, \cdots, x_n$ the positive constants $\xi_1, \cdots, \xi_n$ such that $\xi_i > 1$. Then corresponding to a derivative

$$\frac{\partial^{a_1+\cdots+a_n} u_i}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}$$

we can construct the quantity

$$\zeta_i \xi_1^{a_1} \cdots \xi_n^{a_n}. \tag{17.4}$$

Let us likewise choose r^2 positive constants λ_σ^μ where $\sigma, \mu = 1, \cdots, r$, such that

$$\sum_{\mu=1}^{r} \lambda_\sigma^\mu < 1.$$

We then have the following[41]

LEMMA I. *A set of positive constants ζ, ξ where $(\xi > 1)$ can be selected so that the ratio of any quantity* (17.4) *corresponding to a derivative of class C, to the quantity* (17.4) *corresponding to a derivative of lower rank of the class C, is greater than a given positive number Q.*

LEMMA II. *The determinant*

$$|\delta_\sigma^\mu - \lambda_\sigma^\mu|$$

is positive and the cofactors of order $r-1$ are all positive or zero.

Let $H_{i\sigma\mu}$ be the number of the derivatives of the unknown u_μ of class $B+1$ which appear in the right of an equation with left member $D_i u_\sigma$. Also let $P_{i\sigma j\mu}$ be the absolute values of the constant terms of the power series expansions of $p_{i\sigma j\mu}$, about the values (17.3). Denote the largest of the numbers

$$P_{i\sigma j\mu} H_{i\sigma\mu} / \lambda_\sigma^\mu$$

by Q. Taking the class C in Lemma I to be the class $B+1$ we can then choose positive constants ζ, ξ in accordance with this lemma so that

$$\frac{\lambda_\sigma^\mu \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{H_{i\sigma\mu} \zeta_\mu \xi_1^{j_1} \cdots \xi_n^{j_n}} > P_{i\sigma j\mu}, \tag{17.5}$$

where $D_i u_\sigma$ is of higher rank than $D_j u_\mu$. Evidently the ratio (17.5) is valid also for a derivative $D_j u_\mu$ in the right member of an equation (17.2) which follows the corresponding left member on account of the above property (d). Now let G be an upper bound to the absolute values of the terms of the expansions of p, q about the values (17.3). Then choose r^2

[41] Janet, *loc. cit.*, (6), p. 144 and p. 141. In the proof of Lemma I use is made of the fact that without changing the ordering of the derivatives effected by an assignment of cotes, it is possible to replace the given set of cotes by a set composed entirely of positive integers. Cf. Riquier, *loc. cit.* (5) p. 253.

numbers ε_σ^μ, positive or zero, where $\sigma, \mu = 1, \cdots, r$ so as to satisfy the conditions

$$\frac{\lambda_\sigma^\mu + \varepsilon_\sigma^\mu}{H_{i\sigma\mu}} \cdot \frac{\zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{\zeta_\mu \xi_1^{j_1} \cdots \xi_n^{j_n}} \geqq G \tag{17.6}$$

for all combinations of the indices such that $D_j u_\mu$ is a derivative of class $B+1$ in the right member of an equation (17.2) of which $D_i u_\sigma$ is the left member. Finally choose the r positive constants ν_σ so that

$$\nu_\sigma \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n} > G, \tag{17.7}$$

where $D_i u_\sigma$ is any principle derivative of class $B+1$.

Consider the system

$$D_i Y_\sigma = \sum \frac{1}{H_{i\sigma\mu}} \left(\frac{\lambda_\sigma^\mu + \varepsilon_\sigma^\mu}{1-\tau} - \varepsilon_\sigma^\mu \right) \frac{\zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{\zeta_\mu \xi_1^{j_1} \cdots \xi_n^{j_n}} D_j Y_\mu + \frac{\nu_\sigma \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{1-\tau} \tag{17.8}$$

which has the same form as the system (17.2). Let

$$\tau = \frac{\xi_1 x_1 + \cdots + \xi_n x_n + \sum DY}{\varrho},$$

where the summation $\sum DY$ denotes the sum of all derivatives corresponding to derivatives Du of class not exceeding B, and ϱ is a small positive number such that the function

$$\frac{G}{1 - \dfrac{\sum x + \sum DY}{\varrho}} \tag{17.9}$$

dominates the coefficients p and the terms q in an equation of (17.2). Since $\xi_i > 1$ the function $G/1-\tau$ has the same dominating properties. Hence the terms $q_{i\sigma}$ in equations (17.2) are dominated by the corresponding terms in (17.8) in consequence of the inequalities (17.7). If $G_1 + G_2 \geqq G$ and G_1 exceeds the absolute value of the constant $P_{i\sigma j\mu}$ then

$$\frac{G_1 + G_2}{1-\tau} - G_2$$

dominates the function $p_{i\sigma j\mu}$. Corresponding to a coefficient $p_{i\sigma j\mu}$ in the system (17.2) put

$$G_1 = \frac{\lambda_\sigma^\mu \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{H_{i\sigma\mu} \zeta_\mu \xi_1^{j_1} \cdots \xi_n^{j_n}}, \qquad G_2 = \frac{\varepsilon_\sigma^\mu \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}}{H_{i\sigma\mu} \zeta_\mu \xi_1^{j_1} \cdots \xi_n^{j_n}};$$

then it follows that the p's in (17.2) are dominated by the corresponding coefficients in (17.8) when use is made of (17.5) and (17.6).

Let us put

$$Y_\sigma = \zeta_\sigma \overline{Y}_\sigma$$

and regard the quantities $\bar{Y}_\sigma$ as functions of the single variable y where

$$y = \xi_1 x_1 + \cdots + \xi_n x_n.$$

Denoting the ith derivative of $\bar{Y}_\sigma$ where $i = i_1 + \cdots + i_n$ by $\bar{Y}_\sigma^{(i)}$ we have

$$D_i Y_\sigma = \bar{Y}_\sigma^{(i)} \zeta_\sigma \xi_1^{i_1} \cdots \xi_n^{i_n}.$$

The quantity τ becomes a function $\bar{\tau}$ depending on the variable y and derivatives of $\bar{Y}_\sigma$ of order not exceeding h_σ, where $h_\sigma = B - c_\sigma$, c_σ being the first cote of u_σ. Equations (17.8) then become

$$\bar{Y}_\sigma^{(h_\sigma+1)} = \sum_{\mu=1}^{r} \left(\frac{\lambda_\sigma^\mu + \varepsilon_\sigma^\mu}{1-\bar{\tau}} - \varepsilon_\sigma^\mu \right) \bar{Y}_\mu^{(h_\mu+1)} + \frac{\nu_\sigma}{1-\bar{\tau}}. \tag{17.10}$$

Let us seek to find a solution $\bar{Y}_\sigma(y)$ of this system, satisfying the initial conditions

$$y = 0, \quad \bar{Y}_\sigma = 0, \quad \bar{Y}_\sigma^{(1)} = 0, \cdots, \bar{Y}_\sigma^{(h_\sigma)} = 0 \quad (\sigma = 1, \cdots, r). \tag{17.11}$$

Since $\bar{\tau}$ is equal to zero for the initial values of its arguments, the equations (17.10) can be solved for the derivatives $\bar{Y}_\sigma^{(h_\sigma+1)}$ on account of Lemma II. We thus obtain a system of ordinary equations of the form

$$\bar{Y}_\sigma^{(h_\sigma+1)} = \Omega[y, \bar{Y}, \bar{Y}^{(1)}, \cdots, \bar{Y}^{(h_\sigma)}] \tag{17.12}$$

valid in the neighborhood of the above values (17.11); this system has a unique analytic solution $\bar{Y}_\sigma(y)$ satisfying the initial conditions. Furthermore the coefficients of the power series expansions of the solution $\bar{Y}_\sigma$ of (17.12) about the value $y = 0$, are either positive or zero. The first $h_\sigma + 1$ terms of this series are zero in consequence of (17.11); the coefficients of all other terms in these series are positive. Let us write the equations (17.10) in the form

$$\bar{Y}_\sigma^{(h_\sigma+1)} = \bar{\tau}\, \bar{Y}_\sigma^{(h_\sigma+1)} + \sum_{\mu=1}^{r} (\lambda_\sigma^\mu + \bar{\tau}\, \varepsilon_\sigma^\mu)\, \bar{Y}_\mu^{(h_\mu+1)} + \nu_\sigma. \tag{17.13}$$

The initial values of the derivatives $\bar{Y}_\sigma^{(h_\sigma+1)}$ satisfy the equations

$$[\bar{Y}_\sigma^{(h_\sigma+1)}]_0 - \sum_{\mu=1}^{r} \lambda_\sigma^\mu [\bar{Y}_\mu^{(h_\mu+1)}]_0 = \nu_\sigma; \tag{17.14}$$

from Lemma II and the fact that the ν_σ are all positive the quantities $[\bar{Y}_\sigma^{(h_\sigma+1)}]_0$ determined by (17.14) are all positive. Differentiating the equations (17.13) and evaluating at $y = 0$, we see likewise that the quantities $[\bar{Y}_\sigma^{(h_\sigma+2)}]_0$ are all positive, etc. It follows that the coefficients of

the power series expansions of the corresponding solution $Y_\sigma(x)$ of the system (17.8) are all positive or zero and hence these expansions dominate the corresponding expansions determined by the equations (17.2) or the system U^*.

§ 18. Reduction to Normal Form.

In consequence of condition I of § 15, the derivatives for which the system S is solved in accordance with condition D, are parametric for the system R. Hence the condition D can be replaced by the somewhat stronger condition.

(D^*). The system S is solved for certain derivatives which are parametric for the system R and the functions F which constitute the right members of these equations depend on the independent variables $x_{\alpha+1}, \cdots, x_n$ and derivatives which are not left members of the system R.

Conversely suppose that the conditions A, B, C, D^*, E, F, G, H are satisfied by a system of equations (15.1). By the condition D^* the set (M_i) of monomials associated with unknown u_i as in § 15 will be normal. Making use of the process developed in § 14 we can therefore form a strongly complete set (M_i^*) of monomials having the same multiples as the set (M_i). By differentiation of the equations of (15.1) which involve in their left members derivatives of u_1, we can deduce a system of equations whose left members will be in one to one reciprocal correspondence with the monomials of the strongly complete set (M_1^*). This process is now to be repeated for the derivatives of the functions $u_2, \cdots, u_r$ in turn which appear in the left members of (15.1). Equations so formed by differentiation of an equation of the system R are to be added to R and those formed by differentiation of an equation of S are to be added to S. In this manner we arrive at a system of equations which can be solved for the derivatives in their left members by the application of the process of elimination used in § 15 in the derivation of the system $\mathfrak{E}$; we shall refer to the solved form of these equations as the *extended system* (15.1). From the discussion in § 15 it is obvious that the extended system (15.1) satisfies conditions $A, \cdots, H$ and the condition I is satisfied on account of the method of formation of these equations. Hence the extended system (15.1) is normal.

§ 19. Applications of the General Existence Theorem.

We saw in § 12 that the general existence theorems of § 6 failed to apply when we attempted to take a characteristic surface $C_{n-\alpha}$ as an $(n-\alpha)$-dimensional surface bearing a portion of the arbitrary data of the problem. The existence theoretic problem connected with the

characteristic surface $C_{n-\alpha}$ defined by the equations $x^1 = 0, \cdots, x^\alpha = 0$ for the case of the system (2.1) is the problem of determining a solution $v_k(x)$ of this system such that

$$\Xi_{\beta_1}(x, v) = 0, \cdots, \Xi_{\beta_m}(x, v) = 0 \tag{19.1}$$

over the surface $C_{n-\alpha}$. The theory of normal systems of § 15 has immediate application to this problem. Let us suppose that the combined system (2.1) and (19.1) can be put into a completely integrable normal form (15.1) where equations (2.1) give rise to the system R and equations (19.1) give rise to the system S. The solution $v_k(x)$ so obtained will then satisfy the system (19.1) throughout the subspace $\mathfrak{D}$, i. e. the surface $x^1 = 0, \cdots, x^\alpha = 0$ with the result that this latter surface is an $(n-\alpha)$-dimensional characteristic surface $C_{n-\alpha}$ of type β for the solution $v_k(x)$, provided that there exists a determinant $\Xi_{\beta-1}(x, v)$ which does not vanish over this surface. In accordance with the remarks of § 12 the above theory applies equally well to a vector invariantive system. The invariantive systems composed of (4.3), (4.5), (4.7) for the affine case and (4.4), (4.8), (4.10), (4.11) for the metric case permit an analogous treatment when the equations (19.1) are replaced by the systems

$$\begin{aligned} &\text{(a) Affine case: } U_k(A^\alpha_{\beta\gamma\delta}) = 0, \\ &\text{(b) Metric case: } U_k(g_{\alpha\beta};\ g_{\alpha\beta,\gamma\delta}) = 0 \end{aligned} \tag{19.2}$$

respectively.

In the practical application of this theory to systems of equations the work is usually facilitated by use of a particular assignment of cotes which is said to put the derivatives into *canonical order* and which orders all derivatives uniquely. We have illustrated this assignment of cotes by the accompanying scheme. The ordering of

	x^1	x^2	$\cdots$	x^{n-1}	x^n	v_1	v_2	$\cdots$	v_w
1	1	1	$\cdots$	1	1	0	0	$\cdots$	0
2	1	0	$\cdots$	0	0	0	0	$\cdots$	0
3	0	1	$\cdots$	0	0	0	0	$\cdots$	0
.	.	.	$\cdots$	.	.	.	.	$\cdots$	.
.	.	.	$\cdots$	.	.	.	.	$\cdots$	.
.	.	.	$\cdots$	.	.	.	.	$\cdots$	.
n	0	0	$\cdots$	1	0	0	0	$\cdots$	0
$n+1$	0	0	$\cdots$	0	1	0	0	$\cdots$	0
$n+2$	0	0	$\cdots$	0	0	1	2	$\cdots$	w

derivatives by the canonical assignment of cotes amounts to arranging the derivatives first according to increasing order; those derivatives of the

same order are arranged according to the number of differentiations with respect to x^1; those derivatives of the same order and having the same number of differentiations with regard to x^1 are arranged according to the number of differentiations with regard to x^2; etc. Finally those derivatives corresponding to the same monomial are ordered according to the index k of the unknown v_k by the $(n+2)$nd cote.

As an illustration of the application of the theory of normal systems we can consider the treatment of the above mentioned problem for the case of the system of field equations in Proc. Note V. It is easily seen that the set of monomials corresponding to the derivatives of any one of the unknowns appearing in the left members of the system composed of equations (4.1), (4.2), (4.3), (5.1), (5.3) in Proc. Note V is strongly complete and in fact that this system is normal, and that the derivatives can be put in canonical order; the complete integrability of this system can be shown by the method of § 20. The convergence proof given in § 17 supplies the proof omitted in the former treatment.

In the discussion of the example in § 7 we found that if $g^{11} = 0$ at the initial point $x^i = 0$, the existence theorem did not apply. In view of the general theory presented in § 12, § 13 let us now see whether $g^{11} = 0$ over $x^1 = 0$ will define $x^1 = 0$ as a characteristic surface C_2. In other words in the notation of § 7, if $\alpha = 0$ is it possible to solve the combined system (1)–(20) and (7 c) for L_1 (= 20) derivatives $\partial g_{\alpha\beta,\gamma\delta}/\partial x^1$? Consider the matrix of the coefficients of the derivatives of $u_1, \cdots, u_6$ with respect to x^1 in the combined system (1)–(20) and (7 c); since all sixth order determinants in this matrix contain α as a factor, it is impossible to solve the system (1)–(20) and (7 c) for 20 (= L_1) derivatives of independent components $g_{\alpha\beta,\gamma\delta}$ with respect to x^1. Hence a surface $\Phi = 0$ such that

$$g^{\alpha\beta}\frac{\partial \Phi}{\partial x^\alpha}\frac{\partial \Phi}{\partial x^\beta} = 0 \tag{19.3}$$

over $\Phi = 0$ for some integral $g_{\alpha\beta}(x)$ must be a characteristic surface for the system (7 c).

Let us assume $\alpha = 0$ over $x^1 = 0$ and

$$\begin{vmatrix} \alpha & a \\ a & \beta \end{vmatrix} \neq 0, \qquad \begin{vmatrix} \alpha & a & b \\ a & \beta & d \\ b & d & \gamma \end{vmatrix} \neq 0 \tag{19.4}$$

at the point $x^i = 0$ on this surface. As a consequence of these assumptions we have

$$a \neq 0, \qquad a^2\gamma - 2abd + b^2\beta \neq 0$$

at $x^i = 0$. The determinant of the columns 5, 6, 7, 8 in Table I is

$$8a(a^2\gamma - 2abd + b^2\beta).$$

Hence we shall drop the equations

$$X_{221} = 0, \qquad X_{231} = 0, \qquad X_{241} = 0, \qquad X_{331} = 0$$

from the system (7 c), thus making rows 5–10 in Table II the coefficients of the derivatives in the equations $X_{\alpha\beta 1} = 0$ which remain in (7 c). In Table II the determinant

$$\begin{vmatrix} 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 11 & 12 \end{vmatrix}$$

formed from rows 5, 6, 7, 8, 9, 10 and columns 1, 2, 3, 4, 11, 12 is equal to $4a^2(a^2\gamma - 2abd + b^2\beta)$ disregarding algebraic sign; hereafter a similar notation for determinants will be used

without further explanation. Hence let us solve the remaining equations $X_{\alpha\beta 1} = 0$ for the derivatives of the unknown $u_1, u_2, u_3, u_4, v_5, v_6$ with respect to x^1. The determinant in Table II,

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 11 & 12 \end{vmatrix} = 8a^2 \begin{vmatrix} \alpha & a \\ a & \beta \end{vmatrix} \begin{vmatrix} \alpha & a & b \\ a & \beta & d \\ b & d & \gamma \end{vmatrix}^2,$$

disregarding sign; for expanding in terms of second order minors in rows 5, 6 the above determinant reduces to the product of $4a^2$ and the eighth order determinant which was encountered in § 7. Hence we can solve equations $X_{\alpha\beta\varepsilon} = 0$ where $\varepsilon = 2, 3, 4$ for derivatives of the unknowns $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, v_5, v_6$ with respect to x^ε. Hence we have the following systems of equations

$$\left.\begin{matrix} \frac{\partial u_i}{\partial x^1} \\ \frac{\partial v_j}{\partial x^1} \end{matrix}\right\} = \sum \frac{\partial u}{\partial x^1} + \sum \frac{\partial v}{\partial x^1} + \sum \frac{\partial w}{\partial x^1} + \star, \tag{19a}$$

$$\left.\begin{matrix} \frac{\partial u_i}{\partial x^2} \\ \frac{\partial v_k}{\partial x^2} \end{matrix}\right\} = \sum \frac{\partial u}{\partial x^2} + \sum \frac{\partial v}{\partial x^2} + \sum \frac{\partial w}{\partial x^2} + \star, \tag{19b}$$

$$\left.\begin{matrix} \frac{\partial u_i}{\partial x^3} \\ \frac{\partial v_k}{\partial x^3} \end{matrix}\right\} = \sum \frac{\partial u}{\partial x^3} + \sum \frac{\partial v}{\partial x^3} + \sum \frac{\partial w}{\partial x^3} + \star, \tag{19c}$$

$$\left.\begin{matrix} \frac{\partial u_i}{\partial x^4} \\ \frac{\partial v_k}{\partial x^4} \end{matrix}\right\} = \sum \frac{\partial u}{\partial x^4} + \sum \frac{\partial v}{\partial x^4} + \sum \frac{\partial w}{\partial x^4} + \star, \tag{19d}$$

where $i = 1, 2, 3, 4$; $j = 5, 6$; $k = 1, \cdots, 6$. The derivatives in the right members of each of the above systems (19a)–(19d) involve unknowns not appearing in the left members of that system.

In accordance with the above theory let us now solve the combined systems (19a)–(19d) and (1)–(20) for derivatives in such a manner that the resulting system will satisfy the conditions for a normal system. Eliminate the left members of equations (1)–(4) and (7)–(14) from the right members of (19a) and use (19b), (19c) and (19d) to eliminate their left members from the right of the resulting system; denote the system which results by $\mathfrak{A}$. Then use equations (15)–(20) to eliminate their left members from the right of (19b) and $\mathfrak{A}$. The equation (15) contains the left member $\partial v_6/\partial x^2$ of an equation (19b) in its right member. However the determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 11 & 15 \end{vmatrix}$$

in Table II vanishes identically and hence the derivative $\partial w_1/\partial x^2$ does not appear on the right of the equation (19b) having $\partial v_6/\partial x^2$ for its left member; hence the equation (19b) in question can be used to eliminate its left member from the remaining equations (19b) and $\mathfrak{A}$. Thus we can eliminate the left members of (19c) and (19d) from the right of (19b) and $\mathfrak{A}$ giving the systems B and A respectively. Then use A, B, (19c), (19d) to eliminate their left members from the right of equations (1)–(4) and (7)–(20) and we have the combined system consisting of (1)–(4), (7)–(20) and (19a)–(19d), in solved form.

Let us now consider the problem of introducing equations (5) and (6) into the above system so that equations (1)–(20) and (19a)–(19d) will be in solved form. From Table II it can be seen that the system (7c) contains the two equations

$$(19e)\qquad \begin{aligned} 2a\frac{\partial v_6}{\partial x^1} &= -\alpha\frac{\partial u_5}{\partial x^1} - \beta\frac{\partial v_7}{\partial x^1} + b\frac{\partial w_2}{\partial x^1} + c\frac{\partial w_3}{\partial x^1} + d\frac{\partial w_4}{\partial x^1} + e\frac{\partial w_5}{\partial x^1} + f\frac{\partial w_6}{\partial x^1} + \ast, \\ 2a\frac{\partial v_5}{\partial x^1} &= -\alpha\frac{\partial u_6}{\partial x^1} - \beta\frac{\partial v_8}{\partial x^1} - 2b\frac{\partial w_2}{\partial x^1} - 2d\frac{\partial w_5}{\partial x^1} - \gamma\frac{\partial w_6}{\partial x^1} + \ast. \end{aligned}$$

Consider the form of the corresponding equations in the system A. From the form of equations (10)–(14) it is seen that only the first two terms on the right of each equation (19e) can give rise to derivatives with respect to x^1 and x^2. Hence making use of (7) and (8) the equations (19e) become

$$(19f)\qquad \begin{aligned} 2a\frac{\partial v_6}{\partial x^1} &= -\alpha\frac{\partial u_5}{\partial x^1} - \beta\frac{\partial v_6}{\partial x^2} + \cdots, \\ 2a\frac{\partial v_5}{\partial x^1} &= -\alpha\frac{\partial u_6}{\partial x^1} - \beta\frac{\partial v_5}{\partial x^2} + \cdots, \end{aligned}$$

where the dots in these and following equations denote derivatives with respect to x^3 and x^4 and terms of lower order; when however, one or more derivatives with respect to x^3 are written down explicitly the dots will denote only derivatives with respect to x^4 and terms of lower order. Since the determinants

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 11 & 5 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 6 & 12 \end{vmatrix}$$

in Table II contain α as a factor and since moreover the determinants

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 11 & 6 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 5 & 12 \end{vmatrix}$$

in Table II vanish identically, there are two equations of the system (19b) of the form

$$(19g)\qquad \begin{aligned} \frac{\partial v_5}{\partial x^2} &= \alpha V\frac{\partial u_6}{\partial x^2} + \text{———} + \frac{\partial(v_7, v_8, w_1, \cdots, w_6)}{\partial x^2} + \ast, \\ \frac{\partial v_6}{\partial x^2} &= \text{———} + \alpha V\frac{\partial u_5}{\partial x^2} + \frac{\partial(v_7, v_8, w_1, \cdots, w_6)}{\partial x^2} + \ast, \end{aligned}$$

where the blank indicates that the corresponding derivative does not enter; and the notation employed in the second term of each of these equations as well as in certain of the following equations, denotes a linear sum of the derivatives of the indicated unknowns; here likewise V represents a rational function of the $g_{\alpha\beta}$ although not necessarily the same function in the two equations (19g). Hence if we make use of equations (15)–(20) and the equations (19g), the equations (19f) can be put in the form

$$(19h)\qquad \begin{aligned} 2a\frac{\partial v_6}{\partial x^1} &= -\alpha\frac{\partial u_5}{\partial x^1} + \alpha V\frac{\partial u_5}{\partial x^2} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots, \\ 2a\frac{\partial v_5}{\partial x^1} &= -\alpha\frac{\partial u_6}{\partial x^1} + \alpha V\frac{\partial(u_5, u_6)}{\partial x^2} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots. \end{aligned}$$

Then equations (19h) can be used to put equations (5) and (6) in form

$$(5')\qquad \frac{\partial u_6}{\partial x^2} = \alpha V\frac{\partial u_6}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots,$$

$$(6')\qquad \frac{\partial u_5}{\partial x^2} = \alpha V\frac{\partial u_5}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots$$

respectively. Then (5′) and (6′) can be used to eliminate their left members from the solved form of (1)–(4), (7)–(20) and (19a)–(19d). The resulting equations will be denoted by (1′)–(4′), (7′)–(20′) and (19a′)–(19d′) respectively. Thus we have for these equations

$$\text{(19a′)} \qquad \frac{\partial}{\partial x^1}\begin{Vmatrix} u_1, \cdots, u_4 \\ v_5, v_6 \end{Vmatrix} = \frac{\partial(u_5, u_6)}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots,$$

$$\text{(19b′)} \qquad \frac{\partial}{\partial x^2}\begin{Vmatrix} u_1, \cdots, u_4 \\ v_1, \cdots, v_6 \end{Vmatrix} = aV\frac{\partial(u_5, u_6)}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots,$$

$$\text{(19c′)} \qquad \frac{\partial}{\partial x^3}\begin{Vmatrix} u_1, \cdots, u_4 \\ v_1, \cdots, v_6 \end{Vmatrix} = \cdots,$$

$$\text{(19d′)} \qquad \frac{\partial}{\partial x^4}\begin{Vmatrix} u_1, \cdots, u_4 \\ v_1, \cdots, v_6 \end{Vmatrix} = \cdots,$$

$$\left.\begin{matrix}\text{(1′)–(4′)} \\ \text{(7′)–(14′)}\end{matrix}\right\} \qquad \frac{\partial}{\partial x^1}\begin{Vmatrix} v_1, \cdots, v_4 \\ v_7, v_8 \\ w_1, \cdots, w_6 \end{Vmatrix} = aV\frac{\partial(u_5, u_6)}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots,$$

$$\text{(15′)–(20′)} \qquad \frac{\partial}{\partial x^2}\{w_1, \cdots, w_6\} = aV\frac{\partial(u_5, u_6)}{\partial x^1} + \frac{\partial(v_7, v_8)}{\partial x^2} + \cdots.$$

Together with above system we must consider the equations

$$\text{(19i)} \qquad \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = g_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma} + g_{\beta\sigma}\Gamma^\sigma_{\alpha\gamma} \qquad \begin{pmatrix} \gamma = 1, \cdots, 4 \\ \alpha, \beta = 1, \cdots, 4 \\ \alpha \geqq \beta \end{pmatrix},$$

$$\text{(19j)} \qquad \frac{\partial \gamma_{lm}}{\partial x^\alpha} = \frac{\partial \gamma_{pq}}{\partial x^r} + \star \qquad \begin{pmatrix} m = 0, \cdots, 3 \\ l = 1, \cdots, \gamma_m \\ \alpha = 1, \cdots, m \\ r > \alpha \end{pmatrix},$$

where the notation

$$\begin{aligned} &\gamma_{l0} \sim \Gamma^i_{11}, \\ &\gamma_{l1} \sim \Gamma^i_{12}, \ \Gamma^i_{22}, \\ &\gamma_{l2} \sim \Gamma^i_{13}, \ \Gamma^i_{23}, \ \Gamma^i_{33}, \\ &\gamma_{l3} \sim \Gamma^i_{14}, \ \Gamma^i_{24}, \ \Gamma^i_{34}, \ \Gamma^i_{44} \end{aligned}$$

is used (see § 4). In addition, differentiating the condition $g^{11} = 0$ with respect to x^2, x^3, x^4, we have

$$\text{(19k)} \qquad \frac{\partial g^{11}}{\partial x^\gamma} \equiv -g^{1\sigma}\Gamma^1_{\sigma\gamma} - g^{1\sigma}\Gamma^1_{\sigma\gamma} = 0 \qquad (\gamma = 2, 3, 4);$$

also differentiating a second time, we have

$$\text{(19l)} \qquad \sum_{\sigma=2}^{4} g^{1\sigma}\frac{\partial \Gamma^1_{\sigma\gamma}}{\partial x^\delta} + \star = 0 \qquad (\gamma \leqq \delta;\ \gamma, \delta = 2, 3, 4).$$

If we expand the equations (19l) and then eliminate the left members of (19j), we find that these six equations can be solved to obtain

$$\text{(19m)} \qquad \begin{aligned} \frac{\partial \Gamma^1_{22}}{\partial x^\varepsilon} &= \sum \frac{\partial \gamma_{pq}}{\partial x^r} + \star \qquad (\varepsilon = 2, 3, 4), \\ \frac{\partial \Gamma^1_{23}}{\partial x^\varepsilon} &= \sum \frac{\partial \gamma_{pq}}{\partial x^r} + \star \qquad (\varepsilon = 3, 4), \\ \frac{\partial \Gamma^1_{24}}{\partial x^4} &= \sum \frac{\partial \gamma_{pq}}{\partial x^r} + \star, \end{aligned}$$

where the $\sum$ denotes a linear form in parametric derivatives of (19j), and $r \geqq \varepsilon$ for any equation of this system.

Let us now assign canonical cotes in the following manner:

cote	x^1	x^2	x^3	x^4	$u_1, \cdots, u_4$ $v_1, \cdots, v_6$	u_5, u_6 $w_1, \cdots, w_6$	v_7, v_8
1	1	1	1	1	0	0	0
2	1	0	0	0	0	0	0
3	0	1	0	0	0	0	0
4	0	0	1	0	0	0	0
5	0	0	0	1	0	0	0
6	0	0	0	0	3	2	1

In addition we shall take all six cotes of the $g_{\alpha\beta}$ to be zero and the first five cotes of the $\Gamma^{\alpha}_{\beta\gamma}$ as zero; the sixth cote of the Γ's occurring in the left members of (19m) will be taken to be 1 and the sixth cote of all other Γ's will be given the value zero.

With the above assignment of cotes it is seen that the equations (1′)–(20′), (19a′)–(19d′), (19i) and (19j) constitute the system R and (19m) the system S of a normal system. The arbitrary data predicted by the form of the left members of this normal system can be represented by the following scheme

$$
\begin{aligned}
& u_5,\ u_6 && \sim J(x^1, x^3, x^4) && \text{for } x^2 = 0,\\
& v_7,\ v_8 && \sim K(x^2, x^3, x^4) && \text{for } x^1 = 0,\\
& w_1, \cdots, w_6 && \sim L(x^3, x^4) && \text{for } x^1 = x^2 = 0,\\
(19\text{n}) \quad & \Gamma^i_{11} && \sim P(x^1, x^2, x^3, x^4), &&\\
& \Gamma^i_{12},\ \Gamma^j_{22} && \sim Q(x^2, x^3, x^4) && \text{for } x^1 = 0,\\
& \Gamma^i_{13},\ \Gamma^j_{23},\ \Gamma^i_{33} && \sim R(x^3, x^4) && \text{for } x^1 = x^2 = 0,\\
& \Gamma^i_{14},\ \Gamma^j_{24},\ \Gamma^i_{34},\ \Gamma^i_{44} && \sim S(x^4) && \text{for } x^1 = x^2 = x^3 = 0,
\end{aligned}
$$

where $i = 1, \cdots, 4$; $j = 2, 3, 4$. The remaining unknowns, namely

$$
(19\text{o}) \qquad u_1, \cdots, u_4, v_1, \cdots, v_6, \Gamma^1_{22}, \Gamma^1_{23}, \Gamma^1_{24}, g_{11}, \cdots, g_{44}
$$

can take on arbitrary values at the point $x^i = 0$. That the system in question is completely integrable will be proved in § 20 and hence we can state the following

EXISTENCE THEOREM: *Let the arbitrary functions represented by* (19n) *and the arbitrary constants* (19o) *be assigned subject to the condition that the initial values at* $x^i = 0$ *are such that* (1) *they satisfy equations* (7b), (2) *the determinant* $|(g_{\alpha\beta})_0|$ *does not vanish,* (3) *the inequalities* (19.4) *are satisfied,* (4) *the quantity* $(g^{11})_0$ *is equal to zero, and* (5) *the equations* (19k) *are satisfied. Then there exists one, and only one, set of functions* $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$ *analytic in the neighborhood of* $x^i = 0$ *which gives rise to the given functions indicated by* (19n) *and the given initial values of the constants* (19o), *and which constitutes a set of integrals of equations* (7b) *for which the surface* $x^1 = 0$ *is a characteristic surface.*

It can be observed that the inequalities (19.4) impose no restriction on the integrals $g_{\alpha\beta}(x)$ since a coördinate system can always be selected in which the equation of the characteristic surface C_2 is $x^1 = 0$ and the inequalities (19.4) are satisfied. The above existence theorem will give all the characteristic surfaces C_2, since if the equation (19.3)

is not satisfied over $\Phi = 0$ we will be able to find a coördinate system in which the system (7c), (7d), (19i) and (19j) can be put in regular form at some point P on $\Phi = 0$.

If a surface $x^1 = x^2 = 0$ is to be a two dimensional characteristic surface C_2, either one or both of the quantities g^{11} and $g^{11}\,g^{22} - g^{12}\,g^{12}$ must vanish over $x^1 = x^2 = 0$. If this were not the case we could find a coördinate system by a transformation of the type (19.12) for $a = 2$ in which the quantity (21) in § 7 would not be zero at some point P on $x^1 = x^2 = 0$, and hence the system could be put in regular form in the neighborhood of the point P. Let us divide the discussion into four different cases.

Case I. $\alpha = 0$ *for* $x^1 = x^2 = 0$; $\alpha\beta - a^2 \neq 0$ *at* $x^i = 0$.

Case II. $\alpha = \beta = 0$ *for* $x^1 = x^2 = 0$; $a \neq 0$ *at* $x^i = 0$.

In these cases it is evident that an existence theorem could be stated which would be of the same form as the existence theorem for the three dimensional characteristic surface C_3 except that there would be additional arbitrariness in the arbitrary data corresponding to the functions $I^{\alpha}_{\beta\gamma}$. The surface $x^1 = x^2 = 0$ is a two dimensional characteristic surface C_2 of type 1.

Case III. $\alpha\beta - a^2 = 0$ *for* $x^1 = x^2 = 0$; $\alpha \neq 0$ *at* $x^i = 0$. In this case it is easily seen that a coördinate transformation of the type (12.12) can be made so that the determinant

$$\begin{vmatrix} \bar{\alpha} & \bar{a} & \bar{b} \\ \bar{a} & \bar{\beta} & \bar{d} \\ \bar{b} & \bar{d} & \bar{\gamma} \end{vmatrix} = -\frac{(\bar{\alpha}\,\bar{d} - \bar{a}\,\bar{b})^2}{\bar{\alpha}}$$

does not vanish at some point P on $x^1 = x^2 = 0$. Since at the point P the determinant formed from Table II,

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 15 \end{vmatrix} = 4\,(\alpha\,d - a\,b)^5$$

disregarding sign, it is easily seen that the system can be put into regular form at the point P on $\bar{x}^1 = \bar{x}^2 = 0$ and hence the surface $x^1 = x^2 = 0$ is not a characteristic surface C_2.

Case IV. $\alpha = \alpha\beta - a^2 = 0$ *for* $x^1 = x^2 = 0$; $\beta \neq 0$ *at* $x^i = 0$. Evidently the surface $x^1 = x^2 = 0$ is a characteristic surface C_2 for Case IV but the detailed treatment of the corresponding existence theorem will be omitted.

Case V. $\alpha = \beta = a = 0$ *for* $x^1 = x^2 = 0$. From the form of Table II and the equations (1)–(20) it is seen that for this case it is impossible to solve for either of the derivatives $\partial u_1/\partial x^1$ and $\partial u_1/\partial x^2$ and hence the system cannot be replaced by a normal system having arbitrary data of the form that occurs in Cases I and II.

We shall consider in detail the existence theorem for Case V. Let us suppose that

$$b \neq 0, \quad c \neq 0, \quad d \neq 0, \quad e \neq 0, \quad c\,d - b\,e \neq 0, \tag{19p}$$

at $x^i = 0$; these conditions can always be obtained by a transformation of the allowed type (12.12). Then from

(19q) Table I: $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \end{vmatrix} = -16\,b^2(c\,d - b\,e)$, over $x^1 = x^2 = 0$,

(19r) Table II: $\begin{vmatrix} 1 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 9 & 17 \end{vmatrix} = 8\,b^4(c\,d - b\,e)$, over $x^1 = x^2 = 0$,

(19s) Table II: $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 5 & 8 & 9 & 10 & 15 & 16 & 17 & 18 \end{vmatrix} = 32\,b^5 e\,(c\,d - b\,e)^2$, over $x^1 = x^2 = 0$

no regard being paid to algebraic sign in (19r) and (19s). From (19p) it follows that the above three determinants do not vanish at $x^i = 0$. Since the determinant (19q) does not vanish, we can drop equations

$$X_{231} = 0, \quad X_{241} = 0, \quad X_{331} = 0, \quad X_{341} = 0$$

from the system (7c). We can solve the remaining equations $X_{\alpha\beta 1} = 0$ for the derivatives of the unknowns $u_2, u_3, u_4, u_5, v_2, w_2$ with respect to x^1, since the determinant (19r) does not vanish at $x^i = 0$. Since the determinant (19s) does not vanish at $x^i = 0$, we can solve the remaining equations (7c) for derivatives of the unknowns $u_3, u_4, u_5, v_2, v_3, v_4, w_1, \cdots, w_4$ with respect to x^2, x^3, x^4. Arranging the equations so obtained in a form corresponding to (19a)–(19d), we shall denote them for the moment by (19α), (19β), (19γ), (19δ) respectively. By a process similar to that used above the equations (19α)–(19δ) can be combined with the equations (1), (2), (4)–(10), (12)–(15), (19), (20) so as to obtain a system completely solved for the derivatives in the left members of the above equations (19α)–(20), where equation (15) is written

$$\frac{\partial v_6}{\partial x^2} = -2\frac{\partial w_1}{\partial x^2} - \frac{3}{2}\frac{\partial v_4}{\partial x^3} + \frac{3}{2}\frac{\partial v_3}{\partial x^4} + \ast. \tag{15}$$

It is to be noticed that the equation (19β) having $\partial w_1/\partial x^3$ for left member does not contain $\partial v_6/\partial x^2$ on the right and hence the above elimination is possible.

Now consider the equations

$$\frac{\partial u_2}{\partial x^2} = -\frac{\partial v_3}{\partial x^1} + \frac{\partial u_1}{\partial x^3} + \ast, \tag{3}$$

$$\frac{\partial u_6}{\partial x^3} = \frac{\partial w_3}{\partial x^1} + \frac{\partial u_5}{\partial x^4} + \ast, \tag{11}$$

$$\frac{\partial v_6}{\partial x^3} = -\frac{3}{4}\frac{\partial w_2}{\partial x^2} - \frac{1}{2}\frac{\partial w_1}{\partial x^3} + \frac{3}{4}\frac{\partial v_2}{\partial x^4} + \ast, \tag{16}$$

$$\frac{\partial v_5}{\partial x^3} = \frac{\partial w_3}{\partial x^2} - \frac{2}{3}\frac{\partial w_1}{\partial x^4} + \frac{2}{3}\frac{\partial v_6}{\partial x^4} + \ast, \tag{17}$$

$$\frac{\partial v_7}{\partial x^3} = -\frac{\partial w_4}{\partial x^2} + \frac{\partial v_1}{\partial x^4} + \ast. \tag{18}$$

Consider also the following equations selected from (19α)–(19γ)

$$\frac{\partial v_3}{\partial x^1} = -\frac{\gamma}{2b}\frac{\partial v_1}{\partial x^1} - \frac{c}{b}\frac{\partial v_4}{\partial x^1} - \frac{f}{b}\frac{\partial v_7}{\partial x^1} - \frac{\delta}{2b}\frac{\partial v_8}{\partial x^1} + \ast,$$

$$\frac{\partial w_3}{\partial x^1} = -\frac{d}{b}\frac{\partial w_5}{\partial x^1} - \frac{\gamma}{2b}\frac{\partial w_6}{\partial x^1} + \ast,$$

$$\frac{\partial w_2}{\partial x^2} = -\frac{e}{b}\frac{\partial w_5}{\partial x^2} + V\frac{\partial w_6}{\partial x^2} + \ast,$$

$$\frac{\partial w_3}{\partial x^2} = -\frac{d}{b}\frac{\partial w_5}{\partial x^2} + V\frac{\partial w_6}{\partial x^2} + \ast,$$

$$\frac{\partial w_4}{\partial x^2} = \frac{c}{b}\frac{\partial w_5}{\partial x^2} + V\frac{\partial w_6}{\partial x^2} + \ast,$$

$$\frac{\partial u_3}{\partial x^2} = -\frac{d}{e}\frac{\partial u_2}{\partial x^2} - \frac{(c^2\gamma - 2bcf + b^2\delta)}{2b^2e}\frac{\partial u_6}{\partial x^2} + \frac{\partial(v_1, v_5, v_6, v_7, v_8, w_5, w_6)}{\partial x^2} + \ast,$$

$$\frac{\partial v_2}{\partial x^2} = \frac{\partial(v_1, v_5, v_6, v_7, v_8, w_5, w_6)}{\partial x^3} + \ast,$$

$$\frac{\partial w_1}{\partial x^\varepsilon} = \frac{c}{2b}\frac{\partial v_5}{\partial x^\varepsilon} + \frac{d}{2b}\frac{\partial v_7}{\partial x^\varepsilon} + \frac{e}{2b}\frac{\partial v_8}{\partial x^\varepsilon} + \frac{f}{2b}\frac{\partial w_5}{\partial x^\varepsilon} + \ast, \qquad (\varepsilon = 2, 3).$$

The terms with coefficients which vanish over $x^1 = x^2 = 0$ have not been written down in these equations. Hence equation (3) can be written

$$(19\,\mathfrak{t}) \qquad \frac{\partial u_2}{\partial x^2} = \frac{c\,d}{b\,e}\frac{\partial u_2}{\partial x^2} + \frac{c(c^2\gamma - 2\,b\,c\,f + b^2\delta)}{2\,b^2\,e}\frac{\partial u_6}{\partial x^2} + \frac{\partial(v_1, v_5, v_6, v_7, v_8, w_5, w_6)}{\partial x^2} + \frac{\partial u_1}{\partial x^3} + \frac{\partial(u_2, u_6, v_1, v_5, v_6, v_7, v_8, w_5, w_6)}{\partial x^3} + \cdots.$$

Then making use of the solved form of equations $(19\,\alpha)$–$(19\,\delta')$ and (1), (2), (4)–(10), (12)–(15), (19), (20) we can put the above equations (18), (17), (16), (11), (19 t) in the following form

$$(18') \qquad \frac{\partial v_7}{\partial x^3} = -\frac{c}{b}\frac{\partial v_8}{\partial x^3} + V\frac{\partial w_5}{\partial x^3} + \cdots,$$

$$(17') \qquad \frac{\partial v_5}{\partial x^3} = -\frac{d}{b}\frac{\partial v_8}{\partial x^3} + V\frac{\partial w_5}{\partial x^3} + \cdots,$$

$$(16') \qquad \frac{\partial v_6}{\partial x^3} = \left(\frac{b\,e + c\,d}{2\,b^2}\right)\frac{\partial v_8}{\partial x^3} + V\frac{\partial w_5}{\partial x^3} + \cdots,$$

$$(11') \qquad \frac{\partial u_6}{\partial x^3} = \frac{\partial(v_8, w_5, w_6)}{\partial x^3} + \cdots,$$

$$(3') \qquad \frac{\partial u_2}{\partial x^2} = \frac{c(c^2\gamma - 2\,b\,c\,f + b^2\delta)}{2\,b^2(b\,e - c\,d)}\frac{\partial u_6}{\partial x^2} + \frac{\partial(v_1, v_5, v_7, v_8)}{\partial x^2} + \left(\frac{b\,e}{b\,e - c\,d}\right)\frac{\partial u_1}{\partial x^3} + \frac{\partial(u_2, v_1, v_8, w_5, w_6)}{\partial x^3} + \cdots.$$

These equations can be used to eliminate their left members from the right of the above solved form of the remaining equations $(19\,\alpha)$–(20) and we can write the equations so obtained in the form:

$$(19\,\alpha') \qquad \frac{\partial}{\partial x^1}\begin{Bmatrix} u_2, u_3, u_4, u_5 \\ v_3, w_3 \end{Bmatrix} = V\frac{\partial u_6}{\partial x^1} + \frac{\partial(u_1, u_6, v_1, v_5, v_7, v_8)}{\partial x^2} + \cdots,$$

$$(19\,\beta') \qquad \frac{\partial}{\partial x^2}\begin{Bmatrix} u_3, & u_4, & u_5 \\ v_2, & v_3, & v_4 \\ w_1, & \cdots, & w_4 \end{Bmatrix} = \frac{\partial(u_1, u_6, v_1, v_5, v_7, v_8)}{\partial x^2} + \cdots,$$

$$(19\,\gamma') \qquad \frac{\partial}{\partial x^3}\begin{Bmatrix} u_3, & u_4, & u_5 \\ v_2, & v_3, & v_4 \\ w_1, & \cdots, & w_4 \end{Bmatrix} = \cdots,$$

$$(19\,\delta') \qquad \frac{\partial}{\partial x^4}\begin{Bmatrix} u_3, & u_4, & u_5 \\ v_2, & v_3, & v_4 \\ w_1, & \cdots, & w_4 \end{Bmatrix} = \cdots,$$

$$\left.\begin{matrix}(1'),\ (2') \\ (4')\text{–}(10') \\ (12')\text{–}(14')\end{matrix}\right\} \quad \frac{\partial}{\partial x^1}\begin{Bmatrix} v_1, & v_2 \\ v_4, \cdots, & v_8, & w_1 \\ w_2, & w_4, & w_5, & w_6 \end{Bmatrix} = \frac{\partial(u_1, u_6, v_1, v_5, v_7, v_8)}{\partial x^2} + \cdots,$$

$$(15') \qquad \frac{\partial v_6}{\partial x^2} = -\frac{c}{b}\frac{\partial v_5}{\partial x^2} - \frac{d}{b}\frac{\partial v_7}{\partial x^2} - \frac{e}{b}\frac{\partial v_8}{\partial x^2} + \cdots,$$

$$(19'),\ (20') \qquad \frac{\partial}{\partial x^2}\{w_5, w_6\} = \cdots,$$

where again the terms with coefficients which vanish over $x^1 = x^2 = 0$ have not been written down.

Over the surface $x^1 = x^2 = 0$, we have

$$(19\,\mathfrak{u}) \qquad g^{11} = g^{12} = g^{22} = 0$$

by hypothesis; hence

$$\frac{\partial g^{\mu\nu}}{\partial x^{\varepsilon}} \equiv -g^{\mu\sigma}\Gamma^{\nu}_{\sigma\varepsilon} - g^{\sigma\nu}\Gamma^{\mu}_{\sigma\varepsilon} = 0, \quad (\mu, \nu = 1, 2;\ \varepsilon = 3, 4;\ \mu \leqq \nu) \tag{19v}$$

over $x^1 = x^2 = 0$. Also over $x^1 = x^2 = 0$, we have

$$g^{13}\frac{\partial \Gamma^1_{3\varepsilon}}{\partial x^{\vartheta}} + g^{14}\frac{\partial \Gamma^1_{4\varepsilon}}{\partial x^{\vartheta}} = \ast,$$

$$g^{23}\frac{\partial \Gamma^2_{3\varepsilon}}{\partial x^{\vartheta}} + g^{24}\frac{\partial \Gamma^2_{4\varepsilon}}{\partial x^{\vartheta}} = \ast, \tag{19w}$$

$$g^{13}\frac{\partial \Gamma^2_{3\varepsilon}}{\partial x^{\vartheta}} + g^{14}\frac{\partial \Gamma^2_{4\varepsilon}}{\partial x^{\vartheta}} + g^{23}\frac{\partial \Gamma^1_{3\varepsilon}}{\partial x^{\vartheta}} + g^{24}\frac{\partial \Gamma^1_{4\varepsilon}}{\partial x^{\vartheta}} = \ast$$

$$(\varepsilon \leqq \vartheta;\ \varepsilon, \vartheta = 3, 4).$$

Now eliminate the left members of (19j) from the above system (19w); the system resulting from (19w) with the exception of that equation which results from the last set of equations (19w) where $\varepsilon = \vartheta = 3$, can be solved so as to obtain

$$\frac{\partial \Gamma^i_{33}}{\partial x^{\varepsilon}} = (g)\, w_6 + \sum (g)\, \Gamma\Gamma \qquad (i = 1, 2;\ \varepsilon = 3, 4),$$

$$\frac{\partial \Gamma^i_{34}}{\partial x^4} = (g)\, w_6 + \sum (g)\, \Gamma\Gamma \qquad (i = 1, 2), \tag{19x}$$

$$\frac{\partial \Gamma^i_{44}}{\partial x^4} = (g)\, w_6 + \sum (g)\, \Gamma\Gamma \qquad (i = 1, 2),$$

when account is taken of (19p); in these equations the (g) denotes a rational expression in the components $g_{\alpha\beta}$ and the last terms are quadratic in the $\Gamma^{\alpha}_{\beta\gamma}$. Elimination of the left members of (19x) from the equation corresponding to $\varepsilon = \vartheta = 3$ which was above excluded, gives

$$w_6 = \sum (g)\, \Gamma\Gamma, \tag{19y}$$

in consequence of which (19x) becomes

$$\frac{\partial \Gamma^i_{33}}{\partial x^{\varepsilon}} = \sum (g)\, \Gamma\Gamma, \quad \frac{\partial \Gamma^i_{34}}{\partial x^4} = \sum (g)\, \Gamma\Gamma, \quad \frac{\partial \Gamma^i_{44}}{\partial x^4} = \sum (g)\, \Gamma\Gamma \tag{19z}$$

$$(i = 1, 2;\ \ \varepsilon = 3, 4).$$

Differentiating (19y) with respect to x^3, x^4 and eliminating the left members of (19i), (19j), and (19z) from the resulting equations, we obtain

$$\frac{\partial w_6}{\partial x^{\varepsilon}} = \left[g_{\mu\nu};\ \gamma_{tm};\ \frac{\partial \gamma_{pq}}{\partial x^{\tau}};\ g_{\mu\nu,\sigma\tau} \right] \qquad (\varepsilon = 3, 4;\ \nu \geqq \varepsilon), \tag{19A}$$

where the right member is a rational expression in the quantities indicated.

Cotes will be assigned to the independent variables x, the unknowns $u_1, \cdots, w_6$, the $g_{\alpha\beta}$ and the γ_{tm} as in the above treatment of the 3-dimensional characteristic surface C_2 with the exception of the sixth cotes of the unknowns for which we shall make the following assignments.

unknowns	$g_{\mu\nu}$	γ_{tm}	v_1, w_5, w_6	v_8	v_5, v_7	u_6	v_6	u_2	u_1	u_3, u_4, u_5 v_2, v_3, v_4 $w_1, \cdots, w_4$
cotes	0	0	1	2	3	4	5	6	7	8

The equation (19α') which contains the derivative $\partial u_2/\partial x^1$ in its left member when written more explicitly, is of the form:

$$\frac{\partial u_2}{\partial x^1} = \frac{c(c^2\gamma - 2bcf + b^2\delta)}{2b^2(be - cd)} \frac{\partial u_6}{\partial x^1} + \frac{\partial(u_6, v_1, v_5, v_7, v_8)}{\partial x^2} + \cdots.$$

Hence using this equation and equations (3'), (16'), (15') we can construct two integrability conditions of the form

$$\text{(19B)} \qquad \begin{aligned} &\frac{\partial^2 u_1}{\partial x^1 \partial x^3} = V \frac{\partial^2 u_6}{\partial x^1 \partial x^3} + \frac{\partial^2(u_1, \cdots, w_6)}{\partial x^\mu \partial x^\nu} + \ast, && \begin{pmatrix} \mu \leqq \nu \\ \nu = 4, \text{ if } \mu = 1 \end{pmatrix}, \\ &\frac{\partial^2 v_8}{\partial x^2 \partial x^3} = \frac{\partial^2(u_1, \cdots, w_6)}{\partial x^\varepsilon \partial x^\vartheta} + \ast, && \begin{pmatrix} \varepsilon, \vartheta \neq 1 \\ \varepsilon \leqq \vartheta \\ \vartheta = 4, \text{ if } \varepsilon = 2 \end{pmatrix}. \end{aligned}$$

The condition that we be able to solve for the derivative $\partial^2 v_8/\partial x^2 \partial x^3$ is that $3(be - cd)/2b^2$ be different from zero.

The system R composed of (19α')–(19δ'), (1')–(20'), (19i), (19j), (19B) and the system S composed of (19z), (19A) constitute a normal system which will be shown to be completely integrable in § 20. The strongly complete set of monomials for the unknown u_1 is x^1x^3 and for v_8 is x^1 and x^2x^3; all other sets of monomials consist of a single variable. The calculation of the complementary sets and their multipliers shows that the data involving arbitrary functions can be represented by the following scheme:

$$\text{(19C)} \qquad \begin{aligned} &u_1, v_1 && \sim P(x^2, x^3, x^4) && \text{for } x^1 = 0, \\ &\partial u_1/\partial x^1, u_6 && \sim Q(x^1, x^2, x^4) && \text{for } x^3 = 0, \\ &u_2, v_8, w_5 && \sim R(x^3, x^4) && \text{for } x^1 = x^2 = 0, \\ &\partial v_8/\partial x^2, v_5, v_7 && \sim S(x^2, x^4) && \text{for } x^1 = x^3 = 0, \\ &v_6 && \sim T(x^4) && \text{for } x^1 = x^2 = x^3 = 0, \\ &\Gamma^i_{11} && \sim K(x^1, \cdots, x^4) && \\ &\Gamma^i_{12}, \Gamma^i_{22} && \sim L(x^2, x^3, x^4) && \text{for } x^1 = 0, \\ &\Gamma^i_{13}, \Gamma^i_{23}, \Gamma^j_{33} && \sim M(x^3, x^4) && \text{for } x^1 = x^2 = 0, \\ &\Gamma^i_{14}, \Gamma^i_{24}, \Gamma^j_{34}, \Gamma^j_{44} && \sim N(x^4) && \text{for } x^1 = x^2 = x^3 = 0, \\ & && (i = 1, 2, 3, 4;\ j = 3, 4). && \end{aligned}$$

The unknowns

$$\text{(19D)} \qquad u_3, u_4, u_5, v_2, v_3, v_4, w_1, w_2, w_3, w_4, w_6, \Gamma^k_{33}, \Gamma^k_{34}, \Gamma^k_{44}, g_{11}, \cdots, g_{44} \qquad (k = 1, 2)$$

can take on arbitrary values at the point $x^i = 0$.

EXISTENCE THEOREM: *Let the arbitrary functions represented by* (19C) *and the arbitrary constants* (19D) *be assigned subject to the condition that the initial values at* $x^i = 0$ *are such that* (1) *they satisfy equations* (7b), (2) *the determinant* $|(g_{\alpha\beta})_0|$ *does not vanish,* (3) *the inequalities* (19p) *are satisfied,* (4) *the equations* (19u), (19v) *and* (19y) *are satisfied. Then there exists one, and only one, set of functions* $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$ *analytic in the neighborhood of* $x^i = 0$ *which gives rise to the given functions indicated by* (19C) *and the given initial values* (19D), *and which constitutes a set of integrals of equations* (7b) *for which the surface* $x^1 = x^2 = 0$ *is a characteristic surface* C_2.

It should be observed that for an arbitrary integral $g_{\alpha\beta}(x)$ of the field equations (7b), a characteristic surface of the type specified by the above existence theorem will not exist; characteristic surfaces of this type will only exist for special integrals of the equations (7b).

An analogous treatment could evidently be made for the characteristic curves C_1. We will however not consider the further treatment of these curves as we believe that the above discussion sufficiently illustrates the procedure to be adopted.

§ 20. Criterium of Complete Integrability.

The method of establishing complete integrability which was discussed in § 9 can be extended to the normal systems which were applied to the treatment of the characteristic surface problem for an invariantive system of partial differential equations. In describing this extension let us consider in particular the affine invariantive case for which we have the equations (4.3), (4.5), (4.7) and in addition a system of the type (19.2a). The process of forming the normal system may necessitate the addition of equations obtained by differentiation of the above system; the normal system being thus formed, the construction of its conditions of integrability will necessitate further differentiations. Let us suppose that the construction of the above normal system and its conditions of integrability involves D_1, D_2, D_3, D_4 differentiations of the systems (4.3), (4.5), (4.7), (19.2a) respectively. Now assume that an upper bound L to the number of conditions on the derivatives of the unknowns in the normal system which are imposed by (4.3), (19.2a), and those conditions obtained by D_1 differentiations of (4.3) and by D_4 differentiations of (19.2a), is established in some manner. Let us also assume that the integrability conditions taken at the point $x^1 = 0$ involve M derivatives, parametric for the normal system. Then if it is known that for an unrestricted affine space there are $L+M$ derivatives of the sort appearing in the integrability conditions which can have arbitrary values at a point P of space, the normal system must be completely integrable. If this were not the case there would be less than M arbitrary derivatives at the point $x^i = 0$ taking account of the original equations (4.3), (4.5), (4.7), (19.2a) and those conditions obtained from them by the above differentiations D, and hence there would be less than $L+M$ arbitrary derivatives for an unrestricted space, contrary to hypothesis. Analogous remarks apply to the cases of the metric and vector invariantive systems.

Consider for example the normal system which arose in § 19 in the treatment of the characteristic surface C_2 of the invariantive system (7b). It can be seen from the form of the arbitrary data (19n) and (19o) that the integrability conditions of the normal system (1′)–(20′), (19a′)–(19d′), (19i), (19j), and (19m) involve

(20.1)

$K(4,2)$	or 10	derivatives	$g_{\mu\nu}$,
$4K(4,2)$	„ 40	„	γ_{lm},
$4K(4,1)+7K(3,1)+11K(2,1)+15K(1,1)\ldots$	„ 74	„	$\partial\gamma_{lm}/\partial x^{\mu}$,
$4K(4,2)+7K(3,2)+11K(2,2)+15K(1,2)\ldots$	„ 130	„	$\partial^2\gamma_{lm}/\partial x^{\mu}\partial x^{\nu}$,
$K(4,2)\ K(4,2)-4K(4,3)$	„ 20	„	B_{lm},
$4K(3,1)+6K(2,1)$	„ 24	„	$\partial B_{lm}/\partial x^{\mu}$,
$4K(3,2)+6K(2,2)$	„ 42	„	$\partial^2 B_{lm}/\partial x^{\mu}\partial x^{\nu}$

or a total of 340 parametric derivatives at the point $x^i = 0$; this number 340 is therefore the number M in the above discussion. At a point P of an unrestricted space there are

	$K(4,2)$	or 10	components	$g_{\mu\nu}$,
	$4K(4,2)$	„ 40	„	$\Gamma^i_{\alpha\beta}$,
	$4K(4,3)$	„ 80	„	$\Gamma^i_{\alpha\beta\gamma}$,
(20.2)	$4K(4,4)$	„ 140	„	$\Gamma^i_{\alpha\beta\gamma\delta}$,
	$K(4,2)\ K(4,2)-4K(4,3)$	„ 20	„	$g_{\alpha\beta,\gamma\delta}$,
	$K(4,2)\ K(4,3)-4K(4,4)$	„ 60	„	$g_{\alpha\beta,\gamma\delta\varepsilon}$,
	$K(4,2)\ K(4,4)-4K(4,5)$	„ 126	„	$g_{\alpha\beta,\gamma\delta\varepsilon\mu}$

or a total of 476 components which can take on arbitrary values. When the quantities of the type (20.1) are known at the point $x^i = 0$,[42] the components (20.2) are determined at $x^i = 0$; conversely when the quantities of the type (20.2) are known at $x^i = 0$, the quantities of the type (20.1) are determined at this point. Hence it follows that there are 476 quantities of the type (20.1) which can take on arbitrary values at a point in the unrestricted space.[43] The maximum number of conditions that can be obtained from the system (7b) by a single differentiation is 36 and by two differentiations is 84, account being taken of the identities (7c). By two differentiations of $g^{11} = 0$ over $x^1 = 0$, we obtain 6 conditions and by three differentiations, we obtain 10 conditions. Hence the upper bound L is $36 + 84 + 6 + 10$ or 136 and the number $L + M$ is equal to $136 + 340$ or 476 which was shown above to be equal to the number of arbitrary derivatives of the type (20.1) at a point of the unrestricted space. It follows therefore that the normal system composed of (1′)–(20′), (19a′)–(19d′), (19i), (19j) and (19m) is completely integrable.

A similar argument applies in the case of the existence theorems mentioned in connection with Cases I and II in § 19. The condition $g^{11} = 0$ over $x^1 = 0$ is simply replaced by the condition $g^{11} = 0$ over $x^1 = x^2 = 0$ in Case 1 and by $g^{11} = g^{22} = 0$ over $x^1 = x^2 = 0$ in Case II. Obvious changes in the above numbers would then result.

In the normal system treated under Case V the above argument must be modified to a larger extent since second derivatives appear in the system. The integrability conditions at $x^i = 0$ when expressed in terms of parametric derivatives will involve the following

	$K(4,2)$	or 10	derivatives	$g_{\mu\nu}$,
	$4K(4,2)$	„ 40	„	γ_{lm},
	$4K(4,1)+8K(3,1)+10K(2,1)+12K(1,1)$ ...	„ 72	„	$\partial\gamma_{lm}/\partial x^\mu$,
	$4K(4,2)+8K(3,2)+10K(2,2)+12K(1,2)$ ...	„ 130	„	$\partial^2\gamma_{lm}/\partial x^\mu\partial x^\nu$,
(20.3)	$4K(4,3)+8K(3,3)+10K(2,3)+12K(1,3)$ ...	„ 212	„	$\partial^3\gamma_{lm}/\partial x^\mu\partial x^\nu\partial x^\delta$,
	$4K(4,4)+8K(3,4)+10K(2,4)+12K(1,4)$ ...	„ 322	„	$\partial^4\gamma_{lm}/\partial x^\mu\partial x^\nu\partial x^\delta\partial x^\varepsilon$,
	$K(4,2)\ K(4,2)-4K(4,3)$	„ 20	„	B_{lm},
	$3K(3,1)+5K(2,1)+2+K(1,1)$	„ 22	„	$\partial B_{lm}/\partial x^\mu$,
	$3K(3,2)+5K(2,2)+K(3,1)+K(2,1)+K(1,2)$	„ 39	„	$\partial^2 B_{lm}/\partial x^\mu\partial x^\nu$,
	$3K(3,3)+5K(2,3)+K(3,2)+K(2,2)+K(1,3)$	„ 60	„	$\partial^3 B_{lm}/\partial x^\mu\partial x^\nu\partial x^\delta$

or a total of $927 = M$ parametric derivatives. To the 476 components in (20.2) which can have arbitrary values at a point P of space, we must add the

[42] *loc. cit.* Ann. (1), p. 720.

[43] *loc. cit.* Ann. (1), p. 690.

$4K(4,5)$ ..... or 244 components $\Gamma^{i}_{\alpha\beta\gamma\delta\varepsilon}$,

$4K(4,6)$ ..... „ 336 „ $\Gamma^{i}_{\alpha\beta\gamma\delta\varepsilon\mu}$,

$K(4,2)\,K(4,5)-4K(4,6)$ ..... „ 224 „ $g_{\alpha\beta,\gamma\delta\varepsilon\mu\nu}$,

which makes a total of 1260 components; hence there are 1260 derivatives of the type (20.3) which can take on arbitrary values at a point P of the unrestricted space. The maximum number of conditions imposed by the equations (7b) is $36+84+160$ or 280 since now three differentiations are allowed. From the equations $g^{11}=g^{12}=g^{22}=0$ over $x^1=x^2=0$ we obtain a maximum of $8+12+15+18$ or 53 conditions. Hence $L=280+53$ or 333 and $L+M$ is equal to the above number 1260. The system is therefore completely integrable.

§ 21. A Geometrical Interpretation.

Consider the problem treated in § 19, namely that of determining a solution $v_k(x)$ of the system (2.1) such that the surface $x^1=\cdots=x^\alpha=0$ is a characteristic surface $C_{n-\alpha}$ for the integral $v_k(x)$. Let us suppose that the combined system (2.1) and (19.1) has been put into a completely integrable normal form with the canonical assignment of cotes.

Let us first consider the case of an $(n-1)$-dimensional characteristic surface C_{n-1}. Suppose that some one of the functions belonging to the arbitrary data, let us say $\psi(x)$, is defined over a surface $x^{a_1}=\cdots=x^{a_\nu}=0$ where none of the a_i are equal to 1. Since the assignment of cotes is canonical we can confine ourselves to differentiations with respect to $x^2, x^3, \cdots, x^n$ in determining the power series expansions of the integrals $v_k(x)$ over the surface $x^1=0$. Moreover if we differentiate with respect to x^1 all equations of the system for which this differentiation is possible, we can thereafter confine ourselves to differentiations with respect to $x^2, \cdots, x^n$ so as to determine over the surface $x^1=0$, all first derivatives of the integrals $v_k(x)$. Now take the function $\psi(x)$ of the form

$$(x^1)^\mu\,\psi_\mu(x)+(x^1)^{\mu+1}\,\psi_{\mu+1}(x)+\cdots,$$

where the functions $\psi_i(x)$ are independent of x^1 and $\mu-2$ is equal to the maximum order of any derivative appearing in the normal system. From what we have just said regarding the determination of the integrals $v_k(x)$ and their first derivatives over the surface $x^1=0$ it is clear that the values of these quantities over $x^1=0$ will be independent of the choice of the above functions $\psi_\mu, \psi_{\mu+1}, \cdots$. Thus there exists an infinite number of sets of different integrals $v_k(x)$ such that each set of integrals and their first derivatives assume the same values over the characteristic surface $x^1=0$.

In particular the above discussion applies when the function $\psi(x)$ is defined over the entire x-space; if, however, $\psi(x)$ is defined only over a subspace

then the completely arbitrary functions, i. e. those functions defined over the entire x-space, which are part of the arbitrary data, will be the same for each of the above sets of integrals $v_k(x)$. Under this latter condition, i. e. the definite assignment of data defined over the entire x-space, the values of the $v_k(x)$ and their first derivatives over the surface $x^1 = 0$ will uniquely determine a solution $v_k(x)$ if this surface is not a characteristic surface.

If the arbitrary data of a normal system obtained in connection with an $n - \alpha$ dimensional characteristic surface problem contains a function which is defined over a surface $x^{b_1} = \cdots = x^{b_\gamma} = 0$, where $\gamma \geqq \alpha$ and at least one of the indices $b_i > \alpha$, then we can have an infinite number of sets of integrals $v_k(x)$ such that each set of integrals and their first derivatives assume the same values over the characteristic surface $x^1 = \cdots = x^\alpha = 0$. In particular the arbitrary data defined over the entire x-space and all surfaces of dimensionality greater than $n - \alpha$, may be the same for each of the above sets of integrals $v_k(x)$. Under these conditions the integrals $v_k(x)$ would be determined uniquely if the surface $x^1 = \cdots = x^\alpha = 0$ were not a characteristic surface. Analogous remarks apply to the invariantive systems.

The examples of § 19 will serve to illustrate the above remarks. In the case of the $(n-1)$-dimensional characteristic surface defined by equations (19.3) the arbitrary data (19 n) contains a function $J(x^1 x^3 x^4)$ defined over $x^2 = 0$ so that the above interpretation is possible. The existence theorems mentioned in Case I and Case II will however not permit the corresponding interpretation for the $(n-2)$-dimensional characteristic surface. For Case V we have an arbitrary function $S(x^2, x^4)$ defined over $x^1 = x^3 = 0$ so that the interpretation applies.

PRINCETON UNIVERSITY.

CRITERIA FOR THE IRREDUCIBILITY OF POLYNOMIALS.[1]

By H. L. Dorwart and Oystein Ore.

Introduction. 1. In this paper a special type of criteria for the irreducibility of polynomials with rational integral coefficients will be treated, namely criteria depending upon the arithmetical properties of the values which the polynomial can take for integral arguments. There exist a number of such criteria, given by various authors, and deduced by different methods. The present paper contains a simple and very elementary method of deducing these criteria from a common source; it also gives a series of new criteria and generalizations of the former results, and one obtains a clearer picture of the mutual relationship between the criteria of this type.

The first criterion of this kind was suggested by Schur,[2] who raised the question of the reducibility or irreducibility of the polynomials

$$f_1(x) = (x-a_1)\cdots(x-a_n)-1,$$
$$f_2(x) = (x-a_1)\cdots(x-a_n)+1,$$

where the a_i are rational integers. A partial solution was given in 1909 by Westlund,[3] who found that $f_1(x)$ is always irreducible in the domain of rational integers, and that $f_2(x)$ can be reducible only if it is a perfect square, in which case of course, n must be even. In the same year Flügel[4] determined all forms and degrees for which $f_2(x)$ is reducible. He found that $f_2(x)$ is reducible only in the following two cases of second and fourth degrees

$$(x-c)(x-c+2)+1 = (x-c+1)^2,$$
$$(x-c)(x-c+1)(x-c+2)(x-c+3)+1 = [(x-c)(x-c+3)+1]^2.$$

Pólya[5] found the following general criterion:

[1] Received, May 19, 1932; presented to the Amer. Math. Soc., April 3, 1931.

[2] Problem 226, Archiv. der Math. und Physik, (3), **13** (1908), p. 367.

[3] J. Westlund, "On the irreducibility of certain polynomials", Am. Math. Monthly, **16** (1909), pp. 66, 67.

[4] W. Flügel, Solution to problem 226, Archiv. der Math. und Physik, (3), **15** (1909), p. 271.

[5] Georg Pólya, "Verschiedene Bemerkungen zur Z[illegible]lentheorie", Jahresber. Deutschen Math. Ver., **28** (1919), pp. 31–40.

If for n integral values of x, the integral polynomial $P(x)$ of degree n has values which are different from zero and, without regard to sign, less than

$$\frac{\left(n-\left[\frac{n}{2}\right]\right)!}{2^{n-\left[\frac{n}{2}\right]}}$$

then $P(x)$ is irreducible.

This criterion can immediately be applied to the problem of Schur, even when the highest coefficient is not unity. It can also be applied to the seemingly more general case, where a polynomial $f(x)$ of nth degree takes the values $+1$ and -1, in all n times, for different integral x. It is of interest to observe however, as we show in the following, that this case can never occur except for some trivial cases of the lowest degrees. If $|f(x)| = 1$ for n integral values of x, then we have either $f(x) = +1$ or $f(x) = -1$ for all of these integral values. By a different method Pólya proves, that a polynomial $f(x)$ of odd degree is irreducible if $n \geqq 17$ and $|f(x)| = p$ for n different integral arguments, where p is a rational prime. It should be observed, that the same remarks as for the case $|f(x)| = 1$ also hold for this case, so that the generality of this theorem of Pólya is somewhat reduced.

In a joint paper the three authors A. Brauer, R. Brauer and H. Hopf[6] have treated a number of generalizations of the theorem of Schur, all dealing with polynomials of the type $G(P(x))$, where

$$P(x) = (x - a_1) \cdots (x - a_n)$$

and $G(x)$ is an irreducible polynomial whose degree does not exceed 4.

By applying the methods of Pólya to positively definite polynomials, Ille[7] was able to show the irreducibility of the polynomials

$$a(x - a_1)^2 \cdots (x - a_{n/2})^2 + 1,$$

where a is supposed to be positive, and $n \geqq 16$. A recent paper by Wegner[8] on the irreducibility of

$$P(x)^4 + d,\ n > 5,\ d > 0,\ d \not\equiv 3 \pmod 4$$

should be mentioned here.

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[6] A. Brauer, R. Brauer and H. Hopf, "Über die Irreduzibilität einiger spezieller Klassen von Polynomen", Jahresber. Deutschen Math. Ver,. **35** (1926), pp. 99–112.

[7] Hildegard Ille, „Einige Bemerkungen zu einem von G. Pólya herrührenden Irreduzibilitätskriterium", Jahresber. Deutschen Math. Ver., **35** (1926), pp. 204–8.

[8] Udo Wegner, „Über die Irreduzibilität einer Klasse von ganzen rationalen Funktionen", Jahresber. Deutschen Math. Ver., **40** (1931), pp. 239–241.

I. Polynomials taking the values ± 1.

2. Our first problem is to find all polynomials taking the values ± 1 a certain number of times for rational integral x. Let us assume that the polynomial $f(x)$ takes the value $+1$ at the m different points $a_1, a_2, \cdots, a_m$, then

$$f(x) = (x-a_1)(x-a_2)\cdots(x-a_m)g(x)+1. \tag{1}$$

Let us furthermore assume it takes the value -1 at a point a_{m+1}. From (1) one then obtains

$$g(a_{m+1})(a_{m+1}-a_1)(a_{m+1}-a_2)\cdots(a_{m+1}-a_m) = -2. \tag{2}$$

The differences $a_{m+1}-a_i$ can therefore take only the values $\pm 1, \pm 2$ and one obtains:

THEOREM I. *A polynomial $f(x)$ taking the values $+1$ (-1) $m > 3$ times cannot take the value -1 $(+1)$.*

Let us also consider the exceptional cases $m \leqq 3$. It should be observed, that if a polynomial $f(x)$ takes a set of values at certain points x, then $f(x-\alpha)$ will take the same values at the points $x+\alpha$; in the same way $f(-x)$ will take these values at points $-x$. We shall say that two polynomials $f(x)$ and $g(x)$ are *equivalent* if one can be obtained from the other by a transformation $x-\alpha$ or $-x-\alpha$, where α is an integer.

Without loss of generality we can assume, that the number of times a polynomial takes the value -1 does not exceed the number of times it takes $+1$. For $m = 3$ a polynomial which also takes the value -1 is from (2) easily seen to be equivalent to a polynomial

$$f(x) = (x-1)(x+1)(x+2)(xh(x)+1)+1 \tag{3}$$

where $h(x)$ is an arbitrary polynomial. It is obvious, that $f(1) = f(-1) = f(-2) = 1$, $f(0) = -1$ and that $f(x)$ cannot take the value -1 for any other point. For $m = 2$ a polynomial taking the value -1 must be equivalent to one of the following forms

$$\begin{aligned} &\text{a)}\quad f_1(x) = (x^2-1)(xh(x)+2)+1, \\ &\text{b)}\quad f_2(x) = (x+1)(x-2)(xh(x)+1)+1, \\ &\text{c)}\quad f_3(x) = (x-1)(x-2)(xh(x)-1)+1. \end{aligned} \tag{4}$$

In the first case $f(x)$ cannot take a second value -1; in the other two cases this is possible if $h(1) = 0$ or $h(3) = 0$ respectively, and the corresponding polynomials are therefore

$$f_1(x) = (x+1)(x-2)(x(x-1)\,h(x)+1)+1,$$
$$f_2(x) = (x-1)(x-2)(x(x-3)\,h(x)-1)+1,$$

where $h(x)$ is an arbitrary polynomial. One finds that $f_1(x)$ is equivalent to a polynomial of the form $-f_2(x)$. When finally $m=1$ one sees that a polynomial taking also the value -1 must be equivalent to

$$\begin{aligned} f_1(x) &= x((x-1)\,h(x)-2)+1, \\ f_2(x) &= x((x-2)\,h(x)-1)+1. \end{aligned} \tag{5}$$

3. These results permit us to find all polynomials of n-th degree taking the values ± 1 n times. When $n>4$ these values must either be all $+1$ or all -1 and consequently

$$f(x) = a(x-a_1)\cdots(x-a_n)\pm 1. \tag{6}$$

When $n=4$ the polynomial

$$f(x) = (x-1)(x+1)(x+2)(ax+1)+1 \tag{7}$$

will take $+1$ three times and -1 once. The polynomials

$$\pm f_1(x) = (x+1)(x-2)(ax(x-1)+1)+1 \tag{8}$$

will take $+1$ twice and -1 twice; for $n=3$ the polynomials

$$\begin{aligned} f_1(x) &= (x^2-1)(ax+2)+1, \\ f_2(x) &= (x+1)(x-2)(ax+1)+1, \\ f_3(x) &= (x-1)(x-2)(ax-1)+1 \end{aligned} \tag{9}$$

take $+1$ twice and -1 once. For $n=2$

$$\begin{aligned} f_1(x) &= x(a(x-1)-2)+1, \\ f_2(x) &= x(a(x-2)-1)+1 \end{aligned} \tag{10}$$

take the values ± 1. Any polynomial of n-th degree taking ± 1 n times is equivalent to one of the polynomials (6), (7), (8), (9), (10).

One can easily find all polynomials taking ± 1 *more* times than their degrees. They are found to be equivalent to the following

$$\begin{array}{lll} n=3, & (x-1)(x+1)(x+2)+1 & \text{values } +1,\ -1,\ +1,\ +1, \\ n=2, & \pm((x-1)(x-2)-1) & \text{„ } -1,\ +1,\ +1,\ -1, \\ & 2x^2-1 & \text{„ } +1,\ -1,\ +1, \\ n=1, & 2x-1 & \text{„ } -1,\ +1, \\ & x-1 & \text{„ } -1,\ +1. \end{array} \tag{11}$$

4. These almost trivial results regarding polynomials taking the values ± 1 immediately give a series of theorems on the irreducibility of such polynomials. Let $f(x)$ be a polynomial of n-th degree taking the values ± 1 at m points x. We shall suppose $m > 4$; it then follows from § 2 and § 3, that $f(x)$ can take only the value $+1$ or the value -1 at all the points and that $m \leqq n$. Any factor $g(x)$ of $f(x)$ must now obviously take the values ± 1 at the same points x, and since $m > 4$, $g(x)$ must be of the form

$$g(x) = (x-a_1) \cdots (x-a_m)\, h(x) \pm 1. \tag{12}$$

We can therefore say:

THEOREM II. *A polynomial $f(x)$ of n-th degree taking the values $+1$ at m points $a_1, a_2, \cdots, a_m$ where $4 < m \leqq n$ can have factors only of the form* (12). *The degree of a factor is therefore never less than m, and when $m > n/2$, $f(x)$ is irreducible.*

When $m \leqq 4$ the possible factors can be obtained from § 2 and § 3. It follows from (11) that $f(x)$ can never have a linear factor except when $m \leqq 2$.

Theorem II obviously applies to the polynomials considered by Schur, and also to the slightly more general polynomials

$$f(x) = a(x-a_1) \cdots (x-a_n) \pm 1. \tag{13}$$

For $n \geqq 5$ this polynomial is therefore irreducible. For $n = 4$ it may have two second degree factors each taking ± 1 four times and therefore being equivalent to the second polynomial in (11). For $n = 3$ one of the eventual factors must be linear and take ± 1 three times which is impossible. The case $n = 2$ is easily treated and one finds:

THEOREM III. *The polynomials*

$$f(x) = a(x-a_1) \cdots (x-a_n) \pm 1$$

are always irreducible except when $f(x)$ is equivalent to one of the following three polynomials

$$x(x-1)(x-2)(x-3)+1 = (x(x-3)+1)^2,$$
$$x(x-2)+1 = (x-1)^2, \quad 4(x-1)x+1 = (2x-1)^2.$$

The first two exceptions were found by Flügel.

5. We shall now consider the polynomials studied by Ille, or, since our method is equally applicable to such polynomials, the polynomials of the type

$$f(x) = b_0 P(x)^2 + b_1 P(x) + 1 = F(P(x)), \tag{14}$$

where

$$F(x) = b_0 x^2 + b_1 x + 1$$

and

$$P(x) = (x - a_1) \cdots (x - a_n). \tag{15}$$

Such polynomials take the value $+1$ at n points $a_1, a_2, \cdots, a_n$ while the degree is $2n$. Any factor of $f(x)$ must therefore according to Theorem II for $n \geqq 5$ be at least of degree n and consequently of the form

$$c(x - a_1) \cdots (x - a_n) + 1.$$

One obtains in this case

$$f(x) = (c_1 P(x) + 1)(c_2 P(x) + 1) = c_1 c_2 P(x)^2 + (c_1 + c_2) P(x) + 1$$

and $F(x)$ must be a reducible polynomial.

THEOREM IV. *The polynomials*

$$F(P(x)) = b_0 P(x)^2 + b_1 P(x) + 1,$$

where $F(x) = b_0 x^2 + b_1 x + 1$ *is an irreducible polynomial and*

$$P(x) = (x - a_1) \cdots (x - a_n),$$

are always irreducible, when $n \geqq 5$.

For $n \leqq 4$ there are certain exceptions, which can easily be deduced from the results of § 3. The possible exceptional cases are equivalent to the following:

$n = 4$ $\quad F(x) = -a^2 x^2 + x + 1, \quad P(x) = x(x-1)(x-2)(x-3),$
$\quad F(P(x)) = (aP(x) + x^2 - 3x + 1)(-aP(x) + x^2 - 3x + 1),$

$n = 3$ $\quad F(x) = x^2 - 3x + 1, \quad P(x) = (x-1)x(x+1),$
$\quad F(P(x)) = (x(x-1) - 1)(x(x-1)((x+1)^2 + 1) - 1),$
$\quad F(x) = 8x^2 - 1, \quad P(x) = (x-1)x(x+1),$
$\quad F(P(x)) = (2x^2 - 1)(4x^4 - 6x^2 + 1),$

$n = 2$ $\quad F(x) = -x^2 + 4x + 1, \quad P(x) = x(x-1),$
$\quad F(P(x)) = ((x-1)(x-2) - 1)(-x(x+1) + 1),$
$\quad F(x) = -x^2 + x + 1, \quad P(x) = x(x-2),$
$\quad F(P(x)) = ((x-1)(x-2) - 1)(-x(x-1) + 1),$
$\quad F(x) = -4x^2 + 4x + 1, \quad P(x) = x(x-1),$
$\quad F(P(x)) = (2x^2 - 1)(-2x^2 + 4x - 1).$

Let us now apply these results to the polynomials

$$f(x) = c(x-a_1)^2 \cdots (x-a_n)^2 + 1$$

studied by Ille. These polynomials correspond to the special case $F(x) = cx^2 + 1$ in Theorem 4; it is therefore not necessary to assume, as does Ille, that c is positive, it is sufficient to assume that c is not a negative square number. From Theorem 4 and the preceding list of exceptions it therefore follows

THEOREM 5. *The polynomials*

$$f(x) = c(x-a_1)^2 \cdots (x-a_n)^2 + 1$$

are always irreducible if $c \neq -b^2$ *except when* $f(x)$ *is equivalent to*

$$-8(x-1)^2x^2(x+1)^2 + 1 = (2x^2-1)(-4x^4+6x^2-1).$$

II. Generalization to quadratic imaginary fields.

6. These investigations can be extended to polynomials with coefficients in an algebraic field with a finite number of units, i. e. to quadratic imaginary fields $K(\sqrt{-D})$, and in this way one can obtain various generalizations of the former criteria for irreducibility. If we first assume $D \neq 1$ and $D \neq 3$, the field $K(\sqrt{-D})$ has only the units ± 1. In this case let $f(x)$ be a polynomial with coefficients in $K(\sqrt{-D})$ taking the value $+1$ for m values of x in $K(\sqrt{-D})$. The polynomial must then have the form

$$f(x) = (x-\alpha_1) \cdots (x-\alpha_m)\, h(x) + 1.$$

If $f(x)$ also takes the value -1 for $x = \alpha_{m+1}$ one obtains the condition

$$(\alpha_{m+1}-\alpha_1) \cdots (\alpha_{m+1}-\alpha_m)\, h(\alpha_{m+1}) = -2. \tag{16}$$

Since the integer 2, as one easily verifies, has no proper divisors in $K(\sqrt{-D})$ if $D \neq 2$ and $D \neq 7$, one of the factors in (16) must be ± 2 and the remaining have the values ± 1; this is however only possible when $m \leqq 3$. In the cases $D = 2$ and $D = 7$ one finds the decompositions

$$2 = (-\sqrt{-2})(\sqrt{-2}), \qquad 2 = \frac{1+\sqrt{-7}}{2} \cdot \frac{1-\sqrt{-7}}{2},$$

and one can therefore only deduce, that $m \leqq 4$.

THEOREM 6. *Let* $f(x)$ *be a polynomial in a quadratic field with the only units* ± 1. *When* $f(x)$ *takes the value* $+1$ *for* m *integral values of* x *in*

$(K\sqrt{-D})$, *then* $f(x)$ *cannot take the value* -1 *if* $m>3$ *and* $D\neq 2$ *and* $D\neq 7$. *If* $D=2$ *or* $D=7$ *the same theorem holds when* $m>4$.

When $D=1$ one obtains the field $K(i)$ with the *four* units $\pm 1, \pm i$ and with the decomposition $2=(1-i)(1+i)$. The differences between the four units are all divisors of 2. In the field $K(\sqrt{-3})$ there are *six* units ± 1, $(\pm 1\pm\sqrt{-3})/2$ and their differences are all divisors of 2 or $\sqrt{-3}$, which are both indecomposable numbers in the field. We shall not go into the elementary, but rather exacting considerations, which are necessary in the last two cases. The result is, that a polynomial $f(x)$ in $K(i)$ which takes more than 7 unit values, must always take the same unit value at all points, and the same result holds in $K(\sqrt{-3})$ for polynomials taking more than 8 unit values. For the cases $D\neq 1, 2, 3, 7$ one finds as in the rational case, that a polynomial taking ± 1 more than 4 times must take $+1$ or -1 at all points. For $D=2$ and $D=7$ the same result holds for polynomials taking ± 1 more than 5 times.

We can summarize these results in the following theorem corresponding to Theorem 1 for the rational case:

THEOREM 7. *Let* $N_1=7$, $N_2=5$, $N_3=8$, $N_7=5$ *and* $N_D=4$ *for all other positive integral rational* D *without square factors. If a polynomial* $f(x)$ *with integral coefficients in the quadratic field* $K(\sqrt{-D})$ *takes unit values* ε_i $(i=1, 2, \cdots, m)$ *for* $m>N_D$ *integers in this field, then all units* ε_i *must be the same.*

As in the rational case the exceptional polynomials taking at least two different unit values can easily be determined, but since the number of exceptions, particularly for $D=1$ and $D=3$, is rather large, the list will not be included here.

7. Theorem 7 can be used to deduce criteria for irreducibility in $K(\sqrt{-D})$ in the same way that Theorem 1 was used in the rational case in § 4. Let $f(x)$ be a polynomial of n-th degree with integral coefficients in $K(\sqrt{-D})$ and let us assume that $f(x)$ takes unit values for m integers $\alpha_1, \cdots, \alpha_m$ in the field. We suppose $m>N_D$ so that $f(x)$ takes the same unit value at all m points, and one can obviously assume, without loss of generality, that this unit is ± 1 so that $f(x)$ has the form

$$f(x)=(x-\alpha_1)\cdots(x-\alpha_m)f_1(x)\pm 1. \tag{17}$$

Every factor $g(x)$ of $f(x)$ in $K(\sqrt{-D})$ must also take a unit value at all points α_i and consequently take the form

$$g(x)=(x-\alpha_1)\cdots(x-\alpha_m)g_1(x)+\varepsilon, \tag{18}$$

where ε is a fixed unit. We therefore obtain the following analogue to Theorem 2:

THEOREM 8. *A polynomial $f(x)$ with integral coefficients in the field $K(\sqrt{-D})$ taking unit values at m integral points*

$$\alpha_1, \cdots, \alpha_m, \quad N_D < m$$

in the field, is of the form (17) *and eventual factors of $f(x)$ in $K(\sqrt{-D})$ must have the form* (18). *The degree of a factor is at least m and for $m > n/2$ $f(x)$ is irreducible in $K(\sqrt{-D})$.*

This theorem has several applications, also for irreducibility in the rational domain. A polynomial with rational coefficients having all of its roots in $K(\sqrt{-D})$ must have the form

$$P_D(x) = \prod_{i=1}^{r} (x^2 + a_i x + b_i) \prod_{i=1}^{s} (x - c_i) \tag{19}$$

where

$$a_i^2 - 4b_i = -D k_i^2 \qquad (i = 1, 2, \cdots, r). \tag{20}$$

We shall in the following always suppose that the coefficients a_i, b_i, c_i are rational integers. It then follows:

THEOREM 9. *A polynomial $f(x)$ of n-th degree with rational integral coefficients having the form*

$$f(x) = P_D(x)\, h(x) \pm 1,$$

where $P_D(x)$ is given by (19) *and where*

$$2r + s > N_D, \quad 2r + s > n/2,$$

is irreducible in $K(\sqrt{-D})$.

A special case is for instance polynomials of the form

$$A(x^2 + a_1^2) \cdots (x^2 + a_n^2) \pm 1, \tag{21}$$

which are irreducible in the rational field and in $K(i)$ when $n \geqq 4$. The possible exceptions for $n < 4$ can easily be found. Another immediate consequence of Theorem 8 is the following

THEOREM 10. *Polynomials of the form*

$$f(x) = a(x - a_1) \cdots (x - a_n) \pm 1$$

are irreducible in all quadratic imaginary fields, when $n > 8$.

8. We shall finally apply Theorem 8 to investigations analogous to those in § 5. It follows without difficulty:

THEOREM 11. *Let $\alpha_1, \cdots, \alpha_m$ be m different integers in the field $K(\sqrt{-D})$ and $m > N_D$; let furthermore*

$$P(x) = (x-\alpha_1)\cdots(x-\alpha_m), \tag{22}$$

and $F(x) = \gamma_0 x^2+\gamma_1 x+1$ *be an irreducible polynomial with integral coefficients* γ_0, γ_1, *in* $K(\sqrt{-D})$. *The polynomial*

$$F(P(x)) = \gamma_0 P(x)^2+\gamma_1 P(x)+1 \tag{23}$$

is then irreducible in $K(\sqrt{-D})$.

A special case are the rational polynomials

$$f(x) = a P_D(x)^2+1 \tag{24}$$

where $P_D(x)$ is given by (19); these polynomials are irreducible in the rational field if $2r+s>N_D$ and if a is not of the form $-b^2$ or $-Dc^2$. In the last case one has however

$$f(x) = (c\sqrt{-D}\,P_D(x)+1)(-c\sqrt{-D}\,P_D(x)+1)$$

and each of the factors is irreducible in $K(\sqrt{-D})$ according to Theorem 9. This gives us:

The polynomials (24) *are irreducible in the rational field if* a *is not a negative square number and* $n>N_D$, *where* n *is the degree of* $P_D(x)$.

An example are the polynomials

$$a(x^2+a_1^2)^2\cdots(x^2+a_n^2)^2\pm 1$$

which are irreducible if $a \neq -b^2$ and $n \geqq 4$.

One can obtain a new class of irreducible polynomials in the following way. Let

$$F(x) = A_0 x^4+A_1 x^3+A_2 x^2+A_3 x+1$$

be an irreducible polynomial with integral rational coefficients, and let us suppose $F(x)$ reducible in a quadratic imaginary field $K(\sqrt{-D})$

$$F(x) = (\lambda_0 x^2+\lambda_1 x+1)(\lambda_0' x^2+\lambda_1' x+1).$$

When $P_D(x)$ is a rational polynomial of the form (19), the polynomial

$$\lambda_0 P_D(x)^2+\lambda_1 P_D(x)+1$$

is irreducible in $K(\sqrt{-D})$ according to Theorem 11, when $n=2r+s>N_D$, and one obtains consequently:

THEOREM 12. *If*

$$F(x) = A_0x^4 + A_1x^3 + A_2x^2 + A_3x + 1$$

is a polynomial with integral rational coefficients and irreducible in the rational field K, but reducible in a quadratic imaginary field $K(\sqrt{-D})$, *and if* $P_D(x)$ *is a polynomial of the form* (19) *satisfying the conditions* (20), *then* $F(P_D(x))$ *is an irreducible polynomial in K, when* $n > N_D$ *where n denotes the degree of* $P_D(x)$.

For the case

$$P_D(x) = (x - c_1) \cdots (x - c_n)$$

this theorem gives one of the main results of the paper by A. Brauer, R. Brauer and H. Hopf.

III. Polynomials taking prime values.

9. We shall now apply the same principles as before to determine the form of polynomials taking the values $\pm p$ for a number of integral values, and from the results new types of criteria for irreducibility can be derived.

Let p be a prime, and $f(x)$ a polynomial taking the value $+p$ for m different integral rational values $a_1, \cdots, a_m$. This polynomial then takes the form

$$f(x) = (x - a_1) \cdots (x - a_m)\, h(x) + p. \tag{25}$$

If $f(x)$ also takes the value $-p$ at a point a_{m+1} we obtain from (25) the condition

$$(a_{m+1} - a_1) \cdots (a_{m+1} - a_m)\, h(a_{m+1}) = -2p. \tag{26}$$

The factors $a_{m+1} - a_i$ occurring on the left side in (26) therefore can take only the values ± 1 except for two factors which may take the values ± 2 or $\pm p$ respectively. Now a product of such differences can take only the values ± 1 or ± 2 if the number of factors does not exceed 3 and one obtains the following theorem without any difficulty:

THEOREM 13. *Let p be a prime and let* $f(x)$ *be a polynomial taking the value* $+p\,(-p)$ *at m different integral points. If then* $m > 4$ *the polynomial* $f(x)$ *cannot take the value* $-p\,(+p)$ *for any other integral x.*

For the applications we need this theorem in the following modified form:

THEOREM 14. *Let* $f(x)$ *be a polynomial taking the values* $+p$ *at m different integral points* $a_1, a_2, \cdots, a_m$. *If* $m > 5$, $f(x)$ *must take the same value* $+p$ *or* $-p$ *at all points and consequently have the form*

$$f(x) = (x - a_1) \cdots (x - a_m)\, h(x) \pm p. \tag{27}$$

Theorem 14 is in most cases a consequence of Theorem 13; it only remains to show that a polynomial taking $+p$ four times cannot take $-p$

more than one time or that a polynomial taking $+p$ three times cannot take $-p$ more than two times. In the first case $f(x)$ must have the form

$$f(x) = (x-a_1)(x-a_2)(x-a_3)(x-a_4)h(x)+p$$

and if $f(a_5) = -p$ one obtains the condition

$$(a_5-a_1)(a_5-a_2)(a_5-a_3)(a_5-a_4)h(a_5) = -2p. \tag{28}$$

It is obvious that at least two of the differences on the left must have the values ± 1 and a_5 is therefore the middlepoint between a_1 and a_2, so that $a_2 = a_1+2$, $a_5 = a_1+1$. This shows that $f(x)$ must be equivalent to a polynomial of the form

$$f(x) = (x-1)(x\pm 2)(x\pm p)h(x)+p, \tag{29}$$

where $h(0) = \pm 1$ and here the signs can be so combined that $f(0) = -p$. One sees also that the polynomial (29) cannot take the value $-p$ for any other value than $x = 0$. In the second case let

$$f(x) = (x-a_1)(x-a_2)(x-a_3)h(x)+p$$

be a polynomial taking the value $+p$ at three points a_1, a_2, a_3. If it shall take the value $-p$ at a fourth point a_4 one obtains the condition

$$(a_4-a_1)(a_4-a_2)(a_4-a_3)h(a_4) = -2p. \tag{30}$$

Let us first suppose that two of the left-hand differences are ± 1, for instance

$$|a_4-a_1| = |a_4-a_2| = 1.$$

We can then without loss of generality assume $a_1 = -1$, $a_2 = +1$, $a_4 = 0$. If now $f(x)$ takes the value $-p$ for a fifth value a_5 the differences a_5-a_1 and a_5-a_2 can take only the values $\pm 1, \pm 2, \pm p$. A simple consideration of the possible arrangement of the integral points on the real axis shows, that $p = 3$ is the only possibility and in this case one must have $a_5 = \pm 2$ while $a_3 = \pm 3$, and polynomials taking $+3$ three times and -3 two times can actually be constructed. Any further value -3 is however obviously excluded. In the remaining case of (30) one must have

$$|a_4-a_1| = 1, \qquad |a_4-a_2| = 2, \qquad |a_4-a_3| = p$$

and if $f(x)$ is to take another value $-p$ one finds $p = 3$ or $p = 5$ and in neither case is it possible to find a polynomial $f(x)$ taking a third value

$-p$. The form of the exceptional polynomials taking both values $\pm p$ can be found in the same way as the polynomials taking the values ± 1 were determined in § 2.

10. Let us now consider a polynomial taking the values $\pm p$ at certain points $a_1, a_2, \cdots, a_m$. Let us suppose $f(x)$ reducible, $f(x) = k(x)\,l(x)$; a factor $k(x)$ can then take only the values ± 1 and $\pm p$ at these points. If $k(x)$ takes ± 1 for k_1 points a_i and $\pm p$ for k_p points, then $k_1 + k_p = m$ and at least one of the two numbers k_1 and k_p is $\geqq m/2$. When $m > 10$ and for instance $k_p \geqq m/2 > 5$, then according to Theorem 14 $k(x)$ must have the form

$$k(x) = (x - a_1) \cdots (x - a_{k_p})\,\overline{k}(x) \pm p$$

and its degree is $\geqq m/2$. A similar result is obtained when $k_1 \geqq m/2 > 5$.

THEOREM 15. *A polynomial of the form*

$$f(x) = (x - a_1) \cdots (x - a_m)\,f_1(x) \pm p,$$

where $m > 10$, *can have factors only of degrees* $\geqq m/2$.

When m is equal to the degree n of $f(x)$ we obtain Pólya's result in a more concise form:

THEOREM 16. *A polynomial of the form*

$$f(x) = a(x - a_1) \cdots (x - a_n) \pm p, \tag{31}$$

where $n > 10$, *is irreducible if* n *is odd and when* n *is even it may have only two factors of the degree* $n/2$.

When n is even, for instance $n = 2m$, the possible factors of degree m must be

$$k(x) = b(x - a_1) \cdots (x - a_m) + 1,$$
$$l(x) = c(x - a_1) \cdots (x - a_m) + p,$$

where it is supposed that $f(x)$ takes the value $+p$ for all a_i, as can be done without influencing the generality. From the fact that these factors must take the values $\pm p$, ± 1 respectively for the remaining points $a_{m+1}, \cdots, a_{2m}$ one obtains necessary and sufficient conditions for reducibility. A necessary condition is that $|a|$ be a square, a being the highest coefficient in (31). By a detailed study of the conditions imposed on the possible factors of $f(x)$ for the cases when $n \leqq 10$, it is found that Theorem 16 is true for every n except $n = 3$, in which case $f(x)$ is irreducible unless it is equivalent to one of the following two polynomials

$$(x - 1)(x + 1)(x + p) + p = x(x^2 + px - 1),$$
$$4x(x - 1)(2x + p - 1) + p = (2x - 1)(4x^2 + 2px - 4x - p).$$

A theorem somewhat more general than Theorem 15 is the following:

THEOREM 17. *A polynomial taking the values* ± 1 *or* $\pm p$ *at altogether* $m > 10$ *points cannot have any factors of degree less than* $m/2$.

A possible factor must take the values ± 1 and $\pm p$ at the same m points and it must therefore take either ± 1 or $\pm p$ at at least $m/2$ points, and since $m/2 > 5$, the degree of the factor is at least $m/2$ according to previous theorems.

It is obvious that these investigations can be generalized in various ways. If, for instance, $f(x)$ takes the value $\pm d$ at certain points, then the degree of the possible factors is always greater than a certain limit depending upon the number of primes dividing d. The results can also be generalized to quadratic imaginary fields as before, and one can in this way obtain results generalizing the theorem of Wegner mentioned in the introduction.

INDEPENDENT ARCS OF A CONTINUOUS CURVE.[1]

DEDICATED TO J. R. KLINE.

BY LEO ZIPPIN.[2]

In this paper we give a new proof, for locally compact continuous curves, of a theorem which has been proven for plane continuous curves by N. E. Rutt, for regular curves by K. Menger, and for compact continuous curves by G. Nöbeling.[3] The extension of the field of this theorem is certainly not of conspicuous importance, although it is not altogether a consequence of Nöbeling's result. However, the proof we give is entirely independent of various theorems upon which Nöbeling strongly leans, and *in particular* of the corresponding theorem of Menger for regular curves to which Nöbeling reduces the consideration of his general case. These theorems become immediate applications of, in some instances slight corollaries to, ours.[4] Furthermore, the proof is readily followed and, it is thought, strikingly fundamental. Of the subject of continuous curves, for example, little more is used than their definition and the property that they are arcwise connected. Finally, the proof is thoroughly mathematical inductive. A proof of this nature, for the theorem in question, seems to us altogether desirable.

1. **Introduction.** A locally compact continuous curve C'' is a locally compact, metric, separable space which is locally connected and connected. As is well-known, it is arcwise connected. If we observe that an open connected subset of C'' is, on our definition, itself a space C''[5] there results the local arcwise connectedness of C'' (this is, of course, also well-known). Principally because it facilitates our induction, it will be convenient to *extend* this class of spaces slightly. We introduce a *local continuous curve* C: this shall have all the properties of a space C'' excepting that it is not required to be connected (in the large). Every open connected subset of a space C is a space C'', in particular every com-

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[2] National Research Fellow.

[3] For references, etc., see G. Nöbeling: "Eine Verschärfung des n-Bein-Satzes". Fund. Math., **18** (1931), pp. 23-38.

[4] Exception should be made of that part of Menger's "n-Bein-Satz" which relates to "wachsender Ordnung": see Nöbeling.

[5] i. e. a locally compact continuous curve.

ponent, i. e. every maximal connected subset (it is clear that these are open in a locally connected space). For us the spaces C will have this advantage that every open subset of a space C is a space C. We shall say that a point set K separates two points x and y of C provided these belong to no component of $C-K$. The statement that the null-set does *not* separate x and y we shall interpret to mean, that these belong to a component C_{xy} of C; in consequence of this C will then contain an arc xy. We shall say that a set of arcs xy are independent if they are mutually exclusive excepting for their common endpoints.

PRINCIPAL THEOREM. *If two points x and y of a local continuous curve C are not separated by any set of N points (N finite) then C contains at least $N+1$ independent arcs xy.*

The basis of our proof rests on a purely combinatorial theorem, of independent interest, on linear graphs, and a method of proof which we discovered[6] for the cyclic connectivity theorem of G. T. Whyburn (the case $N = 1$ of our theorem). We propose to begin with a theorem on linear graphs, and it may be well to say an expository word about our precise understanding of this structure.

2. A linear graph is a finite collection of *elements*, whose total number we shall call the *complexity* of the graph, which are either *vertices* or *1-cells*. The concepts vertex and 1-cell are essentially undefined. For each 1-cell we shall require to know which vertices are incident with it and allow that it is incident with none, with one only, or with two vertices which need not be distinct. Here incidence is essentially an undefined relation. We can now define a *chain* in a linear graph as an alternate succession of distinct vertices and 1-cells, of one of four obvious types, such that each 1-cell is incident with the one or two vertices which are adjacent to it (in the order of the succession, understood). A chain beginning with a vertex x_1 and ending with a vertex x_n is called an *arc* $x_1 x_n$. We shall introduce point-set terminology. A subset of the graph is *connected* if every pair of its elements belongs to a chain of the subset. A *component* is a maximal connected subset. The *closure* of a subset H is the set of elements of H *and* the set of vertices of the graph which are incident with 1-cells of H. We shall say that H *separates* two elements x and y if these belong to no chain of its complement.

If x and y are distinct vertices of a linear graph L, there exists a number $r_{xy} \geqq 0$ such that at least one set of r_{xy} elements, *but no set of fewer*, separates x and y in L; we shall call this the *cut-number* in L for x and y. We remark that L cannot contain more than r_{xy} independent arcs xy,

[6] This proof was presented to J. R. Kline in the spring of 1929, and to R. L. Moore in the summer of that year. It has never been published.

i. e. arcs xy which are mutually exclusive except for the common elements x and y. We wish to prove:

THEOREM ON LINEAR GRAPHS.[7] *If x and y are vertices of a linear graph L, then L contains exactly r_{xy} independent arcs xy.*

If our theorem is not true there exists a linear graph *of minimum complexity* for which it does not hold. Let this be L, and write (for convenience) $r_{xy} = r$.[8] Now if L contains a 1-cell xy, let us call this z. If L contains an arc xy which contains at most a single vertex distinct from x and y, let us call this vertex z: in this case our arc xy is an arc $[x, xz, z, zy, y]$.[9] Let L' be the linear graph $L-z$. Clearly if A is any subset of L' separating x and y in L', $A+z$ separates x and y in L. Then r'_{xy} (the cut-number in L' for x and y) cannot be less than $r-1$, and L', being of lower complexity than L, must contain exactly $r-1$ independent arcs xy. Now if z is a 1-cell xy, it is clear that $[x, xy, y]$ is the desired r-th arc. But if z is a vertex, then $[x, xz, z, zy, y]$ is the desired r-th arc. *For* none of the arcs which we have found in L' can contain the 1-cells xz or zy, since with either of these an arc must also contain the vertex z and this is not in L'.

We may then assume that every arc xy contains at least two more distinct vertices; let $x, xp, p, pq, q, \cdots, y$ be any such arc of L. Let L'' be the linear graph $L-pq$, pq being a 1-cell of this arc. It is of lower complexity than L and is contained in L. Then it must contain $r-1$ elements and not fewer which separate x and y in L''. Let these be $a'_1, \cdots, a'_{r-1}$. If one of these, say a'_j, is not a vertex it is a 1-cell. By the first paragraph we must assume that the 1-cell contains at least one vertex distinct from x and y; call this one a_j.[10] If a'_i is a vertex, rename it a_i. Then $A = \sum_1^{r-1} a_k$ separates x and y in L'', for any arc containing a 1-cell a'_j must contain both vertices incident with a'_j, therefore a_j. Now A does not separate x and y in L. Then $L-A$ contains an arc xy and every such arc must contain the 1-cell pq, and therefore the vertices p and q. Then these vertices do not belong to A and, moreover, $p+A$ and $q+A$ must separate x and y in L.

Let X denote the component of $L-(q+A)$ which contains x, and $\bar{Y}$ the

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[7] To J. W. Alexander are due one or two suggestions which have considerably simplified the author's original proof. This form of the theorem is suggested by him. Compare the next, and equivalent theorem (our original one).

[8] We may take it that $r \geqq 2$; for $r = 0, 1$, the theorem is trivial.

[9] Vertices p and q in juxtaposition shall denote the 1-cell pq unless otherwise indicated.

[10] If a 1-cell a'_j is not incident with two distinct vertices it can belong to no arc xy: then the remaining elements (fewer than $r-1$) must separate x and y in L''.

closure of that component Y which contains y. Since no $r-1$ vertices separate x and y in L it is obvious that $\bar{Y}$ contains $q+A$ and that X contains at least r 1-cells. Let L^0 be a linear graph which consists of the linear graph $\bar{Y}$, a new vertex x^0, and r 1-cells x^0q, x^0a_k (it is understood here, and in immediate sequel, that k runs from 1 to $r-1$). Since X contains at least one vertex p distinct from x, it is clear that L^0 is of lower complexity than L. Moreover, no $r-1$ elements of L^0 separate x^0 and y in L^0: otherwise it is apparent that the corresponding elements in $\bar{Y}$ necessarily separate x from y in L. Then L^0 contains r independent arcs x^0y. Then $\bar{Y}$ contains r arcs yq, ya_k, which are *mutually exclusive except for* y. We remark that since X contains p, $\bar{Y}$ does not. Then no arc yq, ya_k, contains the 1-cell pq. In particular the arc yq of $\bar{Y}$ does not contain p, and it contains no vertex a_k.

Let X'' be that component of $L-(p+A)$ which contains x, Y'' the component containing y, and $\bar{X}''$ the closure of X''. We see that the arc yq, above, belongs to Y''. Then X'' does not contain q and is thus necessarily a subset of the component X of the preceding paragraph. But applying the device of that paragraph to $\bar{X}''$ (it will be observed that Y'' contains a vertex q distinct from y and at least r 1-cells) we see that $\bar{X}$ (since it contains $\bar{X}''$) contains r arcs xp, xa_k, which are mutually exclusive except for x. Moreover none of these arcs contains q, therefore none of them contains the 1-cell pq. Then we see at once that the arc xa_k plus the arc ya_k, $k=1, \cdots, r-1$, and the arc xp plus the 1-cell pq plus the arc yq constitute a set of r independent arcs xy. Our theorem is proved.

Now if no N elements of a graph L separate two given vertices x and y, it is clear that $r_{xy} \geqq N+1$. Then we may state the following equivalent theorem:

THEOREM ON LINEAR GRAPHS.[11] *If no N elements of a linear graph L separate two given vertices x and y, then L contains at least $N+1$ independent arcs xy.*

It is not necessary perhaps to assist the reader in anticipating our later application of this theorem by the remark that a point-set K (in particular, then, a rather simple sort of local continuous curve) will be a linear graph provided it can be expressed as the point-set sum of a finite number of points (which we will call its vertices) and a finite set of mutually exclusive open arcs[12] (which we shall call its 1-cells) if we interpret incidence to mean that a given 1-cell (arc) has a given vertex (point) for

[11] Compare the special case of this in H. Whitney: "Congruent Graphs . . ." Amer. Jour. of Math., **54**, p. 160.

[12] An open arc denotes throughout the homeomorph of the open linear interval: $0<t<1$.

an endpoint. Then an *arc* of the linear graph K will be a point-set arc of the point-set K.

We now embark upon the principal theorem:

If x and y are points of a local continuous curve C and are not separated in C by any set of N points (N finite), then C contains at least $N+1$ independent arcs xy.

3. The theorem is trivial for $N=0$; we suppose it proven for $N<r$ and proceed inductively. Let C be a space satisfying the hypothesis of our theorem for $N=r$. Then it contains *at least* r independent arcs xy, $r \geqq 1$. We denote these by $x(m)\,y$, $m = 1, 2, \cdots, r$. Let $C_0 = \sum_1^r x(m)\,y$. If W is any countable collection of points of C_0 such that every convergent[13] subset of W converges to x or to y, the points of W with x and y divide the set C_0 into a countable collection of non-overlapping arcs whose totality we shall call a partition W^* of C_0. We devote the next section to proving

ASSERTION I. *There exists a partition W^* of C_0 such that if $A = \sum \alpha_m$, where $\alpha_m \subset x(m)y$ and is an arc of the partition W^*, then $C - A$ contains at least one arc xy.*

3.1. Let T denote the linear interval: $0 \leqq t \leqq 1$, and let π_m be a homeomorphism (hereafter fixed), $\pi_m\, T = x(m)\,y$, $\pi_m(0) = x$, $\pi_m(1) = y$. Let T_n be the set of points of T such that: $1/2^{n+1} \leqq t \leqq 1/2^n$, or $1-1/2^n \leqq t \leqq 1-1/2^{n+1}$; let $T^n = \sum_1^n T_i$; let S_n be the set of points on the arcs $x(m)\,y$ of C_0 which correspond to T_n under the respective homeomorphisms π_m; and let $S^n = \sum_1^n S_i$. We make the following definition: if H_k is the sum of k point-sets h_s, $s = 1, \cdots, k$, such that each set h_s belongs to an arc $x(m)\,y$ and no arc $x(m)\,y$ contains a point of more than one of these sets (clearly then $k \leqq r$), then H_k is said to be a *representative k-set*. In general, H_k will consist of a set of i points and j arcs. To avoid confusion, we shall designate representative k-sets which consist of points by P_k, those which consist entirely of arcs by Q_k, and reserve H_k for the general case.

1*. *There exists an integer n_1 such that if T_1 be divided into $2n_1$ equal intervals, non-overlapping, and Q_k be any representative k-set consisting of arcs of S_1 each of which corresponds[14] to some subinterval of this division of T_1, then $C - Q_k$ contains at least $r-k+1$ independent arcs xy.*

[13] An infinite set Z is convergent if there exists a point z such that every infinite subset of Z has z for limit point; we say that Z converges to z.

[14] Under the proper homeomorphism π_m: hereafter we suppress this in statement, but it is always to be understood.

Let P_k be any representative k-set of points of S_1. Since $k \leqq r$, no $r-k$ points of $C-P_k$ separate x and y. But $C-P_k$ is an open subset of C, and therefore it is a space C, i. e. a local continuous curve. Then, *by the induction on our theorem*, $C-P_k$ must contain at least $r-k+1$ independent arcs xy. Then there is an open subset U_k of C, $U_k \supset P_k$, such that $C-\overline{U}_k$ contains $r-k+1$ independent arcs xy. Let P_k^* be the space of representative k-sets of S_1.[15] We metrize it as follows: let P_k and P_k' be two *points* of the space P_k^*. Then if these do *not* represent[16] the same set of arcs of C_0, the *distance* of the *points* shall be 1: in symbols $D(P_k, P_k') = 1$. But if P_k and P_k' do represent the same set of arcs, then we may suppose their points (p_s) and (p_s') respectively, $s = 1, \cdots, k$, so enumerated that particular points p_s and p_s' belong to the same arc of C_0: in this case $D(P_k, P_k') = \text{Max.}(d(p_s, p_s'))$.[17] It is readily verified that P_k^* is compact. But each *point* P_k belongs to an open subset U_k^* of P_k^*. Here U_k^* is simply the set of all representative k-sets P_k' such that P_k' belongs to the open set U_k associated above with P_k. Then, as is sufficiently well-known, there exists an ε_k such that any subset of P_k^* of diameter less than ε_k belongs to an open set U_k^*. We determine such an ε_k for each value of $k = 1, 2, \cdots, r$, and let $\varepsilon_1^* = \text{Minimum}(\varepsilon_k)$.

3.2. The homeomorphism π_m are, of course, uniformly continuous, and their number is finite. It follows that there exists an integer n' such that if α is any arc of $x(m)y$, $m = 1, \cdots, r$, and α corresponds under π_m to an interval of T of length less than $1/n'$, then the diameter of α (in the metric of our space C) is less than ε_1^*. Then we may take for the integer n_1 required by 1^* any integer which is greater than n'. The proof of this should be clear from the foregoing. We divide T_1 into $2n_1$ equal, non-overlapping, intervals and let W_1^* be the set of arcs of S_1 which correspond to these intervals. Suppose now that $W_{\varrho-1}^*$ has been defined for $S^{\varrho-1}$ so that if Q_j is any representative j-set of arcs of $W_{\varrho-1}^*$, $C-Q_j$ contains at least $r-j+1$ independent arcs xy.

3.3. Let H_k be a representative k-set consisting of i points, $1 \leqq i \leqq r$, of S_ϱ and j arcs, $0 \leqq j \leqq r-1$, of $W_{\varrho-1}^*$, where $i+j = k \leqq r$. Then H_k is the sum of a representative j-set Q_j of arcs of $W_{\varrho-1}^*$ and a representative i-set P_i of points of S_ϱ. Now $C-Q_j$ contains at least $r-j+1$ independent arcs xy. It follows that $C-H_k$ contains at least $r-k+1$

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[15] This is, of course, a product space. It is readily recognised as the homeomorph of a finite sum of mutually exclusive euclidean k-cubes.

[16] Two k-sets represent the same set of arcs of C_0 if whenever one of them has a point in common with a particular arc $x(m)y$, then the other does also.

[17] Read: The maximum of the set of distances (in the metric associated with C) of the pairs of points p_s and p_s'.

independent arcs xy. Then there exists an open set U_k containing P_i such that $C-(\overline{U}_k+Q_j)$ contains at least $r-k+1$ independent arcs xy. For a fixed i and j, we metrize the space H_k^*, $k=i+j$, of representative k-sets as follows: given two *points* H_k and H_k', if the subsets Q_j and Q_j' of H_k and H_k' respectively are identical *and* if the points (p_s) and (p_s') of H_k and H_k' respectively can be so enumerated that p_s and p_s', for a fixed s, belong to the same arc of C_0, *then* $D(H_k, H_k') = \text{Max.}(d(p_s, p_s'))$. In any other case, $D(H_k, H_k')=1$. As in the argument above, H_k^* is a compact metric space and every *point* of it belongs to an open subset U_k^*. There exists an ε_{ij} such that every subset of H_k^* of diameter less than ε_{ij} belongs to one of these open subsets U_k^*. We determine such an ε_{ij} for every value of i and j subject to: $1 \leqq i \leqq r$, $0 \leqq j \leqq r-1$, $i+j \leqq r$, and let $\varepsilon_\varrho^* = \text{Min.}(\varepsilon_{ij})$. There exists an integer n^* such that the image under any π_m of an interval of T of length less than $1/n^*$ is of diameter (in the C-metric) less than ε_ϱ^*. Let $n_\varrho > n^*$. We divide each *part-interval* of T_ϱ into n_ϱ equal, non-overlapping, subintervals.

3.4. Let W_ϱ^* be the set of arcs of S^ϱ which correspond to the $2\sum_1^\varrho n$ subintervals (determined above) of T^ϱ; clearly each arc of $W_{\varrho-1}^*$ is an arc of W_ϱ^*. Then it is not difficult to see that if Q_j is any representative j-set of arcs of W_ϱ^*, $C-Q_j$ contains at least $r-j+1$ independent arcs xy. Accordingly we may suppose that W_n^* has been defined for every integer n, and we set $W^* = \sum W_n^*$. Then W^* is the desired partition of C_0 and I. is proven. For if Q_r is a representative r-set of arcs of W^*, Q_r must consist of arcs of some W_ϱ^* and by the argument above $C-Q_r$ contains at least one arc xy.

4. In this section we define a countable set of arcs such that in C_0 and their sum we can find the desired $r+1$ arcs xy. The set of representative r-sets of arcs of W^* is countable. We suppose this set enumerated in a simple sequence $A_1, A_2, \cdots, A_n, \cdots$. It will be convenient to introduce a number of definitions whose intentions are readily anticipated. The sets A_n we shall call *cuts*. The component of C_0-A_n containing x will be spoken of as the *x-side* of the cut, the component containing y as the *y-side*. That arc of A_n which lies on a particular arc $x(m)y$ of C_0 will be called the *trace* of the cut on $x(m)y$. If this trace is u_m, the component of $x(m)y-u_m$ containing x will be known as the x-side of the trace, the component containing y as the y-side. If for some infinite subset (A_i) of the cuts the traces on one or more arcs belong to W_n^*, for sufficiently large n, then we shall say that on these arcs the traces are *stationary*, and the totality of these traces we call the *stationary part* of the set (A_i). An arc x^*y^* such that $x^*y^* \cdot C_0 = x^*+y^*$, and

such that x^* is on the x-side, y^* on the y-side, of some cut A_n, will be called a *covering-arc* for A_n. This is equivalent to saying that x^* is on the x-side of some trace of A_n, and y^* is on the y-side of some trace (possibly the same) of A_n. Clearly, if x^*y^* is any covering arc for A_n, A_n does not separate x and y in $C_0+x^*y^*$.

4.1. The diameters of the set of all covering-arcs for the cut A_n have a greatest lower bound $d_n \geqq 0$. Let $x_n^* y_n^*$ be an arbitrary, but hereafter fixed, covering arc for A_n such that diam. $(x_n^* y_n^*) < d_n + 1/n$. It will be important for us to confine our attention to a certain subset of these arcs, in a sense a minimal or non-superfluous subset which we define in (4.3).

4.2. It may happen that there exists an integer n' such that no cut A_i separates x and y in $C_0 + \sum_1^{n'} x_n^* y_n^*$. If this has happened, then our problem can be reduced almost at once to our theorem on linear graphs. But it will be convenient to proceed on the assumption that this does not take place, and to postpone the consideration of this possibility to a later section where it appears in its most natural light. Nothing we shall proceed to do, in the meanwhile, is thereby disturbed. In particular, then, we have supposed that for none of our covering-arcs $x_n^* y_n^*$ does it happen that x_n^* is the point x and y_n^* the point y. For then no cut A_i separates x and y in $C_0 + x_n^* y_n^*$. In this case, of course, we see immediately that $x_n^* y_n^*$ is the desired $(r+1)$th arc!

4.3. Let $B_1 = A_1$, $x_1 y_1 = x_1^* y_1^*$, and $C_1 = C_0 + x_1 y_1$. Suppose B_{n-1}, $x_{n-1} y_{n-1}$, and C_{n-1} to have been defined and that no A_i, $i \leqq s_{n-1}$ (the definition of this integer will presently be clear), separates x and y in C_{n-1}. There exists a least integer s_n such that A_{s_n}, $s_n \geqq n$, separates x and y in C_{n-1}. Let $B_n = A_{s_n}$, $x_n y_n = x_{s_n}^* y_{s_n}^*$, and $C_n = C_{n-1} + x_n y_n$. Then no cut A_i, $i \leqq s_n$, separates x and y in C_n. We suppose C_n defined for all integral values n. We are particularly concerned with the set of arcs $(x_n y_n)$ and their associated cuts (B_n). We remark here that $x_k y_k$ cannot be a covering-arc for $B_{k'}$ if $k' > k$.

We devote the next three sections to proving

ASSERTION II. *Not more than a finite number of the covering-arcs $(x_n y_n)$ are of diameter greater than a preassigned $\varepsilon > 0$, and every convergent subsequence*[18] *converges to x or to y.*

5. It is clear that every subsequence $(x_{i_n} y_{i_n})$, $n \leqq i_n < i_{n+1}$, of $(x_n y_n)$ contains a subsequence $(x_{j_n} y_{j_n})$, $i_n \leqq j_n < j_{n+1}$, such that $x_{j_n} \subset x(m')y$

[18] A subsequence $(x_{i_n} y_{i_n})$ is convergent if there exists a point z such that every subsequence of $(x_{i_n} y_{i_n})$ has z for sequential limiting point: in this case $(x_{i_n} y_{i_n})$ is said to converge to z.

and $y_{j_n} \subset x(m'')y$, where these are particular arcs, possibly the same among the arcs of C_0. Let (B_{j_n}) be the sequence of associated cuts and u_{j_n} and v_{j_n} the traces of B_{j_n} on the respective arcs $x(m')y$ and $x(m'')y$. We wish to show, first, that

2*. *$(x_{j_n} y_{j_n})$ contains a subsequence $(x_{s_n} y_{s_n})$, $j_n \leqq s_n < s_{n+1}$, such that either $x_{s_n} \neq x$ and (u_{s_n}) converges to x, or else $y_{s_n} \neq y$ and (v_{s_n}) converges to y.*

The arguments are by reductio ad absurdum and are given rather concise form.

Case 1. $m' \neq m''$. Since x and y belong to every arc $x(m)y$, we may assume that for some infinite subsequence $(x_{k_n} y_{k_n})$ of arcs of $(x_{j_n} y_{j_n})$, $j_n \leqq k_n < k_{n+1}$, $x_{k_n} \neq x$ *and* $y_{k_n} \neq y$. (Otherwise, for some subsequence, there is a choice of m' and m'' so that $m' = m''$).

a) suppose (u_{k_n}) and (v_{k_n}) are stationary (see 4.). Then for at least two arcs $x_k y_k$ and $x_{k'} y_{k'}$, $k' > k$, $u_{k'} = u_k$ and $v_{k'} = v_k$. It follows that $x_k y_k$ is a covering-arc for $B_{k'}$. This is impossible, by the remark of 4.3.

b_u) suppose now (u_{k_n}) are stationary. In order from x to y on $x(m')y$ there is a first trace u_k. Then x_k is on the x-side of *every* trace u_{k_n}. But from a) we may assume that (v_{k_n}) converges to x: otherwise it will contain a subsequence (v_{s_n}), $k_n \leqq s_n < s_{n+1}$, which converges to y and 2* is at once established. But then it is clear that $x_k y_k$ is a covering-arc for $B_{k'}$, for at least one $k' > k$ and sufficiently large. This is impossible.

b_v) if in b_u) above we read y for x and v for u, we conclude that (v_{j_n}) are not stationary.

Now, however, 2* follows at once unless (u_{k_n}) converges to y and (v_{k_n}) converges to x. But here, finally, it is readily seen that $x_{k_1} y_{k_1}$ is a covering-arc for B_{k_n}, if n is sufficiently large.

Case 2. $m' = m''$, and there is a single set of traces (u_{j_n}). Precisely as in a) above, we see that (u_{j_n}) cannot be stationary.

b_x) There exists a subsequence (u_{k_n}) which converges to x. Then if $x_{k_n} \neq x$, $n = 1, 2, \cdots$, 2* is proven. But if $x_k = x$, where k is a particular k_n, we see that $x_k y_k$ is a covering-arc for $B_{k'}$, $k' > k$ and sufficiently large.

b_y) If in b_x) we read y for x we have at once the last alternative to 2* and the conclusion that also this alternative leads to contradiction.

Then 2* is proven.

5.1. We shall say that a sequence of traces (u_{s_n}) converges *monotonically* to x (to y, resp.) provided $u_{s_{n+1}}$ is on the x-side (y-side, resp.) of u_{s_n} for every n. Now if a set of traces converges to x (resp. to y) it

certainly contains a subset which converges monotonically to x (resp. to y). Therefore we may in 2* read *converges monotonically* for *converges*. If the traces of (B_{s_n}) are stationary on i, $0 \leqq i \leqq r$, of the arcs of $x(m)y$ we have called § 4 the totality of the traces on these i arcs the *stationary part* of (B_{s_n}). Clearly x and y are not limit points of this stationary part. We proceed to prove

$2''_{\mathbf{x}}$: *if* $x_{s_n} \neq x$, $n = 1, 2, \cdots$, *if* $x_{s_n} \subset x(m')y$ for fixed m', and *if* (u_{s_n}) converges monotonically to x, *then* at least one of the arcs of $(x_{s_n} y_{s_n})$ is of diameter less than a preassigned $\varepsilon > 0$.

There exists a compact neighborhood U of x, $\bar{U} \subset V = S(x, \delta/2)$ where $\delta < \varepsilon$ and is so chosen that V does not include y and contains no point of the stationary part of the cuts (B_{s_n}) associated with $(x_{s_n} y_{s_n})$. Let $n_1 > 4/\varepsilon$. For some $n_2 > n_1$, $x_{s_n} \subset U$ if $n > n_2$. Let z_{s_n}, $n > n_2$, be the first point of $x_{s_n} y_{s_n}$ on $\bar{U} - U$. Then the set of points (z_{s_n}) has a limit point z and for a subsequence $(x_{j_n} y_{j_n})$ of arcs of $(x_{s_n} y_{s_n})$, $n_2 < s_n \leqq j_n < j_{n+1}$, the corresponding points (z_{j_n}) converge to z. Let $Z = S(z, \delta')$ where δ' is chosen so that $Z \subset V$ and has no point in common with any arc of $x(m)y$ which does not contain z: in particular Z does not contain x. There is an $n_3 > n_2$ such that if $n_3 \; n' \geqq n_3$ Z will contain an arc $z z_{j_n}$ and $z_{j_n} z_{j_{n'}}$.

Case a. z belongs to no arc $x(m)y$.[19] Let $k = j_{n_3}$ and let k' be any j_n thereafter such that $u_{k'}$ lies on the x-side of x_k. There is in Z an arc $z_k z_{k'}$. Then $x_k z_k + z_k z_{k'} + z_{k'} x_{k'}$ has only the points x_k and $x_{k'}$ in common with C_0 and x_k is on the y-side while $x_{k'}$ is on the x-side of the trace $u_{k'}$ of $B_{k'}$. Therefore this sum contains a covering-arc for $B_{k'}$, and the covering-arc belongs to V.

Case b. z belongs to $x(m')y$. Let k' be a j_n, $n \geqq n_3$, such that $u_{k'}$ lies on the x-side[20] of every point of $\bar{Z} \cdot x(m')y$. In Z there is an arc $z z_{k'}$. Then $x_{k'} z_{k'} + z_{k'} z$ contains a covering-arc for $B_{k'}$, and the covering-arc belongs to V.

Case c. z belongs to an arc $x(m'')y$, $m'' \neq m'$. Let $k^* = n_3$ and let k be the first integer such that u_k belongs to the open arc $x x_{k^*}$.

i. *If it is true for any integer greater than or equal to* k, let k' be the first for which the trace of $B_{k'}$ on $x(m'')y$ belongs to the open arc zy. Now Z contains an arc $z z_{k^*}$ and $x_{k^*} z_{k^*} + z z_{k^*}$ belongs to V and contains a covering-arc for $B_{k'}$. For, x_{k^*} is on the y-side and z on the x-side of this cut.

[19] The reader will observe that this is the only case where z may fail to be a point of $C_0 + \Sigma x_n y_n$. While entirely irrelevant to the present discussion, this will make the entire difference between the principal and a subsequent theorem: see 8.

[20] This is a slight stretching of our definition of "x-side" whose meaning should be clear.

ii. If i. above does not hold, let k' be the first integer such that z is not a point of $B_{k'}$. Such an integer certainly exists, for if the traces of (B_{j_n}) are stationary on $x(m'')y$ z is not a point of any of these traces, and if they are not stationary there exists a trace of indefinitely high subscript which does not contain z. Since i. does not hold, z must be on the y-side of $B_{k'}$. Since (u_{j_n}) converges monotonically to x, $x_{k'+1}$ is on the x-side of $B_{k'}$. But Z contains an arc $zz_{k'+1}$ and this plus $x_{k'+1}z_{k'+1}$ belongs to V and contains a covering-arc for $B_{k'}$.

Now in every possible case we have found in V, and therefore of diameter less than $\frac{1}{2}\varepsilon$, a covering-arc for at least one cut $B_{k'}$, $k' \geqq n_3$, and k' a particular subscript i_n. Therefore $d_{k'} \leqq \frac{1}{2}\varepsilon$ and diam. $(x_{k'}\, y_{k'}) < \frac{1}{2}\varepsilon + 1/k' < 3\varepsilon/4$.[21] Our proof of $2''_x$ is completed.

$2''_y$. (In the statement of, and throughout the argument for, $2''_x$ read y for x and read v for u.) Then in this case also there is at least one arc of diameter less than a preassigned $\varepsilon > 0$.

5.2. We are at once in a position to prove Assertion II. For if there exists an $\varepsilon > 0$ such that infinitely many of the arcs of $(x_n\, y_n)$ are of diameter greater than ε, there exists a subsequence $(x_{i_n}\, y_{i_n})$ such that every arc of this subsequence has a diameter greater than ε. But applying in succession the considerations of 5. and 5.1 we are led to contradiction: we have already shown that no such subsequence exists. Now, further, if a set of traces (u_{s_n}) converges to x (resp. to y) it is clear that the corresponding points x_{s_n} (resp. y_{s_n}) must converge to x (resp. y) or be the point x (resp. y). If, then, we suppose that $(x_n\, y_n)$ contains a convergent subsequence $(x_{j_n}\, y_{j_n})$ which converges to a point z not x or y, we are led to immediate contradiction with 2*. Then Assertion II is completely established.

6. Now it is apparent that $C_0 + \sum_1^{n'} x_n\, y_n$, $n' > 1$, is by no means a point-set which can be regarded as the geometric representation of our concept of a linear graph. For if we choose to designate certain open subarcs of this point-set as its 1-cells, we must be able to assert that our 1-cells are mutually exclusive and that they are in finite number. But we can extract from an arc $x_n\, y_n$ a simple subset which we shall call K_n, and which will be the sum of a finite set of non-overlapping arcs, such that $C_0 + \sum_1^{n'} K_n$ can be interpreted as a linear graph *the while* $\sum_1^{n'} K_n$ *contains a covering-arc* for $B_{n'} = A_{s_{n'}}$ (see 4.3) *and for every* A_i, $i \leqq s_{n'}$. This will be more readily anticipated if we recall, and here emphasize, that if $x'\, y'$

[21] See 4.1. It will be recalled that $k' \geqq n_3 > n_2 > n_1 > 4/\varepsilon$.

is any covering-arc for some cut A_i, then any arc whatever with endpoints x' and y' which has these points only on C_0 is a covering-arc for A_i. We shall agree to say that such another arc is equivalent to $x'y'$. We must define the sets K_n sequentially since their description will depend upon the order in which the arcs $x_n y_n$ are given to us. The underlying idea is very simple.

Let $K_1 = x_1 y_1$; the open arc we shall call the 1-cell and the two endpoints we shall call the vertices of K_1. Suppose now that K_i, $i \leqq n$, n fixed, has been defined in accordance with the intimations of the preceding paragraph: let $K^n = \sum_1^n K_i$. It is clear that $K^n - C_0 \cdot K^n = K^n - \sum_1^n (x_i + y_i) \equiv H^n$ [22] does not contain an arc joining x_{n+1} and y_{n+1}, even if (as may happen) these points belong to K^n. Let x'_{n+1} be the last point (in order from x_{n+1}) of $K^n \cdot x_{n+1} y_{n+1}$ such that H^n contains an arc joining x'_{n+1} to x_{n+1}. Now x'_{n+1} may be the point x_{n+1} but it is not y_{n+1}. Let y'_{n+1} be the last point (in order $y_{n+1} x'_{n+1}$) of $K^n \cdot x'_{n+1} y_{n+1}$, such that H^n contains an arc joining y'_{n+1} to y_{n+1}. Not every point of $x'_{n+1} y'_{n+1}$ can belong to H^n. Let z be a point of this arc which is not a point of H^n. Then on each subarc $z x'_{n+1}$ and $z y'_{n+1}$ (in order from z) there exists a first point z', and there exists a first point z'' respectively, such that $z' + z'' \subset K^n$. Let K be the set of all points z of $x'_{n+1} y'_{n+1}$ such that the corresponding points z' and z'' belong to *different* components of $H^n + x_{n+1} + y_{n+1}$. Then $K_{n+1} = \overline{K}$.

We shall first show that $K^{n+1} = K^n + K_{n+1}$ contains an arc equivalent to $x_{n+1} y_{n+1}$. Let z^* be the last point (in order $x'_{n+1} y'_{n+1}$) of $x'_{n+1} y'_{n+1} \cdot K^n$ such that $H^n + K_{n+1}$ contains an arc joining z^* to x'_{n+1}, and suppose that z^* is not y'_{n+1}. No point of $z^* y'_{n+1}$ can belong to any component of H^n of which z^* is a limit point. Then there exists a point z on this arc such that no point of $z^* z - z^*$ belongs to H^n. Then for this point z, the point z' determined above must be the point z^* and the point z'' must belong to a different component of H^n; consequently the arc $z^* z'' \subset K_{n+1}$, and $H^n + K_{n+1}$ contains an arc joining x'_{n+1} to z'' so that z^* was not the last point. Then $z^* = y'_{n+1}$. Then $H^n + K_{n+1}$ contains an arc joining x'_{n+1} to y'_{n+1} and H^n contains arcs joining x_{n+1} to x'_{n+1} and y_{n+1} to y'_{n+1}, so that our assertion is proved.

Now $K_{n+1} = \overline{K}$ (see last sentence of second paragraph, this section) *consists of a finite set of non-overlapping* arcs of $x'_{n+1} y'_{n+1}$. *For* K certainly consists of a set of mutually exclusive open arcs $(z' z'')$ where the endpoints z' and z'' belong to different components of H^n. If the number of these were infinite, it would follow that they contain a subsequence

[22] I. e. we shall call it H^n.

converging to a point z^0 and that this point is a limit point of at least two distinct components of H^n. But then z^0 must be a point of C_0. But then z^0 would have to be the point x'_{n+1} (and this x_{n+1}) or else z^0 would be y'_{n+1} ($= y_{n+1}$, in this case). *But* we have seen that no point of $x'_{n+1}y'_{n+1}$ belongs to a component which has either of these endpoints for limit point. We may now designate by the 1-cells of K_{n+1} the open arcs of K, and by its vertices the endpoints of these arcs. Now we may suppose that K_i has been defined for all integral values i.

6.1. In part because the argument will assist us in the next and concluding section of this theorem, we shall at this point dispose of an hypothesis which we have carried along from 4.2: i. e. we shall now suppose that there exists an integer n' such that no cut A_i separates x and y in $C_0 + \sum_1^{n'} x_n^* y_n^*$ (see 4.2). Then certainly no cut A_i separates x and y in $C_0 + \sum_1^{n'} x_n y_n$. Then no cut A_i separates x and y in $C_0 + K^{n'}$, where $K^{n'}$ is defined exactly as above. But the point-set $C_0 + K^{n'}$ is a linear graph whose vertices are x and y and the vertices of $K^{n'}$ and whose 1-cells are the components[23] of the complement in $C_0 + K^{n'}$ of this set.

Now C_0 contains r independent arcs $x(m)y$. If any set of r elements, $a'_1, \cdots, a'_r$, of the linear graph $C_0 + K^{n'}$ separate x and y, each arc $x(m)y$ must contain one of these elements. If a'_i is a vertex we shall call it a_i. If a'_j is a 1-cell with at least one vertex distinct from x and from y we shall replace it by this vertex a_j. If a'_k is an element not any of the above, then it is one of our arcs $x(m)y$. Then let a_k denote an arbitrary inner point of it (for definiteness, say π_m $(t = 1/2)$) and let this point also be called a vertex (it is immaterial that this introduces two new 1-cells, since we are dealing with a point-set). Then $A = \sum_1^r a_i$ separates x and y in $C_0 + K^{n'}$. Now if A_i is any cut which contains A, and such a one must obviously exist, A_i separates x and y. Since this is impossible by our definition of n', it follows that no r elements of $C_0 + K^{n'}$ separate x and y, and the principal theorem is now an immediate consequence of our theorem on linear graphs.

7. It will be convenient to have the distance (xy), $d(xy)$, $\geqq 3$.[24] Let U_k and V_k, $k = 1, 2, \cdots$, be spherical neighborhoods of (new) radius $1/k$. Then $\overline{U}_k \cdot \overline{V}_k = 0$. *For a fixed*, but arbitrary, k let n_k be the first integer such that every point of C_0 which is not an inner point of an arc

[23] It is clear that these are open arcs; in the future we shall merely indicate the vertices of our graphs, the 1-cells follow then of themselves.

[24] We merely change the metric of C, multiplying all distances by $3/d(xy)$.

of $W^*_{n_k}$ (one refers to 3.4) is contained in $U_k + V_k$. Let X denote the points of C_0 which correspond (see 3.1) to points t of the subinterval of T: $0 \leqq t \leqq 1/2^{n_k+1}$, and let Y denote the set of points corresponding to: $1 - 1/2^{n_k+1} \leqq t \leqq 1$. Then clearly $X \subset U_k$ and $Y \subset V_k$. There is a first integer n' such that a) if A_i is a cut whose traces belong to $W^*_{n_k}$, then $n' > i$, *and* b) if K_i does not belong to $U_k + V_k$, then $n' > i$. For a) one observes that the set of such cuts A_i is finite. For b) one remembers that $K_i \subset x_i y_i$ and applies II of 5).

No cut A_i, $i < n'$, separates x and y in $L_{n'} = C_0 + K^{n'}$. Let L be a linear graph whose vertices are the *point-set* X, *the point-set* Y, and all points not belonging to $X + Y$ which are endpoints of arcs of $W^*_{n_k}$ or are vertices of $K^{n'}$. The 1-cells of L are, as customarily, the components (open arcs) of the complement in $L_{n'}$ of this set of vertices.[25] Now it follows, exactly as in 6.1 that L contains $r+1$ independent arcs XY. But the open arc of points corresponding to each of these contains an arc $p^k q^k$, where $p^k q^k \cdot (U_k + V_k) = 0$, $p^k \subset \bar{U}_k$, $q^k \subset \bar{V}_k$, and $p^k q^k$ is an arc of the point set $L_{n'}$.[26] Then there exists in $L_{n'}$ at least one "family" of arcs w^k_1 such that $w^k_1 = \sum_1^{r+1} p^k_s q^k_s$ and each arc $p^k_s q^k_s$ is an arc $p^k q^k$ above, and these arcs *are mutually exclusive*. Now let $C^* = C_0 + \sum_1^{\infty} K_n$. Then if $p^k q^k$ is any arc of C^* (an arc $p^n q^n$ is to be understood, as above, as an arc such that $p^n q^n \cdot (U_n + V_n) = 0$, $p^n \subset \bar{U}_n$, $q^n \subset \bar{V}_n$) we see by our choice b) of the integer n' that $p^k q^k$ must belong to $L_{n'}$. But the set of all distinct arcs $p^k q^k$, where two arcs are distinct if they do not coincide as point-sets, of $L_{n'}$ is necessarily finite. *For* the point-set $L_{n'}$ is a linear graph with vertices x, y, the endpoints of arcs of $W^*_{n_k}$, and the vertices of $K_{n'}$. If an arc $p^k q^k$ contains none of these vertices it must belong to a 1-cell of $L_{n'}$ and if two arcs $p^k q^k$ belong to the same 1-cell it is clear that they coincide. Otherwise let p' and q' be the first and last vertices respectively on the arc $p^k q^k$, in that order. Then the subarc $p' q'$ is an *arc* of the linear graph, and the number of these is finite. Then two arcs $p^k q^k$ which coincide on $p' q'$ can be distinct only if the corresponding open arcs $p^k p'$ or if the corresponding open arcs $q^k q'$ belong to different 1-cells. But the number of these possibilities is finite.

Then C^* is a point set such that for every k it contains at most a finite set of distinct arcs $p^k q^k$. Then necessarily it contains at most a finite set of distinct families w^k of such arcs, and we have seen that it always

[25] I. e. the point-set, $X + Y +$ set of point-vertices.

[26] p^k and q^k may not be vertices.

contains at least one such family. But it is further clear that every arc $p^{k'}q^{k'}$, where $k'>k$, contains an arc $p^k q^k$. Then it follows that there must exist a sequence $w^1, w^2, \cdots$, of particular families of arcs such that $w^n \subset w^{n+1}$. Then if $w^n = \sum_1^{r+1} p^n(s)\,q^n$, we may suppose the arcs so enumerated that

$$p^n(s)\,q^n \subset p^{n+1}(s)\,q^{n+1},\ s = 1, 2, \cdots, r+1.$$

Let $\sum_s = \sum_1^{\infty} p^n(s)\,q^n$. Clearly $\sum_s$, $s = 1, \cdots, r+1$, is an open arc and has x and y for limit points. Clearly, also, $\sum_i \cdot \sum_j = 0$; $i \neq j$; $i, j = 1, \cdots, r+1$. Now if the point z, $x \neq z$ and $y \neq z$, belongs to $\sum_s$ it must belong to $\sum_s$. *For* there exists an integer k such that z does not belong to $U_k + V_k$. But in the complement of this point-set, $\sum_s$ coincides with the arc $p^k(s)\,q^k$ and this is closed. Then $x + \sum_s + y$ is an arc $x(s)y$, and the set of arcs $x(s)y$[27] is an independent set of $r+1$ arcs xy. Then our principal theorem is completely established.

REMARK. It may deserve to be pointed out that we might have devoted 6) to proving, and it is easy to show, that $C_0 + \sum_1^{\infty} x_n y_n$ is a regular curve in the sense of Menger-Urysohn no r points of which separate x and y. Then we might have deduced the principal theorem from Menger's "n-Bein Satz". If it is allowed that the theorem on linear graphs is in itself of interest, then we have had to add very little to avoid the use of Menger's theorem and therefore, in a sense, to replace it.

8. Nöbeling[28] has introduced a subset of a continuous curve which he calls a „Gerüst". We have had occasion, independently of him, to use a similar structure (unnamed by us) which we shall follow him by calling a "frame".[29] We have defined it as follows: let C be a local continuous curve and let P be a countable everywhere dense subset of C, $P = (p_i)$, $i = 1, 2, \cdots$. Suppose well-ordered the set of all pairs of points (p_i, p_j) of P. Let $p_i p_j$ be an arc of C such that the diameter of this arc is less than $d_{ij} + 1/n_{ij}$, where d_{ij} is the greatest lower bound of the set of diameters of all arcs $p_i p_j$ of C and n_{ij} is the subscript associated with the pair (p_i, p_j) in the ordering.[30]

[27] These are not to be confused with the arcs $x(m)y$ of C_0.

[28] The considerations of this section were inspired by an inspection of the paper of Nöbeling.

[29] See by the present author: "Generalization of a theorem ...", Amer. Jour. of Math., 54 (1932), top page 177. Our definition was given for generalised continuous curves (completeness replacing compactness). Compare also, this paper, the condition (4.1) on the "covering-arcs".

[30] If no arc exists let $p_i p_j$ denote, for example, $p_i + p_j$.

Let $F = \sum\sum p_i p_j$: then F is a frame of C.

A frame F has the following important property,[31] which may readily be converted into an abstract definition.

If there exists an arc xy in C of diameter less than $\varepsilon > 0$, then there exists in $F+x+y$ an arc xy of diameter less than ε. For if (p_{j_n}) and (p_{k_n}), $n = 1, 2, \cdots$, are subsequences of P converging to x and y respectively, each of these is a Cauchy sequence and it is clear that the diameters of the set of arcs $(p_{j_n} p_{j_{n+1}})$ and $(p_{k_n} p_{k_{n+1}})$ of F converge to zero. Then $C_x = x + \sum_n p_{j_n} p_{j_{n+1}}$ and $C_y = y + \sum_n p_{k_n} p_{k_{n+1}}$ are compact continuous curves in $F+x+y$. Then it follows that for N sufficiently large there exist in C arcs $p_{j_n} p_{k_n}$, $n > N$, of diameter less than ε. Then for N' sufficiently large there exist in F arcs $p_{j_n} p_{k_n}$, $n > N'$, of diameter less than ε. It should be clear that there exists an s sufficiently large so that the sum of the arc $x p_{j_s}$ of C_x, the arc $p_{j_s} p_{k_s}$ of F, and the arc $p_{k_s} y$ of C_y is of diameter less than ε. From this we may draw at once the two conclusions: a) if X and Y are self-compact subsets of C, $F+X+Y$ is locally arcwise connected, b) if z is a point of C and (z_{j_n}) is a sequence of points of F converging to z and Z is an arbitrary neighborhood of z (in C), then there exists an integer n_3 such that for n, $n' \geqq n_3$ Z will contain an arc $z_{j_n} z_{j_{n'}}$ *which is an arc of* F.

Now b) above has this interesting consequence. Suppose that A is a subset of F closed in C which does not separate two points x and y of C. We can show that it does not separate $F+x+y$ between x and y. Otherwise we may write $F - A = H + K$, $H \supset x$, $K \supset y$, $\bar{H} \cdot \bar{K} = 0$ (*these closures being in* F). But clearly $C - A = (\bar{H} - A) + (\bar{K} - A)$, *these closures being in* C. Then $\bar{H} - A$ has a component C_x containing x and $\bar{K} - A$ a component C_y containing y, and since A does not separate x and y, $C_x \cdot C_y$ contains a point z, $z \cdot A = 0$. Therefore z is at non-zero distance from A. Also, it is a limit point of H and of K. Then there is in $F - A$ an arc having a point in H and a point in K. This is impossible. We may conclude: if no N points of C separate x and y, then no N points of an arbitrary frame F separate x and y in $F+x+y$.

Nöbeling has discovered the

THEOREM. *If no* N *points of a local continuous curve*[32] *separate two given points* x *and* y, *and if* F *is an arbitrary frame of* C, *then* $F+x+y$ *contains at least* $N+1$ *independent arcs* xy.

[31] This is remarked somewhere by Nöbeling. There is here a certain overlapping unavoidable for completeness sake. Our argument holds, further, in generalized continuous curves.

[32] He proves it, of course, for compact continuous curves.

We have already proven this theorem. For the only difference to us between $F+x+y$ and C lies in this that the first set, although it is locally compact, is not locally *self-compact,* i. e. *it is not also closed in* C. But in the argument which we have given for the principal theorem this plays no role except in the consideration of *Case a* of 5.1 (see note thereto). And in this case, it is not material to us whether or not z belongs to F; we are concerned solely with the possibility of defining the integer n_3. *That* we have shown to be possible in b) of the preceding paragraphs. Then for this theorem we may follow, word for word, the argument of the principal theorem even using neighborhoods of C, excepting that we choose arcs in $F+x+y$.

8.1. NÖBELING THEOREM. *If* X *and* Y, $X \cdot Y = 0$, *are self-compact subsets of a local continuous curve* C *not separated in* C *by any set of* N *points and if* F *is an arbitrary frame of* C, *then* $F+X+Y$ *contains at least* $N+1$ *independent arcs* XY.[33]

Let C' be a "coalescence" space of C which results from identifying the points of X as a single "point" x and the points of Y as a single "point" y.[34] It is clear that C' is a local continuous curve such that no N points of it separate x and y. Then by our principal theorem, C' contains at least $N+1$ independent arcs $x(m)y$, $m = 0, 1, \cdots, N$. While these do not correspond to arcs of C, it should be clear that the open arcs $<x(m)y>$ correspond to $N+1$ mutually exclusive open curves[35] of C, each of which has at least one limit point in X and one limit point in Y. Then X contains a subset X'' which consists of at most $N+1$ points, and Y a subset Y'' of at most $N+1$ points, such that no N points of C separate X'' from Y''.[36]

Now let C'' be a coalescence space of C which results from identifying the points of X'' as a single point x'' and the points of Y'' as a point y''. We have again a local continuous curve no N points of which separate x'' from y''. It should be reasonably clear that the point set F'' of C'' corresponding to our fixed frame F of C is a frame of C''.[37] Then by the theorem above, $F''+x''+y''$ contains at least $N+1$ independent arcs $x''(s)y''$. The open arcs $<x''(s)y''>$ correspond to $N+1$ mutually exclusive

[33] I. e. arcs mutually exclusive except possibly for their endpoints and such that each has an endpoint in X and an endpoint in Y and these points only in $X+Y$.

[34] C' is a continuous (singular) image of C, with upper semi continuous inverse, actually $(1-1)$ except for the "points" x and y.

[35] I. e. homeomorphs of the open interval: $0<t<1$.

[36] This is Hilfssatz 1 of Nöbeling. Our proof has, admittedly, suppressed some details.

[37] This deserves the remark that although we have defined a frame in a rather precise manner, it admits of abstract definition in terms of the properties we have deduced for it from its definition.

open curves of F, and each of these has a single limiting continuum in X'' and a single limiting continuum in Y''. Since X'' and Y'' are finite point sets it follows that these limiting continua must be points of X'' and Y'' respectively, and that the arcs $x''(s)y''$ above actually correspond to $N+1$ independent arcs $X''Y''$ of $X''+Y''+F$. Since X'' and Y'' belong to X and Y respectively, and the arcs $X''Y''$ are independent and $X \cdot Y = 0$, it follows that $F+X+Y$ contains at least $N+1$ independent arcs XY.

9. We append[38] to the principal theorem a corollary which appears to have escaped attention, and is sufficiently curious.

COROLLARY. *If X and Y, $X \cdot Y = 0$, are self-compact subsets of a local continuous curve C, then in order that C shall contain at least $N+1$ mutually exclusive arcs XY[39] it is necessary and sufficient that* 1) *X and Y each contain at least $N+1$ points,* 2) *no N points whatever of C separate*[40] *every point of X from every point of Y.*

Our conditions are clearly necessary. The sufficiency is accomplished by a simple artifice which is not readily available, unfortunately, if X or Y fails to be locally connected. *For the moment,* let us make the convenient assumption that X and Y are finite point sets. Certainly we may consider that C is immersed in that subset of Hilbert space whose coördinates are all of the form: $(0, 0, x_3, x_4, \cdots)$. Let x^* be the point $(1, 0, 0, \cdots)$ and y^* the point $(0, 1, 0, \cdots, 0, \cdots)$ of Hilbert space. Let X^*, respectively Y^*, be the set of points of the Hilbert space which belong to linear segments joining x^*, respectively y^*, to points of X, respectively Y. It is apparent that $C^* = C+X^*+Y^*$ is a local continuous curve such that no N points of $C^*-(x^*+y^*)$ separate x^* from y^*. Then C^* contains at least $N+1$ independent arcs x^*y^*. Now $X^* \cdot Y^* = 0$, $X^* \cdot C = X$, and $Y^* \cdot C = Y$. It is immediate that C must contain $N+1$ mutually exclusive arcs XY, each of them a suitably chosen subset of some one of the independent arcs x^*y^* above.

Now to dispose of the restriction which we introduced, in course of proof, that X and Y are finite it suffices to show that each contains a finite subset satisfying the conditions of our theorem. We do this easily by a reductio ad absurdum.[41] Let X_n, respectively Y_n, $n = 1, 2, \cdots$, be a finite subset (of at least $N+1$ points) of X, respectively Y, such that X_n (resp. Y_n) belongs to X_{n+1} (resp. Y_{n+1}) and $\sum X_n$ (resp. $\sum Y_n$) is dense in X (resp. Y). If our

[38] Added Sept. 1, 1932.

[39] I. e. mutually exclusive arcs, each having an endpoint in X and Y respectively, and these points only in $X+Y$.

[40] We extend, slightly, the customary sense of "separate": thus if x is a point of X we agree to understand that x separates itself from every point of Y.

[41] The argument differs a little from that of Nöbeling's Hilfssatz.

desideratum is not to be true there must exist point sets Z_n, consisting of precisely N points, such that Z_n separates every point of X_n from every point of Y_n. Without disturbing the course of our argument we may suppose that the point sets Z_n have a sequential limiting set Z; consisting, then, of N or fewer points. (Otherwise we have merely to choose a proper subsequence of the point sets Z_n, and corresponding sequences of the (X_n) and (Y_n), and make a slight change of notation.)[42] Now there exist two points x and y of X and Y respectively such that Z does not separate them in C. Then there exists an open neighborhood U of Z such that $\bar{U}$ does not separate x and y in C, and there exists an integer k such that Z_n belongs to U if $n>k$. But since $\sum_{n>k} X_n$ and $\sum_{n>k} Y_n$, are dense in X and Y respectively, we conclude easily that there exists an integer k' $(>k)$ such that the point set $\bar{U}$ cannot separate every point of X_n from every point of Y_n if $n>k'$. Then Z_n does *not* separate every point of X_n from every point of Y_n, if $n>k'$. The contradiction, with our previous argument establishes the corollary.

[42] That $\{Z_n\}$ is compact follows from the fact that it may be supposed, by an implicit induction, to lie on a fixed set of N arcs XY.

PRINCETON.

ON THE SURFACES WITHOUT TANGENT PLANES.[1]

By S. Saks.[2]

1. The elementary problem of determination of the length of a continuous curve $y = f(x)$ was solved a long time ago by Jordan and Lebesgue. In order that the curve $y = f(x)$ be rectifiable, it is necessary and sufficient that the function $f(x)$ be of bounded variation. If this condition is satisfied and L denotes the length of the curve, then $f(x)$ is differentiable almost everywhere and we have

$$L \geq \int_a^b \{1 + [f'(x)]^2\}^{1/2}\, dx.$$

In order that the equality hold here, it is necessary and sufficient that $f(x)$ be absolutely continuous.

The analogous results for the area of the surface $z = f(x, y)$ have been obtained rather recently. The most complete results in this direction are due to Tonelli [1].[3] We shall recall briefly the fundamental definitions of his theory. Let

$$z = f(x, y) \qquad (0 \leq x, y \leq 1) \tag{1.1}$$

be continuous in the square $K_0 \equiv (0, 0; 1, 1)$, and let $V_1(\xi)$ and $V_2(\eta)$ denote the total variations of the functions $f(\xi, y)$ and $f(x, \eta)$ considered as functions of a single variable y and x respectively, for ξ and η fixed. The function (1.1) is said to be of bounded variation if each of the integrals $\int_0^1 V_1(\xi)\, d\xi$, $\int_0^1 V_2(\eta)\, d\eta$ is finite. The function (1.1) is called absolutely continuous, if it is of bounded variation and if $f(\xi, y)$ and $f(x, \eta)$ are absolutely continuous in y and x respectively for almost all values of ξ and η. These hypotheses imply that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are summable over the square K_0.

Tonelli established the following result: In order that the surface (1.1) have a finite area A it is necessary and sufficient that the function $f(x, y)$ be of bounded variation. If this condition is verified, the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist almost everywhere in the square K_0 and

$$A \geq \int_0^1 \int_0^1 \left[1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]^{1/2} dx\, dy.$$

The equality holds here if and only if $f(x, y)$ is absolutely continuous.

[1] Received, May 27, 1932.

[2] International Research Fellow.

[3] The numbers in brackets refer to the list of papers at the end of this article.

2. So far we have a complete analogy between the case of one variable and that of two variables. The analogy breaks down, however, for the geometric interpretation of the above results. The theorem of Lebesgue concerning the differentiability of a function of bounded variation asserts that every curve $y = f(x)$ with a finite length has almost everywhere a tangent. On the contrary, the fact that a function $z = f(x, y)$ of bounded variation possesses partial derivatives almost everywhere does not imply the existence of a tangent plane as defined below, which requires the total differentiability (in Stolz's sense) of the function z. And we know only (by a theorem of Rademacher [1]-Stepanoff [1]) that if $f(x, y)$ satisfies the Cauchy-Lipschitz condition, or more generally, if almost everywhere on a set E

$$\overline{\lim_{x' \to x,\, y' \to y}} \frac{f(x', y') - f(x, y)}{[(x' - x)^2 + (y' - y)^2]^{1/2}} < \infty, \tag{2.1}$$

then $f(x, y)$ is totally differentiable almost everywhere on E.[4]

The question was naturally raised whether this theorem could be extended, at least partially, to the case of arbitrary continuous functions. R. Caccioppoli [1] has proved already that this extension cannot be complete by exhibiting an example of a surface with a finite area which has no tangent plane at all the points of a set of positive measure. In this paper we obtain a stronger result. We shall prove namely that there exists a *continuous function* $u(x, y)$ *on the square* $K_0 = (0, 0; 1, 1)$, *absolutely continuous and of the total variation less than* 1 *on every straight segment in* K_0, *and such that the surface* $z = u(x, y)$ *has no tangent plane at any point.* It is obvious that such a surface not only has a finite area but is absolutely continuous in the sense of Tonelli.

3. Our proof will be based on the theorem of Baire according to which no complete metric space is of the first category. This method, due essentially to Mazurkiewicz and Banach, has been recently applied to many analogous problems.[5]

Let $u(x, y)$ be an arbitrary function in the square $K_0 = (0, 0; 1, 1)$. We shall denote by $\| u \|$ (the norm of u) the least upper bound of the total variations of $u(x, y)$ on all straight segments in K_0. If E is a set in K_0 then $\| u \|_E$ will denote the least upper bound of the total variations of $u(x, y)$ on all straight segments contained in E. We have

[4] We shall remark incidentally that this theorem may be slightly generalized as follows: *If to every point* (x, y) *of a set* E *there corresponds an angle* $A(x, y)$ *with vertex at* (x, y), *such that the limit superior* (2.1) *is finite when* (x', y') *approaches* (x, y) *remaining in the angle* $A(x, y)$, *then* $f(x, y)$ *is totally differentiable almost everywhere on* E.

[5] Mazurkiewicz [1, 2]; Banach [1]; Auerbach and Banach [1]; Kaczmarz [1].

evidently

$$\| u \|_{K_0} = \| u \|, \text{ and } \| u \|_E \leqq \| u \|_F \text{ if } E \subset F \subset K_0.$$

We denote by U the class of functions $u(x, y)$ continuous on K_0, vanishing on the boundary and such that

(A) $\| u \| < \infty$,

(B) if K is a rectangle in K_0, then $\| u \|_K$ converges to zero with the area of K.

The "distance" between two functions u, v in U will be defined as $\| u - v \|$. It is easily seen that with this definition of the distance the class U is a metric, linear and complete space. Indeed, if $\{u_n\}$ is a sequence of functions of U such that

$$\lim_{m, n} \| u_m - u_n \| = 0,$$

then u_n converges uniformly to a function u which belongs also to the space U, while equality

$$\lim_{n} \| u_n - u \| = 0.$$

We shall prove the following theorem:

There exist U-functions[6] $u(x, y)$ *with the property that through every point* (x_0, y_0) *there pass infinitely many vertical sections of the surface* $z = u(x, y)$ *without tangent lines at* x_0, y_0. *The class of such functions is a residual set*[7] *in the space* U.

The general definition of a tangent plane to a surface $z = f(x, y)$ may be stated as follows: The plane

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0, \qquad z_0 = f(x_0, y_0),$$

is a tangent plane at the point (x_0, y_0, z_0) if the ratio

$$\frac{A(x' - x_0) + B(y' - y_0) + C(z' - z_0)}{[A^2 + B^2 + C^2]^{1/2} [(x' - x_0)^2 + (y' - y_0)^2 + (z' - z_0)^2]^{1/2}}$$

tends to zero as $x' \to x_0$, $y' \to y_0$; $z' = f(x', y')$.

It is easily seen that, if a surface has a tangent plane at a point (x_0, y_0, z_0) then all the vertical section of the surface through (x_0, y_0, z_0) (with the exception at most of one if the tangent plane is perpendicular

[6] "U-function" is an abbreviation for "function belonging to class U".

[7] A set in a metric space is said to be of the 1st category (Baire) if it is a sum of a sequence of nowhere dense sets. A set which is the complement of a set of the 1st category is called residual (Denjoy).

to the plane (xy)) have also a tangent line at this point. Therefore the theorem stated above includes that given at the end of the preceding section.

The functions considered in this paper are all of bounded variation, and even absolutely continuous and of uniformly bounded total variations, along all segments. In connection with this we shall mention an interesting fact recently established by Mr. F. G. Deutsch:[8] Every function of two variables, monotonic with respect to each one separately, is almost everywhere totally differentiable. This theorem may be easily proved by using the generalization of the Rademacher-Stepanoff theorem which was stated in the footnote on p. 115.

4. THEOREM 1. *There exist U-functions* $z = u(x, y)$ *such that at every interior point* (x, y) *of* $K_0 \equiv (0, 0; 1, 1)$,

$$\overline{\lim_{t \to +0}} \left| \frac{1}{t} [u(x + t\cos\theta, y + t\sin\theta) - u(x, y)] \right| = \infty$$

for every θ *with possible exception of a set of values of* θ *of the 1st category. The set* U_1 *of such surfaces is a residual set in the space* U.

Proof. Suppose that an U-function $z = w(x, y)$ does not belong to U_1. There exists then at least one point (x_0, y_0) in the interior of K_0 such that

$$\overline{\lim_{t \to 0}} \left| \frac{1}{t} [w(x_0 + t\cos\theta, y_0 + t\sin\theta) - w(x_0, y_0)] \right| < \infty$$

for a set of values of θ of the 2nd category. Then by a familiar reasoning of Baire's type there exists an integer $n > 0$ and a closed interval (θ_1, θ_2) of length $\geqq 1/n$ with $-\pi \leqq \theta_1 < \theta_2 \leqq \pi$ such that

$$\left| \frac{1}{t} [w(x_0 + t\cos\theta, y_0 + t\sin\theta) - w(x_0, y_0)] \right| \leqq n \tag{4.1}$$

for every t, $0 < t \leqq 1/n$, and every θ in (θ_1, θ_2).

Indeed, denote by P_k, for any integer k, the set of values of θ such that $0 < t \leqq 1/k$ implies

$$\left| \frac{1}{t} [w(x_0 + t\cos\theta, y_0 + t\sin\theta) - w(x_0, y_0)] \right| \leqq k.$$

By assumption $\sum P_k$ is of the 2nd category. All the sets P_k are closed, hence at least one of them, say P_K, contains an interval, say (θ_1, θ_2), $-\pi \leqq \theta_1 < \theta_2 \leqq \pi$. Then any integer $n > \max(K, (\theta_2 - \theta_1)^{-1})$ will satisfy the required condition.

Denote by A_n the set of U-functions $z = w(x, y)$ for which there exists a point (x_0, y_0), $1/n \leqq x_0, y_0 \leqq 1 - 1/n$, satisfying (4.1). Then

[8] Not yet published.

$$CU_1 = \sum_n A_n, \tag{4.2}$$

where CU_1 denotes, as usually, the complement of U_1 in the space U. It is obvious that A_n are closed sets and we shall prove that they are also nowhere dense in U. If this is proved, it will follow from (4.2) that CU_1 is a set of the 1st category, and therefore that U_1 is a residual set.

Let $p_1^n, p_2^n, \cdots, p_{k_n}^n$ be a finite set of interior points of K_0 with the following properties:

(P_1) *If* $1/n \leqq x_0, y_0 \leqq 1-1/n$, $\theta_2 - \theta_1 \geqq 1/n$, $-\pi \leqq \theta_1 < \theta_2 \leqq \pi$, *then there always exists a value* θ_0, $\theta_1 \leqq \theta_0 \leqq \theta_2$, *such that the ray*

$$x = x_0 + t\cos\theta_0, \qquad y = y_0 + t\sin\theta_0$$

contains at least one point p_i^n $(i = 1, 2, \cdots, k_n)$ *for a value* $t = t_0$, $0 < t_0 \leqq 1/n$;

(P_2) *no straight line contains more than two points* p_i^n $(i = 1, 2, \cdots, k_n)$.

In the interior of K_0 draw a system of k_n equal and not-overlapping circles C_i^n $(i = 1, 2, \cdots, k_n)$ with centers at p_i^n. In view of the property (P_2) we may suppose that no straight line has points in common with more than two circles C_i^n $(i = 1, 2, \cdots, k_n)$. Let now $u(x, y)$ be an arbitrary U-function. Denote by σ_n the upper bound of $|u(x'', y'') - u(x', y')|$ for $|x'' - x'| < 1/n$, $|y'' - y'| < 1/n$. Then in the interior of each circle C_i^n define a function $v_n(x, y)$ such that the equation $z = v_n(x, y)$ represent the cone with the base C_i^n and the altitude $2(\sigma_n + n^{-1/2})$; outside of these circles we set $v_n(x, y) = 0$. Since no straight line meets more than two circles C_i^n, the variation of $v_n(x, y)$ on every straight segment in K_0 is $\leqq 4(\sigma_n + n^{-1/2})$. Thus, $v_n(x, y) \subset U$ and we have $\|v_n\| \leqq 4(\sigma_n + n^{-1/2})$. Therefore, on setting

$$u_n = u + v_n$$

we have

$$\lim_n \|u_n - u\| = \lim_n \|v_n\| = 0. \tag{4.3}$$

The second property of u_n which we shall need is as follows:

(P_1^*) *If* $1/n \leqq x_0, y_0 \leqq 1-1/n$, $\theta_2 - \theta_1 \geqq 1/n$, $-\pi \leqq \theta_1 < \theta_2 \leqq \pi$, *then there exist values* t_0 *and* θ_0, $\theta_1 \leqq \theta_0 \leqq \theta_2$, $0 < t_0 \leqq 1/n$ *such that*

$$\left| \frac{1}{t_0} [u_n(x_0 + t_0\cos\theta_0, y_0 + t_0\sin\theta_0) - u_n(x_0, y_0)] \right| \geqq n^{1/2}.$$

Indeed, if

$$v_n(x_0, y_0) \leqq \sigma_n + n^{-1/2} \tag{4.4}$$

then we may choose as the corresponding values of t_0 and θ_0 those which satisfy the condition (P_1). We have then

$$\left|\frac{1}{t_0}[u_n(x_0+t_0\cos\theta_0,\ y_0+t_0\sin\theta_0)-u_n(x_0,\ y_0)]\right|$$
$$\geqq n[u(x_0+t_0\cos\theta_0,\ y_0+t_0\sin\theta_0)+2(\sigma_n+n^{-1/2})-u(x_0,\ y_0)-v_n(x_0,\ y_0)]$$
$$\geqq n[2(\sigma_n+n^{-1/2})-2\sigma_n-n^{-1/2}]=n^{1/2}.$$

If, on the contrary,

$$v_n(x_0,\ y_0)>\sigma_n+n^{-1/2}$$

then (x_0, y_0) belongs to a certain circle C_i^n, say C_j^n. In this case we may put $\theta_0=\theta_1$ and then t_0 may be determined as the value of t for which the ray

$$x=x_0+t\cos\theta_0,\qquad y=y_0+t\sin\theta_0$$

meets the boundary of the circle C_j^n. We prove this by the same simple evaluation as in case (4.4).

Now let N be an arbitrary positive integer. It follows from (P_1^*) that for $n>N^2$ no u_n belongs to A_N. This means, however, in view of (4.3), that every CA_N, no matter what is the positive integer N, is an everywhere dense set. Hence our proof is complete.

5. The surfaces which belong to the class U_1 of the preceding theorem obviously have no total differential at any point, and consequently no tangent plane non-perpendicular to the plane (xy). Now, in order to eliminate the assumption that the tangent plane is non-perpendicular to (xy), we shall use the following:

LEMMA. *The set U_2 of functions $z=u(x, y)$ in the space U such that at every interior point (x, y) of K_0*

$$\overline{\lim_{t\to 0}}\left|\frac{1}{t}[u(x+t\cos\theta,\ y+t\sin\theta)-u(x, y)]\right|<\infty$$

for every value of θ with the exception at most of a set of the 1*st category, is a residual set in U.*

Proof. By the same argument as in the proof of the theorem of § 4 we see that if a function $w(x, y)$ in U does not belong to U_2 then there exist a point (x_0, y_0) and a positive integer n such that

$$(R)\qquad \left|\frac{1}{t}[w(x_0+t\cos\theta,\ y_0+t\sin\theta)-w(x_0, y_0)\right|\geqq 1$$

for every t, $|t|<1/n$, and every θ of an interval (θ_1, θ_2), $-\pi/2\leqq\theta_1<\theta_2\leqq\pi/2$, $\theta_2-\theta_1\geqq 1/n$.

We shall denote by B_n the set of U-functions satisfying the condition (R). Every set B_n is closed and we have

$$(5.1)\qquad C\,U_2=\sum_n B_n.$$

Let $u(x, y)$ be an arbitrary U-function. We shall prove, as in the preceding § 4, that for every N, u is a limit point of a sequence of U-functions which do not belong to B_N. Therefore, every B_N is the complement of an everywhere dense set, and being closed, is nowhere dense. Hence by virtue of (5.1), $C U_2$ will be of the 1st category, and consequently U_2 will be a residual set in the space U.

To obtain the desired result, take an arbitrary integer n. Denote, as in the preceding proof, by σ_n the upper bound of $|u(x'', y'') - u(x', y')|$ for $|x'' - x'|, |y'' - y'| \leqq 1/n$. Divide K_0 in n^2 equal and not-overlapping squares K_i^n $(i = 1, 2, \cdots, n^2)$ and in the interior of each square K_i^n draw a finite number of circles C_{ij}^n $(j = 1, 2, \cdots, s_n)$ so that the following conditions be satisfied:

(S_1) *To every point* (x_0, y_0) *in* K_i^n *and every interval* (θ_1, θ_2), $-\pi/2 \leqq \theta_1 < \theta_2 \leqq \pi/2$, $\theta_2 - \theta_1 \geqq 1/n$, *there corresponds a value* j $(j = 1, 2, \cdots, s_n)$ *such that*

$$\tan \theta_1 \leqq \frac{x - x_0}{y - y_0} \leqq \tan \theta_2,$$

for every point (x, y) *in* C_{ij}^n.

(S_2) *No straight line has points in common with more than two circles* C_{ij}^n $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, s_n)$.

(S_3) $$\| u \|_{C_{ij}^n} \leqq 1/2n.$$

In the interior of each circle C_{ij}^n draw two not-overlapping circles $C_{ij}^{\prime n}$ and $C_{ij}^{\prime\prime n}$ so that

(S_4) $$u(x', y') \leqq u(x'', y'')$$

for every point (x', y') *in* $C_{ij}^{\prime n}$ *and every point* (x'', y'') *in* $C_{ij}^{\prime\prime n}$.

This may be done as follows. If the function $u(x, y)$ is constant on C_{ij}^n then each pair of not-overlapping circles in the interior of C_{ij}^n may be chosen as $C_{ij}^{\prime n}$ and $C_{ij}^{\prime\prime n}$. On the other hand, if u is not constant on C_{ij}^n, then let (x_0', y_0') and (x_0'', y_0'') be two arbitrary points in the interior of this circle such that

$$u(x_0', y_0') < u(x_0'', y_0'').$$

Thus, every pair of sufficiently small circles $C_{ij}^{\prime n}$, $C_{ij}^{\prime\prime n}$, with the centers at (x_0', y_0'), (x_0'', y_0'') satisfies the condition (S_4).

Now, as in the proof of the preceding theorem, we define the auxiliary function $w_n(x, y)$ so that $z = w_n(x, y)$ represent the cones with the bases $C_{ij}^{\prime n}$ and the altitude σ_n on the circles $C_{ij}^{\prime n}$, and the cones with the

bases $C_{ij}^{\prime n}$ and the altitude $-\sigma_n$ on the circles $C_{ij}^{\prime n}$; outside of these circles we set $w_n(x, y) = 0$. The condition (S_2) yields at once

$$\| w_n \| \leqq 8\sigma_n. \tag{5.2}$$

Put

$$u_n = u + w_n. \tag{5.3}$$

Denote by M_{ij}^n and m_{ij}^n respectively the maximum of $u_n(x, y)$ on $C_{ij}^{\prime n}$ and the minimum of $u_n(x, y)$ on $C_{ij}^{\prime\prime n}$. Let, for n, i fixed, M_i^n be the least of the numbers M_{ij}^n, and m_i^n the greatest of the numbers m_{ij}^n. For every point (x, y) in K_i^n, outside of the circles $C_{ij}^{\prime n}$, $C_{ij}^{\prime\prime n}$, we have

$$m_{ij}^n \leqq u_n(x, y) = u(x, y) \leqq M_{ij}^n \qquad (j = 1, 2, \cdots, s_n),$$

and therefore

$$m_i^n \leqq u_n(x, y) = u(x, y) \leqq M_i^n. \tag{5.4}$$

Set now

$$v_n(x, y) = \begin{cases} M_i^n, \text{ if } (x, y) \subset K_i^n \text{ and } u_n(x, y) > M_i^n, \\ u_n(x, y), \text{ if } (x, y) \subset K_i^n \text{ and } m_i^n \leqq u_n(x, y) \leqq M_i^n, \\ m_i^n, \text{ if } (x, y) \subset K_i^n \text{ and } u_n(x, y) < m_i^n. \end{cases}$$
$$(n = 1, 2, \cdots; i = 1, 2, \cdots, n^2).$$

It follows from (5.4) that $u_n(x, y) = v_n(x, y)$ outside of the circles C_{ij}^n. Thus the total variation of $v_n(x, y) - u_n(x, y)$ on any segment l is at most equal to the sum of the total variations of $u_n(x, y)$ on the parts of l in the circles C_{ij}^n. Hence, in view of (S_2), (S_3), (5.2) and (5.3),

$$\| v_n - u_n \| \leqq 8\sigma_n + \frac{1}{n}, \qquad \| v_n - u \| \leqq 16\sigma_n + \frac{1}{n}.$$

Hence

$$\lim_n \| v_n - u \| = 0. \tag{5.5}$$

On the other hand, from (5.4), the condition (S_4) and the definition of v_n it readily follows that, for every n, i $(i = 1, 2, \cdots, n^2)$, all values of $v_n(x, y)$ in K_i^n are assumed on each pair of circles C_{ij}^n, $C_{ij}^{\prime\prime n}$ $(j = 1, 2, \cdots, s_n)$. Hence, we conclude from (S_1) that to every point (x_0, y_0) in K_i^n and to every pair of values $\theta_1, \theta_2, -\pi/2 \leqq \theta_1 < \theta_2 \leqq \pi/2, \theta_2 - \theta_1 \geqq 1/n$, there corresponds a point (x', y') in this K_i^n such that

$$\tan \theta_1 \leqq \frac{x' - x}{y' - y} \leqq \tan \theta_2, \quad v_n(x, y) = v_n(x', y').$$

Therefore, on setting

$$\theta_0 = \arctan \frac{x' - x}{y' - y}, \quad -\frac{\pi}{2} \leqq \theta_0 \leqq \frac{\pi}{2},$$

we have $\theta_1 \leqq \theta_0 \leqq \theta_2$, and the equality

$$v_n(x+t\cos\theta_0,\ y+t\sin\theta_0)-v_n(x,y)=0$$

holds for a certain t of the absolute value $\leqq 1/n$.

Hence v_n does not satisfy the condition (R) and does not belong to B_n, neither to B_N if $n \geqq N$; i. e., for any N, $u(x,y)$ is, according to (5.5), a limit point of a set of U-functions which do not belong to B_N. This, as has been indicated above, proves our lemma.

6. The sum of two sets of the 1st category is of the same category, and the product of two residual sets is again a residual set. Thus, from Theorem 1 and the lemma of the preceding section, it follows

THEOREM 2. *There exist functions in the space U such that at every interior point (x,y) of the square K_0*

$$(6.1)\qquad \begin{aligned} &\varliminf_{t\to 0}\left|\frac{1}{t}[u(x+t\cos\theta,\ y+t\sin\theta)-u(x,y)]\right| < \\ &\varlimsup_{t\to+0}\left|\frac{1}{t}[u(x+t\cos\theta,\ y+t\sin\theta)-u(x,y)]\right| = \infty \end{aligned}$$

for infinitely many values of θ, indeed, for all values of θ except perhaps for those of a set of the 1st category. The set of such functions is a residual set in the space U.

7. In two preceding sections we have considered the space U more restrictive than that of all absolutely continuous functions in the sense of Tonelli. We shall denote the latter by T. However Theorems 1 and 2 remain valid if we substitute there the space T for U, with the distance in the space T defined by

$$\varrho(u,v) \equiv \max_{K_0}|u(x,y)-v(x,y)|$$
$$+\int_0^1\int_0^1\left|\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}\right|dx\,dy+\int_0^1\int_0^1\left|\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}\right|dx\,dy$$

for any two absolutely continuous functions in the square K_0. With this definition the space T is complete and the proofs of Theorems 1 and 2 relative to this space are essentially the same as for the space U.

Another example of application of these theorems is exhibited by the space V of all functions $v(x,y)$ continuous on K_0, vanishing on the boundary and satisfying in addition to the properties (A), (B) of § 3, the following condition:

(C) For every segment l in K_0 the derivative $\dfrac{\partial v}{\partial l}$ exists and vanishes almost everywhere on l.

The definition of the distance for the space V is the same as for the space U.

Theorem 2 for this space shows in particular that *there exist surfaces* $z = v(x, y)$ *with a finite area and with partial derivatives* $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *vanishing almost everywhere on* K_0, *but without a tangent plane* (*and therefore without a total differential*) *at any point.*

The proofs for the space V is practically the same as for the space U. We need merely to slightly modify the definitions of the auxiliary functions, as the functions defined by systems of cones do not belong to the space V. If a circle C of the radius r and the center (x_0, y_0) is given then, instead of the cone of altitude h and base C, we may use now the surface:

$$z = h[1 - u(\sqrt{(x - x_0)^2 + (y - y_0)^2})], \tag{7.1}$$

where $u(t)$ denotes a monotonic non-decreasing function such that $u(0) = 0$, $u(r) = 1$, $u'(t) = 0$ almost everywhere on $(0, r)$. We have $\frac{\partial z}{\partial l} = 0$ almost everywhere along any straight segment l in C; any function defined by the formulas of type (7.1) on a finite number of not-overlapping circles, and vanishing outside these circles, belongs to the space V.

The reader will easily find many other similar examples of functional spaces for which Theorems 1 and 2 remain true.

8. Several problems are suggested by the preceding discussion, of which we mention

(i) Is it possible to replace in the relation (6.1) the $\overline{\lim}_{t \to 0}$ by the $\overline{\lim}_{t \to +0}$ (as in Theorem 1)?

(ii) Is it possible to state in Theorem 2 that relation (6.1) holds for all values of θ except at most of a denumerable set?[9]

The first problem may be related to some known facts concerning the space of continuous functions of a single variable. There were given, since the classical result of Weierstrass, over a dozen examples of continuous functions without either both-sided finite or infinite, or one-sided finite derivatives. On the contrary the existence of a continuous function without one-sided finite or infinite derivative at any point was a much more difficult problem solved only in 1922 by Besicovitch.[10] The difference between the set of functions of Weierstrass and that of Besicovitch may be characterized by the fact that while the former is a residual set in the space of con-

[9] An exceptional denumerable set is necessary. Indeed, every function of bounded variation of two variables has derivatives in infinitely many directions almost everywhere.

[10] See for instance Pepper [1].

tinuous functions, the latter is only of the 1st category.[11] It is possible that the solution of problem (i) would lead to a result of a similar kind.

References.

Auerbach and Banach [1]. Über die Höldersche Bedingung, Studia Math., 3 (1931), 185–188.

Banach [1]. Über die Baire'sche Kategorie gewisser Funktionenmengen, ibid., 174–179.

Caccioppoli [1]. Sul carattere infinitesimale delle superficie quadrabili, Atti della R. Accademia Nazionale dei Lincei, (S. 6), 7 (1928), 901–905.

Kaczmarz [1]. Integrale vom Dini'schen Typus, Studia Math., 3 (1931), 189–199.

Mazurkiewicz [1]. Sur les fonctions non-dérivables, Studia Math., 3 (1931), 95–100.

Mazurkiewicz [2]. Sur l'intégrale $\int_0^1 \frac{1}{t}[f(x+t)+f(x-t)-2f(x)]\,dt$, ibid., 114–118.

Pepper [1]. On continuous functions without a derivative, Fund. Math., 12 (1928), 244–253.

Rademacher [1]. Über partielle und totale Differenzierbarkeit, Math. Ann., 79 (1919), 340–359.

Saks [1]. On the functions of Besicovitch in the space of continuous functions, Fund. Math., 19 (1932), 211–219.

Stepanoff [1]. Über totale Differenzierbarkeit, Math. Ann., 90 (1923), 318–320.

Tonelli [1]. Sulla quadratura delle superficie, Atti della R. Accademia Nazionale dei Lincei, (S. 6), 3 (1926), 357–362, 445–450, 633–638.

[11] Saks [1].

ON THE MINORS OF ABSOLUTELY CONVERGENT DETERMINANTS.

By Leon W. Cohen.*

The study of infinite determinants of the form

$$A = |\delta_{ik} + a_{ik}| \tag{1}$$

subject to the conditions

$$\sum_{i=1}^{\infty} a_{ii} < +\infty; \quad \left[\sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty} |a_{ik}|^q\right]^{p/q}\right]^{1/p} = \sigma < +\infty \qquad (i \neq k);$$
$$\frac{1}{p} + \frac{1}{q} = 1; \quad 1 < p \leqq 2 \tag{2}$$

and the associated system of linear equations was begun by von Koch[1] who considered the case $p = 2$. The extension to the general case was made by Bóbr[2] by applying the theory of limited linear forms. The writer[3] also considered the solution theory and showed the convergence of the series

$$\sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} |A_{ik}|^q\right]^{p/q}; \quad \sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty} |A_{ik}|^p\right]^{q/p} \qquad (i \neq k) \tag{3}$$

where A_{ik} is the minor of the element a_{ik}.

In this note, we consider the minors

$$A_{r_1 \cdots r_s; c_1 \cdots c_s} \qquad (r_i \neq c_j;\ i, j = 1, \cdots, s)$$

obtained by striking out the rows r_i and the columns c_j from A and extend the result of (3) to these minors.

The matrix of the minor $A_{r_1 \cdots r_s; c_1 \cdots c_s}$, which is an absolutely convergent determinant, can, by a finite number of interchanges of rows and of columns, be brought into the following form:

$$\left\| \begin{array}{c|c} \| a_{c_i r_j} \| & \| a_{c_i r} \| \\ (i, j = 1, \cdots, s) & \begin{pmatrix} i = 1, \cdots, s \\ r \neq r_i \end{pmatrix} \\ \hline \| a_{c r_j} \| & \| \delta_{cr} + a_{cr} \| \\ \begin{pmatrix} j = 1, \cdots, s \\ c \neq c_j \end{pmatrix} & (c \neq c_i;\ r \neq r_j) \end{array} \right\|.$$

* Received, January 27, 1932.

[1] H. von Koch, Palermo Rend. 28 (1909), pp. 255–266.

[2] St. Bóbr, Math. Zeit. 10 (1921), pp. 1–10.

[3] L. W. Cohen, Bull. Amer. Math. Soc. 36 (1930), pp. 563–572.

The determinant of this matrix differs from $A_{r_1 \cdots r_s; c_1 \cdots c_s}$ at most in sign. In order to find a bound for $A_{r_1 \cdots r_s; c_1 \cdots c_s}$ we recall that the absolute value of A is at most[4]

$$P = \prod (1 + |p|),$$

where the product is extended of over all products

$$p = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1} \qquad (n = 1, 2, \ldots)$$

with distinct indices which can be formed from the matrix $\| a_{ik} \|$. A term of $A_{r_1 \cdots r_s; c_1 \cdots c_s}$ has, except for sign and a factor which is a term of $\delta_{cr} + a_{cr}$, three possible forms:

$$\prod_{h=1}^{s} a_{c_h j_h}, \tag{4}$$

$$\prod_{t=1}^{u} \left(\prod_{h=k_{t-1}+1}^{k_t - 1} a_{c_h j_h} \cdot a_{c_{k_t} i_1} a_{i_1 i_2} \cdots a_{i_{n_t} j_{k_t}} \right) \prod_{h=k_u+1}^{s} a_{c_h j_h}, \tag{5}$$

$$\prod_{h=1}^{s} a_{c_h i_1} a_{i_1 i_2} \cdots a_{i_{n_h} j_h}, \tag{6}$$

where $j_1, \cdots, j_s$ is a permutation of $r_1, \cdots, r_s$;

$$0 < u < s; \qquad 0 = k_0 < k_1 < \cdots < k_u \leqq s;$$

$$\prod_{h=1}^{0} f(h) = \prod_{h=s+1}^{s} f(h) = 1.$$

Hence we may write the inequality

$$|A_{r_1 \cdots r_s; c_1 \cdots c_s}| \leqq P \left\{ \sum_{(j;r)} \prod_{h=1}^{s} |a_{c_h j_h}| \right.$$

$$+ \sum_{u=1}^{s-1} \sum_{(j;r)} \sum_{1 \leq k_1 < \cdots < k_u \leq s} \prod_{t=1}^{u} \left(\prod_{h=k_{t-1}+1}^{k_t - 1} |a_{c_h j_h}| \sum_{n_t=1}^{\infty} \sum_{i_1 \cdots i_{n_t}=1}^{\infty} |c_{k_t} i_1 \cdots i_{n_t}; j_{k_t}| \right) \tag{7}$$

$$\left. \times \prod_{h=k_u+1}^{s} |a_{c_h j_h}| + \sum_{(j;r)} \prod_{h=1}^{s} \sum_{n_h=1}^{\infty} \sum_{i_1 \cdots i_{n_h}=1}^{\infty} |c_h i_1 \cdots i_{n_h}; j_{n_h}| \right\},$$

where the symbol $\sum_{(j;r)}$ indicates that the sum is to be taken over all permutations $j_1, \cdots, j_s$ of $r_1, \cdots, r_s$ and

[4] F. Riesz, Les systèmes d'équations linéaires ... Paris, (1913), pp. 33–35.

$$|c_h\, i_1 \cdots i_{n_h};\, j_h| = |a_{c_h i_1}\, a_{i_1 i_2} \cdots a_{i_{n_h} j_h}|,$$

if the subscripts are distinct and zero otherwise.

The first reduction we can make on (7) is a consequence of the following:[5]

$$\left[\sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty}|a_{ik}|^p\right]^{q/p}\right]^{1/q} = \tau \leqq \sigma \qquad (i \neq k), \tag{8}$$

$$\sum_{i_1 \cdots i_{n_t}=1}^{\infty} |c_{k_t}\, i_1 \cdots i_{n_t};\, j_{k_t}| \leqq \sigma^{n_t-1}\left[\sum_{i=1}^{\infty}|a_{c_{k_t} i}|^q\right]^{1/q}\left[\sum_{k=1}^{\infty}|a_{ic_{k_t}}|^p\right]^{1/p} \tag{9}$$

$$(i \neq c_{k_t}, j_{k_t} \text{ in the series on the right}).$$

These follow from the Hölder inequality, which we state in the form

$$N_i^{(p)}(\lambda a_i + \mu b_i) \leqq |\lambda|\, N_i^{(p)}(a_i) + |\mu|\, N_i^{(p)}(b_i); \qquad (p>1) \tag{10}$$

where $N_i^{(p)}(a_i) = \left[\sum_{i=1}^{\infty}|a_i|^p\right]^{1/p}$. Using (8), (9) and the notation of (10) in (7), we get, assuming $\sigma<1$

$$\begin{aligned}|A_{r_1 \cdots r_s;\, c_1 \cdots c_s}| \leqq P\Bigg\{&\sum_{(j;r)}\prod_{h=1}^{s}|a_{c_h j_h}| \\ &+\sum_{u=1}^{s-1}\left(\frac{1}{1-\sigma}\right)^u \sum_{(j;r)}\sum_{1\leq k_1<\cdots<k_u\leq s}\prod_{t=1}^{u}\left(\prod_{h=k_{t-1}+1}^{k_t-1}|a_{c_h j_h}|\, N_i^{(q)}(a_{c_{k_t} i})\, N_i^{(p)}(a_{ij_{k_t}})\right) \\ &\times\prod_{h=k_u+1}^{s}|a_{c_h j_h}| + \left(\frac{1}{1-\sigma}\right)^s \sum_{(j;r)}\prod_{h=1}^{s} N_i^{(q)}(a_{c_h i})\, N_i^{(p)}(a_{ij_h})\Bigg\}.\end{aligned} \tag{11}$$

Applying (10), we get because of its homogeneity

$$\begin{aligned}\left[\sum_{c_1 \cdots c_s=1}^{\infty}\left[\sum_{r_1 \cdots r_s=1}^{\infty}|A_{r_1 \cdots r_s;\, c_1 \cdots c_s}|^q\right]^{p/q}\right]^{1/p} \leqq P\Bigg\{&\sum_{(j;r)}\prod_{h=1}^{s} N_{c_h}^{(p)}\, N_{j_h}^{(q)}(a_{c_h j_h}) \\ &+\sum_{u=1}^{s-1}\left(\frac{1}{1-\sigma}\right)^u \sum_{(j;r)}\sum_{1\leq k_1<\cdots<k_u\leq s}\prod_{t=1}^{u}\left(\prod_{h=k_{t-1}+1}^{k_t-1} N_{c_h}^{(p)}\, N_{j_h}^{(q)}(a_{c_h j_h})\right. \\ &\left.\times N_{c_{k_t}}^{(p)}\, N_i^{(q)}(a_{c_{k_t} i})\, N_{j_{k_t}}^{(q)}\, N_i^{(p)}(a_{ij_{k_t}})\right)\prod_{h=k_u+1}^{s} N_{c_h}^{(p)}\, N_{j_h}^{(q)}(a_{c_h j_h}) \\ &+\left(\frac{1}{1-\sigma}\right)^s \sum_{(j;r)}\prod_{h=1}^{s} N_{c_h}^{(p)}\, N_i^{(q)}(a_{c_h i})\, N_{j_h}^{(q)}\, N_i^{(p)}(a_{ij_h})\Bigg\} \qquad (i \neq c_h \neq j_k).\end{aligned} \tag{12}$$

[5] The inequality (8) is given by Bóbr, loc. cit. p. 5. The inequality (9) is given by the writer, loc. cit. p. 566.

Using (2) and (8) in (12), we get

$$\left[\sum_{c_1\cdots c_s=1}^{\infty}\left[\sum_{r_1\cdots r_s=1}^{\infty}|A_{r_1\cdots r_s;\,c_1\cdots c_s}|^q\right]^{p/q}\right]^{1/p}$$

$$\leqq P\left\{\sigma^s\sum_{(j;r)}1+\sum_{u=1}^{s-1}\left(\frac{1}{1-\sigma}\right)^u\sigma^u\tau^u\sigma^{t=1}\sum_{(j;r)}^{u}{}^{(k_t-k_{t-1}-1)+s-k_u}\sum_{1\leqq k_1<\cdots<k_u\leqq s}1\right.$$

$$\left.+\left(\frac{1}{1-\sigma}\right)^s\sigma^s\tau^s\sum_{(j;r)}1\right\} \tag{13}$$

$$=Ps!\,\sigma^s\left(1+\frac{\tau}{1-\sigma}\right)^s \qquad (c_i\neq r_j;\ i,j=1,2,\cdots,s).$$

The assumption made in (11) that $\sigma<1$ can be removed by the familiar artifice of a convergence factor. Because of the convergence required in (2), there is an integer m such that

$$\left[\sum_{i=m+1}^{\infty}\left[\sum_{k=1}^{\infty}|a_{ik}|^q\right]^{p/q}\right]^{1/p}<\frac{1}{2} \qquad (i\neq k). \tag{14}$$

We choose a positive t less than 1 for which

$$t\left[\sum_{i=1}^{m}\left[\sum_{k=1}^{\infty}|a_{ik}|^q\right]^{p/q}\right]^{1/p}<\frac{1}{2} \qquad (i\neq k) \tag{15}$$

and consider the determinant $A'=|\delta_{ik}+a'_{ik}|$ where

$$\delta_{ik}+a'_{ik}=\begin{cases}t(\delta_{ik}+a_{ik}), & i=1,\cdots,m,\\ \delta_{ik}+a_{ik}, & i=m+1,\,m+2,\cdots.\end{cases}$$

From (14), (15) and the Hölder inequality it is clear that (2) holds for A' and that $\sigma'<1$. Further

$$|A_{r_1\cdots r_s;\,c_1\cdots c_s}|\leqq t^{-m}|A'_{r_1\cdots r_s;\,c_1\cdots c_s}|$$

from which it follows that

$$\sum_{c_1\cdots c_s=1}^{\infty}\left[\sum_{r_1\cdots r_s=1}^{\infty}|A_{r_1\cdots r_s;\,c_1\cdots c_s}|^q\right]^{p/q} \qquad (r_i\neq c_j;\ i,j=1,\cdots,s)$$

converges.

If, in (7), we use the inequality[6]

$$\sum_{i_1\cdots i_{n_t}=1}^{\infty}|c_{k_t}i_1\cdots i_{n_t};j_{k_t}|\leqq\tau^{n_t-1}\left[\sum_{i=1}^{\infty}|a_{c_{k_t}i}|^q\right]^{1/q}\left[\sum_{i=1}^{\infty}|a_{ij_{k_t}}|^p\right]^{1/p}, \tag{16}$$

we find by an analogous method that

$$\sum_{r_1\cdots r_s=1}^{\infty}\left[\sum_{c_1\cdots c_s=1}^{\infty}|A_{r_1\cdots r_s;\,c_1\cdots c_s}|^p\right]^{q/p} \qquad (r_i\neq c_j;\ i,j=1,\cdots,s)$$

[6] Loc. cit. (3), p. 567, p. 565.

converges and, if $\tau < 1$, that the sum is at most

$$\left[P s!\, \tau^s \left(1 + \frac{\sigma}{1-\tau}\right)^s \right]^q.$$

The results may be summed up in the

THEOREM. *If the infinite determinant* $A = |\delta_{ik} + a_{ik}|$ *satisfies* (2) *and* $A_{r_1 \cdots r_s; c_1 \cdots c_s}$ *is the minor of* A *obtained by striking out the rows* r_i *and the columns* c_j $(r_i \neq c_j;\ i, j = 1, \cdots, s)$ *then the series*

$$\sum_{c_1 \cdots c_s = 1}^{\infty} \left[\sum_{r_1 \cdots r_s = 1}^{\infty} |A_{r_1 \cdots r_s; c_1 \cdots c_s}|^q \right]^{p/q}$$

$$\sum_{r_1 \cdots r_s = 1}^{\infty} \left[\sum_{c_1 \cdots c_s = 1}^{\infty} |A_{r_1 \cdots r_s; c_1 \cdots c_s}|^p \right]^{q/p} \qquad (r_i \neq c_j;\ i, j = 1, \cdots, s)$$

converge. If $\sigma < 1$, *then the two sums are respectively at most equal to*

$$\left[P \sigma^s s! \left(1 + \frac{\tau}{1-\sigma}\right)^s \right]^p; \qquad \left[P \tau^s s! \left(1 + \frac{\sigma}{1-\tau}\right)^s \right]^q.$$

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ON INTERPOLATION.*

By J. Shohat.

Introduction. Let $y_1, y_2, \cdots, y_n$ and $x_1 < x_2 < \cdots < x_n$ be $2n$ arbitrarily given real quantities. Consider, with Fejér,[1] the polynomial of degree $2n-1$

$$\Omega(x) = \sum_{i=1}^{n}\left[-\frac{\varphi''(x_i)}{\varphi'(x_i)}(x-x_i)+1\right]\left[\frac{\varphi(x)}{(x-x_i)\varphi'(x_i)}\right]^2 y_i \tag{1}$$

$$\left(\varphi(x) \equiv \prod_{i=1}^{n}(x-x_i)\right).$$

We readily see that

$$\Omega(x_i) = y_i, \quad \Omega'(x_i) = 0 \qquad (i = 1, 2, \cdots, n) \tag{2}$$

$$1 = \sum_{i=1}^{n}\left[\frac{-\varphi''(x_i)}{\varphi'(x_i)}(x-x_i)+1\right]\left[\frac{\varphi(x)}{(x-x_i)\varphi'(x_i)}\right]^2. \tag{3}$$

(2) shows that, with $y_i = f(x_i)$, where $f(x)$ is a given function, $\Omega(x)$ may be considered as its interpolational polynomial. Fejér[1] has shown that if we choose in (1) for x_i the zeros of Legendre or trigonometric polynomials of degree n and let n increase indefinitely, then $\Omega(x)$ possesses very important convergence properties. His analysis is based upon the interesting property, that with such a choice all factors $\left[-\frac{\varphi''(x_i)}{\varphi'(x_i)}(x-x_i)+1\right]$ are *positive* in $(-1, 1)$, which property, however, is perhaps not necessary for convergence, according to Fejér.

It is known that Legendre and trigonometric polynomials are members of the family of the so called *classical orthogonal polynomials*—of Jacobi (J), Laguerre (L) and Hermite (H), which have many properties in common. It was, therefore, quite natural to undertake an investigation of (1), regarding its convergence properties, taking for x_i the zeros of *any* of the aforesaid orthogonal polynomials. It turns out—and this is the main object of the present paper—that *with all such polynomials formula* (1) *maintains its convergence properties* (Th. I, § 7) (although the factors $\left[-\frac{\varphi''(x_i)}{\varphi'(x_i)}(x-x_i)+1\right]$ no longer remain all positive). Furthermore, the

* Received, January 19, 1932; presented to the Society, October 1931.

[1] L. Fejér, Über Interpolation, Gött. Nachr. (1916), pp. 66–91.

ordinary Lagrange interpolation formula with such polynomials also shows interesting convergence properties (Th. III, § 12).

Throughout this paper the following notations will be used: $G_n(x)$ to denote an *arbitrary* polynomial of degree $\leqq n$; ε or δ, A—to denote resp. arbitrarily small and arbitrarily large, but fixed, positive quantities; N—a sufficiently large positive integer; τ—a fixed positive quantity independent on x and n (ε, δ, A, τ may be different in different formulae); finally, if a formula bears the number, say, (18—J, L, H) or (27—J), this means resp. a formula pertaining to the polynomials of Jacobi, Laguerre and Hermite, or to Jacobi polynomials only.

1. **Orthogonal Tchebycheff polynomials and mechanical quadratures.** Consider the system

$$\varphi_n(x; p; a, b) \equiv \varphi_n(x; p) \equiv \varphi_n(x) = a_n(x^n - S_n x^{n-1} + \cdots) \quad (n = 0, 1, \cdots;\ a_n(p) \equiv a_n > 0) \tag{4}$$

of orthogonal and normal Tchebycheff polynomials uniquely determined by

$$\int_a^b p(x)\,\varphi_m(x)\,\varphi_n(x)\,dx = \delta_{mn} \qquad (m, n = 0, 1, \cdots), \tag{5}$$

where $p(x)$ is not negative in (a, b) and has moments of all orders $\alpha_n = \int_a^b p(x)\,x^n\,dx$ $(n = 0, 1, \cdots)$, with $\alpha_0 > 0$. (5) leads, it is known, to

$$\begin{gathered}\sqrt{\lambda_{n+2}}\,\varphi_{n+1}(x) = (x - c_{n+1})\,\varphi_n(x) - \sqrt{\lambda_{n+1}}\,\varphi_{n-1}(x) \quad (n \geqq 1);\\ \lambda_{n+2} = \frac{a_n^2}{a_{n+1}^2};\quad c_{n+1} = \int_a^b p(x)\,x\,\varphi_n^2(x)\,dx.\end{gathered} \tag{6}$$

$$\int_a^b p(x)\,\varphi_n(x)\,G_{n-1}(x)\,dx = 0 \qquad (n = 1, 2, \cdots). \tag{7}$$

(7), in turn, making use of the zeros $x_1 < x_2 < \cdots < x_n$ of $\varphi_n(x)$—all real, distinct and between a and b, leads to the following formula of mechanical quadratures of Gauss' type:

$$\begin{gathered}\int_a^b p(x)\,f(x)\,dx = \sum_{i=1}^{n} H_i\,f(x_i) + R_n(f),\text{[2]}\\ H_i = \int_a^b p(x)\,\frac{\varphi_n(x)\,dx}{(x - x_i)\,\varphi_n'(x_i)} = \int_a^b p(x)\left[\frac{\varphi_n(x)}{(x - x_i)\,\varphi_n'(x_i)}\right]^2 dx > 0,\end{gathered} \tag{8}$$

with the remarkable property

$$R_n(G_{2n-1}) \equiv 0 \qquad (n = 1, 2, \cdots), \tag{9}$$

[2] All integrals are generally taken in the sense of Riemann.

from which follows (taking $f(x) \equiv G_{2n-1}(x) \equiv 1$):

$$\sum_{i=1}^{n} H_i = \int_a^b p(x)\,dx = \alpha_0. \tag{10}$$

In (7) and hereafter $f(x)$ denotes a function defined on (a, b) as follows: it is finite for all x under consideration and bounded in (a, b), if (a, b) is finite; it is bounded in any finite subinterval of (a, b), if the latter is infinite, i. e. $|f(x)| \leqq M_A \equiv M$, for $|x| \leqq A$ (M_A may increase indefinitely with A).

The comparison of (1) and (8) suggests a close relation between these two formulae and a possibility of *transforming* (1) *by introducing therein the H_i explicitly.* In fact, *this transformation, carried out for the classical orthogonal polynomials, proved to be the essential point in the discussion which follows.*

2. Some properties of the classical orthogonal polynomials.

$$(a, b) = (-1, 1), \quad (0, \infty), \quad (-\infty, \infty),$$

$$p(x) = (1+x)^{\alpha-1}(1-x)^{\beta-1}, \quad x^{\alpha-1}e^{-x}, \quad e^{-x^2} \qquad (\alpha, \beta > 0), \tag{11—I, L, H}$$

$$x_1 \to a, \quad x_n \to b \qquad (n \to \infty);$$

$$R_n(f) \to 0 \qquad (n \to \infty) \tag{12—I, L, H}$$

for all functions $f(x)$ as described above.[3]

$$|\varphi_n(x)| < \tau \quad (-1+\varepsilon \leqq x \leqq 1-\varepsilon); \quad |\varphi_n(\pm 1)| \sim n^{\alpha}_{\beta} {}^{-1/2}; \tag{13—I}$$

$$|\varphi_n(x)| < \tau n^{-1/4} \quad (\varepsilon \leqq x \leqq A); \quad |\varphi_n(0)| \sim n^{(\alpha-1)/2}; \tag{13—L}$$

$$|\varphi_n(x)| < \tau n^{-1/4} \quad (|x| \leqq A); \tag{13—H}$$

$$a_n < \tau \cdot 2^n; \quad a_n = [\Gamma(n+1)\,\Gamma(n+\alpha)]^{-1/2}; \tag{14—I, L, H}$$

$$a_n = \pi^{-1/4}\,2^{n/2}\,[\Gamma(n+1)]^{-1/2};$$

$$A(x)\,\varphi_n''(x) + B(x)\,\varphi_n'(x) + C_n\,\varphi_n(x) = 0; \tag{15—I, L, H}$$

$$A(x) = 1-x^2, \quad x, \quad 1,$$

$$B(x) = \alpha-\beta-(\alpha+\beta)x, \quad \alpha-x, \quad -2x, \tag{16—I, L, H}$$

$$C_n = n(n+\alpha+\beta-1), \quad n, \quad 2n.$$

The differential equation (15), where $A(x)$ *and* $B(x)$ *do not depend on* n, is of great importance. Thus, the theory of linear differential equations of 2nd order shows at once that

[3] a) Stieltjes, Quelques recherches sur les quadratures dites mécaniques, Œuvres, vol. 1, pp. 377-396; b) J. Chokhate (J. Shohat), Sur la convergence des quadratures mécaniques ···, C. R., vol. 186 (1928), pp. 344-346.

(17—I, L, H) $$p(x) = \left(\frac{1}{A(x)}\right) e^{\int \frac{B(x)}{A(x)} dx}.$$

Differentiate (15) and use again (17). We get the fundamental relation:

(18—I, L, H) $$\varphi_n'(x; p) = n \frac{a_n(p)}{a_{n-1}(Ap)} \varphi_{n-1}(x; Ap).$$

Using (6, 7, 16, 18), we get:

$$A(x)\varphi_{n-1}(x; Ap) =$$

$$H\varphi_{n+1}(x; p) + K\varphi_n(x; p) + L\varphi_{n-1}(x; p) \quad (H, K, L\text{—const.}),$$

(19—J, L, H) $$H = -\frac{a_{n-1}(Ap)}{a_{n+1}(p)},\ 0,\ 0;\quad L = \frac{a_{n-1}(p)}{a_{n-1}(Ap)},$$

$$A(x_i)\varphi_{n-1}(x_i; Ap) = \varphi_{n-1}(x_i; p)\left[-H\sqrt{\frac{\lambda_{n+1}}{\lambda_{n+2}}} + L\right];$$

(20—J) $$\varphi_{n-1}(x_i; Ap) = \left[\frac{\varphi_{n-1}(x_i; p)}{A(x_i)}\right]\left[\frac{a_{n-1}(Ap)\, a_{n-1}(p)}{a_n^2(p)} + \frac{a_{n-1}(p)}{a_{n-1}(Ap)}\right];$$

(21—J) $$\varphi_n'(x_i; p) = \left[\frac{n\varphi_{n-1}(x_i; p)}{A(x_i)}\right]\left[\frac{a_{n-1}(p)}{a_n(p)} + \frac{a_n(p)\, a_{n-1}(p)}{a_{n-1}^2(Ap)}\right];$$

(22—L) $$\varphi_{n-1}(x_i; Ap) = \frac{a_{n-1}(p)}{a_{n-1}(Ap)\, A(x_i)}\varphi_{n-1}(x_i; p);$$

(23—L) $$\varphi_n'(x_i; p) = \left[\frac{n\, a_n(p)\, a_{n-1}(p)}{a_{n-1}^2(Ap)\, A(x_i)}\right]\varphi_{n-1}(x_i; p);$$

(23—H) $$\varphi_n'(x_i; p) = \left[\frac{n\, a_n(p)}{a_{n-1}(p)}\right]\varphi_{n-1}(x_i; p).$$

By virtue of (14),

(24—J, L, H) $$\varphi_n'(x_i) = \frac{n\tau_n}{A(x_i)}\varphi_{n-1}(x_i)\ (\tau_n < \tau),\ \frac{[n(n+\alpha-1)]^{1/2}}{A(x_i)}\varphi_{n-1}(x_i),$$

$$(2n)^{1/2}\varphi_{n-1}(x_i).$$

3. Expression of H_i and transformation of $\Omega(x)$. The general theory of orthogonal Tchebycheff polynomials leads to the following expression for H_i:[4]

(25) $$H_i = \frac{1}{K_n(x_i)} \quad \left(K_n(x) \equiv \sum_{i=0}^{n} \varphi_i^2(x)\right),$$

$$K_{n-1}(x) = \left(\frac{a_{n-1}}{a_n}\right)[\varphi_n'(x)\varphi_{n-1}(x) - \varphi_{n-1}'(x)\varphi_n(x)];$$

[4] Cf., for ex., J. Shohat, On a Certain Formula of Mechanical Quadratures ..., Transactions of the American Math. Soc., v. 31 (1930), pp. 448-463.

(26) $$H_i = \frac{1}{\frac{a_{n-1}}{a_n}\varphi_n'(x_i)\,\varphi_{n-1}(x_i)}.$$

Making use of (14, 22—J, 23—L, 24), we get the following *expression for H_i, in case of the classical orthogonal polynomials*:

(27—J, L, H)
$$H_i = \frac{n\,l_n}{A(x_i)\,\varphi_n'^2(x_i)},$$

(J) $$l_n = \frac{a_n^2(p)}{a_{n-1}^2(Ap)} < \tau \qquad (i = 1, 2, \cdots, n);$$

(L, H) $$H_i = \frac{1}{x_i\,\varphi_n'^2(x_i)}, \quad \frac{2}{\varphi_n'^2(x_i)}.$$

We are in a position now to transform (1):

(28—J, L, H) $$\Omega(x) = \sum_{i=1}^{n}\left[\frac{B(x_i)}{A(x_i)}(x-x_i)+1\right]\left[\frac{\varphi_n(x)}{(x-x_i)\,\varphi_n'(x_i)}\right]^2 f(x_i)$$ (see (15));

$$\Omega(x) = \sigma_n \sum_{i=1}^{n} H_i\,b_i(x)\,h_i^2(x)\,f(x_i),$$

(29—J, L, H) $$\sigma_n = \frac{l_n}{n} < \frac{\tau}{n},\ 1,\ \frac{1}{2}; \quad b_i(x) = B(x_i)(x-x_i)+A(x_i);$$

$$h_i(x) \equiv \frac{\varphi_n(x)}{x-x_i}.$$

(29) *gives the desired expression of the interpolation polynomial $\Omega(x)$.* Before we proceed with its investigation, we wish to indicate some applications. We get, combining (27, 8, 5) with the relation

$$\int_a^b p(x)\,(x-x_i)\left(\frac{\varphi_n(x)}{x-x_i}\right)^2 dx = 0 \quad (i = 1, 2, \cdots, n;\ \text{see (7)}):$$

(30—J, L, H)
$$\int_a^b p(x)\,x\,h_i^2(x)\,dx = \frac{n\,l_n\,x_i}{A(x_i)},\quad 1,\quad 2x_i,$$

$$\int_a^b p(x)\,h_i^2(x)\,dx = \frac{n\,l_n}{A(x_i)},\quad \frac{1}{x_i},\quad 2, \qquad (i = 1, 2, \cdots, n),$$

$$\int_a^b p(x)\,x^2\,h_i^2(x)\,dx = \frac{1+nx_i^2\,l_n}{A(x_i)},\quad 1+x_i,\quad 1+2x_i^2;$$

$$\text{(31—L, H)}\qquad \int_a^b p(x)\,x^3\,h_i^2(x)\,dx = c_{n+1} + 2x_i + x_i^2,\quad 2x_i + 2x_i^3 \quad \text{(see (6))};$$

$$\text{(32—L, H)}\qquad \begin{aligned} \int_a^b p(x)\,h_i(x)\,dx &= \frac{1}{\varphi_n'(x_i)},\ \frac{2}{\varphi_n'(x_i)}, \\ \int_a^b p(x)\,x\ h_i(x)\,dx &= \frac{1}{\varphi_n'(x_i)},\ \frac{2x_i}{\varphi_n'(x_i)}. \end{aligned} \qquad (i = 1, 2, \cdots, n).$$

4. **Expression for the difference $f(x) - \Omega(x)$.** Making use of (3,29) and letting $y_i = f(x_i)$ ($f(x)$ as defined above), we get:

$$\text{(33—J, L, H)}\qquad f(x) - \Omega(x) = \sigma_n \sum_{i=1}^{n} H_i\,b_i(x)\,h_i^2(x)\,[f(x) - f(x_i)],$$

where we take hereafter

$$\text{(34—J, L, H)}\qquad |x| \leqq 1, \qquad 0 \leqq x \leqq A, \qquad |x| \leqq A.$$

We proceed to prove that, *under certain conditions to be stated explicitly below, we can choose h_1 so that*

$$\text{(35—J, L, H)}\qquad b_i(x) \geqq 0 \quad \textit{for} \quad |x - x_i| \leqq h_1,$$

with h_1 independent on x and n ($i = 1, 2, \cdots, n$).

(i) *Polynomials of Jacobi.* Here

$$\text{(36)}\qquad |x_i| < 1 \quad (i = 1, 2, \cdots, n); \qquad x_1 \to -1, \quad x_n \to 1 \quad (n \to \infty),$$

$$\text{(37)}\qquad b_i(x) = [(\alpha - \beta) - (\alpha + \beta)x_i]\,(x - x_i) + 1 - x_i^2 \quad (i = 1, 2, \cdots, n).$$

Let $\alpha \neq \beta$. We may assume $\alpha > \beta$ (replace, otherwise, x by $-x$). We find: $x = 0$ implies

$$b_i(x) > 0 \quad \text{if} \quad |x_i| < \frac{1}{2(\alpha + \beta) + 1};$$

$0 < x \leqq 1 - \varepsilon$ implies

$$b_i(x) > 0 \quad \text{if} \quad x_i \leqq 0,$$

or

$$0 < x_i \leqq x \quad \text{and} \quad |x - x_i| < \frac{1 - (1 - \varepsilon)^2}{2(\alpha + \beta) + 1},$$

or

$$x_i > x \quad \text{and} \quad x_i \geqq \frac{\alpha - \beta}{\alpha + \beta},$$

and so on, for all x_i and x under consideration.

(35—J) is thus established for $-1 + \varepsilon \leqq x \leqq 1 - \varepsilon$ and for any $\alpha, \beta\ (> 0)$. Moreover, making use of (36), we find that (35—J) *holds for* $-1 \leqq x \leqq 1$

if and only if $\alpha, \beta \leqq 1$ (in which case *all* $b_i(x) \geqq 0$), and that $\frac{\beta}{\alpha} > 1$ *implies* $b_i\binom{+1}{-1} < 0$ *for* $i = \overset{n}{1}$ $(n \geqq N)$, *and no such* h_1, *as required by* (35), *exists.*

(ii) *Polynomials of Laguerre and Hermite.* By a reasoning quite similar to that employed in (i), we find that (35—L) *holds for* $\varepsilon \leqq x \leqq A$ *and arbitrary* $\alpha(>0)$. Moreover, (35—L) *holds for* $0 \leqq x \leqq A$ *if and only if* $\alpha \leqq 1$; $\alpha > 1$ *implies* $b_i(0) < 0$ *for* $n \geqq N$; *and no such* h_1, *as required by* (35), *exists.* (35—H) *holds for* $|x| \leqq A$.

Let x *be a point of continuity of* $f(x)$. Choose h_2 so that

$$|f(x) - f(x')| \leqq \delta \quad \text{for} \quad |x - x'| \leqq h_2. \tag{38}$$

Such h_2, *independent on* x, exists, even if x be allowed to vary over an arbitrarily fixed, but finite, interval of continuity of $f(x)$. We rewrite (33), following Fejér, as

$$\begin{aligned} f(x) - \Omega(x) &= \sigma_n \sum_{|x-x_i| \leqq h_3} H_i\, b_i(x)\, h_i^2(x)\, [f(x) - f(x_i)] \\ &\quad + \sigma_n \sum_{|x-x_i| > h_3} (\text{id.}) \quad 0 < h_3 \leqq \min(h_1, h_2). \end{aligned} \tag{39—J, L, H}$$

It is in this form that we shall investigate the convergence properties of $\Omega(x)$.

5. **Investigation of** $\sigma_n \sum'' \equiv \sigma_n \sum_{|x-x_i| > h_3} H_i\, b_i(x)\, h_i^2(x)\, [f(x) - f(x_i)]$.

(i) *Polynomials of Jacobi.* Here

$$\sigma_n < \frac{\tau}{n}, \quad |f(x)| \leqq M, \quad |f(x_i)| \leqq M \quad (i = 1, 2, \cdots, n). \tag{40}$$

Hence,

$$|b_i(x)| < \tau \quad (i = 1, 2, \cdots, n); \quad \sigma_n |\sum''| \leqq \frac{2\tau M}{n h_3^2}\, \varphi_n^2(x) \sum_{|x-x_i| > h_3} H_i.$$

$$\sigma_n |\sum''| \leqq \frac{2\tau M}{n h_3^2}\, \varphi_n^2(x) \sum_{i=1}^{n} H_i < \frac{\tau \varphi_n^2(x)}{n h_3^2} \qquad \text{(see (10)).} \tag{41}$$

Assume, first, $-1 + \varepsilon \leqq x \leqq 1 - \varepsilon$, and use (13—J):

$$\sigma_n |\sum''| < \frac{\tau}{n h_3^2} < \delta \ (n \geqq N) \quad \textit{uniformly in} \ (-1+\varepsilon, 1-\varepsilon). \tag{42}$$

Secondly, take $x = \pm 1$ and use again (13—J):

$$\sigma_n |\sum''|_{x=\pm 1} < \frac{\tau n^{2\left(\frac{\beta}{\alpha}-1\right)}}{h_3^2} < \delta \ (n \geqq N), \quad \text{if} \ \frac{\beta}{\alpha} < 1. \tag{42—bis}$$

(ii) *Polynomials of Laguerre.* Here $\sigma_n = 1$, $0 \leqq x \leqq A$, $|f(x)| \leqq M$. Subdivide $\sum''$ into two parts:

$$\sum'' = \sum_{|x_i| \leqq A}'' + \sum_{|x_i| > A}'' \equiv \sum_1 + \sum_2 \qquad (|x - x_i| > h_3). \tag{43}$$

In $\sum_1$ we have (see (16—L)):

$$|b_i(x)| < \tau, \quad |f(x_i)| \leqq M \qquad (i = 1, 2, \cdots, n),$$

and making use of (13—L),

$$|\sum_1| < \tau n^{-1/2} h_3^{-2} \sum_{i=1}^{n} H_i < \tau n^{-1/2} h_3^{-2} < \delta \ (n \geqq N) \tag{44}$$

uniformly in (ε, A).

Regarding $\sum_2$, we have:

$$|\sum_2| < \tau n^{-1/2} h_3^{-2} \sum_{i=1}^{n} H_i \left[(\alpha + x_i)(A + x_i) + x_i\right] \left[M + |f(x_i)|\right]. \tag{45}$$

Let $f(x)$ be subject to the following condition:

$$\int_0^\infty e^{-x} x^{\alpha+1} |f(x)| \, dx \text{ exists.} \tag{46}$$

((46) is certainly satisfied, if $\int_0^\infty e^{-x} x^{\alpha-1} f^2(x) \, dx$ exists). By virtue of (12—L),

$$\lim_{n \to \infty} \sum_{i=1}^{n} H_i \left[(\alpha + x_i)(A + x_i) + x_i\right] \left[M + |f(x_i)|\right]$$
$$= \int_0^\infty e^{-x} x^{\alpha-1} \left[(\alpha + x)(A + x) + x\right] \left[M + |f(x)|\right] dx.$$

Hence,

$$|\sum_2| < \tau n^{-1/2} h_3^{-2} < \delta \ (n \geqq N) \quad \textit{uniformly in } (\varepsilon, A), \tag{47}$$

$$\sigma_n |\sum'''| < \delta \ (n \geqq N) \quad \textit{uniformly in } (\varepsilon, A) \quad \text{(under condition (46))}.^5 \tag{48}$$

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[5] For the more restricted class of functions $f(x)$ such that, for $x \geqq$ a sufficiently large positive quantity x_0, $|f(x)| \leqq x^m$, m denoting an arbitrarily large, but fixed, positive integer, we can establish (48), *without making any use of* (12). It suffices to consider the same subdivision (43), where A is now replaced by x_0. (44) then holds true, as before, and as to Σ_2, we write, by virtue of (9) $(n \geqq N)$,

$$|\sum_2| < \tau n^{-1/2} h_3^{-2} \sum_{i=1}^{n} H_i \left[(\alpha + x_i)(x_0 + x_i) + x_i\right] \left[M + x_i^m\right]$$
$$= \tau n^{-1/2} h_3^{-2} \int_0^\infty e^{-x} x^{\alpha-1} \left[(\alpha + x)(x_0 + x) + x\right] \left[M + x^m\right] dx \left(n \geqq \frac{m+3}{2}\right) < \tau n^{-1/2} h_3^{-2} < \delta.$$

In particular, (48) holds true, if $f(x) \equiv 0$ for $x > x_0$.

For $x = 0$ the above considerations, combined with (13—L), yield:

$$\sigma_n \left| \sum'' \right|_{x=0} < \tau n^{\alpha-1} h_3^{-2} < \delta \ (n \geqq N), \quad \text{if } \alpha < 1. \tag{49}$$

(iii) *Polynomials of Hermite.* Here $\sigma_n = \frac{1}{2}$, $|x| \leqq A$, $|f(x)| \leqq M$, and we write:

$$\sum'' = \sum_{1 \atop |x_i| \leqq A}'' + \sum_{2 \atop |x_i| > A}'' \qquad (|x - x_i| > h_3). \tag{50}$$

Considerations quite similar to those employed in (ii), making use of (12—H, 13—H), lead to

$$\sigma_n \left| \sum'' \right| < \delta \qquad (n \geqq N) \ \textit{uniformly in} \ (-A, A), \tag{51}$$

for all $f(x)$ such that

$$\int_{-\infty}^{\infty} e^{-x^2} x^2 |f(x)| \, dx \ \text{exists.}^6 \tag{52}$$

6. **Investigation of** $\sigma_n \sum' \equiv \sigma_n \sum_{|x - x_i| \leqq h_3} H_i \, b_i(x) \, h_i^2(x) \, [f(x) - f(x_i)]$. The preceding discussion shows that, under the same conditions resp. as in (41, 42, 48, 49, 51) and with the same uniformity in x,

$$\sigma_n \sum_{|x - x_i| > h} H_i \, |b_i(x)| \, h_i^2(x) \, |f(x_i)| \to 0 \qquad (n \to \infty) \tag{53—J, L, H}$$

($h > 0$ arbitrarily fixed, limited by the bounds of the x_i only),

$$\sigma_n \sum_{|x - x_i| \leqq h} H_i \, b_i(x) \, h_i^2(x) \to 1 \qquad (n \to \infty). \tag{54—J, L, H}$$

In fact,

$$1 \equiv \sigma_n \sum_{|x - x_i| \leqq h} H_i \, b_i(x) \, h_i^2(x) + \sigma_n \sum_{|x - x_i| > h} (\text{id.}) \qquad \text{(see (3, 29))}.$$

We are now prepared to formulate

7. **The convergence properties of** $\Omega(x)$. Since, by (3, 37) and one of the formulae (41, 42, 48, 49, 51),

$$|f(x) - \Omega(x)|$$
$$\leqq \sigma_n \delta \sum_{|x - x_i| \leqq h_3} H_i \, b_i(x) \, h_i^2(x) + \sigma_n \sum_{|x - x_i| > h_3} H_i \, |b_i(x)| \, h_i^2(x) \, [|f(x)| + |f(x_i)|],$$

we arrive, through (53, 54), to

THEOREM I. (i) *The interpolation polynomial* $\Omega(x)$, *where* x_i *are the zeros of Jacobi polynomial with arbitrary* α, β (> 0) *and* $y_i = f(x_i)$, *converges,*

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[6] Here again (see the preceding footnote) we can establish (51), *without using* (12) but with the use of (9) only, for all $f(x)$ such that $|f(x)| \leqq x^{2m}$ $(|x| \geqq x_0$, m-positive integer).

as $n \to \infty$, *to the value of* $f(x)$ *at any point of continuity of* $f(x)$ (*assumed to be bounded*) *inside* $(-1, 1)$, *the convergence being uniform on any interval of continuity of* $f(x)$ *wholly inside* $(-1, 1)$. *Moreover, if* $\frac{\beta}{\alpha} < 1$, $\Omega\binom{+1}{-1} \to f\binom{+1}{-1}$ *resp.*[7] (ii) *If* x_i *are the zeros of Laguerre polynomial with arbitrary* α (>0), $\Omega(x)$ *converges to the value of* $f(x)$ *at any finite point* $x > 0$, *where* $f(x)$ *is continuous, provided,* $f(x)$ *is bounded in any finite interval, and* $\int_0^\infty e^{-x} x^{\alpha+1} |f(x)|\, dx$ *exists, the convergence being uniform in any finite interval* (ε, A) *of continuity of* $f(x)$. *Moreover,* $\alpha < 1$ *implies* $\Omega(0) \to f(0)$. (iii) *With* x_i*—zeros of Hermite polynomial,* $\Omega(x) \to f(x)$ *at any finite point of continuity of* $f(x)$, *assumed to be bounded in any finite interval, provided,* $\int_{-\infty}^{\infty} e^{-x^2} x^2 |f(x)|\, dx$ *exists, the convergence being uniform on any finite interval of continuity of* $f(x)$.

We add the following remarks.

a) For Jacobi polynomials we have the following inequalities:[8]

$$|\varphi_n(x)| \leqq \varphi_n(1) \quad \left(\beta \geqq \alpha \geqq \frac{1}{2}\right),$$

(55)

$$|\varphi_n(x)| \leqq |\varphi_n(-1)| \quad \left(\alpha \geqq \beta \geqq \frac{1}{2}\right) \quad (-1 \leqq x \leqq 1),$$

(55—bis) $$|\varphi_n(x)| < \pi \sqrt{\frac{2}{\pi}} \quad (-1 \leqq x \leqq 1) \quad \left(\alpha = \beta < \frac{1}{2}\right),$$[9]

which, combined with Th. 1 (i) leads to

THEOREM II. *If* $f(x)$ *be continuous in the whole interval* $(-1, 1)$, $\Omega(x)$, *where* x_i *are the zeros of Jacobi polynomial with* $\alpha, \beta < 1$, *converges, as* $n \to \infty$, *to* $f(x)$ *uniformly for* $-1 \leqq x \leqq 1$. *With the same* α, β (53—J, 54—J) *hold uniformly in* $(-1, 1)$.

b) The mechanism of convergence: $\Omega(x) \to f(x)$ acts "locally", i. e. *the said convergence at a certain point* $x = \xi$[10] *depends on the values of* $f(x)$ *in the immediate neighborhood of this point only.* In fact, Th. I, combined with (53—J, L, H), gives:

(56—J, L, H) $$f(x) = \lim_{n \to \infty} \sum_{|x - x_i| \leqq h} H_i\, b_i(x)\, h_i^2(x) f(x_i).$$

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[7] With $\alpha = \beta = 1$ or $\frac{1}{2}$, we get the results of Fejér (l. c.).

[8] For which I am obliged to a friendly communication of Dr. E. G. Kogbetliantz.

[9] S. Bernstein, Sur les polynomes orthogonaux ..., Journal des Math., 10 (1931), pp. 219–277; p. 252.

[10] Throughout this and next section x is confined resp. to the intervals given in Th. I.

Apply now (56) to $F(x)$ defined as follows:

$$F(X) \equiv f(x) \quad \text{for} \quad |x-\xi| \leqq \delta \quad \text{and} \quad \equiv 0$$

elsewhere ($f(x)$ is continuous at $x = \xi$), and we get:

$$\lim_{n\to\infty} \sum_{|\xi - x_i| \geq \delta} H_i\, b_i(x)\, h_i^2(x) f(x_i) = 0 \qquad (|x-\xi| > \delta). \tag{57—J, L, H}$$

c) $\Omega(x)$ is "stable" with regard to $f(x)$ (cf. Fejér, l. c.), i. e., writing $\Omega_f(x)$: $|f(x)| \leqq M$, $|f(x) - \varphi(x)| \leqq \delta$ *imply resp.* $|\Omega_f(x)| < \tau M$, $|\Omega_f(x) - \Omega_\varphi(x)| < \tau\delta$ *for all* n (τ *independent on* M, n *and* δ). In fact, the above discussion shows that

$$|\Omega_f(x)| \leqq M\sigma_n \Big[\sum_{|x-x_i| \leqq h_3} H_i\, b_i(x)\, h_i^2(x) + \sum_{|x-x_i| > h_3} H_i\, h_i^2(x)\, |b_i(x)| \Big] < \tau M$$

(if $|f(x)| \leqq M$).

On the other hand,

$$\Omega_f(x) - \Omega_\varphi(x) \equiv \Omega_{f-\varphi}(x).$$

8. **Application.** An almost immediate consequence of the convergence properties of $\Omega(x)$ is the following statement. *The zeros of the classical orthogonal polynomials are everywhere dense in* $(-1, 1)$, $(0, A)$, $(-A, A)$ *resp.* For assume the existence of an interval (c, d)—part resp. of the above interval—which does not contain any x_i ($n \geqq N$). Then, by (3), with x *inside* (c, d),

$$1 \equiv \sigma_n \sum_{|x-x_i| > \delta} H_i\, h_i^2(x)\, b_i(x) \qquad (c + \delta < x < d - \delta),$$

which, in view of (53), is impossible.

Since (53), for $f(x) \equiv 1$, has been established by means of (9) only (see footnotes on pp. 137—8), it follows that the above property of the zeros of the classical orthogonal polynomials (of utmost importance for the convergence of the corresponding mechanical quadratures) has thus been established, *without using the theory of convergence of the said quadratures.*

9. **Lagrange interpolation formula and Tchebycheff polynomials.** Consider Lagrange interpolation formula

$$M(x) = \sum_{i=1}^{n} \frac{\varphi_n(x)}{(x-x_i)\,\varphi_n'(x_i)} f(x_i), \tag{58}$$

x_i *being the zeros of Tchebycheff polynomial* $\varphi_n(x) \equiv \varphi_n(x;\, a, b;\, p)$. Rewrite (58) as

$$M(x) = \sum_{i=1}^{n} \frac{\varphi_n(x)}{(x-x_i) \cdot \left(\dfrac{a_{n-1}}{a_n}\right) \varphi_n'(x_i)\, \varphi_{n-1}(x_i)} \cdot \frac{a_{n-1}}{a_n} \varphi_{n-1}(x_i),$$

$$M(x) = \sum_{i=1}^{n} H_i\, h_i(x)\, \varphi_{n-1}(x_i) f(x_i) \qquad \text{(see (26))}. \tag{59}$$

Hereafter, $f(x)$ is assumed to be continuous in a certain finite subinterval (c, d) $(a \leqq c < d \leqq b)$.

Denote by $P_{n-1}(x)$ the polynomial, of degree $\leqq n-1$, of the best approximation $(= E_{n-1})$, in Tchebycheff sense, to $f(x)$ on (c, d), i. e.

$$E_{n-1} = \max |f(x) - P_{n-1}(x)| \leqq \max |f(x) - G_{n-1}(x)| \text{ on } (c, d). \tag{60}$$

We have then,

$$P_{n-1}(x) = \sum_{i=1}^{n} \frac{h_i(x)}{\varphi_n'(x_i)} P_{n-1}(x_i),$$

$$f(x) - M(x) = f(x) - P_{n-1}(x) - \sum_{i=1}^{n} \frac{h_i(x)}{\varphi_n'(x_i)} [f(x_i) - P_{n-1}(x_i)], \tag{61}$$

$$f(x) - M(x) = f(x) - P_{n-1}(x) - \frac{a_{n-1}}{a_n} \sum_{i=1}^{n} H_i\, h_i(x)\, \varphi_{n-1}(x_i)\, [f(x_i) - P_{n-1}(x_i)].$$

Hence, making use of Cauchy inequality and of (60),

$$|f(x) - M(x)| \leqq E_{n-1} \left\{ 1 + \frac{a_{n-1}}{a_n} \sqrt{\sum_{i=1}^{n} H_i\, h_i^2(x) \sum_{i=1}^{n} H_i\, \varphi_{n-1}^2(x_i)} \right\} \qquad (c \leqq x \leqq d).$$

Furthermore, by (8,9),

$$\sum_{i=1}^{n} H_i\, \varphi_{n-1}^2(x_i) = \int_a^b p(x)\, \varphi_{n-1}^2(x)\, dx = 1,$$

and we thus get the following *expression for the error of Lagrange interpolation formula, which holds for any system of orthogonal Tchebycheff polynomials $\varphi_n(x;\ a, b;\ p)$:*[11]

$$\begin{aligned} &|f(x) - M(x)| \leqq E_{n-1} \left\{ 1 + \frac{a_{n-1}}{a_n} \sqrt{\sum_{i=1}^{n} H_i\, h_i^2(x)} \right\} && (a, b \text{ arbitrary}), \\ &|f(x) - M(x)| \leqq E_{n-1} \left\{ 1 + \frac{b-a}{2} \sqrt{\sum_{i=1}^{n} H_i\, h_i^2(x)} \right\} && (c \leqq x \leqq d) \\ & && (a, b \text{ finite}), \end{aligned} \tag{62}$$

since, for the finite interval (a, b), $\dfrac{a_{n-1}}{a_n} = \sqrt{\lambda_{n+1}} < \dfrac{b-a}{2}$.[12]

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[11] $p(x)$ can be replaced by $d\psi(x)$, $\psi(x)$—bounded and non—decreasing in (a, b).

[12] J. Chokhate, Sur le développement de l'intégrale $\int_a^b \frac{p(y)\, dy}{x-y} \ldots$, Rendiconti Circolo Matematico Palermo, v. 47 (1923), pp. 25–46; p. 33.

The problem under consideration is now reduced to

10. **Estimating the sum** $\sum_{i=1}^{n} H_i h_i^2(x)$. We subject $p(x)$ to the following condition:

(63) $$p(x) \geqq p_0 > 0 \text{ for } c' \leqq x \leqq d' \qquad (c \leqq c' < d' \leqq d).$$

Then we write:

$$\sum_{i=1}^{n} H_i h_i^2(x) = \frac{dQ}{dx}, \quad Q(x) \equiv \int_{c'}^{x} p(t) \sum_{i=1}^{n} H_i h_i^2(t)\, dt,$$

$$Q(x) \leqq \frac{1}{p_0} \int_{c'}^{d'} p(t) \sum_{i=1}^{n} H_i h_i^2(t)\, dt \leqq \frac{1}{p_0} \int_{a}^{b} p(t) \sum_{i=1}^{n} H_i h_i^2(t)\, dt$$

$$= \frac{1}{p_0} \sum_{i=1}^{n} H_i^2 [\varphi_n'(x_i)]^2 \qquad \text{(by (8); } c' \leqq x \leqq d').$$

By Markoff-Bernstein Theorem, applied to the polynomial $Q(x)$, of degree $2n-1$,

(64) $$\sum_{i=1}^{n} H_i h_i^2(x) < \tau n^{\sigma} \sum_{i=1}^{n} H_i^2 [\varphi_n'(x_i)]^2$$

$(\sigma = 1 \text{ for } c'+\varepsilon \leqq x \leqq d'-\varepsilon; \quad \sigma = 2 \text{ for } c' \leqq x \leqq d')$.[13]

By (8, 10),

$$\sum_{i=1}^{n} \frac{H_i}{\alpha_0} < 1, \qquad \frac{H_i}{\alpha_0} < 1, \qquad H_i^2 < H_i \alpha_0,$$

(65) $$\sum_{i=1}^{n} H_i^2 [\varphi_n'(x_i)]^2 < \alpha_0 \sum_{i=1}^{n} H_i [\varphi_n'(x_i)]^2 = \alpha_0 \int_a^b p(x) [\varphi_n'(x)]^2\, dx.$$

We thus get, *under condition* (63):

(66) $$|f(x) - M(x)| < E_{n-1} \left\{ 1 + \tau \frac{a_{n-1}}{a_n} n^{\sigma/2} \sqrt{\int_a^b p(x) [\varphi_n'(x)]^2\, dx} \right\}$$

(a, b arbitrary; σ, x as in (64)).

$$|f(x) - M(x)| < E_{n-1} \left\{ 1 + \tau n^{\sigma/2} \sqrt{\int_a^b p(x) [\varphi_n'(x)]^2\, dx} \right\} \quad (a, b \text{ finite}).$$

Let us assume we are given the development

(67) $$\varphi_n'(x) = \sum_{i=0}^{n-1} f_i \varphi_i(x) \qquad \left(f_i = \int_a^b p(x) \varphi_n'(x) \varphi_i(x)\, dx \right).$$

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[13] For the polynomials of Laguerre and Hermite (see (27—L, H), (10)) $\sum_{i=1}^{n} H_i^2 [\varphi_n'(x_i)]^2 < \tau$, so that

(L, H) $$|f(x) - M(x)| < \tau E_{n-1} n^{\sigma/2+1}, \quad < \tau E_{n-1} n^{(\sigma+1)/2} \qquad (\sigma, x \text{ as in (64)}).$$

However, better results will be obtained below in a different way.

Then, by (5), (66),

$$|f(x) - M(x)| < E_{n-1}\left\{1 + \tau n^{\sigma/2}\frac{a_{n-1}}{a_n}\sqrt{\sum_{i=0}^{n-1} f_i^2}\right\} \tag{68}$$

(a, b arbitrary; σ, x as in (64)).

As to the coefficients f_i in (67), they can be found in some cases by means of the following device. We write, assuming that

$p'(x)$ exists and is integrable in (a, b):

$$f_i = f_i + 0 = \int_a^b p(x)\,\varphi_n'(x)\,\varphi_i(x)\,dx + \int_a^b p(x)\,\varphi_i'(x)\,\varphi_n(x)\,dx \tag{69}$$

$$(0 \leqq i \leqq n-1),$$

$$f_i = \int_a^b p(x)\,[\varphi_i(x)\,\varphi_n(x)]'\,dx \qquad (i = 0, 1, \cdots, n-1), \tag{70}$$

$$f_i = p(x)\,\varphi_i(x)\,\varphi_n(x)\Big|_a^b - \int_a^b p'(x)\,\varphi_i(x)\,\varphi_n(x)\,dx \qquad (0 \leqq i \leqq n-1), \tag{71}$$

and this enables us to evaluate or estimate f_i, whenever *a simple relation exists between* $p(x)$ *and* $p'(x)$.[14]

In order to illustrate, assume further that

$$\frac{p'(x)}{p(x)} \text{ is bounded on } (a, b). \tag{72}$$

Then, by Schwartz inequality and (5),

$$\int_a^b p'(x)\,\varphi_i(x)\,\varphi_n(x)\,dx = \int_a^b p(x)\frac{p'(x)}{p(x)}\,\varphi_i(x)\,\varphi_n(x)\,dx = O(1)$$

$$(i = 0, 1, \cdots),$$

$$f_i = p(x)\,\varphi_i(x)\,\varphi_n(x)\Big|_a^b + O(1). \tag{73}$$

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[14] In case of Legendre polynomials, for ex., $p'(x) \equiv 0$, so that we get at once:

$$f_i = 0 \ (i - n \equiv 0, \text{ mod. } 2), \qquad f_i = [(2i+1)(2n+1)]^{1/2} \ (i - n \equiv 1, \text{ mod. } 2).$$

For Hermite polynomials, $p(\pm\infty) = 0$, $p'(x) = -2xp(x)$, so that

$$f_i = 0 \ (i < n-1), \ \varphi_{n-1}'(x) = 2\left(\frac{a_{n-1}}{a_n}\right)\varphi_n(x).$$

This not only fully agrees with (18—H) (where $A(x) \equiv 1$), but gives, comparing:

$$2\left(\frac{a_{n-1}}{a_n}\right) = n\frac{a_n}{a_{n-1}},$$

and thus we get, *without any further consideration*,

$$\lambda_{n+1} = \left[\frac{a_{n-1}}{a_n}\right]^2 = \frac{n}{2}, \qquad a_n = \pi^{-1/4}\,2^{n/2}\,[\Gamma(n+1)]^{-1/2}.$$

If we assume, in addition, that in (63) $c' = c = a$, $d' = d = b$, i. e.

$$p(x) \geqq p_0 > 0 \quad \text{for} \quad a \leqq x \leqq b, \tag{74}$$

then,[15]

$$\begin{aligned} &|\varphi_n(a)|, \quad \varphi_n(b) < \tau (n+1)^{1/2} \qquad (n = 0, 1, \cdots); \\ &|f_i| < \tau [(i+1)(n+1)]^{1/2} \quad (i = 0, 1, \cdots, n-1). \end{aligned} \tag{75}$$

11. **Estimating the sum $\sum_{i=1}^{n} H_i h_i^2(x)$ for classical orthogonal polynomials.** The sum in question is closely related to the sum $\sum_{i=1}^{n} H_i b_i(x) h_i^2(x)$ we have been dealing with in the preceding sections. We know (see (29, 35, 53, 54)) that for certain x

$$\sigma_n \sum_{i=1}^{n} H_i |b_i(x)| h_i^2(x) \to 1 \qquad (n \to \infty); \tag{77—J, L, H}$$

$$\sum_{i=1}^{n} H_i |b_i(x)| h_i^2(x) < \tau n, \quad < \tau, \quad < \tau. \tag{78—J, L, H}$$

It follows, *if under certain conditions*

$$b_i(x) \geqq b_0 > 0 \text{ for all } x_i \quad (b_0 \text{ independent on } n \text{ and } x), \tag{79}$$

then, *under the same conditions,* (see (14, 62))

$$\sum_{i=1}^{n} H_i h_i^2(x) < \tau n, \quad < \tau, \quad < \tau. \tag{80—J, L, H}$$

Replace in the expression of $b_i(x)$ x_i by z. A study of the zeros of the polynomial $b(z)$ thus obtained[16] readily leads to the following conclusions: *the inequalities* (79, 80) *hold if and only if*

$$-1 + \varepsilon \leqq x \leqq 1 - \varepsilon \qquad (\alpha, \beta (>0) \text{ arbitrary}), \tag{81—J}$$

$$\begin{aligned} &(\sqrt{\alpha} - 1)^2 + \varepsilon \leqq x \leqq (\sqrt{\alpha} + 1)^2 - \varepsilon \qquad (\alpha \geqq 1) \\ &\varepsilon \leqq x \leqq (\sqrt{\alpha} - 1)^2 - \varepsilon \qquad (\alpha > 1), \end{aligned} \tag{81—L or}$$

$$-\sqrt{2} + \varepsilon \leqq x \leqq \sqrt{2} - \varepsilon. \tag{81—H}$$

12. **Another method for estimating $|f(x) - M(x)|$.** The first of the formulae (61) gives, by virtue of (60),

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[15] J. Shohat, A Simple Method for Normalizing Tchebycheff Polynomials ..., Bulletin Amer. Math. Soc., vol. 33 (1927), pp. 427–433; p. 432.

[16] In view of (11), we must have: for Jacobi polynomials $b(\pm 1) \geqq b_0$; for Laguerre polynomials $b(0) \geqq b_0$ and $\xi_{1,2}$—zeros of $b(z)$—either real and negative or imaginary; for Hermite polynomials-imaginary $\xi_{1,2}$.

$$|f(x) - M(x)| \leqq E_{n-1}\left\{1 + \sum_{i=1}^{n}\left|\frac{h_i(x)}{\varphi_n'(x_i)}\right|\right\} \qquad (c \leqq x \leqq d). \tag{82}$$

In order to obtain an estimate of the right-hand member of (82), apply (58) to $\varphi_{n-1}(x)$:

$$\frac{\varphi_{n-1}(x)}{\varphi_n(x)} = \sum_{i=1}^{n}\frac{1}{x - x_i}\,\frac{\varphi_{n-1}(x_i)}{\varphi_n'(x_i)}, \tag{83}$$

which, when differentiated and combined with (25, 26), leads to the *fundamental* relation:

$$K_{n-1}(x) = \sum_{i=1}^{n}\left[\frac{h_i(x)}{\varphi_n'(x_i)}\right]^2 K_{n-1}(x_i) = \sum_{i=1}^{n}\left[\frac{h_i(x)}{\varphi_n'(x_i)}\right]^2 \cdot \left(\frac{1}{H_i}\right). \tag{84}$$

We have now, by Cauchy inequality,

$$\sum_{i=1}^{n}\left|\frac{h_i(x)}{\varphi_n'(x_i)}\right| \leqq \sqrt{K_{n-1}(x)\int_a^b p(t)\,dt} \quad (x \text{ arbitrary; see (8,10)}). \tag{85}$$

Hence, *for any system of orthogonal polynomials corresponding to a finite or infinite interval* (a, b) *and for any* $f(x)$ *continuous on a finite subinterval* (c, d)

$$|f(x) - M(x)| < \tau E_{n-1}\sqrt{K_{n-1}(x)} \qquad (c \leqq x \leqq d). \tag{86}$$

In case $p(x)$ satisfies (63) ($a \leqq c'$, $d' \leqq b$), a, b *being finite,*

$$K_n(x) = O(n^{\sigma})^{17} \tag{87}$$
$$(\sigma = 1 \text{ for } c' + \varepsilon \leqq x \leqq d' - \varepsilon;\ \sigma = 2 \text{ for } c' \leqq x \leqq d').$$

$$|f(x) - M(x)| < \tau E_{n-1}\, n^{\sigma/2} \quad \text{(under condition (63))}, \tag{88}$$
(a, b finite, $f(x)$ continuous in (c, d); σ, x as in (87); $c \leqq c'$; $d' \leqq d$).

For Laguerre and Hermite polynomials we substitute into (86) resp.:

$$K_n(x) = O(n^{1/2}) \text{ for } \varepsilon \leqq x \leqq A, \quad K_n(0) \sim n^{\alpha}. \tag{89—L}$$

$$K_n(x) = O(n^{1/2}) \text{ for } -A \leqq x \leqq A \qquad \text{(see (13))}. \tag{89—H}$$

As to E_{n-1}, its order, with respect to n, is known from the theory of polynomial approximation of continuous functions. If, for example, $f^{(i)}(x)$ $(i \geqq 0)$ is continuous in a certain interval or satisfies therein Lipschitz condition of order γ, then in the same interval $E_n = O(n^{-i})$, $O(n^{-i-\gamma})$ resp. Thus *formula* (86) *enables us to state, in very general cases, for what*

[17] J. Shohat, On a General Formula ..., Transactions of the American Math. Soc., vol. 29 (1927), pp. 569–583; 579-580.

classes of continuous functions Lagrange interpolation formula (58) *converges and to estimate its degree of convergence.*

Various theorems could be formulated in this way. It suffices to state for the classical orthogonal polynomials

THEOREM III. *Take in Lagrange interpolation formula* (58) *for* x_i *the zeros of the polynomial of Jacobi, or Laguerre, or Hermite. Then correspondingly:* (J) $M(x) \to f(x)$ $(n \to \infty)$ *uniformly in any interval lying wholly inside* $(-1, 1)$, *where* $f(x)$ *satisfies Lipschitz condition of order* $> \frac{1}{2}$, *and uniformly over the whole interval* $(-1, 1)$, *provided,* $\alpha, \beta \leqq 1$ *and* $f'(x)$ *is therein continuous.*[18] (L) $M(x) \to f(x)$ *uniformly over any finite interval wholly to the right of* 0, *if* $f(x)$ *satisfies therein Lipschitz condition of order* $> \frac{1}{4}$. *If* $f(x)$ *is continuous in* $(0, \varepsilon)$ *and* $E_n \cdot n^{\alpha} \to 0$ $(n \to \infty)$, *then* $M(0) \to f(0)$.[19] (H) $M(x) \to f(x)$ *uniformly over any finite interval, where* $f(x)$ *satisfies Lipschitz condition of order* $> \frac{1}{4}$.[20]

[18] If, for instance, $\alpha \leqq 1$, $\beta > 1$, the uniform convergence holds in $(-1, 1-\varepsilon)$.

[19] Thus, for $\alpha = 1$, $M(x) \to f(x)$ uniformly over any finite interval $(0, A)$, where $f'(x)$ is continuous.

[20] (ADDED IN PROOF.) In a recent paper, „Über gewisse Polynome ...", Math. Zeitschrift, vol. 35 (June, 1932), pp. 579–602, G. Szegö discusses the convergence properties of a generalized Fejér interpolation formula for the same polynomials (J, L, H) (first part of the present paper). His method is different (no use being made of mechanical quadratures), and his $f(x)$—the function to be interpolated—is more restricted.

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DER MASSBEGRIFF IN DER THEORIE DER KONTINUIERLICHEN GRUPPEN.*

VON ALFRED HAAR.

1. Der Ausgangspunkt der Lieschen Theorie der kontinuierlichen Gruppen, die sog. Infinitesimaltransformation, wird bekanntlich mittels eines Differentiationsprozesses gewonnen; deshalb ist die Liesche Theorie in ihrer ursprünglichen Form nur auf solche Gruppen anwendbar, welche durch solche Gleichungen dargestellt sind, die die fraglichen Differentiationsbedingungen erfüllen. Dieser Theorie steht eine andere gegenüber, die von Hurwitz in einer berühmten Arbeit[1] angebahnt wurde, welche man treffend als eine Integrationstheorie der kontinuierlichen Gruppen bezeichnet hat; diese Theorie wurde insbesondere im letzten Jahrzehnt durch eine Reihe von wichtigen Arbeiten gefördert, von denen wir hier nur die schönen Arbeiten von Schur und Weyl erwähnen.

Es liegt daher der Gedanke nahe, die Frage zu untersuchen, ob man *in jeder N-gliedrigen kontinuierlichen Gruppe einen invarianten Integrationsprozeß einführen kann* (für Liesche Gruppen folgt dies unmittelbar aus einer Bemerkung von Hurwitz in der angeführten Arbeit). Diese Frage ist offenbar damit gleichwertig, ob man *in der Gruppenmannigfaltigkeit einen Inhalts- bzw. Maßbegriff einführen kann, der invariant gegenüber den Transformationen der Gruppe ist*, d. h. der zu solchen Punktmengen der Gruppenmannigfaltigkeit, die durch irgendwelche Transformationen der Gruppe ineinander übergehen, denselben Inhalt bzw. Maß zuordnet.

In der vorliegenden Arbeit soll gezeigt werden, daß bei jeder N-gliedrigen kontinuierlichen Gruppe ein solcher Inhalts- bzw. Maßbegriff tatsächlich vorhanden ist. Unsere Untersuchungen gelten sogar für noch allgemeinere kontinuierliche Gruppen; wir werden im wesentlichen nur annehmen, daß die *Gruppenmannigfaltigkeit metrisch, separabel und im Kleinen kompakt ist.* § 1 enthält die Konstruktion des Inhaltsbegriffes; im § 2 wird gezeigt, daß dieser Begriff alle wesentlichen Eigenschaften des klassischen Inhaltsbegriffes der Euklidischen Ebene besitzt (und außerdem natürlich invariant gegenüber der Gruppe ist); auf diese Resultate gestützt, wird im § 3 das Analogon des Lebesgueschen Maßbegriffes begründet. § 4 enthält einige Anwendungen, insbesondere den invarianten Integrationsprozeß und die

* Received, July 26, 1932.

[1] Göttinger Nachrichten, 1897, S. 71—90.

Übertragung derjenigen schönen Sätze, die Peter und Weyl über die Darstellungen der geschlossenen (kompakten) Gruppen bewiesen haben[2].

§ 1. Der Inhalt.

2. Die Elemente einer vorgelegten N-gliedrigen Gruppe $\mathfrak{G}$ deuten wir als Punkte einer N-dimensionalen Mannigfaltigkeit, die ebenfalls mit $\mathfrak{G}$ bezeichnet werden soll. Zwei Eigenschaften dieser Mannigfaltigkeiten werden im folgenden insbesondere herangezogen: 1° *es besitzt jeder Punkt eine Umgebung, die eineindeutig, stetig abbildbar ist auf die offene Kugel des N-dimensionalen Euklidischen Raumes;* und 2° *die Mannigfaltigkeit selbst ist durch eine höchstens abzählbare Menge solcher Umgebungen überdeckbar.* In den folgenden Untersuchungen wird (mit Rücksicht auf spätere Anwendungen) eine etwas allgemeinere Klasse von kontinuierlichen Gruppen behandelt[3]; wir werden nämlich nur voraussetzen, daß die Elemente der Gruppe $\mathfrak{G}$ Punkte eines *metrischen, separablen Raumes* sind, der außerdem *im Kleinen kompakt* ist; die letzte Eigenschaft bedeutet bekanntlich, daß jeder Punkt eine in $\mathfrak{G}$ kompakte Umgebung (d. h. eine Umgebung, deren abgeschlossene Hülle in $\mathfrak{G}$ kompakt ist) besitzt. Man beachte, daß über den Zusammenhang von $\mathfrak{G}$ keine Annahmen gemacht werden (so daß gewisse sog. gemischt-kontinuierliche Gruppen nicht ausgeschlossen werden), statt dessen soll nur angenommen werden, daß die Gruppenmannigfaltigkeit *in sich dicht* ist[4].

Die drei *Gruppenpostulate* sollen in der üblichen Form zugrunde gelegt werden:

1° eine *assoziative Verknüpfung,* vermöge deren zu zwei beliebigen Gruppenelementen A und B ihr Produkt $C = AB$ definiert ist;

2° die Existenz des *Identitätselementes* E $(AE = EA = A)$;

3° die Existenz der *Inversen* $(A^{-1}A = AA^{-1} = E)$.

Daraus folgt bereits, daß (falls A ein bestimmtes Gruppenelement ist) die Transformation $X \to XA$, die irgendeinem Element X das Element XA

[2] Die vorliegende Arbeit ist (bis auf unwesentliche Abänderungen) die Übersetzung einer der Ungarischen Akademie der Wissenschaften in ihrer Sitzung am 18. April 1932 vorgelegten, in ungarischer Sprache verfaßten Abhandlung.

[3] Über den allgemeinen Begriff der kontinuierlichen Gruppe vgl. O. Schreier, Abhandlungen aus dem Math. Seminar der Hamburg. Univ., Bd. 4 und Bd. 5; ferner E. Cartan, Mémorial des Sciences Mathématiques, Fasc. 42. Paris 1930.

[4] Diese letzte Bedingung gewährleistet erst die Kontinuität der Gruppe; sie ist für das Folgende zwar nicht unerläßlich, wir wollen sie aber annehmen, um triviale Fälle auszuschließen. Die N-gliedrige kontinuierliche Gruppe erfüllt offenbar die obigen Bedingungen. — Da in der vorliegenden Arbeit die Begriffe: kompakt, offen, abgeschlossen, stets relativ zu der Gruppenmannigfaltigkeit $\mathfrak{G}$ zu verstehen sind, so wollen wir der Kürze halber diese Bezeichnungen statt kompakt, offen, abgeschlossen *in* $\mathfrak{G}$ benutzen.

zuordnet, eine eineindeutige Abbildung der Gruppenmannigfaltigkeit auf sich ist; diese Abbildungen bilden offenbar eine zu $\mathfrak{G}$ isomorphe Gruppe.

Die *Stetigkeit* der Gruppe formuliert man, wie folgt: Wenn die Gruppenelemente $A_1, A_2, \cdots, A_n, \cdots$ gegen das Element A konvergieren, d. h. wenn die Entfernungen $A A_n$ des metrischen Raumes $\mathfrak{G}$ gegen Null streben, und in gleicher Weise die Elemente $B_1, B_2, \cdots, B_n, \cdots$ gegen B konvergieren, so sollen 1° $A_1^{-1}, A_2^{-1}, \cdots, A_n^{-1}, \cdots$ gegen A^{-1}—, und 2° $A_1 B_1, A_2 B_2, \cdots, A_n B_n, \cdots$ gegen AB konvergieren[5]. Daraus folgt, daß die obigen Abbildungen $X \to XA$ topologische Abbildungen der Gruppenmannigfaltigkeit auf sich sind[6].

Die *Kongruenz* wird folgendermaßen festgelegt: Ist $\mathfrak{M}$ eine Punktmenge (in $\mathfrak{G}$) und A ein beliebiges Gruppenelement, so bezeichnen wir mit $\mathfrak{M}A$ diejenige Punktmenge, die aus $\mathfrak{M}$ vermöge der Abbildung $X \to XA$ entsteht; d. h. ist X das allgemeine Element von $\mathfrak{M}$, so ist XA das allgemeine Element von $\mathfrak{M}A$. Die beiden Punktmengen $\mathfrak{M}$ und $\mathfrak{M}A$ sollen kongruent heißen: $\mathfrak{M} \cong \mathfrak{M}A$. Zufolge der Gruppeneigenschaften *ist die Kongruenz ein reflexiver, symmetrischer, transitiver Begriff;* wegen der Stetigkeit der Gruppe folgt, daß, *wenn* $\mathfrak{M}_1 \cong \mathfrak{M}_2$ *und* $\mathfrak{M}_1$ *offen (bzw. abgeschlossen oder kompakt) ist,* $\mathfrak{M}_2$ *ebenfalls offen (bzw. abgeschlossen oder kompakt) sein muß*[7].

3. Es sollen vorab einige einfache Tatsachen festgestellt werden, die bei der Definition des Inhaltes Verwendung finden. Es seien $\mathfrak{A}$ und $\mathfrak{B}$ offene und kompakte Teilmengen von $\mathfrak{G}$, $\overline{\mathfrak{A}}$ bzw. $\overline{\mathfrak{B}}$ ihre abgeschlossene Hüllen[8]. Ist $\overline{B}$ ein beliebiger Punkt von $\overline{\mathfrak{B}}$, so kann man in mannigfacher Weise eine der Menge $\mathfrak{A}$ kongruente Menge angeben, die $\overline{B}$ enthält (z. B. $\mathfrak{A} A^{-1} \overline{B}$, wo A irgendein Element von $\mathfrak{A}$ ist). Die abgeschlossene, kompakte Menge $\overline{\mathfrak{B}}$ ist daher in der Summe eines Systems solcher Mengen enthalten, die der offenen Menge $\mathfrak{A}$ kongruent, also ebenfalls offen sind. Die Übertragung des bekannten Borel-Heineschen Satzes auf metrische, separable Räume[9] liefert die Tatsache, daß $\overline{\mathfrak{B}}$ bereits in einer *Summe von endlich vielen dieser offenen Mengen enthalten ist;* man kann daher umsomehr $\overline{\mathfrak{B}}$ vermöge endlich vielen, der *abgeschlossenen* Menge $\overline{\mathfrak{A}}$ kongruenten Mengen überdecken. *Wir bezeichnen mit* $h(\overline{\mathfrak{B}}; \overline{\mathfrak{A}})$ *die kleinste*

[5] Vgl. Schreier a. a. O.[3] und Cartan a. a. O.[3], S. 6.

[6] Die durch die Transformationen $X \to XA$ der Gruppenmannigfaltigkeit gelieferte Darstellung der Gruppe ist bekanntlich einfach transitiv im Lieschen Sinne.

[7] Würde man die Mengen $\mathfrak{M}$ und $A\mathfrak{M}$ als kongruent bezeichnen, so erhielte man einen von dem im folgenden dargelegten verschiedenen Maßbegriff; auf den Zusammenhang dieser beiden Maßbegriffe komme ich in einer späteren Arbeit zurück.

[8] Wir bezeichnen im folgenden stets die abgeschlossene Hülle einer in $\mathfrak{G}$ offenen, kompakten Menge $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \cdots, \mathfrak{M}$ mit $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}, \overline{\mathfrak{C}}, \cdots, \overline{\mathfrak{M}}$.

[9] Vgl. F. Hausdorff, Mengenlehre, 2. Aufl., S. 130.

Anzahl der bei einer solchen Überdeckung von $\overline{\mathfrak{B}}$ *nötigen,* $\overline{\mathfrak{A}}$ *kongruenten Mengen;* d. h. $\overline{\mathfrak{B}}$ ist enthalten in der Vereinigungsmenge von $h(\overline{\mathfrak{B}}; \overline{\mathfrak{A}})$ Mengen, welche $\overline{\mathfrak{A}}$ kongruent sind, es kann aber durch eine geringere Anzahl solcher Mengen nicht überdeckt werden.

Die so definierten positiven ganzen Zahlen $h(\overline{\mathfrak{B}}; \overline{\mathfrak{A}})$ besitzen die folgenden Eigenschaften[10]:

1° $h(\overline{\mathfrak{B}}; \overline{\mathfrak{B}}) = 1$;

2° *ist* $\mathfrak{B}'$ *eine abgeschlossene Teilmenge von* $\overline{\mathfrak{B}}$ $(\overline{\mathfrak{B}'} < \overline{\mathfrak{B}})$, *so ist* $h(\overline{\mathfrak{B}'}; \overline{\mathfrak{A}}) \leqq h(\overline{\mathfrak{B}}; \overline{\mathfrak{A}})$;

3° *ist* $\overline{\mathfrak{B}_1} + \overline{\mathfrak{B}_2}$ *die Vereinigungsmenge der beiden abgeschlossenen kompakten Mengen* $\overline{\mathfrak{B}_1}$ *und* $\overline{\mathfrak{B}_2}$, *so ist*

$$h(\overline{\mathfrak{B}_1} + \overline{\mathfrak{B}_2}; \overline{\mathfrak{A}}) \leqq h(\overline{\mathfrak{B}_1}; \overline{\mathfrak{A}}) + h(\overline{\mathfrak{B}_2}; \overline{\mathfrak{A}});$$

4° *ist* $\mathfrak{B}_1 \cong \mathfrak{B}_2$, *so ist*

$$h(\overline{\mathfrak{B}_1}; \overline{\mathfrak{A}}) = h(\overline{\mathfrak{B}_2}; \overline{\mathfrak{A}});$$

5° *sind* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ *offene, kompakte Mengen, so ist offenbar*

$$h(\overline{\mathfrak{B}}; \overline{\mathfrak{A}}) \leqq h(\overline{\mathfrak{B}}; \overline{\mathfrak{C}})\, h(\overline{\mathfrak{C}}; \overline{\mathfrak{A}});$$

6° *wenn die offenen, kompakten Mengen* $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n, \cdots$ *nur einen einzigen gemeinsamen Punkt* (A) *besitzen und (für jedes* n*)* $\mathfrak{A}_{n+1}$ *Teilmenge von* $\mathfrak{A}_n$ *ist, wenn ferner die abgeschlossenen Hüllen* $\overline{\mathfrak{B}'}$ *und* $\overline{\mathfrak{B}''}$ *der beiden offenen, kompakten Mengen* $\mathfrak{B}'$ *und* $\mathfrak{B}''$ *keinen gemeinsamen Punkt besitzen, so ist für hinreichend große* n

$$h(\overline{\mathfrak{B}'} + \overline{\mathfrak{B}''}; \overline{\mathfrak{A}_n}) = h(\overline{\mathfrak{B}'}; \overline{\mathfrak{A}_n}) + h(\overline{\mathfrak{B}''}; \overline{\mathfrak{A}_n}).$$

Man zeigt diese letzte Eigenschaft, wie folgt: Wäre die Behauptung nicht richtig, so würde es, bei jedem n, eine der Menge $\overline{\mathfrak{A}_n}$ kongruente Menge — etwa $\overline{\mathfrak{A}_n}\, C_n$ — geben, welche sowohl Punkte von $\overline{\mathfrak{B}'}$, wie auch Punkte von $\overline{\mathfrak{B}''}$ enthielte. Es seien B_n' bzw. B_n'' solche Punkte von $\overline{\mathfrak{B}'}$ bzw. $\overline{\mathfrak{B}''}$ und A_n' bzw. A_n'' diejenigen Punkte von $\overline{\mathfrak{A}_n}$, die bei der Transformation $X \to X C_n$ in B_n' bzw. B_n'' übergehen: $A_n' C_n = B_n'$, $A_n'' C_n = B_n''$. Es konvergieren nun einerseits die Punktfolgen A_n' und A_n'' mit wachsendem n gegen A; andererseits kann man aber — zufolge der angenommenen Kompaktheit und Abgeschlossenheit von $\overline{\mathfrak{B}'}$ und $\overline{\mathfrak{B}''}$ — eine Teilfolge $n_1 < n_2 < \cdots < n_k < \cdots$ der natürlichen Zahlen derart herauswählen, daß jede der Punktfolgen B_{n_k}' bzw. B_{n_k}'' gegen einen Punkt B' bzw. B'' von $\overline{\mathfrak{B}'}$ bzw. $\overline{\mathfrak{B}''}$ konvergieren. Auf Grund der Stetigkeitseigenschaften der vorgelegten Gruppe würde daraus die Konvergenz der Elementenfolge $C_{n_k} = A_{n_k}'^{-1} B_{n_k}'$

[10] Die hier auftretenden Mengen $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{B}', \mathfrak{B}_1, \mathfrak{B}_2, \cdots$ sind natürlich offene, kompakte Teilmengen von $\mathfrak{G}$.

$= A_{n_k}''^{-1} B_{n_k}''$ folgen, und es würde sich als Häufungspunkt $A^{-1}B' = A^{-1}B''$ ergeben; dies ist aber unmöglich, da die beiden Mengen $\overline{\mathfrak{B}}'$ und $\overline{\mathfrak{B}}''$ keinen gemeinsamen Punkt besitzen, also B' von B'' verschieden sein muß.

4. Nach diesen Vorbereitungen wählen wir eine beliebige offene, kompakte Menge $\mathfrak{E}$ zum Ausgangspunkt der weiteren Untersuchungen, die wir als *Einheitsmenge* bezeichnen; um den Inhaltsbegriff einzuführen, könnte man außerdem noch eine willkürliche Folge von ineinander geschachtelten offenen, kompakten Mengen $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n, \cdots$ $(\mathfrak{A}_{n+1} < \mathfrak{A}_n)$ heranziehen, die nur einen einzigen gemeinsamen Punkt haben. Der Einfachheit halber betrachten wir die *abzählbar vielen offenen Kugeln um einen festen Punkt A*, deren Durchmesser gleich $\frac{1}{n}$ ist[11], und bezeichnen diese offenen, kompakten Mengen mit $\mathfrak{K}_n$ $(n = 1, 2, 3, \cdots)$; $\overline{\mathfrak{K}}_n$ sei — wie stets — die abgeschlossene Hülle von $\mathfrak{K}_n$.

Es sei nun $\mathfrak{B}$ irgendeine offene, kompakte Menge; wir betrachten die Zahlenfolge:

$$l_n(\overline{\mathfrak{B}}) = \frac{h(\overline{\mathfrak{B}}; \overline{\mathfrak{K}}_n)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_n)} \qquad (n = 1, 2, 3, \cdots); \tag{1}$$

man erkennt unschwer, daß *diese* (von positiven rationalen Zahlen gebildete) *Folge beschränkt und ihre untere Grenze von Null verschieden ist.* In der Tat folgt aus der Eigenschaft 5° der Funktion $h(\overline{\mathfrak{B}}; \overline{\mathfrak{A}})$ einerseits (für $\mathfrak{A} = \mathfrak{K}_n$, $\mathfrak{C} = \mathfrak{E}$)

$$l_n(\overline{\mathfrak{B}}) \leqq h(\overline{\mathfrak{B}}; \overline{\mathfrak{E}}) \qquad (n = 1, 2, 3, \cdots),$$

und andererseits, wenn man in der fraglichen Ungleichung $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ bzw. durch $\mathfrak{K}_n$, $\mathfrak{E}$, $\mathfrak{B}$ ersetzt,

$$l_n(\overline{\mathfrak{B}}) \geqq \frac{1}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{B}})} \qquad (n = 1, 2, 3, \cdots).$$

Man kann daher aus der Folge (1) eine gegen einen positiven Grenzwert konvergierende Teilfolge

$$l_{n_1}(\overline{\mathfrak{B}}),\ l_{n_2}(\overline{\mathfrak{B}}),\ \cdots,\ l_{n_k}(\overline{\mathfrak{B}}),\ \cdots$$

herausgreifen; ist $\overline{\mathfrak{B}}' \cong \overline{\mathfrak{B}}$, so strebt (wegen der Eigenschaft 4°) $l_{n_k}(\overline{\mathfrak{B}}')$ natürlich gegen denselben Grenzwert. Mittels des wohlbekannten Diagonalverfahrens kann man ferner zu irgendwelchen vorgelegten, abzählbar unendlich vielen, offenen, kompakten Mengen

$$\mathfrak{B}_1,\ \mathfrak{B}_2,\ \cdots,\ \mathfrak{B}_m,\ \cdots$$

eine Teilfolge $n_1 < n_2 < \cdots < n_k < \cdots$ der natürlichen Zahlen derart bestimmen, daß für alle $m = 1, 2, 3, \cdots$ die Grenzwerte

[11] Wir beschränken uns natürlich auf solche Kugeln um A, die in einer kompakten Umgebung dieses Punktes liegen.

$$\lim_{k=\infty} l_{n_k}(\overline{\mathfrak{B}}_m) = \lim_{k=\infty} \frac{h(\overline{\mathfrak{B}}_m; \overline{\mathfrak{K}}_{n_k})}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_{n_k})} \tag{2}$$

existieren. Die Mengen $\mathfrak{B}_m$ wählen wir nun folgendermaßen: Wir wählen zuerst (was wegen der angenommenen Separabilität von $\mathfrak{G}$ möglich ist) eine abzählbare, in $\mathfrak{G}$ überall dichte Punktmenge $A_1, A_2, \cdots, A_\nu, \cdots$ und betrachten jede *in $\mathfrak{G}$ kompakte Kugel mit rationalem Durchmesser, deren Mittelpunkt einer dieser Punkte A_ν ist.* Die Gesamtheit dieser Kugeln ist offenbar abzählbar. *Bildet man in jeder möglichen Weise die Vereinigungsmenge von endlich vielen Kugeln dieser Art, so erhält man ebenfalls abzählbar viele Punktmengen;* jede dieser Punktmengen ist offenbar kompakt und offen. Wir bezeichnen im folgenden die auf diese Weise gebildeten Mengen in irgendeiner Reihenfolge mit $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_m, \cdots$ und denken uns die Teilfolge $n_1 < n_2 < \cdots < n_k < \cdots$ derart bestimmt, daß für diese Mengen die Grenzwerte (2) existieren.

Der Einfachheit halber modifizieren wir in geringem Maße die bisherigen Bezeichnungen, indem wir die Kugeln

$$\mathfrak{K}_{n_1}, \mathfrak{K}_{n_2}, \cdots, \mathfrak{K}_{n_k}, \cdots$$

(um A) von nun an mit

$$\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_k, \cdots$$

bezeichnen; die übrigen Kugeln um A lassen wir fortan außer acht. In dieser Schreibweise *existiert der Grenzwert*

$$\lim_{k=\infty} l_k(\overline{\mathfrak{B}}_m) = \lim_{k=\infty} \frac{h(\overline{\mathfrak{B}}_m; \overline{\mathfrak{K}}_k)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_k)},$$

falls $\mathfrak{B}_m$ eine der soeben festgelegten offenen, kompakten Mengen oder eine Menge, die einer solchen kongruent ist, bedeutet.

5. Den Grenzwert der Folge

$$l_k(\overline{\mathfrak{C}}) = \frac{h(\overline{\mathfrak{C}}; \overline{\mathfrak{K}}_k)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_k)} \qquad (k = 1, 2, 3, \cdots) \tag{2'}$$

werden wir (falls er existiert) als *Inhalt* der offenen, kompakten Menge $\mathfrak{C}$ bezeichnen, wenn noch außerdem die Menge $\mathfrak{C}$ eine gewisse weitere Eigenschaft besitzt, die wir vorab angeben wollen. Wir führen zu diesem Ende den Begriff der *Nullmenge* ein. Eine Teilmenge $\mathfrak{N}$ von $\mathfrak{G}$ soll *Nullmenge heißen, wenn es zu jeder beliebig kleinen positiven Größe ε eine $\mathfrak{N}$ enthaltende offene, kompakte Menge $\mathfrak{H}_\varepsilon$ gibt, für die*

$$\limsup_{k=\infty} \frac{h(\overline{\mathfrak{H}}_\varepsilon; \overline{\mathfrak{K}}_k)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_k)} \leqq \varepsilon \tag{3}$$

ausfällt. Aus dieser Definition folgt unmittelbar, daß *die Teilmengen einer Nullmenge, die Vereinigungsmenge endlich vieler Nullmengen und die einer*

Nullmenge kongruenten Mengen wiederum Nullmengen sind. Wir werden sehen, daß der Grenzwert der Folge (2′) stets vorhanden ist, falls der *Rand* der offenen, kompakten Menge $\mathfrak{C}$ (d.h. die Menge $\overline{\mathfrak{C}} - \mathfrak{C}$) eine Nullmenge ist.

Wir müssen zuerst zeigen, daß es solche Mengen $\mathfrak{C}$ überhaupt gibt. Zu diesem Ende betrachten wir irgendeinen Punkt A_ν der in 4. benutzten, überall dichten Menge $A_1, A_2, \cdots, A_\nu, \cdots$, und es sei $\mathfrak{K}(r)$ die Kugel um A_ν mit dem Radius r. (Wir beschränken uns natürlich wiederum auf solche Werte von r, für die $\mathfrak{K}(r)$ noch kompakt in $\mathfrak{G}$ ist.) Ist r eine rationale Zahl, so ist $\mathfrak{K}(r)$ eine derjenigen Mengen, die wir oben mit $\mathfrak{B}_m$ bezeichnet haben; daher konvergiert für solche r die Folge

$$(2'') \qquad l_n(\overline{\mathfrak{K}}(r)) = \frac{h(\overline{\mathfrak{K}}(r); \overline{\mathfrak{K}}_n)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_n)} \qquad (n = 1, 2, 3, \cdots)$$

gegen einen Grenzwert. Andererseits ist für alle n (vgl. die Eigenschaft 2° von h) die Funktion

$$\lambda_n(r) = l_n(\overline{\mathfrak{K}}(r))$$

eine nicht abnehmende Funktion von r. Da die Folge dieser monotonen Funktionen $\lambda_n(r)$ bei jedem rationalen r konvergiert, so strebt diese Funktionenfolge — nach einem wohlbekannten Satze — mit etwaiger Ausnahme von abzählbar vielen Argumentenwerten, für jedes r gegen eine Funktion $\lambda(r)$. Da die durch diesen Grenzübergang gewonnene Funktion $\lambda(r)$ ebenfalls nicht abnehmend ist, so ist sie, wiederum mit etwaiger Ausnahme von abzählbar vielen Stellen, stetig. Indem wir diese beiden abzählbaren Mengen der Veränderlichen r ausschließen, erreichen wir, daß *für jeden anderen Wert von r einerseits der Limes*

$$\lim_{n=\infty} \lambda_n(r) = \lambda(r)$$

existiert und behaupten andererseits, daß *der Rand jeder Kugel* $\mathfrak{K}(r)$ (um A), *deren Radius r nicht ausgeschlossen wurde, eine Nullmenge ist.* In der Tat kann man, wegen der angenommenen Stetigkeit von $\lambda(r)$, an einer solchen Stelle r zu jeder positiven Zahl ε solche Werte r', r'', r''' von r finden, welche ebenfalls nicht ausgeschlossen wurden, der Ungleichung $r' < r'' < r < r'''$ genügen und so beschaffen sind, daß

$$0 \leqq \lambda(r''') - \lambda(r') < \varepsilon$$

wird. Nun haben einerseits die abgeschlossenen Mengen $\overline{\mathfrak{K}}(r''') - \mathfrak{K}(r'')$ und $\overline{\mathfrak{K}}(r')$ keinen gemeinsamen Punkt; daher ist (zufolge der Eigenschaft 6° von h) für hinreichend große Werte von n

$$h((\overline{\mathfrak{K}}(r''') - \mathfrak{K}(r'')) + \overline{\mathfrak{K}}(r'); \overline{\mathfrak{K}}_n) = h(\overline{\mathfrak{K}}(r''') - \mathfrak{K}(r''); \overline{\mathfrak{K}}_n) + h(\overline{\mathfrak{K}}(r'); \overline{\mathfrak{K}}_n);$$

andererseits ist aber, da $\overline{\mathfrak{K}}(r''') - \mathfrak{K}(r'') + \overline{\mathfrak{K}}(r')$ Teilmenge von $\overline{\mathfrak{K}}(r''')$ ist (vgl. die Eigenschaft 2° von h), für jedes n

$$h(\overline{\mathfrak{K}}(r''') - \mathfrak{K}(r'') + \overline{\mathfrak{K}}(r'); \overline{\mathfrak{K}}_n) \leqq h(\overline{\mathfrak{K}}(r'''); \overline{\mathfrak{K}}_n).$$

Daher gilt für hinreichend große Werte von n die Ungleichung

$$h(\overline{\mathfrak{K}}(r''') - \mathfrak{K}(r''); \overline{\mathfrak{K}}_n) \leqq h(\overline{\mathfrak{K}}(r'''); \overline{\mathfrak{K}}_n) - h(\overline{\mathfrak{K}}(r'); \overline{\mathfrak{K}}_n),$$

d. h. es ist

$$\limsup_{n=\infty} \frac{h(\overline{\mathfrak{K}}(r''') - \mathfrak{K}(r''); \overline{\mathfrak{K}}_n)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_n)} \leqq \lambda(r''') - \lambda(r') < \varepsilon.$$

Da die offene, kompakte Menge $\overline{\mathfrak{K}}(r''') - \mathfrak{K}(r'')$ den Rand von $\mathfrak{K}(r)$ enthält, so zeigt diese Ungleichung, daß der Rand der Kugel $\mathfrak{K}(r)$ in der Tat eine Nullmenge ist.

Damit ist nicht nur gezeigt, daß *es offene, kompakte Mengen gibt, deren Rand eine Nullmenge ist,* sondern auch, daß *jede Kugel, deren Mittelpunkt in der betrachteten überall dichten Menge* $A_1, A_2, \cdots, A_\nu, \cdots$ *liegt — mit Ausnahme von höchstens abzählbar unendlich vielen — diese Eigenschaft besitzt,* ferner, daß *für jede solche Kugel der Grenzwert der Folge* (2'') *existiert.*

6. Zu weiteren Mengen dieser Art gelangen wir folgendermaßen: Es sei $\mathfrak{A}$ eine beliebige offene, kompakte Menge und $\overline{\mathfrak{A}}'$ irgendeine abgeschlossene Teilmenge von $\mathfrak{A}$. Der Rand von $\mathfrak{A}$ und $\overline{\mathfrak{A}}'$ haben keinen gemeinsamen Punkt; es sei η die Entfernung der Menge $\overline{\mathfrak{A}}'$ vom *Rande* von $\mathfrak{A}$ (die ebenfalls abgeschlossen ist). Wir betrachten diejenigen Punkte der (in $\mathfrak{G}$) überall dichten Menge $A_1, A_2, \cdots, A_\nu, \cdots$, die der Menge $\overline{\mathfrak{A}}'$ angehören; unser soeben erhaltenes Resultat lehrt, daß man um jeden dieser Punkte eine Kugel angeben kann, deren Radius (etwa) zwischen $\frac{\eta}{2}$ und $\frac{3\eta}{4}$ liegt, und deren Rand eine Nullmenge ist. Die Gesamtheit dieser Kugeln überdeckt natürlich die abgeschlossene Menge $\overline{\mathfrak{A}}'$[12]; man kann daher endlich viele dieser Kugeln derart herausgreifen, daß $\overline{\mathfrak{A}}'$ bereits in ihrer Vereinigungsmenge enthalten ist. Diese Vereinigungsmenge $\mathfrak{C}$ ist eine offene, kompakte Menge, deren Rand wiederum eine Nullmenge ist, da der Rand von $\mathfrak{C}$ nur aus solchen Punkten besteht, die dem Rande einer der endlich vielen Kugeln angehört, aus welchen $\mathfrak{C}$ aufgebaut ist. Da ferner diese Kugeln sowie ihre Ränder Teilmengen der offenen Menge $\mathfrak{A}$ sind, so folgt, daß nicht nur $\mathfrak{C}$, sondern auch die abgeschlossene Hülle $\overline{\mathfrak{C}}$ Teilmenge von $\mathfrak{A}$ ist.

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[12] da in beliebiger Nähe jedes Punktes von $\overline{\mathfrak{A}}'$ ein Punkt der Punktmenge $A_1, A_2, \cdots, A_\nu, \cdots$ liegt.

Wiederholt man diese Schlußweise mit der Modifikation, daß man um jeden innerhalb $\overline{\mathfrak{A}}'$ liegenden Punkt der überall dichten Menge $A_1, A_2, \cdots, A_\nu, \cdots$ nicht eine Kugel, deren Rand eine Nullmenge ist, sondern eine Kugel mit rationalem Radius $\left(\text{zwischen } \frac{\eta}{2} \text{ und } \frac{3\eta}{4}\right)$ konstruiert, so führt dieses Verfahren offenbar zu einer solchen offenen Menge, die wir in 4. mit $\mathfrak{B}_m$ bezeichnet haben, für die also

$$\lim_{n=\infty} l_n(\overline{\mathfrak{B}}_m)$$

existiert; und es ist außerdem — wie oben — $\overline{\mathfrak{A}}' < \mathfrak{B}_m$, $\overline{\mathfrak{B}}_m < \mathfrak{A}$.

Wir haben somit das folgende Resultat gewonnen:

I. *Es sei $\mathfrak{A}$ eine offene, kompakte Menge und $\overline{\mathfrak{A}}'$ eine abgeschlossene Teilmenge von $\mathfrak{A}$; dann existiert*

einerseits eine $\overline{\mathfrak{A}}'$ enthaltende offene Menge $\mathfrak{C}$, die samt ihrem Rande Teilmenge von $\mathfrak{A}$ ist ($\overline{\mathfrak{A}}' < \mathfrak{C}$, $\overline{\mathfrak{C}} < \mathfrak{A}$) und deren Rand eine Nullmenge ist; und

andererseits eine $\overline{\mathfrak{A}}'$ enthaltende offene Menge $\mathfrak{B}$, die samt ihrem Rande Teilmenge von $\mathfrak{A}$ ist ($\overline{\mathfrak{A}}' < \mathfrak{B}$, $\overline{\mathfrak{B}} < \mathfrak{A}$) und für die der Grenzwert $\lim_{n=\infty} l_n(\overline{\mathfrak{B}})$ vorhanden ist.

Der wesentliche Inhalt des ersten Teiles dieses Satzes ist die Tatsache, daß die durch Nullmengen berandeten offenen Mengen in gewissem Sinne *überall dicht* liegen, indem jede offene Menge durch solche Mengen beliebig genau angenähert werden kann.

Wir werden zeigen, daß, wenn *der Rand einer offenen, kompakten Menge $\mathfrak{C}$ eine Nullmenge ist, so existiert der Grenzwert*

$$\lim_{n=\infty} l_n(\overline{\mathfrak{C}}) = \lim_{n=\infty} \frac{h(\overline{\mathfrak{C}}; \overline{\mathfrak{K}}_n)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_n)} = I(\mathfrak{C}),$$

und dies wird die Definition des *Inhaltes solcher Mengen* sein.

§ 2. Eigenschaften des Inhaltes.

7. Den soeben ausgesprochenen Satz beweisen wir, wie folgt:

Es sei $\mathfrak{C}$ eine offene, kompakte Menge, deren Rand eine Nullmenge ist. Nach der Definition der Nullmengen kann man zu jeder noch so kleinen positiven Größe ε eine den Rand von $\mathfrak{C}$ enthaltende offene, kompakte Menge $\mathfrak{H}_\varepsilon$ finden, für die

$$\limsup_{n=\infty} \frac{h(\overline{\mathfrak{H}}_\varepsilon; \overline{\mathfrak{K}}_n)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_n)} \leqq \varepsilon$$

ausfällt. Den Durchschnitt von $\mathfrak{C}$ und $\mathfrak{H}_\varepsilon$ bezeichnen wir mit $\mathfrak{C}''$ ($\mathfrak{C}'' = \mathfrak{C}\mathfrak{H}_\varepsilon$), und es sei $\overline{\mathfrak{C}}' = \mathfrak{C} - \mathfrak{C}''$; $\mathfrak{C}''$ ist eine offene, $\overline{\mathfrak{C}}'$ eine ab-

geschlossene Menge. Da $\mathfrak{C}''$ Teilmenge von $\mathfrak{H}_\varepsilon$ ist, so folgt (vgl. Eigenschaft 2° von h)

$$\limsup_{n=\infty} \frac{h(\overline{\mathfrak{C}}''; \overline{\mathfrak{K}}_n)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_n)} \leqq \varepsilon. \tag{4}$$

Nach dem zweiten Teile unseres Satzes I kann man eine offene, $\overline{\mathfrak{C}}'$ enthaltende Menge $\mathfrak{B}$ angeben, deren abgeschlossene Hülle Teilmenge von $\mathfrak{C}$ ist, und für die der Grenzwert $\lim_{n=\infty} l_n(\mathfrak{B})$ existiert. Daher ist für jedes n

$$h(\overline{\mathfrak{C}}; \overline{\mathfrak{K}}_n) \geqq h(\overline{\mathfrak{B}}; \overline{\mathfrak{K}}_n);$$

da ferner die Vereinigungsmenge von $\mathfrak{B}$ ($> \overline{\mathfrak{C}}'$) und $\mathfrak{C}''$ die ursprüngliche offene Menge $\mathfrak{C}$ ist, so ist (vgl. Eigenschaft 3° von h)

$$h(\overline{\mathfrak{C}}; \overline{\mathfrak{K}}_n) \leqq h(\overline{\mathfrak{B}}; \overline{\mathfrak{K}}_n) + h(\overline{\mathfrak{C}}''; \overline{\mathfrak{K}}_n).$$

Die erste dieser Ungleichungen liefert die Beziehung

$$\liminf_{n=\infty} \frac{h(\overline{\mathfrak{C}}; \overline{\mathfrak{K}}_n)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_n)} \geqq \lim_{n=\infty} l_n(\overline{\mathfrak{B}});$$

die zweite aber — mit Rücksicht auf (4) — die Beziehung

$$\limsup_{n=\infty} \frac{h(\overline{\mathfrak{C}}; \overline{\mathfrak{K}}_n)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_n)} \leqq \lim_{n=\infty} l_n(\overline{\mathfrak{B}}) + \varepsilon. \tag{5}$$

Aus der durch Subtraktion gewonnenen Ungleichung

$$\limsup_{n=\infty} \frac{h(\overline{\mathfrak{C}}; \overline{\mathfrak{K}}_n)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_n)} - \liminf_{n=\infty} \frac{h(\overline{\mathfrak{C}}; \overline{\mathfrak{K}}_n)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_n)} < \varepsilon$$

schließt man unmittelbar (da ε beliebig klein sein kann) auf die Existenz des Grenzwertes

$$\lim_{n=\infty} \frac{h(\overline{\mathfrak{C}}; \overline{\mathfrak{K}}_n)}{h(\overline{\mathfrak{E}}; \overline{\mathfrak{K}}_n)} = \lim_{n=\infty} l_n(\overline{\mathfrak{C}}) = I(\mathfrak{C}),$$

womit die Behauptung bewiesen ist.

Diese positive Größe $I(\mathfrak{C})$ soll als Inhalt der offenen, kompakten Menge $\mathfrak{C}$ (deren Rand eine Nullmenge ist) bezeichnet werden. Dieselbe Zahl könnte man auch als Inhalt von $\overline{\mathfrak{C}}$ und jeder Teilmenge von $\overline{\mathfrak{C}}$, die $\mathfrak{C}$ enthält, definieren. Unser Beweis zeigt, daß $I(\mathfrak{C})$ die obere Grenze der durch die Formel $\lim_{n=\infty} l_n(\overline{\mathfrak{B}})$ gelieferten Größen ist, falls $\mathfrak{B}$ eine beliebige solche Teilmenge von $\mathfrak{C}$ ist, für die dieser Grenzwert existiert.

Es soll noch hervorgehoben werden, daß *der Inhaltsbegriff nicht für alle offenen Mengen eingeführt ist, für die der fragliche Grenzwert existiert,*

sondern nur für diejenigen, deren Rand eine Nullmenge ist[13]; man wird notgedrungen zu dieser Einschränkung geführt, da widrigenfalls für den Inhalt die klassischen Eigenschaften dieser Begriffsbildung nicht gültig wären.

8. Wir wollen zeigen, daß der soeben eingeführte Inhaltsbegriff in der Gruppenmannigfaltigkeit von derselben Art ist, wie der Jordan-Cantorsche Inhalt in der Ebene. Die fundamentalen Eigenschaften von $J(\mathfrak{C})$ sind die folgenden:

1° *Der Inhalt ist stets größer als Null;* dies ist eine unmittelbare Folge der in 4. bewiesenen Tatsache, daß im Falle irgendeiner offenen, kompakten Menge die untere Grenze der Folge $l_n(\overline{\mathfrak{C}})$ von Null verschieden ist.

2° *Sind* $\mathfrak{C}_1$ *und* $\mathfrak{C}_2$ *kongruente Mengen* ($\mathfrak{C}_1 \cong \mathfrak{C}_2$) *und besitzt* $\mathfrak{C}_1$ *einen Inhalt* (d. h. ist der Rand der offenen, kompakten Menge $\mathfrak{C}_1$ eine Nullmenge), *so besitzt auch* $\mathfrak{C}_2$ *einen Inhalt, und es ist* $J(\mathfrak{C}_1) = J(\mathfrak{C}_2)$. Denn aus der für $\mathfrak{C}_1$ gemachten Annahme folgt (vgl. 2. und 5.), daß auch $\mathfrak{C}_2$ offen, kompakt und der Rand von $\mathfrak{C}_2$ eine Nullmenge ist; daher existiert $J(\mathfrak{C}_2)$. Die Eigenschaft 4° von h liefert unmittelbar $J(\mathfrak{C}_1) = J(\mathfrak{C}_2)$.

3° *Besitzt jede der offenen, kompakten Mengen* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n$ *einen Inhalt, so besitzt der Durchschnitt* ($\mathfrak{C}_1\,\mathfrak{C}_2 \cdots \mathfrak{C}_n$) *und die Vereinigungsmenge* ($\mathfrak{C}_1 + \mathfrak{C}_2 + \cdots + \mathfrak{C}_n$) *ebenfalls einen Inhalt;* da nämlich jeder Randpunkt dieser Mengen dem Rande einer der Mengen $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n$ angehört, so sind die Ränder von $\mathfrak{C}_1\,\mathfrak{C}_2 \cdots \mathfrak{C}_n$ und $\mathfrak{C}_1 + \mathfrak{C}_2 + \cdots + \mathfrak{C}_n$ (vgl. 5.) sicherlich Nullmengen. Die Eigenschaft 3° von h lehrt übrigens, daß $J(\mathfrak{C}_1 + \mathfrak{C}_2) \leqq J(\mathfrak{C}_1) + J(\mathfrak{C}_2)$ ist.

4° *Besitzt jede der elementenfremden (offenen, kompakten) Mengen* $\mathfrak{C}_1, \mathfrak{C}_2$ *einen Inhalt, so ist*

$$J(\mathfrak{C}_1 + \mathfrak{C}_2) = J(\mathfrak{C}_1) + J(\mathfrak{C}_2).$$

Beweis. Da der Rand von $\mathfrak{C}_1$ eine Nullmenge ist, so kann man — wie oben — zu jedem positiven ε eine offene Menge $\mathfrak{B}$ finden, deren abgeschlossene Hülle $\overline{\mathfrak{B}}$ Teilmenge von $\mathfrak{C}_1$ ist und so beschaffen ist, daß einerseits der Grenzwert $\lim_{n=\infty} l_n(\overline{\mathfrak{B}})$ existiert, und andererseits (vgl. die Ungleichung (5))

$$J(\mathfrak{C}_1) \leqq \lim_{n=\infty} l_n(\overline{\mathfrak{B}}) + \varepsilon \tag{5'}$$

ist. Da die abgeschlossenen Mengen $\overline{\mathfrak{B}}$ und $\overline{\mathfrak{C}}_2$ keinen gemeinsamen Punkt besitzen, so ist (vgl. Eigenschaft 6° von h) für hinreichend große n

$$h(\overline{\mathfrak{B}} + \overline{\mathfrak{C}}_2; \overline{\mathfrak{K}}_n) = h(\overline{\mathfrak{B}}; \overline{\mathfrak{K}}_n) + h(\overline{\mathfrak{C}}_2; \overline{\mathfrak{K}}_n).$$

[13] Es ist daher wohl möglich, daß die in 4. eingeführten Mengen $\mathfrak{B}_m$ (oder die Einheitsmenge $\mathfrak{E}$), die uns zum Ausgangspunkt unserer Inhaltsdefinition dienten, keinen Inhalt besitzen, obwohl der Grenzwert $\lim_{n=\infty} l_n(\overline{\mathfrak{B}}_m)$ sicherlich vorhanden ist.

Da aber $\mathfrak{B}+\mathfrak{C}_2$ Teilmenge von $\mathfrak{C}_1+\mathfrak{C}_2$ ist, so gilt (für alle n)

$$h(\overline{\mathfrak{B}}+\overline{\mathfrak{C}}_2;\ \overline{\mathfrak{K}}_n) \leqq h(\overline{\mathfrak{C}}_1+\overline{\mathfrak{C}}_2;\ \overline{\mathfrak{K}}_n);$$

daher ist für hinreichend große n

$$h(\overline{\mathfrak{C}}_1;\ \overline{\mathfrak{K}}_n)+h(\overline{\mathfrak{C}}_2;\ \overline{\mathfrak{K}}_n) \geqq h(\overline{\mathfrak{C}}_1+\overline{\mathfrak{C}}_2;\ \overline{\mathfrak{K}}_n) \geqq h(\overline{\mathfrak{B}};\ \overline{\mathfrak{K}}_n)+h(\overline{\mathfrak{C}}_2;\ \overline{\mathfrak{K}}_n).$$

Daraus folgt aber unmittelbar — mit Rücksicht auf (5′) —

$$I(\mathfrak{C}_1)+I(\mathfrak{C}_2) \geqq I(\mathfrak{C}_1+\mathfrak{C}_2) \geqq I(\mathfrak{C}_1)+I(\mathfrak{C}_2)-\varepsilon;$$

d. h. unsere obige Behauptung.

Auf Grund dieser Eigenschaft ergibt sich unschwer der folgende allgemeinere Satz:

4° *Besitzt jede der offenen Mengen* $\mathfrak{C}_1$ *und* $\mathfrak{C}_2$ *einen Inhalt, so gilt die Beziehung*

$$I(\mathfrak{C}_1)+I(\mathfrak{C}_2) = I(\mathfrak{C}_1+\mathfrak{C}_2)+I(\mathfrak{C}_1\,\mathfrak{C}_2);$$

dabei bezeichnet (wie oben) $\mathfrak{C}_1+\mathfrak{C}_2$ bzw. $\mathfrak{C}_1\,\mathfrak{C}_2$ die Summe bzw. den Durchschnitt der beiden Mengen $\mathfrak{C}_1$ und $\mathfrak{C}_2$.

Beweis. Es sei $\mathfrak{C}'$ die *offene* Menge, gebildet von den *inneren* Punkten von $\mathfrak{C}_1-\mathfrak{C}_1\,\mathfrak{C}_2$ (offener Kern); der Rand von $\mathfrak{C}'$ ist eine Nullmenge, da er aus Randpunkten von $\mathfrak{C}_1$ und $\mathfrak{C}_2$ besteht. $\mathfrak{C}'$ und $\mathfrak{C}_2$ besitzen keinen gemeinsamen Punkt; daher ist

$$I(\mathfrak{C}'+\mathfrak{C}_2) = I(\mathfrak{C}')+I(\mathfrak{C}_2); \tag{6}$$

in derselben Weise erkennt man die Richtigkeit der Beziehung

$$I(\mathfrak{C}'+\mathfrak{C}_1\,\mathfrak{C}_2) = I(\mathfrak{C}')+I(\mathfrak{C}_1\,\mathfrak{C}_2). \tag{7}$$

Nun ist die offene Menge $\mathfrak{C}'+\mathfrak{C}_2$ zwar nicht identisch mit $\mathfrak{C}_1+\mathfrak{C}_2$, da $\mathfrak{C}'+\mathfrak{C}_2$ diejenigen Punkte von $\mathfrak{C}_1$, die dem Rande von $\mathfrak{C}_2$ angehören, nicht enthält; da aber jeder solche Punkt Häufungspunkt von $\mathfrak{C}_2$ ist, so folgt, daß die abgeschlossenen Hüllen von $\mathfrak{C}_1+\mathfrak{C}_2$ und $\mathfrak{C}'+\mathfrak{C}_2$ übereinstimmen. Daher ist

$$h(\overline{\mathfrak{C}_1+\mathfrak{C}_2};\ \overline{\mathfrak{K}}_n) = h(\overline{\mathfrak{C}'+\mathfrak{C}_2};\ \overline{\mathfrak{K}}_n)$$

und folglich auch

$$I(\mathfrak{C}_1+\mathfrak{C}_2) = I(\mathfrak{C}'+\mathfrak{C}_2).$$

Ersetzt man in dieser Überlegung $\mathfrak{C}_2$ durch $\mathfrak{C}_1\,\mathfrak{C}_2$, so erhält man in gleicher Weise

$$I(\mathfrak{C}'+\mathfrak{C}_1\,\mathfrak{C}_2) = I(\mathfrak{C}_1+\mathfrak{C}_1\,\mathfrak{C}_2) = I(\mathfrak{C}_1).$$

Man erhält daher aus den Beziehungen (6) und (7) die Gleichungen

$$I(\mathfrak{C}_1+\mathfrak{C}_2) = I(\mathfrak{C}')+I(\mathfrak{C}_2), \qquad I(\mathfrak{C}_1) = I(\mathfrak{C}')+I(\mathfrak{C}_1\,\mathfrak{C}_2),$$

woraus durch Subtraktion unsere Behauptung folgt.

5° *Besitzt jede der unendlich vielen offenen Mengen* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n, \cdots$ *einen Inhalt, und ist* (für jedes n) $\mathfrak{C}_{n+1}$ *Teilmenge von* $\mathfrak{C}_n$, *so ist, falls der Durchschnitt dieser Mengen* $\mathfrak{C}_n$ *aus einem einzigen Punkt* C *besteht,*

$$\lim_{n=\infty} I(\mathfrak{C}_n) = 0.$$

Beweis. Wir wählen innerhalb einer beliebigen offenen Menge $\mathfrak{H}$, die einen Inhalt besitzt, k *verschiedene* Punkte $H^{(1)}, H^{(2)}, \cdots, H^{(k)}$. Vermöge der Abbildungen

$$X \to X(C^{-1} H^{(p)}) \qquad (p = 1, 2, \cdots, k) \tag{8}$$

entstehen aus der Menge $\mathfrak{C}_n$ die ihr kongruenten Mengen $\mathfrak{C}_n^{(1)}, \mathfrak{C}_n^{(2)}, \cdots, \mathfrak{C}_n^{(k)}$, die bzw. die Punkte $H^{(1)}, H^{(2)}, \cdots, H^{(k)}$ enthalten. Man erkennt unschwer, daß die so erhaltenen Mengen $\mathfrak{C}_n^{(p)}$ $(p = 1, 2, \cdots, k)$ für hinreichend große n elementenfremd sind. In der Tat, es würden im entgegengesetzten Falle für beliebig große n solche Punkte C_n' und C_n'' von $\mathfrak{C}_n$ geben, die durch zwei verschiedene Abbildungen (8) in denselben Punkt überführt werden:

$$C_n' C^{-1} H^{(p)} = C_n'' C^{-1} H^{(q)} \qquad (p \leqq q);$$

da C_n' und C_n'' mit wachsendem n gegen C streben, so würde dies der Verschiedenheit der $H^{(1)}, H^{(2)}, \cdots, H^{(k)}$ widersprechen. Folglich sind für hinreichend große n alle $\mathfrak{C}_n^{(p)}$ elementenfremd und in $\mathfrak{H}$ enthalten; daher ist

$$k I(\mathfrak{C}_n) = \sum_{p=1}^{k} I(\mathfrak{C}_n^{(p)}) = I(\mathfrak{C}_n^{(1)} + \mathfrak{C}_n^{(2)} + \cdots + \mathfrak{C}_n^{(k)}) \leqq I(\mathfrak{H});$$

d. h. es ist für hinreichend große n

$$I(\mathfrak{C}_n) \leqq \frac{1}{k} I(\mathfrak{H}),$$

woraus die Behauptung unmittelbar folgt.

Man könnte diese Eigenschaften des Inhaltes auch für nicht offene Mengen beweisen; wir unterlassen dies in Hinsicht darauf, daß diese Tatsachen in den allgemeinen Eigenschaften des Maßbegriffes (den wir im nächsten Paragraphen begründen) enthalten sind.

Zusammenfassend sind wir zu dem folgenden Satze gelangt:

II. *In jeder* N*-gliedrigen kontinuierlichen Gruppe* $\mathfrak{G}$ *kann man einen Inhaltsbegriff definieren, vermöge dessen zu gewissen offenen, kompakten Mengen* $\mathfrak{C}$ *eine positive Größe* $I(\mathfrak{C})$ *zugeordnet ist; dieser Begriff besitzt dieselben Eigenschaften, wie der Jordan-Cantorsche Inhalt der gewöhnlichen Geometrie, d. h. es gelten, falls jede der Mengen* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n, \cdots$ *einen Inhalt besitzt, die folgenden Tatsachen:*

ist $\mathfrak{C}_1$ *kongruent* $\mathfrak{C}_2$, *so ist* $I(\mathfrak{C}_1) = I(\mathfrak{C}_2)$;
ist $\mathfrak{C}_1$ *Teilmenge von* $\mathfrak{C}_2$, *so ist* $I(\mathfrak{C}_1) \leqq I(\mathfrak{C}_2)$;
es gilt das Gesetz der Additivität:

$$I(\mathfrak{C}_1) + I(\mathfrak{C}_2) = I(\mathfrak{C}_1 + \mathfrak{C}_2) + I(\mathfrak{C}_1 \mathfrak{C}_2);$$

ist (für jedes n) $\mathfrak{C}_{n+1}$ *in* $\mathfrak{C}_n$ *enthalten, so ist* $\lim_{n=\infty} I(\mathfrak{C}_n) = 0$, *falls die Mengen* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n, \cdots$ *nur einen einzigen gemeinsamen Punkt haben;*

ist $\mathfrak{A}$ *eine beliebige offene, kompakte Menge und* $\overline{\mathfrak{A}}'$ *eine beliebige abgeschlossene Teilmenge von* $\mathfrak{A}$, *so existiert eine* $\overline{\mathfrak{A}}'$ *enthaltende Menge* $\mathfrak{C}$, *deren abgeschlossene Hülle Teilmenge von* $\mathfrak{A}$ *ist und die einen Inhalt besitzt*[14];

besitzt die offene, kompakte Menge $\mathfrak{C}$ *einen Inhalt, so existiert zu jeder positiven Größe* ε *eine offene Menge* $\mathfrak{C}'$, *deren abgeschlossene Hülle in* $\mathfrak{C}$ *enthalten ist* ($\overline{\mathfrak{C}}' < \mathfrak{C}$) *von der Beschaffenheit, daß* $I(\mathfrak{C}') > I(\mathfrak{C}) - \varepsilon$, *sowie eine die abgeschlossene Hülle von* $\mathfrak{C}$ *enthaltende offene, kompakte Menge* $\mathfrak{C}''$ ($\mathfrak{C}'' > \overline{\mathfrak{C}}$) *von der Beschaffenheit, daß* $I(\mathfrak{C}'') < I(\mathfrak{C}) + \varepsilon$ *ist*[15].

Unser Satz bleibt richtig, wenn die zugrunde gelegte kontinuierliche Gruppe $\mathfrak{G}$ *nicht N-gliedrig, jedoch so beschaffen ist, daß ihre Elemente einen metrischen, in sich dichten, separablen und im Kleinen kompakten Raum bilden.*

§ 3. Das Analogon des Lebesgueschen Maßes.

9. Auf Grund des Vorangehenden kann man ohne Schwierigkeiten in der Gruppenmannigfaltigkeit einen Maßbegriff einführen, der alle wesentlichen Eigenschaften des Lebesgueschen Maßes besitzt und außerdem invariant gegenüber Kongruenz ist[16]. In voller Analogie zu dem klassischen Verfahren Lebesgues definieren wir zu diesem Ende zuerst das *äußere Maß* einer in $\mathfrak{G}$ kompakten Punktmenge $\mathfrak{M}$; um die fragliche Analogie klar hervortreten zu lassen, wollen wir im folgenden eine *offene, kompakte Menge* $\mathfrak{C}$, *die einen Inhalt im obigen Sinne besitzt, als Intervall bezeichnen.*

[14] Vgl. Satz I, S. 155.

[15] Die Richtigkeit dieser letzten Eigenschaft erkennt man leicht folgendermaßen: Ist $\mathfrak{H}$ eine den Rand von $\mathfrak{C}$ enthaltende offene Menge, für die die Ungleichung (3) statthat, so sei *erstens* $\mathfrak{A} = \mathfrak{C}$, $\overline{\mathfrak{A}}' = \mathfrak{C} - \mathfrak{C}\mathfrak{H}$; ist $\mathfrak{C}'$ eine offene, kompakte Menge, die einen Inhalt besitzt von der Beschaffenheit, daß $\overline{\mathfrak{A}}' < \mathfrak{C}'$ und $\overline{\mathfrak{C}}' < \mathfrak{A} = \mathfrak{C}$ ist (vgl. Satz I), so ist offenbar

$$I(\mathfrak{C}') > I(\mathfrak{C}) - \varepsilon.$$

Setzt man *zweitens* $\mathfrak{A} = \mathfrak{C} + \mathfrak{C}\mathfrak{H}$, $\overline{\mathfrak{A}}' = \overline{\mathfrak{C}}$, so gelangt man in derselben Weise zu einer Menge $\mathfrak{C}''$ ($> \overline{\mathfrak{C}}$), für die

$$I(\mathfrak{C}'') < I(\mathfrak{C}) + \varepsilon$$

ist.

[16] In meiner in [2] erwähnten ungarischen Arbeit habe ich, um den Maßbegriff einzuführen, das bekannte de la Vallée-Poussinsche Verfahren eingeschlagen.

Da die abgeschlossene Hülle $\bar{\mathfrak{M}}$ ebenfalls kompakt ist, so kann man (auf Grund des Borel-Heineschen Satzes) $\mathfrak{M}$ jedenfalls in ein System von endlich vielen Intervallen einschließen. Wir betrachten irgendein abzählbares System von Intervallen $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n, \cdots$ von der Beschaffenheit, daß jedes Element von $\mathfrak{M}$ in der Vereinigungsmenge dieser Intervalle enthalten ist, und definieren das äußere Maß $m_a(\mathfrak{M})$ als untere Grenze der so erhaltenen Summen $I(\mathfrak{C}_1)+I(\mathfrak{C}_2)+I(\mathfrak{C}_3)+\cdots$; $m_a(\mathfrak{M})$ ist stets endlich und $\geqq 0$; ist $\mathfrak{M}'$ in der kompakten Menge $\mathfrak{M}$ enthalten, so ist $m_a(\mathfrak{M}') \leqq m_a(\mathfrak{M})$; für zwei beliebige kompakte Mengen $\mathfrak{M}_1$ und $\mathfrak{M}_2$ gilt

$$m_a(\mathfrak{M}_1+\mathfrak{M}_2) \leqq m_a(\mathfrak{M}_1)+m_a(\mathfrak{M}_2);$$

sind die abgeschlossenen Hüllen $\bar{\mathfrak{M}}_1$ und $\bar{\mathfrak{M}}_2$ elementenfremd, so ist offenbar

$$m_a(\mathfrak{M}_1+\mathfrak{M}_2) = m_a(\mathfrak{M}_1)+m_a(\mathfrak{M}_2).$$

Man erkennt ferner unschwer, daß das *äußere Maß eines Intervalls* $\mathfrak{C}$ *mit dessen Inhalt übereinstimmt.* In der Tat folgt die Ungleichung $m_a(\mathfrak{C}) \leqq I(\mathfrak{C})$ unmittelbar aus der Definition von m_a; andererseits kann man aber zu jedem positiven ε ein Intervall $\mathfrak{C}'$, dessen Hülle $\bar{\mathfrak{C}}'$ in $\mathfrak{C}$ enthalten ist, derart angeben, daß $I(\mathfrak{C}') > I(\mathfrak{C}) - \varepsilon$ wird. Ist nun $\mathfrak{C}$ (und damit auch $\bar{\mathfrak{C}}'$) in der Vereinigungsmenge der Intervalle $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n, \cdots$ enthalten, so kann man die abgeschlossene Menge $\bar{\mathfrak{C}}'$ bereits durch endlich viele dieser Intervalle überdecken; daher ist jedenfalls $I(\mathfrak{C}')$ nicht größer als die Summe der Inhalte dieser endlich vielen Intervallen und also um so mehr

$$I(\mathfrak{C}) - \varepsilon < I(\mathfrak{C}') \leqq I(\mathfrak{C}_1)+I(\mathfrak{C}_2)+\cdots+I(\mathfrak{C}_n)+\cdots,$$

woraus unmittelbar

$$m_a(\mathfrak{C}) \geqq I(\mathfrak{C}) - \varepsilon,$$

also die Behauptung folgt. In gleicher Weise erkennt man, daß auch *für die abgeschlossene Hülle* $\bar{\mathfrak{C}}$ *eines Intervalls die Beziehung* $m_a(\bar{\mathfrak{C}}) = I(\mathfrak{C})$ gilt. Ist nämlich (für ein beliebig kleines positives ε) $\mathfrak{C}''$ ein $\bar{\mathfrak{C}}$ enthaltendes Intervall von der Beschaffenheit, daß $I(\mathfrak{C}'') < I(\mathfrak{C}) + \varepsilon$ ist, so ist

$$m_a(\bar{\mathfrak{C}}) \leqq m_a(\mathfrak{C}'') = I(\mathfrak{C}'') \leqq I(\mathfrak{C}) + \varepsilon,$$

woraus mit Rücksicht auf

$$m_a(\bar{\mathfrak{C}}) \geqq m_a(\mathfrak{C}) = I(\mathfrak{C})$$

die Beziehung $m_a(\bar{\mathfrak{C}}) = I(\mathfrak{C})$ folgt[17].

[17] Daraus folgt auch, daß, falls $\mathfrak{A}$ eine das Intervall $\mathfrak{C}$ enthaltende Punktmenge bedeutet, die andererseits in der abgeschlossenen Hülle $\bar{\mathfrak{C}}$ dieses Intervalls enthalten ist, die Beziehung $m_a(\mathfrak{A}) = m_a(\mathfrak{C}) = m_a(\bar{\mathfrak{C}})$ gilt.

Ist endlich das Intervallsystem

$$\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n, \cdots,$$

in dessen Vereinigungsmenge $\mathfrak{M}$ enthalten ist, so beschaffen, daß

$$m_a(\mathfrak{M}) \geqq I(\mathfrak{C}_1)+I(\mathfrak{C}_2)+\cdots+I(\mathfrak{C}_n)+\cdots+\varepsilon$$

ausfällt, so gelangt man, indem man von jedem $\mathfrak{C}_n$ diejenigen Punkte wegläßt, welche in einem der vorangehenden Intervalle $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_{n-1}$ enthalten sind und sodann die offenen Kerne der so erhaltenen Mengen bildet, zu *elementenfremden* Intervallen

$$\mathfrak{C}_1 = \mathfrak{C}_1', \ \mathfrak{C}_2', \ \mathfrak{C}_3', \cdots, \mathfrak{C}_n', \cdots$$

von der Beschaffenheit, daß jeder Punkt von $\mathfrak{M}$ innerhalb oder auf dem Rande eines dieser Intervalle liegt ($\mathfrak{M} < \overline{\mathfrak{C}}_1'+\overline{\mathfrak{C}}_2'+\cdots+\overline{\mathfrak{C}}_n'+\cdots$) und es ist offenbar

$$m_a(\mathfrak{M}) \geqq I(\mathfrak{C}_1')+I(\mathfrak{C}_2')+\cdots+I(\mathfrak{C}_n')+\cdots+\varepsilon.$$

Andererseits ist aber

$$m_a(\mathfrak{M}) \leqq I(\mathfrak{C}_1')+I(\mathfrak{C}_2')+\cdots+I(\mathfrak{C}_n')+\cdots,$$

da man zu jeder der abgeschlossenen Hüllen $\overline{\mathfrak{C}}_n'$ ein sie enthaltendes Intervall $\mathfrak{C}_n'' > \overline{\mathfrak{C}}_n'$ angeben kann, dessen Inhalt beliebig wenig von dem Inhalt von $\mathfrak{C}_n'$ abweicht, und da diese Intervalle $\mathfrak{C}_n''$ eine Überdeckung von $\mathfrak{M}$ bewirken. *Das äußere Maß $m_a(\mathfrak{M})$ ist daher auch als die untere Grenze der Summen $I(\mathfrak{C}_1')+I(\mathfrak{C}_2')+\cdots+I(\mathfrak{C}_n')+\cdots$ definierbar, wobei die Intervalle $\mathfrak{C}_n'$ paarweise elementenfremd und so beschaffen sind, daß $\mathfrak{M}$ in der Vereinigung ihrer abgeschlossenen Hüllen enthalten ist.*

10. Der Begriff der *Meßbarkeit* wird genau wie in der Lebesgueschen Theorie definiert. Ist $\mathfrak{C}$ ein $\mathfrak{M}$ enthaltendes Intervall, so wird $I(\mathfrak{C})-m_a(\mathfrak{C}-\mathfrak{M})$ als *inneres Maß* $m_i(\mathfrak{M})$ von $\mathfrak{M}$ (in bezug auf $\mathfrak{C}$) bezeichnet. $\mathfrak{M}$ ist *meßbar*, wenn $m_a(\mathfrak{M}) = m_i(\mathfrak{M})$ ist. *Die Meßbarkeit von $\mathfrak{M}$ ist von der Wahl des $\mathfrak{M}$ umfassenden Intervalls $\mathfrak{C}$ unabhängig.* Man erkennt dies, indem man zeigt, daß, falls $\mathfrak{C}'$ ein $\overline{\mathfrak{C}}$ enthaltendes Intervall ist ($\mathfrak{C}' > \overline{\mathfrak{C}} > \mathfrak{M}$), jede der Gleichungen

$$\begin{aligned} m_a(\mathfrak{C}-\mathfrak{M})+m_a(\mathfrak{M}) &= I(\mathfrak{C}), \\ m_a(\mathfrak{C}'-\mathfrak{M})+m_a(\mathfrak{M}) &= I(\mathfrak{C}') \end{aligned}$$

eine Konsequenz der anderen ist[18]. In der Tat, ist $\mathfrak{C}''$ ein Intervall, dessen abgeschlossene Hülle $\overline{\mathfrak{C}}''$ in $\mathfrak{C}'-\overline{\mathfrak{C}}$ enthalten ist und dessen Inhalt

$$I(\mathfrak{C}'') \geqq I(\mathfrak{C}')-I(\mathfrak{C})-\varepsilon$$

[18] da der allgemeine Fall leicht auf diesen zurückführbar ist.

ist, wo ε eine beliebig kleine positive Größe bedeutet, so findet man unmittelbar

$$m_a(\mathfrak{C}' - \mathfrak{M}) \geqq m_a(\bar{\mathfrak{C}}'' + (\mathfrak{C} - \mathfrak{M})) = m_a(\bar{\mathfrak{C}}'') + m_a(\mathfrak{C} - \mathfrak{M}),$$

da $\bar{\mathfrak{C}}''$ und die abgeschlossene Hülle von $\mathfrak{C} - \mathfrak{M}$ elementenfremd sind, und folglich

$$m_a(\mathfrak{C}' - \mathfrak{M}) - m_a(\mathfrak{C} - \mathfrak{M}) \geqq I(\mathfrak{C}'') \geqq I(\mathfrak{C}') - I(\mathfrak{C}) - \varepsilon. \tag{9}$$

Andererseits ist aber

$$m_a(\mathfrak{C}' - \mathfrak{M}) \leqq m_a(\mathfrak{C}' - \mathfrak{C}) + m_a(\mathfrak{C} - \mathfrak{M}),$$

und da die Menge $\mathfrak{C}' - \mathfrak{C}$ das Intervall $\mathfrak{C}' - \bar{\mathfrak{C}}$ enthält, in der abgeschlossenen Hülle dieses Intervalls aber enthalten ist, so ist

$$m_a(\mathfrak{C}' - \mathfrak{C}) = I(\mathfrak{C}' - \bar{\mathfrak{C}}) = I(\mathfrak{C}') - I(\mathfrak{C}),$$

und folglich

$$m_a(\mathfrak{C}' - \mathfrak{M}) - m_a(\mathfrak{C} - \mathfrak{M}) \leqq I(\mathfrak{C}') - I(\mathfrak{C}). \tag{9'}$$

Der Vergleich der Ungleichungen (9) und (9′) liefert die Behauptung[19].

11. Es soll noch gezeigt werden, daß dieser Maßbegriff dieselben Eigenschaften besitzt, wie das Lebesguesche Maß; zu diesem Ende leiten wir für die Meßbarkeit einer Menge $\mathfrak{M}$ (von $\mathfrak{G}$) eine notwendige und hinreichende Bedingung ab, die wörtlich mit dem in der Lebesgueschen Theorie auftretenden entsprechenden Kriterium übereinstimmt.

Wir sagen, daß ein System von Intervallen die Menge $\mathfrak{M}$ überdeckt, falls jeder Punkt von $\mathfrak{M}$ innerhalb oder auf dem Rande eines dieser Intervalle liegt. Das fragliche Kriterium lautet dann, wie folgt:

Damit die in dem Intervall $\mathfrak{C}$ enthaltende Menge $\mathfrak{M}$ *meßbar sei, ist notwendig und hinreichend,* daß zu jeder noch so kleinen positiven Größe ε ein abzählbares System von paarweise elementenfremden Intervallen $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n, \cdots$ existiert, deren Gesamtheit $\mathfrak{M}$ überdeckt, sowie ein ebenfalls abzählbares System von elementenfremden Intervallen $\mathfrak{C}_1', \mathfrak{C}_2', \cdots, \mathfrak{C}_n', \cdots$, das $\mathfrak{C} - \mathfrak{M}$ überdeckt, von der folgenden Beschaffenheit: In jedem der Intervalle

$$\mathfrak{C}_1, \mathfrak{C}_1', \mathfrak{C}_2, \mathfrak{C}_2', \cdots, \mathfrak{C}_n, \mathfrak{C}_n', \cdots \tag{10}$$

bilden diejenigen Punkte, die bereits in einem vorangehenden Intervall enthalten sind, wiederum ein Intervall; für das so erhaltene Intervallsystem

$$\mathfrak{C}_1'', \mathfrak{C}_2'', \cdots, \mathfrak{C}_p'', \cdots$$

sei die Summe $\sum_{p=1}^{\infty} m(\mathfrak{C}_p'') = \sum_{p=1}^{\infty} I(\mathfrak{C}_p'')$ kleiner als ε.

[19] Ist $\mathfrak{C}'$ ein das Intervall $\mathfrak{C}$ enthaltendes Intervall $(\mathfrak{C}' > \mathfrak{C})$, so zeigt die Beziehung

$$I(\mathfrak{C}) + I(\mathfrak{C}' - \bar{\mathfrak{C}}) = m_a(\mathfrak{C}) + m_a(\mathfrak{C}' - \mathfrak{C}) = I(\mathfrak{C}'),$$

daß jedes Intervall meßbar ist.

Man erkennt leicht, daß diese Bedingung hinreichend ist; denn wenn man von jedem der Intervalle (10) die fraglichen Punkte fortläßt, so gelangt man zu einer Überdeckung von $\mathfrak{C}$ durch elementenfremde Intervalle; daher ist

$$(11) \qquad \sum_{n=1}^{\infty} (m(\mathfrak{C}_n) + m(\mathfrak{C}'_n)) = m(\mathfrak{C}) + \sum_{p=1}^{\infty} m(\mathfrak{C}''_p) < m(\mathfrak{C}) + \varepsilon.$$

Da ferner

$$m_a(\mathfrak{M}) \leqq \sum_{n=1}^{\infty} m(\mathfrak{C}_n), \qquad m_a(\mathfrak{C} - \mathfrak{M}) \leqq \sum_{n=1}^{\infty} m(\mathfrak{C}'_n)$$

ist, so ergibt sich

$$m_a(\mathfrak{M}) + m_a(\mathfrak{C} - \mathfrak{M}) \leqq m(\mathfrak{C}) + \varepsilon,$$

woraus die Meßbarkeit von $\mathfrak{M}$ unmittelbar folgt. Umgekehrt, ist $\mathfrak{M}$ meßbar, so kann man $\mathfrak{M}$ und $\mathfrak{C} - \mathfrak{M}$ durch ein System $\mathfrak{C}_n$ bzw. $\mathfrak{C}'_n$ von elementenfremden Intervallen derart überdecken, daß (für beliebig kleines ε)

$$\sum_{n=1}^{\infty} m(\mathfrak{C}_n) \leqq m_a(\mathfrak{M}) + \frac{\varepsilon}{2}, \qquad \sum_{n=1}^{\infty} m(\mathfrak{C}'_n) \leqq m_a(\mathfrak{C} - \mathfrak{M}) + \frac{\varepsilon}{2},$$

also

$$\sum_{n=1}^{\infty} m(\mathfrak{C}_n) + \sum_{n=1}^{\infty} m(\mathfrak{C}'_n) \leqq m_a(\mathfrak{M}) + m_a(\mathfrak{C} - \mathfrak{M}) + \varepsilon = m(\mathfrak{C}) + \varepsilon$$

wird. Wenn man aus diesen Intervallen $\mathfrak{C}_1, \mathfrak{C}'_1, \mathfrak{C}_2, \mathfrak{C}'_2, \cdots, \mathfrak{C}_n, \mathfrak{C}'_n, \cdots$ in der obigen Weise die Intervalle $\mathfrak{C}''_p$ bildet, so liefert (11) unmittelbar $\sum_{p=1}^{\infty} m(\mathfrak{C}''_p) \leqq \varepsilon$.

Im Besitze dieses Kriteriums kann man die Beweise über die Grundeigenschaften des Lebesgueschen Maßes, wie man diese etwa in dem bekannten Werke von de la Vallée-Poussin findet[20], wörtlich auf unseren Fall übertragen; es ergeben sich insbesondere die bekannten Tatsachen über die Additivität des Maßbegriffes.

12. J. v. Neumann und F. Rieß (denen ich ein vorläufiges Manuskript der vorangehenden Untersuchungen übergeben habe) verdanke ich die interessante Bemerkung, daß man den Maßbegriff mit Hilfe der Carathéodoryschen Theorie der Maßfunktionen direkt aus den Ergebnissen des § 1 herleiten kann, ohne den Inhaltsbegriff zu entwickeln. Man kann dabei — nach einer brieflichen Mitteilung von Herrn J. v. Neumann — wie folgt verfahren:

Es sei wiederum $\overline{\mathfrak{B}}$ irgendeine der in 4. eingeführten abgeschlossenen Mengen, für die also der Grenzwert

$$\lim_{n=\infty} l_n(\overline{\mathfrak{B}}) = l(\overline{\mathfrak{B}})$$

[20] Cours d'Analyse, 2. Aufl., Bd. I, S. 244—249.

existiert. Für eine beliebige offene kompakte Menge $\mathfrak{A}$ bezeichnen wir mit $\mu(\mathfrak{A})$ die *obere Grenze* aller Größen $l(\overline{\mathfrak{B}})$, welche solchen Mengen $\overline{\mathfrak{B}}$ entsprechen, die in $\mathfrak{A}$ enthalten sind. Man zeigt unschwer, daß für zwei offene Mengen $\mathfrak{A}_1$, $\mathfrak{A}_2$ die Beziehung

$$\mu(\mathfrak{A}_1+\mathfrak{A}_2) \leqq \mu(\mathfrak{A}_1)+\mu(\mathfrak{A}_2) \tag{12}$$

statthat. Denn ist $\overline{\mathfrak{B}}<\mathfrak{A}_1+\mathfrak{A}_2$, so gibt es solche abgeschlossene Mengen $\overline{\mathfrak{A}}_1'$ und $\overline{\mathfrak{A}}_2'$, daß $\overline{\mathfrak{A}}_1'<\mathfrak{A}_1$, $\overline{\mathfrak{A}}_2'<\mathfrak{A}_2$ und $\overline{\mathfrak{B}}<\overline{\mathfrak{A}}_1'+\overline{\mathfrak{A}}_2'$ ist. Auf Grund des Satzes I kann man andererseits zwei abgeschlossene Mengen $\overline{\mathfrak{B}}_1$, $\overline{\mathfrak{B}}_2$ (für die $l(\overline{\mathfrak{B}}_1)$ und $l(\overline{\mathfrak{B}}_2)$ existiert) derart angeben, daß $\overline{\mathfrak{A}}_1'<\overline{\mathfrak{B}}_1<\mathfrak{A}_1$, $\overline{\mathfrak{A}}_2'<\overline{\mathfrak{B}}_2<\mathfrak{A}_2$ ist; daher ist in der Tat

$$l(\overline{\mathfrak{B}}) \leqq l(\overline{\mathfrak{B}}_1+\overline{\mathfrak{B}}_2) \leqq l(\overline{\mathfrak{B}}_1)+l(\overline{\mathfrak{B}}_2) \leqq \mu(\mathfrak{A}_1)+\mu(\mathfrak{A}_2),$$

woraus (12) unmittelbar folgt. Man erkennt auch — auf Grund der Eigenschaft 6° von $\mathfrak{H}$ — daß, falls $\mathfrak{A}_1$ und $\mathfrak{A}_2$ elementenfremd sind,

$$\mu(\mathfrak{A}_1+\mathfrak{A}_2) = \mu(\mathfrak{A}_1)+\mu(\mathfrak{A}_2) \tag{12'}$$

ist. Wenn ferner die offene kompakte Menge $\mathfrak{A}$ die Vereinigungsmenge der Mengen $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n, \cdots$ ist, wobei (für alle n) $\mathfrak{A}_n$ in $\mathfrak{A}_{n+1}$ enthalten ist, so ist

$$\lim_{n=\infty} \mu(\mathfrak{A}_n) = \mu(\mathfrak{A}). \tag{12''}$$

In der Tat, da aus $\mathfrak{A}'<\mathfrak{A}''$ die Ungleichung $\mu(\mathfrak{A}') \leqq \mu(\mathfrak{A}'')$ folgt, so existiert der in (12″) linksstehende Grenzwert, und man hat nur zu zeigen, daß er nicht kleiner als $\mu(\mathfrak{A})$ sein kann. Wäre dies nämlich der Fall, so könnte man eine solche Teilmenge $\overline{\mathfrak{B}}$ von $\mathfrak{A}$ angeben, daß die Ungleichung $l(\overline{\mathfrak{B}})>\mu(\mathfrak{A}_n)$ für alle n richtig wäre. Daraus folgt, daß diese Menge $\mathfrak{B}$ nicht in $\mathfrak{A}_n$ enthalten, also $\overline{\mathfrak{B}}-\mathfrak{A}_n\overline{\mathfrak{B}}$ nicht leer ist. Da andererseits $\overline{\mathfrak{B}}-\mathfrak{A}_{n+1}\overline{\mathfrak{B}}$ offenbar Teilmenge von $\overline{\mathfrak{B}}-\mathfrak{A}_n\overline{\mathfrak{B}}$ ist, so würde daraus folgen, daß die unendlichvielen abgeschlossenen Mengen $\overline{\mathfrak{B}}-\mathfrak{A}_n\overline{\mathfrak{B}}$ $(n=1,2,3,\cdots)$ einen nicht leeren Durchschnitt besitzen, im Widerspruch mit der Tatsache, daß $\overline{\mathfrak{B}}-\mathfrak{A}\overline{\mathfrak{B}}$ (wegen $\overline{\mathfrak{B}}<\mathfrak{A}$) leer ist.

Der Vergleich von (12) und (12″) (wenn man noch $\mathfrak{A}_n$ durch $\mathfrak{A}_1+\mathfrak{A}_2+\cdots+\mathfrak{A}_n$ ersetzt) führt zu der Ungleichung

$$\mu(\mathfrak{A}_1+\mathfrak{A}_2+\cdots) \leqq \mu(\mathfrak{A}_1)+\mu(\mathfrak{A}_2)+\cdots; \tag{12'''}$$

und man erkennt unmittelbar, daß, falls $\mathfrak{A}_1 \cong \mathfrak{A}_2$ ist, $\mu(\mathfrak{A}_1)=\mu(\mathfrak{A}_2)$ gilt.

Ist endlich $\mathfrak{M}$ eine beliebige kompakte Menge, so setze man $\mu(\mathfrak{M})$ gleich der unteren Grenze aller Größen $\mu(\mathfrak{A})$, wobei $\mathfrak{A}$ irgendeine offene, kompakte Menge bedeutet, welche $\mathfrak{M}$ enthält. Die Beziehungen (12)—(12‴)

zeigen, daß die so definierte Maßfunktion μ alle Bedingungen erfüllt, welche in der Carathéodoryschen Theorie der Maßfunktionen auftreten[21]. Daher führt dieses μ zu einem Maßbegriff, dem alle Eigenschaften des klassischen Lebesgueschen Maßes zukommen und außerdem gegenüber Kongruenz invariant ist.

§ 4. Anwendungen.

13. Als erste Anwendung weisen wir auf die Tatsache hin, daß bei Gruppen von der behandelten Art zwei meßbare Mengen $\mathfrak{M}$ und $\mathfrak{M}'$ (beispielsweise zwei offene, kompakte Mengen) nur dann *„zerlegungsgleich"* sein können, falls sie dasselbe Maß besitzen; $\mathfrak{M}$ und $\mathfrak{M}'$ werden bekanntlich als zerlegungsgleich bezeichnet, wenn man jede dieser Mengen in elementenfremde Teilmengen

$$\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 + \cdots, \qquad \mathfrak{M}' = \mathfrak{M}_1' + \mathfrak{M}_2' + \cdots$$

derart zerlegen kann, daß (für jedes n) $\mathfrak{M}_n \cong \mathfrak{M}_n'$ ist. Dies ist eine unmittelbare Folge der Invarianz des Maßbegriffes gegenüber Kongruenz, d. h. gegenüber den Transformationen $X \to XA$ der Gruppenmannigfaltigkeit.

14. Auf Grund des Inhalts- bzw. Maßbegriffes kann man in wohlbekannter Weise einen *invarianten Integrationsprozeß* auf der Gruppenmannigfaltigkeit einführen; man gelangt zu dem Analogon des Riemannschen bzw. des Lebesgueschen Integrals, je nachdem man den Inhalt oder das Maß zugrunde legt. In der letzteren Auffassung heiße eine in unserer Gruppenmannigfaltigkeit $\mathfrak{G}$ definierte *reelle Funktion* $f(X)$ *meßbar*, wenn die Punktmenge $\mathfrak{M}_{ab}$, in der $a \leqq f(X) \leqq b$ ausfällt, für alle reellen Größenpaare a, b meßbar ist. Man kann dann analog wie in der Lebesgueschen Theorie zu jeder in einer meßbaren Punktmenge $\mathfrak{M}$ definierten meßbaren Funktion $f(X)$ die wohlbekannten Summen einführen und das Integral

$$\int_{\mathfrak{M}} f(X)\, dX$$

als Grenzwert dieser Summen definieren; ist $f(X)$ meßbar und beschränkt, so folgt die Existenz des fraglichen Grenzwertes wörtlich, wie in den klassischen Fällen. Aus der Invarianz des Maßbegriffes gegenüber Kongruenz schließt man unmittelbar, daß für jedes Gruppenelement A die Beziehung

$$\int_{\mathfrak{M}A} f(XA)\, dX = \int_{\mathfrak{M}} f(X)\, dX$$

gilt, die die *Invarianz dieses Integrationsprozesses* ausdrückt[22].

[21] C. Carathéodory, Vorlesungen über reelle Funktionen, S. 238 und S. 258.

[22] $\mathfrak{M}A$ bedeutet dabei — wie stets — die Punktmenge, die durch die Transformation $X \to XA$ aus $\mathfrak{M}$ entsteht.

Ist $f(X)$ eine beliebige, in der Gruppenmannigfaltigkeit $\mathfrak{G}$ definierte reelle oder komplexwertige Funktion von der Beschaffenheit, daß die Integrale

$$\int_{\mathfrak{G}} f(X)\,dX \quad \text{und} \quad \int_{\mathfrak{G}} |f(X)|^2\,dX$$

existieren, so gilt daher (für jedes Gruppenelement A) die Beziehung

$$(13)\quad \int_{\mathfrak{G}} f(XA)\,dX = \int_{\mathfrak{G}} f(X)\,dX, \qquad \int_{\mathfrak{G}} |f(XA)|^2\,dX = \int_{\mathfrak{G}} |f(X)|^2\,dX;$$

daher ist die Funktionaloperation O_A, die der Funktion $f(X)$ die Funktion $f(XA)$ zuordnet,

$$O_A(f(X)) = f(XA)$$

eine *orthogonale lineare Transformation.* Führt man in wohlbekannter Weise auf $\mathfrak{G}$ ein (vollständiges) orthogonales Funktionensystem ein, so gelangt man, indem man die Beziehungen zwischen den Fourierschen Koeffizienten der Funktionen $f(X)$ und $f(XA)$ aufstellt, zu *einem System von unendlichen orthogonalen Matrizen, das eine treu isomorphe Darstellung der vorgelegten Gruppe $\mathfrak{G}$ liefert.*

15. Ist die Gruppenmannigfaltigkeit $\mathfrak{G}$ kompakt, so kann man ohne Schwierigkeiten die schöne Theorie von F. Peter und H. Weyl[23] über die Darstellung der geschlossenen Lieschen Gruppen auf den vorliegenden Fall übertragen, da in diesen Untersuchungen lediglich nur der invariante Integrationsprozeß benutzt wird. Um dies kurz anzudeuten, nehmen wir an, daß die Gruppenmannigfaltigkeit $\mathfrak{G}$ (die natürlich metrisch und separabel vorausgesetzt wird) nicht nur im Kleinen kompakt, sondern kompakt ist, und bezeichnen mit $f(X)$ irgendeine in $\mathfrak{G}$ definierte stetige (reelle) Funktion[24] (Gruppenzahl). Wir führen — wie in der angeführten Arbeit — die durch das Integral

$$F(XY^{-1}) = \int_{\mathfrak{G}} f(XT^{-1})\,f(YT^{-1})\,dT$$

dargestellte stetige Funktion ein, die offenbar nur von der Verbindung XY^{-1} der Veränderlichen abhängt, da — mit Rücksicht auf (13) — die Beziehung

$$\int_{\mathfrak{G}} f(XT^{-1})\,f(YT^{-1})\,dT = \int_{\mathfrak{G}} f(X(TY)^{-1})\,f(Y(TY)^{-1})\,dT$$
$$= \int_{\mathfrak{G}} f(XY^{-1}\,T^{-1})\,f(T^{-1})\,dT$$

[23] Mathematische Annalen, Bd. 97, S. 737—755.

[24] Wir beschränken uns der Einfachheit halber auf reelle Funktionen, da es uns nur darauf ankommt, zu zeigen, daß die Peter-Weylsche Theorie auf den vorliegenden Fall übertragbar ist.

gilt. Um die Eigenwerte und die Eigenfunktionen des offenbar symmetrischen Kernes $F(XY^{-1})$ zu konstruieren, kann man etwa — wie in der Arbeit von Peter und Weyl — die Iterierten

$$F_1 = F, \quad F_{\nu+1}(XY^{-1}) = \int_{\mathfrak{G}} F_\nu(XT^{-1})\, F(TY^{-1})\, dT \qquad (\nu = 1, 2, 3, \cdots)$$

und deren Spuren σ_ν bilden; die Quotienten $\sigma_\nu : \sigma_{\nu-1}$ streben gegen den größten Eigenwert λ, wie in E. Schmidts berühmter Dissertation gelehrt wird. Die Anwendung der bekannten Sätze über symmetrische beschränkte Funktionaloperationen auf diejenige Operation, die einer beliebigen (quadratisch integrierbaren) Funktion $\varphi(X)$ die Funktion $\int_{\mathfrak{G}} F(XY^{-1})\, \varphi(Y)\, dY$ zuordnet, ergibt ebenfalls die Existenz der Eigenwerte des betrachteten Kerns.

Es sei λ ein ν-facher Eigenwert von $F(XY^{-1})$; die zugehörigen Eigenfunktionen, d. h. die sämtlichen Lösungen der Integralgleichung

$$\varphi(X) = \lambda \int_{\mathfrak{G}} F(XY^{-1})\, \varphi(Y)\, dY$$

können bekanntlich in der Form

$$\varphi(X) = c_1 \varphi_1(X) + c_2 \varphi_2(X) + \cdots + c_\nu \varphi_\nu(X)$$

dargestellt werden, wobei die c beliebige Konstanten sind, $\varphi_1(X)$, $\varphi_2(X)$, $\cdots$, $\varphi_\nu(X)$ aber ein (in $\mathfrak{G}$) normiertes Orthogonalsystem bildet:

$$\int_{\mathfrak{G}} \varphi_p(X)\, \varphi_q(X)\, dX = \delta_{pq} = \begin{matrix} 1 \text{ für } p = q \\ 0 \text{ für } p \gtrless q \end{matrix} \qquad (p, q = 1, 2, \cdots, \nu).$$

Man erkennt unmittelbar, daß (für alle p und für jedes Gruppenelement A) auch $\varphi_p(XA^{-1})$ eine zu λ gehörende Eigenfunktion ist, da (mit Rücksicht auf (13))

$$\varphi_p(XA^{-1}) = \lambda \int_{\mathfrak{G}} F(XA^{-1}Y^{-1})\varphi_p(Y)dY = \lambda \int_{\mathfrak{G}} F(XA^{-1}(YA^{-1})^{-1})\varphi_p(YA^{-1})dY$$
$$= \lambda \int_{\mathfrak{G}} F(XY^{-1})\, \varphi(YA^{-1})\, dY$$

ist. Daher ist

$$\varphi_p(XA^{-1}) = c_{p1}(A)\, \varphi_1(X) + c_{p2}(A)\, \varphi_2(X) + \cdots + c_{p\nu}(A)\, \varphi_\nu(X),$$

wobei die

$$c_{pq}(A) = \int_{\mathfrak{G}} \varphi_p(XA^{-1})\, \varphi_q(X)\, dX \qquad (p, q = 1, 2, \cdots, \nu)$$

in $\mathfrak{G}$ definierte stetige Funktionen sind. Aus der Orthogonalität der $\varphi_p(X)$ ergibt sich unmittelbar

$$\int_{\mathfrak{G}} \varphi_p(XA^{-1})\,\varphi_q(XA^{-1})\,dX = \delta_{pq};$$

daher ist die Matrix ν-ter Ordnung

$$C(A) = (c_{pq}(A)) \qquad (p, q = 1, 2, \cdots, \nu)$$

eine orthogonale Matrix; da ferner

$$\varphi_p(X(AB)^{-1}) = \varphi_p(XB^{-1}A^{-1}) = \sum_{r=1}^{\nu} c_{pr}(A)\,\varphi_r(XB^{-1})$$

$$= \sum_{r,q=1}^{\nu} c_{pr}(A)\,c_{rq}(B)\,\varphi_q(X) = \sum_{q=1}^{\nu} c_{pq}(AB)\,\varphi_q(X)$$

ist, so folgt die Matrizenrelation

$$C(A)\,C(B) = C(AB),$$

d. h., *es liefern die (endlichen) orthogonalen Matrizen $C(A)$ eine Darstellung der vorgelegten kompakten Gruppe.*

Die so erhaltenen Darstellungen sind im allgemeinen keine treu isomorphe Darstellungen der Gruppe. Man könnte aber unschwer zeigen, daß, falls die stetige Funktion $f(X)$ so beschaffen ist, daß für jedes vom Einheitselement E verschiedene Gruppenelement X der Funktionswert $f(X) \neq f(E)$ ausfällt, die *Gesamtheit* der zu allen Eigenwerten von $F(XY^{-1})$ gehörenden Darstellungen eine treu isomorphe Darstellung der vorgelegten Gruppe liefert.

Es sei noch darauf hingewiesen, daß auch die wichtigen Sätze über die irreduziblen Gruppendarstellungen, die Peter und Weyl in der erwähnten Arbeit entwickeln, auf den vorliegenden Fall übertragbar sind.

SZEGED, UNGARN.

DIE EINFÜHRUNG ANALYTISCHER PARAMETER IN TOPOLOGISCHEN GRUPPEN.*

VON J. V. NEUMANN.

Einleitung.

1. Hilbert formulierte 1900 die folgende Frage[1]: Sei G eine durch n Parameter stetig beschriebene Gruppe von stetigen Transformationen einer m-parametrigen Mannigfaltigkeit M, ist es dann möglich, in G und M solche Parameter einzuführen, daß alle die die Transformationen beschreibenden Funktionen stetig differentiierbar, oder sogar analytisch[2], ausfallen? Anders ausgedrückt: ist auf solche G, bei geeigneter Parameterwahl, die Liesche Theorie anwendbar? (Vgl. a. a. O. Anm. [1], [2].)

Brouwer gab 1909/10 durch schwierige und scharfsinnige topologische Untersuchungen[3] alle G, M-Typen für $m = 1, 2$ an, wodurch die Hilbertsche Frage für $m = 1$ bejahend beantwortet und für $m = 2$ weitgehend gefördert wurde.

Ein weiter unten anzugebendes Gegenbeispiel lehrt, daß die Hilbertsche Frage bei (in M) intransitivem G im allgemeinen zu verneinen ist. (Für $m = 1$ zeigt Brouwers Liste, daß nur transitive Gruppen existieren; unser Gegenbeispiel entspricht dem einfachsten der übrigen Fälle: $n = 1$, $m = 2$.) Wir werden daher im folgenden G stets als (in M) transitiv voraussetzen.

Unter dieser Voraussetzung werden wir für alle geschlossenen (in der topologischen Terminologie: kompakten) G zeigen, daß die Hilbertsche Frage zu bejahen ist. Für nichtkompakte G liegen vorläufig keine ähnlicherweise abschließenden Resultate vor.

Die vorliegenden Untersuchungen wurden durch die in Anm. [13] angeführte Arbeit von A. Haar angeregt, und sie sind auf die Resultate derselben aufgebaut. Der Verfasser möchte Herrn Haar an dieser Stelle seinen Dank aussprechen, daß er ihm dieselben noch vor der Publikation zur Verfügung stellte.

2. Ein wichtiger Spezialfall des Hilbertschen Problems ist $G = M$. Denn wenn wir die Elemente von G mit a, b, x, y bezeichnen und das

* Received July 26, 1932.

[1] Kongreßvortrag, gehalten am internationalen Mathematikerkongreß in Paris, 1900. Abgedruckt in den Gött. Nachr., 1900. Es handelt sich hier um das „Problem 5".

[2] F. Schur bewies, Math. Ann. **41** (1893), S. 509—538, daß für alle jenen G, die in M transitiv sind, und für die eine Parameterwahl möglich ist, bei der alle genannten Funktionen zweimal stetig differentiierbar sind, auch eine solche möglich ist, bei der sie alle analytisch sind.

[3] L. E. J. Brouwer, Math. Ann. **67** (1909), S. 246—267, und **69** (1910), S. 181—203.

Kompositionsgesetz in G mit $a \cdot b$, so bestimmt jedes feste Element a von G eine Transformation von G: $x \to x \cdot a$[4]. Die Frage lautet dann: Ist es möglich, in G n derartige Parameter einzuführen, daß die das Kompositionsgesetz beschreibenden Funktionen alle stetig differentiierbar bzw. analytisch ausfallen? (Natürlich sind stets Parameter „im kleinen", d. h. in einer geeignet gewählten Umgebung der Einheit von G, bzw. eines beliebig gegebenen Punktes von M, gemeint.) Wir werden als erstes dieses Problem erledigen und dann das allgemeinere (beliebiges M) darauf zurückführen.

Es genügt natürlich zu zeigen, daß G auf eine Gruppe G' mit der genannten Eigenschaft homöomorph[5] und isomorph[6] abgebildet werden kann.

Eine Menge G' von k-dimensionalen komplexen Matrizen von nichtverschwindender Determinante ($k = 1, 2, \cdots$), die die Einheitsmatrix E_k enthält, und mit den Matrizen A, B auch die Inverse A^{-1} und ihr Matrizenprodukt $A \cdot B$, ist eine Gruppe mit der Kompositionsregel $A \cdot B$. Sie kann topologisiert werden, indem wir die Matrizen A als Punkte des k^2-dimensionalen komplexen, d. h. des $2k^2$-dimensionalen reellen, euklidischen Raumes ansehen (vgl. a. a. O. Anm. [7]), S. 5—6). Wir nehmen ferner an, daß jeder Häufungspunkt (d. i. Häufungsmatrix) von G' mit nichtverschwindender Determinante auch zu G' gehört. Dann kann G' in einer geeigneten Umgebung der Einheit homöomorph auf h ($= 0, 1, \cdots, 2k^2$) reelle Parameter $\alpha_1, \cdots, \alpha_h$ (die in einer gewissen Umgebung von $0, \cdots, 0$ variieren) bezogen werden[7], ja die Matrixelemente der Matrizen von G' sind sogar analytische Funktionen von $\alpha_1, \cdots, \alpha_h$. Infolgedessen ist das Kompositionsgesetz von G' (d. i. das Multiplikationgesetz der Matrizen, das in den Matrixelementen algebraisch ist) in den Parametern $\alpha_1, \cdots, \alpha_h$ analytisch. D. h. ein solches G' hat die von uns gewünschten Eigenschaften.

Ist nun G geschlossen (d. h. kompakt), und besitzt es eine stetige Darstellung $a \to D(a)$ durch k-dimensionale Matrizen, die auch treu (d. h. eineindeutig) ist, so können wir so schließen: Die Menge G' aller $D(a)$ (a durchläuft G) ist wegen der Kompaktheit von G abgeschlossen, und auch die inverse Abbildung ist stetig, d. h. die Darstellung ist eine homöomorphe Abbildung. Daß sie isomorph ist, ist klar. Da unsere Behauptung nach obigem für G' gilt[8], gilt sie somit auch für G.

[4] Übt man zuerst $x \to x \cdot a$ aus und dann $x \to x \cdot b$, so wird $x \to x \cdot (a \cdot b)$. G ist offenbar transitiv in sich selbst.

[5] D. h. ein-eindeutig und gleichzeitig mit der Umkehrung stetig.

[6] D. h. das Kompositionsgesetz von G in dasjenige von G' überführend.

[7] Dies bewies der Verf., Math. Zeitschr. **30** (1929), S. 3—42. Vgl. insbesondere Satz I auf S. 27.

[8] Eigentlich fehlt noch $Det\,(D(a)) \neq 0$. Wegen $D(a) \cdot D(a^{-1}) = D(1)$ steht dies fest, wenn $D(1)$ die Einheitsmatrix ist. Dies ist aber stets erreichbar (vgl. z. B. a. a. O. Anm. [7], in Anm. [12], S. 28).

Wir brauchen also nur die Existenz treuer stetiger Darstellungen für die geschlossenen G nachzuweisen. Dieser, auch an und für sich nicht uninteressante, Satz wird daher der Hauptgegenstand unserer Betrachtungen sein.

Die genannte Darstellung $D(a)$ wird übrigens aus lauter unitären Matrizen bestehen. Daher können wir den letztgenannten Satz, unter Berücksichtigung des in Anm. [7] zitierten Satzes, auch so formulieren: Die einzigen (im kleinen) n-parametrigen ($n = 0, 1, 2, \cdots$) geschlossenen (d. h. kompakten) Gruppen sind die abgeschlossenen Gruppen unitärer Matrizen des k-dimensionalen (komplexen) euklidischen Raumes ($k = 0, 1, 2, \cdots$) sowie deren homöomorph-isomorphe Bilder.

3. Um den eigentümlichen, analytisch-gruppentheoretisch-topologischen Charakter der zu verwendenden Methode zu veranschaulichen, seien jene Resultate der neueren gruppentheoretischen und topologischen Literatur, auf der die zu führenden Beweise beruhen, hier zusammengestellt. Es sind die folgenden:

a) A. Haar bewies vor kurzem, daß in jeder topologischen Gruppe[9], welche separabel[10] und im kleinen kompakt[11] ist, ein rechtsinvariantes Lebesguesches äußeres Maß[12] definiert werden kann[13]. Infolgedessen kann nach bekanntem Vorgang auch eine rechtsinvariante Lebesguesche Integration $\int_G f(x)\, dv_x$ (dv_x symbolisiert das „Volumelement" für die Variable x) definiert werden[14]. Hierbei ist zu erwähnen, daß bereits F. Peter und H. Weyl, unter Vervollkommnung von Ansätzen von Hurwitz, eine solche Integration für solche n-parametrige Gruppen konstruierten, deren Kompositionsgesetz auf stetig differentiierbare Funktionen der Parameter führt[15]; das Wichtige an Haars Konstruktion ist aber gerade, daß sie ganz allgemein und rein mengentheoretisch, ohne jede

[9] D. h. eine Gruppe, in der ein Umgebungsbegriff definiert ist, derart, daß die Komposition $a \cdot b$ und die Inverse a^{-1} stetige Funktionen von a, b bzw. von a sind. Vgl. für die Grundbegriffe der Topologie etwa Hausdorff, „Mengenlehre", Berlin und Leipzig (1927), § 40, S. 226—230.

[10] Vgl. a. a. O. Anm. [9], S. 125, und (10) auf S. 229.

[11] D. h. jeder Punkt hat eine Umgebung, deren abgeschlossene Hülle kompakt ist. Da die Abbildung $x \to x \cdot a$ homöomorph ist, genügt es (bei Gruppen), dies für die Einheit zu verlangen.

[12] Vgl. Carathéodory, „Reelle Funktionen", Berlin und Leipzig (1918), S. 237—274; oder (für beliebige topologische Räume) in der Arbeit des Verf., Ann. of Math., **33** (1932), S. 574—586, Def. 2, 4 auf S. 574, 576. „Rechtsinvarianz" besagt, daß die Abbildung $x \to x \cdot a$ maßtreu ist.

[13] A. Haar, Ann. of Math., **34** (1933), S. 147–169, insbesondere § 3.

[14] Z. B. als Stieltjessches Integral $\int_{-\infty}^{+\infty} \lambda\, d(\text{Maß}_x[f(x) \leqq \lambda])$.

[15] Math. Ann. **97** (1928), S. 737—755, insbesondere S. 737.

Beziehung zu einer Parameterdarstellung, direkt zu einem Maßbegriff führt. Unsere Überlegungen, die vom Haarschen Maß zu den analytischen Parametern führen, können also als eine Art Umkehrung der Peter-Weylschen angesehen werden, bei denen das Umgekehrte geleistet wird — wenn auch die Schwierigkeiten von anderer Art sind.

b) Bei geschlossenem (kompaktem) G ist das Haarsche Maß von ganz G, wie man leicht einsieht, endlich, kann also $= 1$ normiert werden. Alsdann sind die Operatoren- und Eigenwertmethoden und -sätze anwendbar, die Peter und Weyl a. a. O. Anm. [15] verwenden, wodurch eine vollständige Theorie der (stetigen) Darstellungen von G durch unitäre endlich vieldimensionale Matrizen entsteht.

c) Bis hierher wurde bloß benutzt, daß G eine topologische Gruppe ist. Unter Benutzung der Tatsache, daß es (im kleinen) homöomorph auf n Parameter bezogen werden kann, wird durch topologische Betrachtungen gezeigt, daß unter den in b) genannten Darstellungen auch treue vorkommen müssen. Hierbei werden in erster Linie zwei „dimensionstheoretische" Sätze benutzt: Brouwers Satz von der Gebietsinvarianz und Lebesgues „Pflastersteinsatz" bzw. der damit zusammenhängende „Allgemeine Zerlegungssatz" der Dimensionstheorie von K. Menger und Urysohn[16].

d) Bei den topologischen Betrachtungen werden auch die in Anm. [7] zitierten Resultate des Verf. über die Parameterdarstellung von Gruppen unitärer Matrizen mehrfach wesentlich herangezogen.

Die Einteilung der Arbeit ist diese: In Teil I wird die Existenz der treuen stetigen Darstellung bewiesen, in Teil II dies auf das Hilbertsche Problem angewendet und das Gegenbeispiel für (in M) intransitive G angegeben.

I. Einführung analytischer Parameter in n-parametrigen geschlossenen Gruppen G.

1. G sei eine topologische Gruppe, d. h. eine, in der ein Umgebungsbegriff definiert ist, derart, daß $a \cdot b$ und a^{-1} in a, b bzw. a stetig sind (vgl. Anm. [9]). Wir nennen G n-parametrig ($n = 1, 2, \cdots$), wenn jedes a von G eine Umgebung $U(a)$ hat, die einer offenen Punktmenge $K(a)$ im n-dimensionalen (reellen) euklidischen Raume R_n (der Zahlen n-tupel $\alpha_1, \cdots, \alpha_n$) homöomorph ist. Nach Anm. [9] genügt es, mit Hinblick auf die homöomorphe Abbildung $x \to x \cdot a$, diese Forderung nur für $a = 1$ (d. i. die Gruppeneinheit) zu stellen.

[16] L. E. J. Brouwer, Math. Ann. **70** (1913), S. 161—165; H. Lebesgue, Fund. Math. **2** (1921), S. 256—285, sowie E. Sperner, Hamb. Abh. **6** (1928), S. 265—272; K. Menger, „Dimensionstheorie", Berlin und Leipzig (1928), S. 251—266.

Aus demselben Grunde können wir alle $K(a)$ als einander gleich annehmen. Eine Translation von $K(a)$ im R_n erlaubt es, das Bild von a nach $0, \cdots, 0$ zu verlegen; eine eventuelle Verkleinerung von $U(a)$ und $K(a)$, $K(a)$ als Kugel mit dem Mittelpunkt $0, \cdots, 0$ anzunehmen; eine nachfolgende Ähnlichkeitstransformation in R_n, den Radius von $K(a)$ gleich 1 zu normieren. Also wird $K(a)$ gleich K_1: $\alpha_1^2 + \cdots + \alpha_n^2 < 1$. Wir bilden noch die (abgeschlossene) Kugel $\overline{K}_{\frac{1}{2}}$: $\alpha_1^2 + \cdots + \alpha_n^2 \leqq \frac{1}{2}$. Bei der Abbildung $U(a) \to K(a) = K_1$ habe $\overline{K}_{\frac{1}{2}}$ das Urbild $\overline{V}(a)$.

$\overline{K}_{\frac{1}{2}}$ ist kompakt und separabel, also auch $\overline{V}(a)$; wenn also G durch endlich viele $\overline{V}(a)$ überdeckt werden kann, so ist es kompakt und separabel; wenn es durch abzählbar viele gelingt, ist es Vereinigung abzählbar vieler kompakter Mengen und separabel. Umgekehrt: da a innerer Punkt von $\overline{V}(a)$ ist, genügen zur Überdeckung von G, falls es kompakt ist, endlich viele $\overline{V}(a)$ und, falls es separabel ist, abzählbar viele. Wir haben also gezeigt:

G ist dann und nur dann kompakt bzw. separabel, wenn es durch endlich bzw. abzählbar viele $\overline{V}(a)$ überdeckt werden kann. Wenn es separabel ist, ist es allenfalls Vereinigung abzählbar vieler kompakter Mengen.

Wir werden im folgenden die Analytisierungsfrage für die „geschlossenen", d. h. kompakten, G lösen, für die „offenen", d. h. bloß separablen, G bleibt sie unerledigt. Im folgenden wird also G durchweg als n-parametrig und kompakt vorausgesetzt.

2. In G definierte stetige Funktionen, mit komplexen Zahlen als Werten, bezeichnen wir mit $f(x)$, $g(x)$, $k(x, y)$, $\varphi(x)$; die aufs Haarsche Maß gegründete Integration mit $\int_G f(x)\, dv_x$ (vgl. Einl. 3., insbesondere a) und Anm. [13], [14]). Wie a. a. O. gesagt wurde, ist das Maß von G endlich, da dies für jede kompakte Menge der Fall ist; durch Multiplizieren aller Maße (und Integrale) mit dem konstanten Faktor $\frac{1}{Maß\ G}$ können wir daher erreichen, daß $Maß\ G = 1$ wird[17]. Nun können wir die in Einl. 3. b) zitierten Betrachtungen von Peter und Weyl auf unser G übertragen. (Vgl. auch Haar, a. a. O. Anm. [13], S. 167—169.)

Jedem stetigen Kern $k(x, y)$ ordnen wir den Integraloperator I_k zu, der durch

$$I_k f(x) = \int_G k(x, y) f(y)\, dv_y$$

[17] Dies ist die einzige Stelle, wo die Kompaktheit von G wirklich wesentlich benötigt wird.

definiert ist. Die Methode von E. Schmidts Dissertation[18] ist auf dieses I_k wörtlich anwendbar, wenn es die Hermitesche Symmetrie hat, d. h. wenn $k(y, x) = \overline{k(x, y)}$ gilt. Infolgedessen treffen insbesondere die folgenden Behauptungen zu:

Die von O verschiedenen Eigenwerte von I_k bilden eine Folge $\lambda_1, \lambda_2, \cdots$ (die Anzahl ihrer Elemente ist 0, 1, 2, $\cdots$ oder abzählbar unendlich). Für jedes λ_ν gibt es unter den Eigenfunktionen, d. h. den Lösungen von $I_k f = \lambda_\nu f$, nur endlich viele linear unabhängige, ihre Maximalanzahl sei N_ν $(= 1, 2, \cdots)$. N_ν solche Eigenfunktionen können sogar normiert-orthogonal[19] gewählt werden: $f_1^{(\nu)}, \cdots, f_{N_\nu}^{(\nu)}$. Jede Funktion von der Form $I_k f$ (f beliebig) kann durch Linearaggregate endlich vieler $f_l^{(\nu)}$ $(\nu = 1, 2, \cdots, l = 1, \cdots, N_\nu)$ in G gleichmäßig beliebig gut approximiert werden.

Sei nun $\varphi(x)$ eine stetige Funktion in G; wir setzen $k(x, y) = \varphi(x y^{-1})$, $I_k = T_\varphi$, und wenden das soeben Gesagte auf T_φ an. Hierzu ist allerdings $\varphi(y x^{-1}) = \overline{\varphi(x y^{-1})}$ notwendig, d. h. $\varphi(x^{-1}) = \overline{\varphi(x)}$.

T_φ hat die folgende Eigentümlichkeit: wenn es $f(x)$ in $g(x)$ überführt, so führt es $f(x \cdot a)$ in $g(x \cdot a)$ über (man ersetze in der definierenden Formel für $I_k = T_\varphi$ die Integrationsvariable y durch $y \cdot a^{-1}$ und beachte die Rechtsinvarianz sowie $x \cdot (y \cdot a^{-1})^{-1} = (x \cdot a) \cdot y^{-1}$). Also ist mit $f(x)$ auch $f(x \cdot a)$ Eigenfunktion von T_φ, und zwar vom selben Eigenwert; also ist $f_l^{(\nu)}(x \cdot a)$ Linearaggregat der $f_1^{(\nu)}(x), \cdots, f_{N_\nu}^{(\nu)}(x)$. Wir wollen jedes endliche normiert-orthogonale Funktionensystem $f_1(x), \cdots, f_N(x)$, für welches jedes $f_l(x \cdot a)$ $(l = 1, \cdots, N,\ a$ in $G)$ Linearaggregat der $f_1(x), \cdots, f_N(x)$ ist, ein Darstellungssystem nennen. Die $f_1^{(\nu)}(x), \cdots, f_{N_\nu}^{(\nu)}(x)$ sind somit Darstellungssysteme, und wir erkennen: Es gibt eine Folge (von 0, 1, 2, $\cdots$ oder abzählbar unendlich vielen) Darstellungssystemen, so daß jede Funktion $T_\varphi f$ (φ fest, f beliebig) durch Linearaggregate endlich vieler ihrer Funktionen beliebig gut approximierbar ist.

Gäbe es eine Folge $\varphi_1, \varphi_2, \cdots$ von φ's, so daß jedes f durch $T_{\varphi_s} g$'s $(s = 1, 2, \cdots, g$ beliebig) beliebig gut approximierbar ist, so erhielten wir durch Zusammenfassung der Darstellungssystemfolgen von $\varphi_1, \varphi_2, \cdots$ eine neue Darstellungssystemfolge, derart, daß jedes f durch Linearaggregate endlich vieler ihrer Funktionen beliebig gut approximierbar ist. Hieraus folgt noch, daß diese Folge nicht leer und auch nicht endlich sein kann.

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[18] Math. Ann. **63** (1907), S. 433—476.

[19] Ein Funktionensystem $f_1, f_2, \cdots$ heißt normiert-orthogonal, wenn

$$\int_a f_j(x)\,\overline{f_l(x)}\,dv_x \begin{cases} = 1, & \text{für } j = l \\ = 0, & \text{für } j \neq l \end{cases}$$

gilt.

Eine Folge $\varphi_1, \varphi_2, \cdots$, für die $T_{\varphi_s} f$ für $s \to \infty$ gleichmäßig in G gegen f konvergiert (für jedes stetige f), hat die obige Eigenschaft. Wenn $U_1, U_2, \cdots$ sich auf 1 (Gruppeneinheit von G) zusammenziehende Umgebungen von 1 sind, und $\varphi_s(x)$ stets reell und $\geqq 0$, aber außerhalb von $U_s = 0$ ist und $\int_G \varphi_s(x)\, dv_x = 1$ gilt, sowie natürlich $\varphi(x^{-1}) = \varphi(x)$ $(= \overline{\varphi(x)})$, so leisten diese $\varphi_1, \varphi_2, \cdots$ offenbar das Gewünschte. Solche φ_s sind aber zu den U_s leicht angebbar: Zunächst genügt $\int_G \varphi_s(x)\, dv_x > 0$ (statt $= 1$), da wir sonst $\varphi_s(x)$ durch $\dfrac{\varphi_s(x)}{\int_G \varphi_s(x)\, dv_x}$ ersetzen. Ferner ist $\varphi_s(x^{-1}) = \varphi_s(x)$ unnötig, da wir sonst $\varphi_s(x)$ durch $\varphi_s(x) + \varphi_s(x^{-1})$ ersetzen, gleichzeitig U_s so weit verringernd, daß für alle x des neuen U_s x, x^{-1} beide zum alten gehören. Weiter können wir U_s in $\overline{V}(1)$ liegend annehmen, sein bei $\overline{V}(1) \to \overline{K}(\frac{1}{2})$ entstehendes Bild sei $\overline{U}$. $\overline{U}$ enthält gewiß eine Kugel mit dem Mittelpunkt $0, \cdots, 0$, deren Radius sei $\varepsilon_s > 0$. Liegt x in U_s, so heiße die Distanz seines Bildpunktes (in $\overline{K}(\frac{1}{2})$) von $0, \cdots, 0$ $d_s(x)$. Dann erfüllt

$$\varphi_s(x) \begin{cases} = \text{Max } (\varepsilon_s - d_s(x), 0), & \text{für } x \text{ in } U_s \\ = 0, & \text{sonst} \end{cases}$$

die übriggebliebenen Bedingungen.

Somit existiert die oben beschriebene Folge von Darstellungssystemen, wir nennen sie $f_1^{(\mu)}(x), \cdots, f_{M_\mu}^{(\mu)}(x)$ $(\mu = 1, 2, \cdots)$.

3. $f_l^{(\mu)}(x \cdot a)$ ist Linearaggregat der $f_1^{(\mu)}(x), \cdots, f_{M_\mu}^{(\mu)}(x)$, also ist

$$f_l^{(\mu)}(x \cdot a) = \sum_{j=1}^{M_\mu} \alpha_{jl}^{(\mu)}(a)\, f_j^{(\mu)}(x) \qquad (\mu = 1, 2, \cdots, l = 1, \cdots, M_\mu,\ a \text{ in } G).$$

Da die $f_j^{(\mu)}(x)$ normiert-orthogonal sind, ist

$$\alpha_{jl}^{(\mu)}(a) = \int_G f_l^{(\mu)}(x \cdot a)\, \overline{f_j^{(\mu)}(x)}\, dv_x,$$

so daß $\alpha_{jl}^{(\mu)}(a)$ stetig von a abhängt. Ferner ist die Matrix $A^{(\mu)}(a) = \{\alpha_{jl}^{(\mu)}(a)\}_{j,l=1,\cdots,M_\mu}$ unitär, und es gilt offenbar $A^{(\mu)}(a \cdot b) = A^{(\mu)}(a)\, A^{(\mu)}(b)$. Die Matrizen $A^{(\mu)}(a)$ (μ fest) bilden also eine stetige Darstellung von G.

Wenn für a, b von G $A^{(\mu)}(a) = A^{(\mu)}(b)$ für jedes $\mu = 1, 2, \cdots$ gilt, so ist nach Definition stets $f_l^{(\mu)}(x \cdot a) = f_l^{(\mu)}(x \cdot b)$. Daher muß $f(x \cdot a) = f(x \cdot b)$ für jedes f gelten, das Linearaggregat endlich vieler $f_l^{(\mu)}$ oder durch solche beliebig gut approximierbar ist — d. h. für jedes stetige f. $x = b^{-1}$ ergibt: für jedes stetige f ist $f(b^{-1}a) = f(1)$, also muß $b^{-1}a = 1$, $a = b$ sein[20].

[20] Ist $c \neq 1$, so sei U eine c nichtenthaltende Umgebung von 1. Konstruieren wir f zu U so, wie weiter oben φ_s zu U_s, so wird $f(1) > 0$, $f(c) = 0$.

D. h.: Die Gesamtheit der Darstellungen $A^{(1)}(a)$, $A^{(2)}(a)$, $\cdots$ ist treu — aber wir brauchen eine einzelne (endlich vieldimensionale!) Darstellung, die es ist!

Sei $M_1 + \cdots + M_\mu = L_\mu$, reihen wir $A^{(1)}(a), \cdots, A^{(\mu)}(a)$ an der Diagonale einer L_μ-dimensionalen Matrix aneinander und besetzen die übrigen Felder mit Nullen, so entsteht eine L_μ-dimensionale Matrix $B^{(\mu)}(a)$ (vgl. Abb.); tun

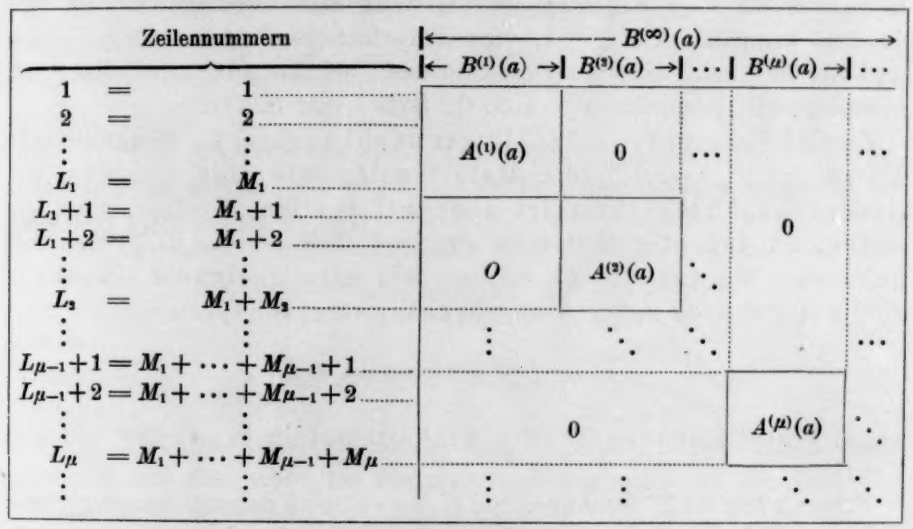

ABBILDUNG.

wir dasselbe mit allen $A^{(1)}(a)$, $A^{(2)}(a)$, $\cdots$, so entsteht eine unendlich vieldimensionale, aber zeilen- und kolonnenfinite Matrix $B^{(\infty)}(a)$ (vgl. Abb.). Die $B^{(\mu)}(a)$ sind offenbar unitäre Darstellungen von G, $B^{(\infty)}(a)$ ebenfalls. $B^{(\mu)}(a)$ hängt stetig von a ab, $B^{(\infty)}(a)$ ebenfalls, falls für unendliche Matrizen die Konvergenz durch die Konvergenz aller Matrizenelemente definiert wird[21]. Schließlich ist $B^{(\mu)}(a)$ der L_μ-te Abschnitt von $B^{(\infty)}(a)$. Die Menge aller $B^{(\mu)}(a)$ (a durchläuft ganz G) heiße $\mathfrak{B}^{(\mu)}$, diejenige aller $B^{(\infty)}(a)$ $\mathfrak{B}^{(\infty)}$; $\mathfrak{B}^{(\mu)}$ ist stetiges Bild von G, $\mathfrak{B}^{(\infty)}$ nach dem weiter oben Gesagten ein-eindeutiges und stetiges Bild — da G kompakt ist, muß daher auch die Inverse der letzteren Abbildung stetig sein.

Im folgenden werden wir zeigen, daß schon ein $\mathfrak{B}^{(\mu)}$ ein-eindeutiges Bild von G, d. h. schon ein $B^{(\mu)}(a)$ treu ist — womit das Programm von Einl. 2 erfüllt sein wird. Dazu werden wir aber die bisher kaum benutzte Beziehung von G zum n-dimensionalen (reellen) euklidischen Raume, d. i. die

[21] Diese Topologie der unendlichen Matrizen $\{\alpha_{jl}\}_{j,l=1,2,\ldots}$ kann aus verschiedenen Entfernungsbegriffen (vgl. a. a. O. Anm. [9], S. 94) hergeleitet werden, z. B. aus

$$\text{Entf.}\,(\{\alpha_{jl}\},\{\beta_{jl}\}) = \operatorname*{Min}_{L=1,2,\ldots}\left\{\operatorname{Max}\left[\operatorname*{Max}_{j,l=1,\ldots,L}(|\alpha_{jl}-\beta_{jl}|),\frac{1}{L}\right]\right\}.$$

Abbildung von $U(1)$ auf $K(1)$, wirklich heranziehen müssen. (Bisher benutzten wir eigentlich nur, daß G topologisch, separabel und kompakt ist.)

4. Da $B^{(\mu)}(a)$ eine Darstellung ist, bildet es die Gruppe G auf eine Gruppe ab, d. h. $\mathfrak{G}^{(\mu)}$ ist eine Matrizengruppe. Wegen $1^2 = 1$ ist das Bild von 1 das eigene Quadrat, und da es unitär ist, muß es die L_μ-dimensionale Einheitsmatrix sein: E_{L_μ}. Da $B^{(\mu)}(a)$ stetig und G kompakt ist, ist auch das Bild kompakt, d. h. $\mathfrak{G}^{(\mu)}$ ist eine abgeschlossene Matrizenmenge. Somit sind die in Anm. [7] zitierten Resultate des Verf. auf $\mathfrak{G}^{(\mu)}$ anwendbar, insbesondere gilt folgendes (vgl. a. a. O., Satz I, auf S. 27):

Es gibt $P_\mu (= 0, 1, \cdots, 2L_\mu^2)$ linear unabhängige, L_μ-dimensionale Matrizen $U_1^{(\mu)}, \cdots, U_{P_\mu}^{(\mu)}$ (die Matrizenelemente sind komplex, die lineare Unabhängigkeit ist aber mit reellen Koeffizienten gemeint), so daß alle Matrizen $\exp(\alpha_1 U_1^{(\mu)} + \cdots + \alpha_{P_\mu} U_{P_\mu}^{(\mu)})$ [22] zu $\mathfrak{G}^{(\mu)}$ gehören. Werden die $\alpha_1, \cdots, \alpha_{P_\mu}$ auf eine geeignete Umgebung $K^{(\mu)}$ des Punktes $0, \cdots, 0$ beschränkt, so erschöpfen die

$$\exp(\alpha_1 U_1^{(\mu)} + \cdots + \alpha_{P_\mu} U_{P_\mu}^{(\mu)})$$

sogar eine Umgebung $\mathfrak{K}^{(\mu)}$ der Einheitsmatrix E_{L_μ} in $\mathfrak{G}^{(\mu)}$ ein-eindeutig.

Fassen wir ein $U_p^{(\mu)}$ ins Auge. Sei $\varepsilon_1, \varepsilon_2, \cdots$ eine gegen 0 konvergierende Folge von Zahlen >0. $\exp(\varepsilon_s U_p^{(\mu)})$ gehört zu $\mathfrak{G}^{(\mu)}$, es ist also wenigstens einem $B^{(\mu)}(a)$ gleich, etwa für $a = a_p$. Die $B^{(\mu+1)}(a_p)$ liegen in $\mathfrak{G}^{(\mu+1)}$, sie haben daher einen Häufungspunkt. Wenn wir $\varepsilon_1, \varepsilon_2, \cdots$ durch eine Teilfolge ersetzen, können wir also erreichen, daß die $B^{(\mu+1)}(a_p)$ konvergieren. Auch das monotone Abnehmen der $\varepsilon_1, \varepsilon_2, \cdots$ ist durch Aussondern einer Teilfolge erreichbar. Wir dürfen daher annehmen, daß beides für die ursprünglichen $\varepsilon_1, \varepsilon_2, \cdots$ der Fall war. Es ist

$$\varepsilon_s - \varepsilon_{s+1} \to 0, \qquad \varepsilon_s - \varepsilon_{s+1} > 0,$$

und

$$B^{(\mu)}(a_s\, a_{s+1}^{-1}) = B^{(\mu)}(a_s)\, B^{(\mu)}(a_{s+1})^{-1}$$
$$= \exp(\varepsilon_s U_p^{(\mu)}) \exp(\varepsilon_{s+1} U_p^{(\mu)}) = \exp((\varepsilon_s - \varepsilon_{s+1}) U_p^{(\mu)}),$$
$$B^{(\mu+1)}(a_s\, a_{s+1}^{-1}) = B^{(\mu+1)}(a_s)\, B^{(\mu+1)}(a_s)^{-1} \to E_{L_{\mu+1}},$$

wenn wir also $\varepsilon_1, \varepsilon_2, \cdots$ durch $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \cdots$ ersetzen, so tritt $a_s' = a_s\, a_{s+1}^{-1}$ mit $B^{(\mu+1)}(a_s') \to E_{L_{\mu+1}}$ an die Stelle von a_s. Wir dürfen daher annehmen, daß $B^{(\mu+1)}(a_s) \to E_{L_{\mu+1}}$ schon für die ursprünglichen $\varepsilon_1, \varepsilon_2, \cdots$ der Fall war.

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[22] Ist A eine endlich vieldimensionale Matrix, so ist $\exp(A)$ als $\lim_{h\to\infty} \sum_{s=0}^{h} \frac{A^s}{s!}$ definiert. Vgl. a. a. O. Anm. [7], S. 7—15.

Fast alle $B^{(\mu+1)}(a_s)$, $B^{(\mu)}(a_s)$ gehören also zu $\mathfrak{R}^{(\mu+1)}$ bzw. $\mathfrak{R}^{(\mu)}$, für diese ist

$$B^{(\mu+1)}(a_s) = \exp(\alpha_{s,1} U_1^{(\mu+1)} + \cdots + \alpha_{s,P^{(\mu+1)}} U_{P_{\mu+1}}^{(\mu+1)}),$$
$$\alpha_{s,1}, \cdots, \alpha_{s,P^{(\mu+1)}} \text{ aus } K^{(\mu+1)}.$$

Für $s \to \infty$ strebt dieser Ausdruck gegen $E_{L_{\mu+1}}$, also $\alpha_{s,1}, \cdots, \alpha_{s,P_{\mu+1}}$ gegen $0, \cdots, 0$, also strebt

$$\alpha_s = |\alpha_{s,1}| + \cdots + |\alpha_{s,P_{\mu+1}}| > 0$$

(wäre es $= 0$, so wäre $\alpha_{s,1} = \cdots = \alpha_{s,P_{\mu+1}} = 0$, $B^{(\mu+1)}(a_s) = E_{L_{\mu+1}}$, also erst recht $\exp(\varepsilon_s U_p^{(\mu)}) = B^{(\mu)}(a_s) = E_{L_\mu}$, also $\varepsilon_s = 0$, entgegen der Annahme) auch gegen 0. Sei

$$\beta_{s,1} = \frac{\alpha_{s,1}}{\alpha_s}, \cdots, \beta_{s,P_{\mu+1}} = \frac{\alpha_{s,P_{\mu+1}}}{\alpha_s},$$

dann ist

$$|\beta_{s,1}| + \cdots + |\beta_{s,P_{\mu+1}}| = 1.$$

Für eine Teilfolge konvergieren die $\beta_{s,1}, \cdots, \beta_{s,P_{\mu+1}}$; wir dürfen daher annehmen, daß dies schon für die ursprünglichen $\varepsilon_1, \varepsilon_2, \cdots$ der Fall war. Die Limites seien bzw. $\bar{\beta}_1, \cdots, \bar{\beta}_{P_{\mu+1}}$, es ist

$$|\bar{\beta}_1| + \cdots + |\bar{\beta}_{P_{\mu+1}}| = 1.$$

Nennen wir den L_μ-Abschnitt von $U_q^{(\mu+1)}$ $\tilde{U}_q^{(\mu+1)}$. Da $B^{(\mu)}(a_s)$ der L_μ-Abschnitt von $B^{(\mu+1)}(a_s)$ ist, d. h. $\exp(\varepsilon_s U_p^{(\mu)})$ der von

$$\exp(\alpha_s(\beta_{s,1} U_1^{(\mu+1)} + \cdots + \beta_{s,P_{\mu+1}} U_{P_{\mu+1}}^{(\mu+1)})),$$

ist

$$\exp(\varepsilon_s U_p^{(\mu)}) = \exp(\alpha_s(\beta_{s,1} \tilde{U}_1^{(\mu+1)} + \cdots + \beta_{s,P_{\mu+1}} \tilde{U}_{P_{\mu+1}}^{(\mu+1)})).$$

Sobald ε_s, α_s klein genug sind, d. h. für fast alle s, folgt hieraus

$$\varepsilon_s U_p^{(\mu)} = \alpha_s(\beta_{s,1} \tilde{U}_1^{(\mu+1)} + \cdots + \beta_{s,P_{\mu+1}} \tilde{U}_{P_{\mu+1}}^{(\mu+1)}).$$

Somit konvergiert für $s \to \infty$

$$\frac{\varepsilon_s}{\alpha_s} U_p^{(\mu)} \text{ gegen } \bar{\beta}_1 \tilde{U}_1^{(\mu+1)} + \cdots + \bar{\beta}_{P_{\mu+1}} \tilde{U}_{P_{\mu+1}}^{(\mu+1)}.$$

Da sowohl $U_p^{(\mu)}$ als auch der Limes $\neq 0$ ist, folgt hieraus, daß $\frac{\varepsilon_s}{\alpha_s}$ gegen einen Limes $\gamma \neq 0$ strebt; dann muß aber

$$U_p^{(\mu)} = \frac{\bar{\beta}_1}{\gamma} \tilde{U}_1^{(\mu+1)} + \cdots + \frac{\bar{\beta}_{P_{\mu+1}}}{\gamma} \tilde{U}_{P_{\mu+1}}^{(\mu+1)}$$

sein, d. h. $U_p^{(\mu)}$ ist der L_μ-Abschnitt von $\frac{\bar{\beta}_1}{\gamma} U_1^{(\mu+1)} + \cdots + \frac{\bar{\beta}_{P_{\mu+1}}}{\gamma} U_{P_{\mu+1}}^{(\mu+1)}$.

Somit ist jedes $U_p^{(\mu)}$, $p = 1, \cdots, P_\mu$, L_μ-Abschnitt eines Linearaggregats der $U_q^{(\mu+1)}$, $q = 1, \cdots, P_{\mu+1}$. Diese Linearaggregate müssen linear unabhängig sein, weil es schon ihre L_μ-Abschnitte $U_p^{(\mu)}$ sind, also ist $P_\mu \leqq P_{\mu+1}$. Ferner können wir die $U_q^{(\mu+1)}$, $q = 1, \cdots, P_{\mu+1}$, so umbezeichnen (nämlich linear transformieren), daß diese P_μ Linearaggregate gerade bzw. den $U_1^{(\mu+1)}, \cdots, U_{P_\mu}^{(\mu+1)}$ gleich werden. Nehmen wir diese Transformation nacheinander für $\mu = 1, 2, \cdots$ vor, so erreichen wir allgemein: $U_p^{(\mu)}$ ist der L_μ-Abschnitt von $U_p^{(\mu+1)}$. Dies werde im folgenden vorausgesetzt.

5. Wir betrachten für jedes $\nu \geqq \mu$ $\exp(x_1 U_1^{(\nu)} + \cdots + x_{P_\mu} U_{P_\mu}^{(\nu)})$ und nennen eines derjenigen a, deren $B^{(\nu)}(a)$ diesem Ausdruck gleich ist, $a_\nu(x_1 \cdots x_{P_\mu})$. Die $a_\nu(x_1 \cdots x_{P_\mu})$ haben für $\nu \to \infty$ einen Häufungspunkt $a(x_1 \cdots x_{P_\mu})$, weil G kompakt ist. Ferner ist für $\nu \geqq \mu_1$ $(\geqq \mu)$ der L_{μ_1}-Abschnitt von $B^{(\infty)}(a_\nu(x_1 \cdots x_{P_\mu}))$ derselbe wie derjenige von

$$B^{(\nu)}(a_\nu(x_1 \cdots x_{P_\mu})) = \exp(x_1 U_1^{(\nu)} + \cdots + x_{P_\mu} U_{P_\mu}^{(\nu)}),$$

und zwar gleich

$$B^{(\mu_1)}(a_{\mu_1}(x_1 \cdots x_{P_\mu})) = \exp(x_1 U_1^{(\mu)} + \cdots + x_{P_\mu} U_{P_\mu}^{(\mu_1)}),$$

also von ν unabhängig. Daher ist dies auch der L_{μ_1}-Abschnitt von $B^{(\infty)}(a(x_1 \cdots x_{P_\mu}))$, d. h. fast alle $B^{(\infty)}(a_\nu(x_1 \cdots x_{P_\mu}))$ haben denselben L_{μ_1}-Abschnitt wie $B^{(\infty)}(a(x_1 \cdots x_{P_\mu}))$. Wegen $L_{\mu_1} \to \infty$ bedeutet das, daß $B^{(\infty)}(a_\nu(x_1 \cdots x_{P_\mu}))$ für $\nu \to \infty$ gegen $B^{(\infty)}(a(x_1 \cdots x_{P_\mu}))$ konvergiert (vgl das Ende von 3. sowie Anm. [21]), also auch, daß $a_\nu(x_1 \cdots x_{P_\mu})$ gegen $a(x_1 \cdots x_P)$ konvergiert (vgl. ebendort). Da der L_{μ_1}-Abschnitt von $B^{(\infty)}(a(x_1 \cdots x_{P_\mu}))$

$$\exp(x_1 U_1^{(\mu_1)} + \cdots + x_{P_\mu} U_{P_\mu}^{(\mu_1)})$$

ist, also stetig in $x_1, \cdots, x_{P_\mu}$, ist jedes Matrixelement von $B^{(\infty)}(a(x_1 \cdots x_{P_\mu}))$ stetig in $x_1, \cdots, x_{P_\mu}$ also (vgl. das Ende von 3.) auch dieses selbst sowie $a(x_1 \cdots x_{P_\mu})$. Liegt $x_1, \cdots, x_{P_\mu}$ in $K^{(\mu)}$, so hängt schon der L_μ-Abschnitt von $B^{(\infty)}(a(x_1 \cdots x_{P_\mu}))$ ein-eindeutig von $x_1, \cdots, x_{P_\mu}$ ab, um so mehr dieses selbst sowie $a(x_1 \cdots x_{P_\mu})$. Da jeder L_{μ_1}-Abschnitt von $B^{(\infty)}(a(0 \cdots 0))$ gleich $E_{L_{\mu_1}}$ ist, ist dieses die Einheitsmatrix, also $a(0 \cdots 0) = 1$. Wenn wir also $x_1, \cdots, x_{P_\mu}$ auf eine hinreichend kleine Umgebung $\tilde{K}^{(\mu)}$ von $0, \cdots, 0$ beschränken, so liegt $a(x_1 \cdots x_{P_\mu})$ jedenfalls in $U(1)$ (in G).

Also bildet $a(x_1 \cdots x_{P_\mu})$ ein Gebiet $\tilde{K}^{(\mu)}$ im P_μ-dimensionalen euklidischen Raume ein-eindeutig und stetig auf eine Teilmenge von $U(1)$ ab, das seiner-

seits ein-eindeutiges und stetiges Bild eines Gebietes $K(1)$ im n-dimensionalen euklidischen Raume ist. Aus Brouwers Satz von der Gebietsinvarianz (vgl. a. a. O. Anm. [16]) folgt daher $P_\mu \leqq n$. Also ist $P_1 \leqq P_2 \leqq \cdots \leqq n$, von einem gewissen $\bar{\mu}$ an sind also die P_μ konstant: $P_{\bar{\mu}} = P_{\bar{\mu}+1} = \cdots = P \leqq n$.

6. Sei $\mu \geqq \bar{\mu}$. $\mathfrak{G}^{(\mu)}$ ist in einer Umgebung des Punktes E_{L_μ} mit dem Gebiet $K^{(\mu)}$ im P-dimensionalen Raume homöomorph, also P-dimensional; wegen der Gruppeneigenschaft gilt das für jeden Punkt, also ist es als topologischer Raum P-dimensional (vgl. Mengers in Anm. [16] zitiertes Werk, S. 79—80). Als kompakte P-dimensionale Menge im $2L_\mu^2$-dimensionalen euklidischen Raume der L_μ-dimensionalen Matrizen, fällt es unter den „Allgemeinen Zerlegungssatz" der Dimensionstheorie (a. a. O. S. 156): Zu jedem $\varepsilon > 0$ können endlich viele Mengen $\mathfrak{H}_1, \cdots, \mathfrak{H}_{\bar{t}}$ gefunden werden (natürlich von μ, ε abhängend), so daß 1. jedes $\mathfrak{H}_t$ einen Diameter $\leqq \varepsilon$ hat[23], 2. je $P+2$ Mengen $\mathfrak{H}_t$ keinen gemeinsamen Punkt haben, 3. $\mathfrak{H}_1 + \cdots + \mathfrak{H}_t = \mathfrak{G}^{(\mu)}$ ist. Nun sei $\mathfrak{H}_t'$ die Menge aller $B^{(\infty)}(a)$ mit $B^{(\mu)}(a)$ in $\mathfrak{H}_t$. Wieder gilt: 2'. je $P+2$ Mengen $\mathfrak{H}_t'$ haben keinen gemeinsamen Punkt, 3'. $\mathfrak{H}_1' + \cdots + \mathfrak{H}_t' = \mathfrak{G}^{(\infty)} =$ *Menge aller* $B^{(\infty)}(a)$. Ferner ist die Entfernung von $B^{(\infty)}(a)$ und $B^{(\infty)}(b)$ im Sinne von Anm. [21] $\leqq \operatorname{Max}\left(\frac{1}{\mu}, d\right)$, wenn d die Entfernung ihrer L_μ-Abschnitte, d. h. von $B^{(\mu)}(a)$, $B^{(\mu)}(b)$ ist. Somit ist der Diameter von $\mathfrak{H}_t' \leqq \operatorname{Max}\left(\frac{1}{\mu}, \bar{d}_t\right)$, wenn $\bar{d}_t$ der Diameter von $\mathfrak{H}_t$ ist, d. h. $\leqq \operatorname{Max}\left(\frac{1}{\mu}, \varepsilon\right)$. Für hinreichend großes $\mu\left(\mu \geqq \frac{1}{\varepsilon}\right)$ gilt also auch: 1'. jedes $\mathfrak{H}_t'$ hat einen Diameter $\leqq \varepsilon$. Nach der „Umkehrung des allgemeinen Zerlegungssatzes" (a. a. O. S. 157) ist daher $\mathfrak{G}^{(\infty)} \leqq P$-dimensional. Da es aber G homöomorph ist, ist es, wie dieses, n-dimensional. Also ist $P \geqq n$.

Somit ist $P = n$, d. h. $P_{\bar{\mu}} = P_{\bar{\mu}+1} = \cdots = n$.

7. Die in 5. aufgestellte homöomorphe Abbildung von $x_1, \cdots, x_{P_\mu}$ auf $a(x_1, \cdots, x_{P_\mu})$ bildet nach dem soeben Bewiesenen für $\mu \geqq \bar{\mu}$ (wir halten μ fest) die $x_1, \cdots, x_n$ auf die $a(x_1 \cdots x_n)$ ab, u. zw. ein Gebiet $\tilde{K}^{(\mu)}$ des n-dimensionalen euklidischen Raumes auf eine Teilmenge U von $U(1)$, das einem Gebiet des n-dimensionalen euklidischen Raumes homöomorph ist. Aus Brouwers Satz von der Gebietsinvarianz (vgl. a. a. O. Anm. [16]) folgt daher, daß dieses Bild U innere Punkte besitzt.

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[23] Um besseren Anschluß an die in Anm. [21] definierte Entfernung bei unendlich vieldimensionalen Matrizen zu wahren, definieren wir bei P-dimensionalen Matrizen die Entfernung so (das ist der in a. a. O. Anm. [7] verwendeten, und zur gewöhnlichen Topologie führenden, topologisch gleichwertig):

$$\text{Entf.}\left(\{\alpha_{jl}\}, \{\beta_{jl}\}\right) = \operatorname*{Max}_{j,l=1,\cdots,P}\left(|\alpha_{jl} - \beta_{jl}|\right).$$

Seien a, b Punkte von U, $B^{(\mu)}(a) = B^{(\mu)}(b)$ $(\mu \geqq \bar{\mu})$. Es ist

$$a = a(x_1 \cdots x_n), \qquad B^{(\mu)}(a) = \exp(x_1 U_1^{(\mu)} + \cdots + x_n U_n^{(\mu)}),$$

ebenso

$$b = b(y_1 \cdots y_n), \qquad B^{(\mu)}(b) = \exp(y_1 U_1^{(\mu)} + \cdots + y_n U_n^{(\mu)}),$$

also

$$\exp(x_1 U_1^{(\mu)} + \cdots + x_n U_n^{(\mu)}) = \exp(y_1 U_1^{(\mu)} + \cdots + y_n U_n^{(\mu)}),$$

und da $x_1, \cdots, x_n$ und $y_1, \cdots, y_n$ zu $\bar{K}^{(\mu)}$ gehören,

$$x_1 = y_1, \cdots, x_n = y_n.$$

Also ist $a = b$. Für $a \neq b$ ist somit

$$B^{(\mu)}(a) \neq B^{(\mu)}(b).$$

Die Menge der $x \cdot c$, x in U, heiße $U \cdot c$. Ist $\bar{a}$ innerer Punkt von U, so ist stets a innerer Punkt von $U \cdot (\bar{a}^{-1} \cdot a)$, d. h. jeder Punkt von G ist innerer Punkt eines $U \cdot c$. Da G kompakt ist, genügen endlich viele $U \cdot c$, etwa $U \cdot c_1, \cdots, U \cdot c_{\bar{r}}$, um ganz G zu überdecken. Da auch in $U \cdot c$ aus $a \neq b$ $B^{(\mu)}(a) \neq B^{(\mu)}(b)$ folgt (denn $a \cdot c^{-1}, b \cdot c^{-1}$ gehören zu U, also ist wegen $a \cdot c^{-1} \neq b \cdot c^{-1}$ $B^{(\mu)}(a \cdot c^{-1}) \neq B^{(\mu)}(b \cdot c^{-1})$, $B^{(\mu)}(a) \cdot B^{(\mu)}(c)^{-1} \neq B^{(\mu)}(b) \cdot B^{(\mu)}(c)^{-1}$, $B^{(\mu)}(a) \neq B^{(\mu)}(b)$), erkennen wir: in jedem $U \cdot c_r$ $(r = 1, \cdots, \bar{r})$ kommt höchstens ein a mit $B^{(\mu)}(a) = E_{L_\mu}$ vor, insgesamt gibt es also $\leqq \bar{r}$ solche a.

Nennen wir die Menge der a mit $B^{(\mu)}(a) = E_{L_\mu}$ $\mathfrak{E}_\mu$, sie ist offenbar eine Untergruppe von G, $\mathfrak{E}_1 \supseteq \mathfrak{E}_2 \supseteq \cdots$. Nach obigem ist sie für $\mu \geqq \bar{\mu}$ endlich. Die Elementzahlen der endlichen $\mathfrak{E}_{\bar{\mu}} \supseteq \mathfrak{E}_{\bar{\mu}+1} \supseteq \cdots$ sind monoton-nichtwachsend, also von einer gewissen Stelle $\mu = \bar{\bar{\mu}}$ an konstant. Die $\mathfrak{E}_{\bar{\mu}} \supseteq \mathfrak{E}_{\bar{\mu}+1} \supseteq \cdots$ haben alle gleich viele Elemente, also ist $\mathfrak{E}_{\bar{\mu}} = \mathfrak{E}_{\bar{\mu}+1} = \cdots$. Wenn a zu $\mathfrak{E}_{\bar{\mu}}$ gehört, so gehört es demnach zu allen $\mathfrak{E}_\mu$, $\mu \geqq \bar{\bar{\mu}}$, mithin sind alle $B^{(\mu)}(a) = E_{L_\mu}$ (für $\mu \geqq \bar{\bar{\mu}}$, also für alle μ). Daher ist auch $B^{(\infty)}(a)$ die Einheitsmatrix, also $= B^{(\infty)}(1)$, aber $B^{(\infty)}(x)$ ist ein-eindeutig (vgl. 3.), also $a = 1$. Da somit $\mathfrak{E}_{\bar{\mu}}$ nur die 1 enthält, ist für $a \neq 1$ $B^{(\bar{\mu})}(a) \neq E_{L_\mu}$, also für $a \neq b$ $B^{(\bar{\mu})}(a) \neq B^{(\bar{\mu})}(b)$. Also ist $B^{(\bar{\mu})}(a)$ ein-eindeutig, d. h. treu.

Hieraus folgt, daß $B^{(\bar{\mu})}(a)$ eine homöomorphe Abbildung ist, G ist $\mathfrak{G}^{(\bar{\mu})}$ homöomorph und isomorph, so daß $B^{(\bar{\mu})}(a)$ und $\mathfrak{G}^{(\bar{\mu})}$ die Eigenschaften der in Einl. 2. erörterten $D(a)$, G' besitzen. Wir formulieren das Resultat:

SATZ I. *Die einzigen topologischen Gruppen, die (im Sinne von* I., 1.) *n-parametrig und kompakt sind, sind die abgeschlossenen Gruppen aus unitären Matrizen endlich vieldimensionaler (komplexer*[24]*) euklidischer Räume sowie die diesen homöomorph-isomorphen Gruppen.*

[24] Man kann sich auch aufs Reelle beschränken: dazu muß die Dimensionszahl des betreffenden Raumes verdoppelt und die imaginäre Einheit i durch die Matrix $\begin{Bmatrix} 0, & 1 \\ -1, & 0 \end{Bmatrix}$ ersetzt werden.

II. Transformationsgruppen G topologischer Räume M.

1. Eine topologische Gruppe G heiße eine stetige Transformationsgruppe von M, wenn jedem a von G eine homöomorphe (d. h. ein-eindeutige und stetige) Abbildung von M auf M zugeordnet ist: $F_a(\xi)$ (ξ durchläuft M); dabei soll noch

1. die Funktionalgleichung

$$F_b(F_a(\xi)) = F_{a \cdot b}(\xi)$$

gelten,

2. $F_a(\xi)$ ist als Funktion von a (bei festgehaltenem ξ) stetig.

Wir setzen im folgenden G n-parametrig und kompakt sowie in M transitiv voraus, über M machen wir dagegen keine weiteren Annahmen.

Somit ist G mit einer abgeschlossenen Gruppe unitärer Transformationen eines endlich viel- (etwa k-) dimensionalen Raumes homöomorph und isomorph (Satz I), und wir können es vorläufig mit der betr. Gruppe identifizieren. Ferner zeichnen wir ein Element ξ_0 von M aus und bilden für jedes ξ von M die Menge aller a von G mit $F_a(\xi_0) = \xi$: $H(\xi)$. $H(\xi)$ ist offenbar abgeschlossen und wegen der Transitivität nicht leer; $H(\xi_0)$ hat die Gruppeneigenschaft, ist also eine abgeschlossene Untergruppe von G; die $H(\xi)$ sind die Rechtsnebengruppen von $H(\xi_0)$. Jedes a von G gehört genau einem $H(\xi)$ an: dem mit $\xi = F_a(\xi_0)$, und demnach hängt ξ stetig von a ab.

Wir wenden den in I., 4. für die dortigen $\mathfrak{G}^{(\omega)}$ herangezogenen Satz auf G und $H(\xi_0)$ an, dadurch mögen die (k-dimensionalen) Matrizen $U_1', \cdots, U_n'$ und $U_1, \cdots, U_l$ entstehen. Aus Brouwers Satz von der Invarianz der Dimensionszahl (a. a. O. Anm. [16]) folgt, da G gleichzeitig n- und n'-parametrig ist, $n = n'$. Da $H(\xi_0)$ Untergruppe von G ist, sind die $U_1, \cdots, U_l$ Linearaggregate der $U_1', \cdots, U_n'$. Somit ist $l \leqq n$, und wir können die $U_1', \cdots, U_n'$ so umbezeichnen (nämlich linear transformieren), daß

$$U_1' = U_1, \cdots, U_l' = U_l$$

wird. Wir lassen daher die Akzente überhaupt fort: zu G gehören $U_1, \cdots, U_n$, zu $H(\xi_0)$ $U_1, \cdots, U_l$ ($l \leqq n$).

2. Wir betrachten die k-dimensionalen Matrizen $\exp(n_1 U_{l+1} + \cdots + n_{n-l} U_n)$ ($n_1, \cdots, n_{n-l}$ reell) und behaupten: es gibt eine Umgebung U^* von $0, \cdots, 0$ im (reellen) $n - l$-dimensionalen euklidischen Raume, so daß für $u_1, \cdots, u_{n-l}$ und $v_1, \cdots, v_{n-l}$ aus U^* und a, b aus $H(\xi_0)$

$$\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n) \cdot a = \exp(v_1 U_{l+1} + \cdots + v_{n-l} U_n) \cdot b$$

die Folge

$$u_1 = v_1, \cdots, u_{n-l} = v_{n-l}$$

(und damit auch $a = b$) hat. Letzteres kann auch

$$\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n)^{-1} \cdot \exp(v_1 U_{l+1} + \cdots + v_{n-l} U_n) = a \cdot b^{-1}$$

geschrieben werden, d. h. es besagt, daß die linke Seite zu $H(\xi_0)$ gehört.

Wäre die obige Behauptung falsch, so gäbe es zwei Folgen $u_1^{(\nu)}, \cdots, u_{n-l}^{(\nu)}$ und $v_1^{(\nu)}, \cdots, v_{n-l}^{(\nu)}$ $(\nu = 1, 2, \cdots)$, beide gegen $0, \cdots, 0$ konvergierend, so daß für kein ν

$$u_1^{(\nu)} = v_1^{(\nu)}, \cdots, u_{n-l}^{(\nu)} = v_{n-l}^{(\nu)}$$

ist, und für jedes ν

$$\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n)^{-1} \exp(v_1 U_{l+1} + \cdots + v_{n-l} U_n)$$

zu $H(\xi_0)$ gehört. Da dieser Ausdruck gegen E_k konvergiert, kann er nach dem in I., 4. verwendeten Satze für fast alle ν in der Form $\exp(w_1^{(\nu)} U_1 + \cdots + w_l^{(\nu)} U_l)$ geschrieben werden, wobei auch $w_1^{(\nu)}, \cdots, w_l^{(\nu)}$ gegen $0, \cdots, 0$ konvergiert. Die vorliegende Gleichheit kann auch

$$\exp(w_1^{(\nu)} U_{l+1} + \cdots + u_{n-l}^{(\nu)} U_n) \cdot \exp(w_1^{(\nu)} U_1 + \cdots + w_l^{(\nu)} U_l) = \exp(v_1^{(\nu)} U_{l+1} + \cdots + v_{n-l}^{(\nu)} U_n)$$

geschrieben werden, d. h. der Ausdruck

$$\exp(x_{l+1} U_{l+1} + \cdots + x_n U_n) \cdot \exp(x_1 U_1 + \cdots + x_l U_l)$$

hat bei Ersetzung der Variablenreihe $x_1, \cdots, x_n$ durch $w_1^{(\nu)}, \cdots, w_l^{(\nu)}, u_1^{(\nu)}, \cdots, u_{n-l}^{(\nu)}$ und durch $0, \cdots, 0, v_1^{(\nu)}, \cdots, v_{n-l}^{(\nu)}$ denselben Wert. Wenn er, falls $x_1, \cdots, x_n$ auf eine hinreichend kleine Umgebung von $0, \cdots, 0$ beschränkt werden, $x_1, \cdots, x_n$ eindeutig festlegt, so hat das Obige für fast alle ν $u_1^{(\nu)} = v_1^{(\nu)}, \cdots, u_{n-l}^{(\nu)} = v_{n-l}^{(\nu)}$ zur Folge, d. h. einen Widerspruch mit der Annahme.

Wir entwickeln den obigen Ausdruck in eine (überall konvergente) Potenzreihe[25] und deuten alle Glieder von Graden $\geqq 2$ nur durch *** an. Er ist dann gleich:

$$1 + x_1 U_1 + \cdots + x_n U_n + ***.$$

Die Matrizen $U_1, \cdots, U_n$ sind (reell) linear unabhängig. Wir ergänzen sie zu einem vollständigen linear unabhängigen System, indem wir $U_{n+1}, \cdots, U_{2k^2}$ irgendwie demgemäß hinzufügen. Den obigen Ausdruck minus 1 stellen wir dann als Linearaggregat von $U_1, \cdots, U_{2k^2}$ dar, die Koeffizienten sind dann eindeutig bestimmt, u. zw. sind diejenigen von $U_1, \cdots, U_n$, wenn wir alle Glieder von Graden $\geqq 2$ durch *** andeuten, gleich

$$x_1 + ***, \cdots, x_n + ***.$$

[25] Die Potenzreihenbetrachtung ließe sich vermeiden, aber am Ende von 3. kommt ohnehin eine solche vor.

Wenn wir zeigen können, daß die Gleichungen

$$x_1 + *** = y_1, \cdots, x_n + *** = y_n$$

nach $x_1, \cdots, x_n$ (falls diese auf eine hinreichend kleine Umgebung von $0, \cdots, 0$ beschränkt werden) aufgelöst werden können, so sind wir am Ziele. Sie können aber offenbar durch $y_1, \cdots, y_n$-Potenzreihen für $x_1, \cdots, x_n$ aufgelöst werden. Damit ist die Existenz der Umgebung U^* von $0, \cdots, 0$ mit der weiter oben behaupteten Eigenschaft bewiesen.

Diese Eigenschaft kann auch so formuliert werden: Wenn $u_1, \cdots, u_{n-l}$ in U^* liegt, so liegt jedes $\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n)$ in einer anderen Menge $H(\xi)$. Das zugehörige $\xi = \xi(u_1, \cdots, u_{n-l})$ in M ist somit eindeutige Funktion von $u_1, \cdots, u_{n-l}$ und auch stetige Funktion derselben, da es stetig vom Ausdruck $\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n)$ abhängt.

3. In dem in I., 4. verwendeten Satze kam eine Umgebung $\mathfrak{R}^{(\mu)}$ von 1 in $\mathfrak{G}^{(\mu)}$ und eine Umgebung $K^{(\mu)}$ von $0, \cdots, 0$ im P_μ-dimensionalen Raume vor; wir ersetzen jetzt $\mathfrak{G}^{(\mu)}$ durch $H(\xi_0)$, P_μ durch $n-l$, dann mögen $\mathfrak{R}^{(\mu)}$, $K^{(\mu)}$ $\mathfrak{H}$, H heißen. Wenn $u_1, \cdots, u_{n-l}$ in U^* liegt und $w_1, \cdots, w_l$ in H, so nimmt

$$a(u_1 \cdots u_{n-l} w_1 \cdots w_l) = \exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n) \cdot \exp(w_1 U_1 + \cdots + w_l U_l)$$

denselben Wert nicht zweimal an: Denn dazu müßten die $\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n)$ im selben $H(\xi)$ liegen, also die $u_1, \cdots, u_{n-l}$ dieselben sein; alsdann müßte $\exp(w_1 U_1 + \cdots + w_l U_l)$ denselben Wert haben, also auch $w_1, \cdots, w_l$ dieselben sein. Somit bildet es die beschriebene $u_1, \cdots, u_{n-l} w_1, \cdots, w_l$-Menge homöomorph auf eine Teilmenge von G ab. Da aber die erstere Menge ein Gebiet des n-dimensionalen (reellen) euklidischen Raumes ist und da das Bild Teilmenge einer Umgebung der 1 in G ist, welche letztere einem ebensolchen Gebiet homöomorph angenommen werden kann (G ist n-parametrig!), muß nach dem Brouwerschen Satz von der Gebietsinvarianz (vgl. a. a. O. Anm.[16], übrigens ist er hier, wo es sich durchweg um analytische Abbildungen handelt, selbstverständlich) das Bild ebenfalls offen sein (in G). Somit ist es eine Umgebung W der (dazugehörigen) 1.

Wenn ein ξ von M nahe genug bei ξ_0 liegt, so ist es ein $\xi(u_1 \cdots u_{n-l})$, $u_1, \cdots, u_{n-l}$ in U^*. Sonst gäbe es nämlich eine gegen ξ_0 konvergente Folge $\xi^{(\nu)}$ ($\nu = 1, 2, \cdots$), in der das nie der Fall ist. Nun ist $\xi^{(\nu)}$ jedenfalls $= F_{a^{(\nu)}}(\xi_0)$, eine Teilfolge der $a^{(\nu)}$, konvergiert, da G kompakt ist; also können wir annehmen, daß es die $a^{(\nu)}$-Folge selbst tut. Der Limes sei a_0, dann ist $F_{a_0}(\xi_0)$ des Limes der $\xi^{(\nu)} = F_{a^{(\nu)}}(\xi_0)$, d. h. ξ_0, so daß a_0 zu $H(\xi_0)$ gehört. Also ist auch $\xi^{(\nu)} = F_{a^{(\nu)} a_0^{-1}}(\xi_0)$, und $a^{(\nu)} a_0^{-1}$ konvergiert gegen 1; wir können daher annehmen, daß es schon die $a^{(\nu)}$ taten, d. h. $a_0 = 1$. Dann liegen fast alle $a^{(\nu)}$ in der Menge der

$$\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n)^{-1} \cdot \exp(v_1 U_{l+1} + \cdots + v_{n-l} U_n) = a \cdot b^{-1}$$

geschrieben werden, d. h. es besagt, daß die linke Seite zu $H(\xi_0)$ gehört.

Wäre die obige Behauptung falsch, so gäbe es zwei Folgen $u_1^{(\nu)}, \cdots, u_{n-l}^{(\nu)}$ und $v_1^{(\nu)}, \cdots, v_{n-l}^{(\nu)}$ $(\nu = 1, 2, \cdots)$, beide gegen $0, \cdots, 0$ konvergierend, so daß für kein ν

$$u_1^{(\nu)} = v_1^{(\nu)}, \cdots, u_{n-l}^{(\nu)} = v_{n-l}^{(\nu)}$$

ist, und für jedes ν

$$\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n)^{-1} \exp(v_1 U_{l+1} + \cdots + v_{n-l} U_n)$$

zu $H(\xi_0)$ gehört. Da dieser Ausdruck gegen E_k konvergiert, kann er nach dem in I., 4. verwendeten Satze für fast alle ν in der Form $\exp(w_1^{(\nu)} U_1 + \cdots + w_l^{(\nu)} U_l)$ geschrieben werden, wobei auch $w_1^{(\nu)}, \cdots, w_l^{(\nu)}$ gegen $0, \cdots, 0$ konvergiert. Die vorliegende Gleichheit kann auch

$$\exp(w_1^{(\nu)} U_{l+1} + \cdots + u_{n-l}^{(\nu)} U_n) \cdot \exp(w_1^{(\nu)} U_1 + \cdots + w_l^{(\nu)} U_l) = \exp(v_1^{(\nu)} U_{l+1} + \cdots + v_{n-l}^{(\nu)} U_n)$$

geschrieben werden, d. h. der Ausdruck

$$\exp(x_{l+1} U_{l+1} + \cdots + x_n U_n) \cdot \exp(x_1 U_1 + \cdots + x_l U_l)$$

hat bei Ersetzung der Variablenreihe $x_1, \cdots, x_n$ durch $w_1^{(\nu)}, \cdots, w_l^{(\nu)}, u_1^{(\nu)}, \cdots, u_{n-l}^{(\nu)}$ und durch $0, \cdots, 0, v_1^{(\nu)}, \cdots, v_{n-l}^{(\nu)}$ denselben Wert. Wenn er, falls $x_1, \cdots, x_n$ auf eine hinreichend kleine Umgebung von $0, \cdots, 0$ beschränkt werden, $x_1, \cdots, x_n$ eindeutig festlegt, so hat das Obige für fast alle ν $u_1^{(\nu)} = v_1^{(\nu)}, \cdots, u_{n-l}^{(\nu)} = v_{n-l}^{(\nu)}$ zur Folge, d. h. einen Widerspruch mit der Annahme.

Wir entwickeln den obigen Ausdruck in eine (überall konvergente) Potenzreihe[25] und deuten alle Glieder von Graden $\geqq 2$ nur durch *** an. Er ist dann gleich:

$$1 + x_1 U_1 + \cdots + x_n U_n + ***.$$

Die Matrizen $U_1, \cdots, U_n$ sind (reell) linear unabhängig. Wir ergänzen sie zu einem vollständigen linear unabhängigen System, indem wir $U_{n+1}, \cdots, U_{2k^2}$ irgendwie demgemäß hinzufügen. Den obigen Ausdruck minus 1 stellen wir dann als Linearaggregat von $U_1, \cdots, U_{2k^2}$ dar, die Koeffizienten sind dann eindeutig bestimmt, u. zw. sind diejenigen von $U_1, \cdots, U_n$, wenn wir alle Glieder von Graden $\geqq 2$ durch *** andeuten, gleich

$$x_1 + ***, \cdots, x_n + ***.$$

[25] Die Potenzreihenbetrachtung ließe sich vermeiden, aber am Ende von 3. kommt ohnehin eine solche vor.

Wenn wir zeigen können, daß die Gleichungen

$$x_1 + *** = y_1, \cdots, x_n + *** = y_n$$

nach $x_1, \cdots, x_n$ (falls diese auf eine hinreichend kleine Umgebung von $0, \cdots, 0$ beschränkt werden) aufgelöst werden können, so sind wir am Ziele. Sie können aber offenbar durch $y_1, \cdots, y_n$-Potenzreihen für $x_1, \cdots, x_n$ aufgelöst werden. Damit ist die Existenz der Umgebung U^* von $0, \cdots, 0$ mit der weiter oben behaupteten Eigenschaft bewiesen.

Diese Eigenschaft kann auch so formuliert werden: Wenn $u_1, \cdots, u_{n-l}$ in U^* liegt, so liegt jedes $\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n)$ in einer anderen Menge $H(\xi)$. Das zugehörige $\xi = \xi(u_1, \cdots, u_{n-l})$ in M ist somit eineindeutige Funktion von $u_1, \cdots, u_{n-l}$ und auch stetige Funktion derselben, da es stetig vom Ausdruck $\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n)$ abhängt.

3. In dem in I., 4. verwendeten Satze kam eine Umgebung $\mathfrak{R}^{(\mu)}$ von 1 in $\mathfrak{G}^{(\mu)}$ und eine Umgebung $K^{(\mu)}$ von $0, \cdots, 0$ im P_μ-dimensionalen Raume vor; wir ersetzen jetzt $\mathfrak{G}^{(\mu)}$ durch $H(\xi_0)$, P_μ durch $n-l$, dann mögen $\mathfrak{R}^{(\mu)}$, $K^{(\mu)}$ $\mathfrak{H}$, H heißen. Wenn $u_1, \cdots, u_{n-l}$ in U^* liegt und $w_1, \cdots, w_l$ in H, so nimmt

$$a(u_1 \cdots u_{n-l} w_1 \cdots w_l) = \exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n) \cdot \exp(w_1 U_1 + \cdots + w_l U_l)$$

denselben Wert nicht zweimal an: Denn dazu müßten die $\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n)$ im selben $H(\xi)$ liegen, also die $u_1, \cdots, u_{n-l}$ dieselben sein; alsdann müßte $\exp(w_1 U_1 + \cdots + w_l U_l)$ denselben Wert haben, also auch $w_1, \cdots, w_l$ dieselben sein. Somit bildet es die beschriebene $u_1, \cdots, u_{n-l} w_1, \cdots, w_l$-Menge homöomorph auf eine Teilmenge von G ab. Da aber die erstere Menge ein Gebiet des n-dimensionalen (reellen) euklidischen Raumes ist und da das Bild Teilmenge einer Umgebung der 1 in G ist, welche letztere einem ebensolchen Gebiet homöomorph angenommen werden kann (G ist n-parametrig!), muß nach dem Brouwerschen Satz von der Gebietsinvarianz (vgl. a. a. O. Anm.[16], übrigens ist er hier, wo es sich durchweg um analytische Abbildungen handelt, selbstverständlich) das Bild ebenfalls offen sein (in G). Somit ist es eine Umgebung W der (dazugehörigen) 1.

Wenn ein ξ von M nahe genug bei ξ_0 liegt, so ist es ein $\xi(u_1 \cdots u_{n-l})$, $u_1, \cdots, u_{n-l}$ in U^*. Sonst gäbe es nämlich eine gegen ξ_0 konvergente Folge $\xi^{(\nu)}$ ($\nu = 1, 2, \cdots$), in der das nie der Fall ist. Nun ist $\xi^{(\nu)}$ jedenfalls $= F_{a^{(\nu)}}(\xi_0)$, eine Teilfolge der $a^{(\nu)}$, konvergiert, da G kompakt ist; also können wir annehmen, daß es die $a^{(\nu)}$-Folge selbst tut. Der Limes sei a_0, dann ist $F_{a_0}(\xi_0)$ des Limes der $\xi^{(\nu)} = F_{a^{(\nu)}}(\xi_0)$, d. h. ξ_0, so daß a_0 zu $H(\xi_0)$ gehört. Also ist auch $\xi^{(\nu)} = F_{a^{(\nu)} a_0^{-1}}(\xi_0)$, und $a^{(\nu)} a_0^{-1}$ konvergiert gegen 1; wir können daher annehmen, daß es schon die $a^{(\nu)}$ taten, d. h. $a_0 = 1$. Dann liegen fast alle $a^{(\nu)}$ in der Menge der

$$\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n) \cdot \exp(w_1 U_1 + \cdots + w_l U_l),$$

$u_1, \cdots, u_{n-l}$ in U^*, $w_1, \cdots, w_l$ in H. $a^{(\nu)}$, also auch dieses

$$\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_{l+1}),$$

gehört zu $H(\xi^{(\nu)})$, daher ist

$$\xi^{(\nu)} = \xi(u_1, \cdots, u_{n-l}),$$

entgegen der Voraussetzung.

$\xi(u_1, \cdots, u_{n-l})$ bildet somit U^* homöomorph auf eine Teilmenge von M ab, von der ξ_0 innerer Punkt ist. Sei O eine in ihr enthaltene Umgebung von ξ_0, wegen der Homöomorphie der Abbildung ist das Urbild U_0 von O eine (Relativ-) Umgebung von $0, \cdots, 0$ in U^* — d. h. eine Umgebung von $0, \cdots, 0$.

Damit ist die Umgebung O von ξ_0 in M homöomorph auf die $n-l$ Parameter $u_1, \cdots, u_{n-l}$ in U_0 bezogen. In einer Umgebung $\mathfrak{R}$ von 1 in G ist, wie wir wissen, G durch $a = \exp(\alpha_1 U_1 + \cdots + \alpha_n U_n)$ homöomorph auf die n Parameter $\alpha_1, \cdots, \alpha_n$ in K bezogen. Das Kompositionsgesetz von G lautet:

$$\exp(\alpha_1 U_1 + \cdots + \alpha_n U_n) \cdot \exp(\beta_1 U_1 + \cdots + \beta_n U_n) = \exp(\gamma_1 U_1 + \cdots + \gamma_n U_n)$$

(alles in K), u. zw. ist diese Gleichung in hinreichender Nähe von $0, \cdots, 0$ eindeutig nach $\gamma_1, \cdots, \gamma_n$ auflösbar. Das Transformationsgesetz von G in M lautet (soweit die linke Seite in W liegt):

$$\exp(u_1 U_{l+1} + \cdots + u_{n-l} U_n) \cdot \exp(\alpha_1 U_1 + \cdots + \alpha_n U_n) = \exp(v_1 U_{l+1} + \cdots + v_{n-l} U_n) \cdot \exp(w_1 U_1 + \cdots + w_l U_l)$$

($u_1, \cdots, u_{n-l}$ und $v_1, \cdots, v_{n-l}$ in U^*, $w_1, \cdots, w_l$ in H), u. zw. ist auch diese Gleichung in hinreichender Nähe von $0, \cdots, 0$ eindeutig nach $v_1, \cdots, v_{n-l}, w_1, \cdots, w_l$ auflösbar (es interessieren uns nur $v_1, \cdots, v_{n-l}$). Wir haben noch zu zeigen, daß $\gamma_1, \cdots, \gamma_n$ von $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ und $v_1, \cdots, v_{n-l}, w_1, \cdots, w_l$ von $u_1, \cdots, u_{n-l}, \alpha_1, \cdots, \alpha_n$ analytische Funktionen sind. Die erste Aufgabe ist Spezialfall der zweiten: mit $l = 0$, $\alpha_1, \cdots, \alpha_n$ für $u_1, \cdots, u_n$, $\beta_1, \cdots, \beta_n$ für $\alpha_1, \cdots, \alpha_n$, $\gamma_1, \cdots, \gamma_n$ für $v_1, \cdots, v_n$; daher genügt es, diese zu betrachten. Wenn wir in der obigen Gleichung beide Seiten durch ihre (überall konvergenten) Potenzreihen ersetzen und alle Glieder von Graden $\geqq 2$ nur durch *** andeuten, so entsteht die Matrixgleichung:

$$1 + \alpha_1 U_1 + \cdots + \alpha_l U_l + (\alpha_{l+1} + u_1) U_{l+1} + \cdots + (\alpha_n + u_{n-l}) U_n + \cdots + *** = 1 + w_1 U_1 + \cdots + w_l U_l + v_1 U_{l+1} + \cdots + v_{n-l} U_n + ***,$$

woraus (da $U_1, \cdots, U_n$ linear unabhängig sind) jedenfalls Zahlengleichungen

$$\alpha_1 + *** = w_1 + ***, \cdots, \alpha_l + \cdots = w_l + ***,$$
$$\alpha_{l+1} + u_1 + *** = v_1 + ***, \cdots, \alpha_n + u_{n-l} + *** = v_{n-l} + ***$$

folgen. Diese können offenbar durch $u_1, \cdots, u_{n-l}, \alpha_1, \cdots, \alpha_n$-Potenzreihen für $v_1, \cdots, v_{n-l}, w_1, \cdots, w_l$ aufgelöst werden, womit die Analytizität dieser Größen bewiesen ist.

Unser Resultat lautet also:

SATZ II. *Die topologische Gruppe G sei (im Sinne von* I., 1.) *n-parametrig und kompakt und (im Sinne von* II., 1.) *eine stetige Transformationsgruppe des topologischen Raumes M*[26]. *Ferner sei G in M transitiv. Es ist möglich, eine Umgebung W von 1 in G und eine Umgebung U_0 eines beliebigen ξ_0 in M anzugeben, so daß W und U_0 homöomorph auf n bzw. m Parameter ($m = 0, 1, \cdots, n$*[27]*) bezogen werden können (die gewisse Umgebungen von $0, \cdots, 0$ im n- bzw. m-dimensionalen [reellen] euklidischen Raume durchlaufen), so daß sowohl das Kompositionsgesetz von G, als auch das Transformationsgesetz von M durch G in diesen Parametern analytisch ausfällt.*

4. Wir lassen jetzt die Annahme der Kompaktheit von G und seiner Transitivität in M fallen. Dagegen verschärfen wir die übrigen Annahmen: M sei in einer geeigneten Umgebung eines jeden seiner Punkte homöomorph auf m Parameter beziehbar, und $F_a(\xi)$ sei als Zweivariablenfunktion von a, ξ stetig[28]. Es soll gezeigt werden, daß es u. U. unmöglich ist, die Parameter für a und ξ so zu wählen, daß $a \cdot b$ und $F_a(\xi)$ analytisch oder auch nur stetig differentiierbar ausfallen.

Es wird, wie in Einl. 1 erwähnt wurde, $n = 1$, $m = 2$ sein, u. zw. sei G die reelle Zahlengerade, d. h. a eine beliebige reelle Zahl, das Kompositionsgesetz die Addition $a + b$. M sei die reelle u, v-Ebene. $\varphi(r)$ sei eine für alle $r > 0$ definierte, stetige (reelle) Funktion, die zu a gehörige Transformation von M sei eine Drehung mit dem Mittelpunkt 0, 0, aber um einen von r abhängigen Winkel auf jedem Kreise $u^2 + v^2 = r^2$: um $\varphi(r)\, a$. D. h.

$$F_a(u, v) = u \cos\{\varphi(\sqrt{u^2+v^2})\, a\} - v \sin\{\varphi(\sqrt{u^2+v^2})\, a\},$$
$$u \sin\{\varphi(\sqrt{u^2+v^2})\, a\} + v \cos\{\varphi(\sqrt{u^2+v^2})\, a\}.$$

Durch geeignete Wahl von $\varphi(r)$ werden wir die gewünschten Beispiele gewinnen.

[26] Man beachte, daß die Beziehung von M zu Parametern nicht Voraussetzung ist; sie wird vielmehr bewiesen!

[27] m tritt an Stelle des $n - l$ im Beweise.

[28] Im Falle der Kompaktheit und Transitivität wurde beides durch Satz I bewiesen, die a, ξ-Stetigkeit insbesondere dadurch, daß $F_a(\xi)$ sogar analytisch gewählt werden konnte.

Wenn $\varphi(1) = 0$ für alle $r > 0, \leqq r_0$ ($r_0 > 0$, fest) gilt, aber für kein $r > r_0$, so liegt in jeder Umgebung des Punktes ξ_0 gleich $r_0, 0$ ein Gebiet, in dem $F_a(\xi) = \xi$ ist, aber auch ein anderes Gebiet, wo dies nicht gilt. Da eine analytische Funktion ein solches Verhalten nicht zeigen kann, ist eine Parameterwahl, bei der $F_a(\xi)$ analytisch wird, unmöglich. Somit erledigt z. B. $\varphi(r) = \text{Max}(0, r - r_0)$ die Annahme der Analytizität.

Etwas schwieriger ist der Fall der stetigen Differentiierbarkeit. Hier schließen wir so, daß wir annehmen, daß in je einer Umgebung von 0 bzw. 0, 0 solche Parameter a' für a und u', v' für u, v eingeführt wurden, daß $a' \cdot b'$ und $F_{a'}(u', v')$ stetig differentiierbar sind, und daraus folgern, daß $\varphi(r)$ gewisse Eigenschaften besitzen muß. Alsdann wählen wir $\varphi(r)$ so, daß es dieselben nicht besitzt, womit auch die Annahme der stetigen Differentiierbarkeit entfällt.

5. Das Kompositionsgesetz für a' heiße $a' \circ b'$, die Abbildung von a auf a' werde durch die ein-eindeutige und stetige Funktion $a = f(a')$ vermittelt. $f(a')$ ist somit monoton (wachsend oder fallend), nach einem Satze von Lebesgue[29] gibt es also Stellen, wo es differentiierbar ist.

$a' \cdot b'$ ist als a'-Funktion differentiierbar, $a' \circ b'^{-1}$ auch; hätte also $a' \circ b'$ für ein $a' = a'_0$ die Ableitung 0, so hätte sie $(a' \circ b') \circ b'^{-1} = a'$ dort auch. Aber dies ist unmöglich, da a' stets die Ableitung 1 hat. Es ist

$$f(a' \circ b') = f(a') + f(b'),$$

sei $f(a')$ für $a' = a'_1$ differentiierbar; da $a' \circ b'$ die a'-Ableitung $\neq 0$ hat, ist auch $f(a')$ für $a' = a'_1 \cdot b'$ differentiierbar. D. h. $f(a')$ ist überall differentiierbar. Dabei ist

$$\frac{\partial}{\partial a'} f(a')\Big|_{a'=a'_1 \cdot b'} = \frac{\frac{\partial}{\partial a'} f(a')\Big|_{a'=a'_1}}{\frac{\partial}{\partial a'} (a' \circ b')\Big|_{a'=a'_1}},$$

also

$$\frac{\partial}{\partial a'} f(a')\Big|_{a'=\bar{a}'} = \frac{\frac{\partial}{\partial a'} f(a')\Big|_{a'=a'_1}}{\frac{\partial}{\partial a'} (a' \circ [\bar{a}' \circ a_1'^{-1}])\Big|_{a'=a'_1}},$$

also $f(a')$ stetig differentiierbar. Da stets $\frac{\partial}{\partial a'} f(a') \neq 0$ ist (käme $\frac{\partial}{\partial a'} f(a') = 0$ überhaupt vor, so müßte es wegen der obigen Formel immer gelten, d. h. $f(a')$ wäre konstant), ist es auch die Inverse. Wir können daher statt a'

[29] Vgl. z. B. a. a. O. Anm. [12], Satz 7 auf S. 573.

wieder den Parameter a einführen, $F_{a'}(u', v')$ bleibt stetig differentiierbar, d. h. wir können annehmen, daß von vornherein $a' = a$ war.

Wenn die zwei Komponenten von $\frac{\partial}{\partial a} F_a(u', v')$[30] für $a = \bar{a}$ und ein gewisses u', v' verschwinden, so müssen sie es wegen

$$F_{a+b}(u', v') = F_b(F_a(u', v'))$$

auch für $a = \bar{a} + b$ tun, d. h. für alle a. Dann ist $F_a(u', v')$ von a unabhängig, was $u, v = 0, 0$, also $u', v' = 0, 0$ nach sich zieht. Für $u', v' \neq 0, 0$ sind also die zwei Komponenten von $\frac{\partial}{\partial a} F_a(u', v')$ stets $\neq 0, 0$. $\varphi(\sqrt{u^2+v^2})$ ist in u, v, also auch in u', v' stetig; wir nehmen an, daß es stets >0 ist, dann ist auch $\frac{2\pi}{\varphi(\sqrt{u^2+v^2})}$ in u', v' stetig. Da aber identisch

$$F_{\frac{2\pi}{\varphi(\sqrt{u^2+v^2})}}(u', v') = u', v'$$

gilt, so folgt aus dem soeben festgestellten Verhalten von $\frac{\partial}{\partial a} F_a(u, v)$, daß $\frac{2\pi}{\varphi(\sqrt{u^2+v^2})}$ für $u', v' \neq 0, 0$ (nach u', v') stetig differentiierbar ist, also $\varphi(\sqrt{u^2+v^2})$ auch.

Dem Kreis $u^2+v^2 = r^2$ $(r>0, <r_1$, wobei $u^2+v^2<r_1^2$ eine Umgebung von $0, 0$ ist, in der die Parameterdarstellung noch gilt) entspricht in der u', v'-Ebene eine Jordankurve J_r. Aus den Lagen der Kreise folgt, daß $0, 0$ und jedes J_s mit $s<r$ im Inneren von J_r liegt. $\varphi(\sqrt{u^2+v^2})$ ist auf J_r konstant.

Sei C das Maximum von $\frac{\partial}{\partial u'} \varphi(\sqrt{u^2+v^2})$ und $\frac{\partial}{\partial v'} \varphi(\sqrt{u^2+v^2})$ im Inneren von $J_{\frac{r_1}{2}}$. Sei weiter $0<r_2<r_3<\frac{r_1}{2}$, r_2, r_3 fest. Wir wählen ein u_2', v_2' auf J_{r_2} und ein u_3', v_3' auf J_{r_3}, ihre Verbindungsstrecke sei S. Seien nun $\nu-1$ beliebige Zahlen $\varrho_1, \cdots, \varrho_{\nu-1}$ mit $r_2<\varrho_1<\cdots<\varrho_{\nu-1}<r_3$ gegeben, wir setzen noch $\varrho_0 = r_2$, $\varrho_\nu = r_3$. S schneidet jedes ϱ_μ $(\mu = 1, \cdots, \nu-1)$, etwa in $u'^{(\mu)}, v'^{(\mu)}$, u. zw. können wir $u^{(\mu)}, v^{(\mu)}$ zwischen $u^{(\mu-1)}, v^{(\mu-1)}$ und u_3, v_3 wählen, was für alle $\mu = 2, 3, \cdots, \nu$ geschehen soll. Naturgemäß

$$u'^{(0)}, v'^{(0)} = u_2', v_2', \qquad u'^{(\nu)}, v'^{(\nu)} = u_3', v_3'.$$

[30] $F_a(u', v')$ ist ein Punkt der u', v'-Ebene, besteht also aus zwei reellen Funktionen.

Dann liegen die $u^{(\mu)}$, $v^{(\mu)}$ in der Reihenfolge ihrer μ auf S, und es ist:

$$|\varphi(\varrho_\mu) - \varphi(\varrho_{\mu-1})| \leqq 2C \cdot \text{Entf.}\,(u'^{(\mu)}, v'^{(\mu)}; u'^{(\mu-1)}, v'^{(\mu-1)}),$$

$$\sum_{\mu=1}^{\nu} |\varphi(\varrho_\mu) - \varphi(\varrho_{\mu-1})| \leqq 2C \cdot \sum_{\mu=1}^{\nu} \text{Entf.}\,(u'^{(\mu)}, v'^{(\mu)}; u'^{(\mu-1)}, v'^{(\mu-1)})$$

$$= 2C \cdot \text{Entf.}\,(u'^{(\nu)}, v'^{(\nu)}; u'^{(0)}, v'^{(0)}) = C \cdot \text{Entf.}\,(u_3', v_3'; u_2', v_2').$$

Wir haben also, unter der Voraussetzung, daß für alle $r>0$, $<r_1$ $\varphi(r)>0$ ist, folgendes gezeigt: Es gibt zwei r_2, r_3, $0<r_2<r_3<\frac{r_1}{2}$, so daß für alle Systeme $\varrho_0, \varrho_1, \cdots, \varrho_\nu$ mit $r_2 = \varrho_0 < \varrho_1 < \cdots < \varrho_{\nu-1} < \varrho_\nu = r_3$

$$\sum_{\mu=1}^{\nu} |\varphi(\varrho_\mu) - \varphi(\varrho_{\mu-1})|$$

unterhalb einer festen (d. h. nur von r_2, r_3 abhängigen) Schranke bleibt. Wenn wir also ein stetiges $\varphi(r)$ angeben können, derart, daß stets $\varphi(r)>0$ ist und für jedes $r_2<r_3$ die obere Grenze der

$$\sum_{\mu=1}^{\nu} |\varphi(\varrho_\mu) - \varphi(\varrho_{\mu-1})| \qquad (r_2 = \varrho_1 < \varrho_2 < \cdots < \varrho_{\nu-1} < \varrho_\nu = r_3)$$

gleich ∞ ist, so sind wir am Ziele. D. h. es wird ein stetiges positives $\varphi(r)$ gesucht, das in jedem Intervalle von unendlicher Variation ist, und Beispiele solcher $\varphi(r)$ sind bekannt[31].

[31] Vgl. z. B. a. a. O. Anm. [12], S. 590–594; man bilde mit der dort konstruierten, „nirgends differentiierbaren Funktion von Weierstraß“ $w(x)$ $\varphi(r) = 1 + w(r)$. Zur Unendlichkeit der Variation vgl. S. 590 ebendort.

PRINCETON, N. J.

AN ABSTRACT APPROACH TO MANIFOLDS.[1]

By A. W. Tucker.

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Introduction.

The treatment of combinatorial Topology here presented had its origin in attempts to answer two questions:

Under what general conditions can aggregates of cells be regarded for combinatorial purposes as single cells?

To what extent are the criteria defining combinatorial manifolds forced upon the topologist?

Consideration of these questions was evoked—in the first case, directly, and in the second, indirectly—by Lefschetz' *Topology*.[2] As a natural consequence the present treatment has much in common with that text. On it our terminology and notation are based, although new standpoints have forced the invention of many—probably too many—new terms. To assist the reader in the latter respect an index of terms has been included.

Contact is made with such signal contributions to topological literature as Veblen's book,[3] Alexander's fundamental paper,[4] and van Kampen's thesis.[5]

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[1] Received August 6, 1932.—A preliminary account of this paper appeared as a note *On combinatorial Topology*. Proc. Nat. Acad. 18 (1932), pp. 86–89.

[2] Lefschetz, S., *Topology*. Am. Math. Soc. Colloquium Publications, XII (1930).

[3] Veblen, O., *Analysis Situs*. Am. Math. Soc. Colloquium Publications, V Pt. II (1922); (second edition, 1931).

[4] Alexander, J. W., *Combinatorial Analysis Situs*. Trans. Am. Math. Soc. 28 (1926), pp. 301–29.

[5] van Kampen, E. R., *Die kombinatorische Topologie und die Dualitätssätze*. The Hague (1929).

The abstract method of attack—an early source for which is the work of Dehn and Heegaard[6]—allies itself with Weyl's important exposition,[7] elements of which de Rham has incorporated in the first chapter of his thesis,[8] and with a pair of interesting papers by Mayer.[9]

The first few sections of our paper are preparatory. In § 1 cells are introduced as undefined elements which fit together in accordance with certain postulated rules; by orientation of the cells the way is prepared for an algebra of chains. §§ 2 and 3 begin the development of the matrix technique which plays such an important part in the subsequent work. Consistent use of matrix method is a characteristic of the paper. In § 4 the complex is defined, and the homology groups thus brought into being are described in terms of certain canonical bases.

§ 5 presents the idea of cell duality, and § 6 the complementary notions of the open and the closed, as well as a definition of sub-complex. The paper systematically exploits these concepts as powerful instruments for organizing and extending the theory. The reciprocations afforded by dual and complement combine with one another, for the dual of an open (closed) sub-set is closed (open). The definition of sub-complex is made as general as the complex will permit, with a result that most of the sub-sets we have occasion to consider are sub-complexes. § 7 utilizes matrix manipulation to exhibit the inter-relations between a complex and complementary open and closed sub-complexes of that complex. In § 8 there is a discussion of connectedness and circuits, featuring a pair of simple separation theorems.

The next three sections pertain to the first question proposed at the beginning of this introduction. From the "block" of § 9—itself an important generalization of open and closed sub-sets— it is but a step to the "cell-block" of § 10, which furnishes a specific answer to the question proposed. Another step leads to a combinatorial characterization of subdivision in § 11.

By the introduction of "w-multiplication" in § 12 our algebra of chains becomes more nearly an algebra in the technical sense of the word. This multiplication is a common root for products, joins, and intersections, and so is a salient unifying feature. In terms of it § 13 sets forth formulas for the homology characters of a product complex.

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[6] Dehn, M., and Heegaard, P., *Analysis Situs.* Encykl. der M. W. III 1^1, A B 3 (1907), pp. 153–220.

[7] Weyl, H., *Análisis situs combinatorio.* Revista Mat.-Hisp.-Am. 5 (1923), p. 209ff.

[8] de Rham, G., *Sur l'Analysis situs des variétés à n dimensions.* J. de Math. IX s., 10 (1931), pp. 115–200.

[9] Mayer, W., *Über abstrakte Topologie. Abstrakte Topologie II.* Monatsh. für Math. u. Phys. 36 (1929), pp. 1–42 and 219–258, respectively.

At this stage the second of our initial questions is broached. An abstract treatment of intersections in § 14, relating them in particular to subdivision, provides a line of attack by means of which §§ 15 and 16 obtain criteria which determine the absolute manifold. Concerning these criteria it is to be noticed that the conditions yielding duality relations seem to be less restrictive than those yielding invariance under subdivision as well. § 17 records various properties of the absolute manifold, in connection with which it presents the Kronecker index and looping coefficients.

In §§ 18 and 19 the simplex and the simplicial complex are introduced through the medium of joins. § 20 discusses the "focal" subdivision obtained by replacing a cell by the join of its boundary cells with a new vertex, and § 21 the resultant regular subdivision which is possible if all the cells of the complex admit focal subdivision. These considerations of subdivision apply even to a complex having a "fringe" of openness—an important extension.

The question of manifolds and the criteria determining them now recurs with the combined forces of intersections and regular subdivisions brought into action. In §§ 22 and 23 modified conditions are developed which lead to interesting results, involving in particular the concept of manifold with regular boundary. § 24 shows how manifolds, their intersections, etc. may be realized geometrically.

The remaining sections are mainly concerned with the purely combinatorial part of what is known in general topology as the theory of transformations, coincidences, and fixed points. § 25 ushers in the subject; § 26 applies it to certain correspondences arising from welds and subdivision. In § 27 certain results on "fixed cells" are obtained following a simple matrix proof of the generalized Euler-Poincaré formula. § 28 gives (combinatorial) reasons why correspondences can be handled through the medium of products, intersections, Kronecker indices, etc.

For his impetus in topology the writer is indebted to the lectures of Professor Lefschetz and the seminars of Professor Alexander. In particular he expresses his thanks for the stimulus and guidance he has received from Professor Lefschetz in the preparation of this paper.

Abstract foundations.

1. **Cells. Chains.** We deal with a finite set of elements, called *cells*. With each cell $|E_p|$ is associated a non-negative integer p, called its *dimension*. Between any two cells $|E_p|$, $|E_q|$ there may exist a relation: $|E_p| \prec |E_q|$, read $|E_p|$ *is on the boundary of* $|E_q|$. We postulate that

$$|E_p| \not\prec |E_q| \quad \text{if} \quad p \geqq q, \tag{1.1}$$

and that $\prec$ is transitive in that

(1.2) $\quad |E_p| \prec |E_r|$ if $|E_p| \prec |E_q|$ and $|E_q| \prec |E_r|$.

Based on the p-cells, i. e. the cells of dimension p, is a free Abelian group having $+$ as its rule of combination. The elements of this group are called *p-chains*. The identity element is the null chain, denoted by 0 irrespective of the value of p. Other conventions are as usual in addition. To describe the explicit sense in which the group of p-chains is based on the p-cells we postulate that *to each p-cell corresponds a unique pair of p-chains whose sum is the null chain* and that *by choosing one p-chain from each said pair we obtain a complete set of independent generators for the group*. The chosen p-chain of the pair corresponding to a p-cell $|E_p|$ is called the *oriented* p-cell E_p, the choice itself being termed *orientation*.

With every pair, E_{p+1}, E_p of oriented cells of consecutive dimensions is associated an integer, called their *incidence number*. Change of orientation of either cell of the pair changes the sign of the incidence number. We postulate that the incidence number of E_{p+1} and E_p is zero if

(1.3) $\quad |E_p| \not\prec |E_{p+1}|$.

Intuitively the incidence number of E_{p+1} and E_p describes the algebraic number of times E_p is on the boundary of E_{p+1}. Suppose, for example, the opposite sides of a rectangle are matched so as to give rise to a cylinder or Möbius strip. The 1-cell formed by the matched sides appears 0 or ± 2 times on the boundary of the 2-cell. The two 0-cells formed by the matched vertices appear ± 1 times on the boundaries of the 1-cells. Orientation of course determines the ambiguities of sign.

The novelty, if any, in the foundations (cf. Mayer, l. c. [9], pp. 2—4) we have laid down occurs in the relation $\prec$. The usual procedure has been to consider E_p on the boundary of E_q only if E_q, E_{q-1}, $\cdots$, E_{p+1}, E_p existed such that the incidence number of each pair of consecutive cells is not zero. That this is sometimes inadequate is shown by the above cylinder example, since the 1-cell obtained by matching sides would not be on the boundary of the 2-cell. Another example would be the configuration consisting of a 2-cell formed by the surface of a sphere with one point omitted and a 0-cell which is the omitted point, for there would be no 1-cells to use to show the 0-cell is on the boundary of the 2-cell.

2. **A set of cells.** From now on we assume that our cells are oriented. Accordingly we drop the use of the absolute value sign even in such expressions as (1.1) where orientation plays no part.

Consider a set of cells K. Let α_p denote the number of cells having dimension p, and if $\alpha_p > 0$ let $E_p^1, E_p^2, \cdots, E_p^{\alpha_p}$ denote these cells. The dimension n of K is the maximum p for which $\alpha_p > 0$.

Any p-chain generated by cells of K, and so expressible as

$$C_p = r_1 E_p^1 + r_2 E_p^2 + \cdots + r_{\alpha_p} E_p^{\alpha_p}, \tag{2.1}$$

the r's being integers, is said to be *on* K. In accordance with our foundations the set of p-chains on K is a free Abelian group G_p. Let $\{C_p^j\}$ be a *basis* for G_p, by which we mean a set of elements of G_p in terms of which every element may be uniquely expressed. (With non-free Abelian groups, which we shall encounter later, uniqueness can still be had by using residues, partly or wholly, as coefficients). There is at least one basis for G_p, viz., $\{E_p^i\}$. If $\{D_p^k\}$, like $\{C_p^j\}$, is a basis for G_p, the members of $\{C_p^j\}$ can be uniquely expressed in terms of the members of $\{D_p^k\}$. The matrix of coefficients used in doing this we denote simply by $[\{C_p^j\}/\{D_p^k\}]$. In conformity with this notation we denote an individual coefficient by $[C_p^j/D_p^k]$; accordingly

$$C_p^j = [C_p^j/D_p^k]\, D_p^k.^{10} \tag{2.2}$$

Conversely

$$D_p^k = [D_p^k/C_p^l]\, C_p^l. \tag{2.3}$$

Substituting (2.3) in (2.2), and vice versa,

$$[C_p^j/D_p^k]\,[D_p^k/C_p^l] = \delta_l^j \qquad (= 0 \text{ if } j \neq l,\ = 1 \text{ if } j = l), \tag{2.4}$$

$$[D_p^k/C_p^l]\,[C_p^l/D_p^m] = \delta_m^k. \tag{2.5}$$

Hence the matrices $[\{C_p^j\}/\{D_p^k\}]$, $[\{D_p^k\}/\{C_p^l\}]$, are inverse square matrices with determinants $= \pm 1$. Conversely, if $[\{D_p^k\}/\{C_p^l\}]$ is a square matrix with determinant $= \pm 1$, $\{D_p^k\}$ is a basis if $\{C_p^l\}$ is. In particular, we now know that every basis for G_p has the same number of members, viz., α_p.

A p-chain which may be taken as a member of a basis for G_p is called a *basic* p-chain of K. Clearly a necessary condition that a chain be basic

[10] We make the convention that if one of the indices $d, e, \cdots, m$ appears twice in the same term — once up and once down — it is to be summed. No attempt is made to associate an index with always the same range, but the range in each particular case should be clear from the context.

In accordance with this convention the sign / in $[C_p^j/D_p^k]$ is intended to lower the following index k. (The sign : will also be similarly used.) In (2.2) $[C_p^j/D_p^k]$ plays a rôle like that of the partial derivative appearing in the simple tensor transformation of a contravariant vector.

is that the coefficients expressing it in terms of any basis have no common divisor. (As a matter of fact this condition is also sufficient.)

3. Incidence matrices. The p-chain

$$F(E_{p+1}) = s_i E_p^i \tag{3.1}$$

where s_i is the incidence number of E_{p+1} and E_p, is called the *boundary chain* of E_{p+1} on K. If $\alpha_p = 0$ we agree that $F(E_{p+1}) = 0$. Passing on to $C_{p+1} = t_h E_{p+1}^h$ we have

$$F(C_{p+1}) = t_h F(E_{p+1}^h) \tag{3.2}$$

as the boundary chain of C_{p+1} on K.

Examination of (3.2) indicates that the set $\{F(C_{p+1})\}$ of all boundary p-chains on K is a sub-group F_p of G_p. In the language of groups (3.2) establishes a homomorphism (multiple isomorphism) between G_{p+1} and F_p.

A chain C_p on K such that $F(C_p) = 0$ is called a *p-cycle* on K. Clearly the set of all p-cycles on K is a sub-group Γ_p of G_p.

Supposing $\{C_{p+1}^h\}$ and $\{C_p^i\}$ bases for G_{p+1} and G_p, respectively, we can certainly express the boundary chains of the members of $\{C_{p+1}^h\}$ in terms of the members of $\{C_p^i\}$. The integral coefficients used in doing this form a matrix $[\{C_{p+1}^h\} : \{C_p^i\}]$, called a pth *incidence matrix* of K. An individual coefficient being $[C_{p+1}^h : C_p^i]$, we have

$$F(C_{p+1}^h) = [C_{p+1}^h : C_p^i]\, C_p^i. \tag{3.3}$$

When we wish to exhibit $[\{C_{p+1}^h\} : \{C_p^i\}]$ in detail we write it as

(3.4)

	C_p^1	C_p^2	.....	$C_p^{\alpha_p}$
C_{p+1}^1	x	x	.....	x
C_{p+1}^2	x	x	.....	x
⋮	⋮	⋮		⋮
$C_{p+1}^{\alpha_{p+1}}$	x	x	.....	x

Of course $[\{E_{p+1}^h\} : \{E_p^i\}]$ is a pth incidence matrix of K. By (3.1) each element $[E_{p+1}^h : E_p^i]$ is the incidence number of E_{p+1}^h and E_p^i. Hereafter we shall use this convenient notation for incidence numbers.

Let $\{D_{p+1}^g\}$ and $\{D_p^j\}$ also be bases for G_{p+1} and G_p, respectively. Like (3.3) we have

$$F(D_{p+1}^g) = [D_{p+1}^g : D_p^j]\, D_p^j, \tag{3.5}$$

and so, using (2.2) etc.,

$$[D^g_{p+1}: D^j_p] = [D^g_{p+1}/C^h_{p+1}]\,[C^h_{p+1}: C^i_p]\,[C^i_p/D^j_p], \tag{3.6}$$

$$[C^h_{p+1}: C^i_p] = [C^h_{p+1}/D^g_{p+1}]\,[D^g_{p+1}: D^j_p]\,[D^j_p/C^i_p]. \tag{3.7}$$

Accordingly $[\{C^h_{p+1}\}:\{C^i_p\}]$, $[\{D^g_{p+1}\}:\{D^j_p\}]$ have the same rank ϱ_p and the same G. C. D.'s, which means that we have proved:

(3.8) *All pth incidence matrices of K are equivalent, i. e., have the same numbers of rows and columns, the same rank and the same G. C. D.'s.*

Let b_p denote a basis for G_p. If the i_1th member of b_p be replaced by its negative or by itself plus an integer λ times the i_2th member ($i_2 \neq i_1$), a new basis b'_p is obtained. The passage from b_p to b'_p is called an *elementary* change of basis. The inverse change is clearly also elementary. (Cf. the "generating transformations" of Alexander, l. c. [4], p. 317).

Let $[b_{p+1}: b_p]$ denote the pth incidence matrix determined by b_{p+1} and b_p. The change to $[b_{p+1}: b'_p]$ may be described in terms of the (3.4) type of representation as a change of sign in the elements of the i_1th column or the *subtraction* from each element of the i_2th column of λ times the corresponding element of the i_1th column. Similarly the change from $[b_p: b_{p-1}]$ to $[b'_p: b_{p-1}]$ may be described as a change of sign in the elements of the i_1th row or the *addition* to each element of the i_1th row of λ times the corresponding element of the i_2th row.

Focusing attention on a pth incidence matrix we see that by elementary changes of the bases of G_{p+1} and G_p we may achieve the matrix changes which secure the well-known reduction to a matrix containing solely diagonal elements and zeros. Instead of working from the upper left corner as is usual, however, let us work from the lower left. In this way passage is made to a pth incidence matrix $[\{C^h_{p+1}\}:\{C^i_p\}]$ having the form

(3.9)

	C^1_p	C^2_p	C^3_p	.....	$C^{\alpha_p}_p$
C^1_{p+1}	0	0	0	.....	
⋮	⋮	⋮	⋮		⋮
$C^{\alpha_{p+1}-2}_{p+1}$	0	0	x	.....	0
$C^{\alpha_{p+1}-1}_{p+1}$	0	x	0	.....	0
$C^{\alpha_{p+1}}_{p+1}$	x	0	0	.....	0

the x's here being the *invariant factors* of the matrix. Such a matrix we shall call *canonical*. Hence

(3.10) *By elementary changes of basis passage may be made from any pair of bases for G_{p+1} and G_p to a pair whose incidence matrix is canonical.*

4. **The complex.** We call a set of cells a *complex* if every boundary chain on the set is a cycle on the set; i. e., K is a complex if

$$F(F(C_{p+1})) = 0 \tag{4.1}$$

for an arbitrary C_{p+1} on it. (Cf. Mayer's *Komplexring*, l. c. [9], pp. 2-4.) Since every such C_{p+1} is linearly expressible in terms of a basis $\{C^h_{p+1}\}$, (4.1) is equivalent to

$$F(F(C^h_{p+1})) = 0, \tag{4.2}$$

h and p being arbitrary. However, in terms of bases $\{C^i_p\}$, $\{C^j_{p-1}\}$

$$F(F(C^h_{p+1})) = [C^h_{p+1} : C^i_p]\,[C^i_p : C^j_{p-1}]\,C^j_{p-1}. \tag{4.3}$$

Accordingly

$$[C^h_{p+1} : C^i_p]\,[C^i_p : C^j_{p-1}] = 0 \qquad (\textit{all } p) \tag{4.4}$$

is a necessary and sufficient condition that K be a complex.

The importance of (4.1) is that F_p becomes a sub-group of Γ_p, thus giving rise to a difference (factor) group $H_p = \Gamma_p - F_p$, called the pth *homology group* of K. A cycle belonging to F_p is named, for obvious reasons, a *bounding* cycle. Two cycles belonging to the same co-set of F_p, i. e., differing by a bounding cycle, are said to be homologous on K—in symbols

$$C_p \sim D_p \qquad (\text{on } K).$$

Hence $C_p \sim 0$ denotes a bounding cycle.

Choose bases b_{p+1}, b_p such that $[b_{p+1} : b_p]$ is canonical. Then no matter what basis b_{p-1} is taken for G_{p-1} we know from (4.4) that the first ϱ_p rows of $[b_p : b_{p-1}]$ consist of zeros. By elementary changes in the basis of G_{p-1} and in the last $\alpha_p - \varrho_p$ members of the basis of G_p we may pass to bases b'_{p-1}, b^*_p such that $[b^*_p : b'_{p-1}]$ is canonical. Since b_p and b^*_p differ only in their last $\alpha_p - \varrho_p$ members, $[b_{p+1} : b^*_p] = [b_{p+1} : b_p]$ and so is canonical. Hence

(4.5) *If $[b_{p+1} : b_p]$ is canonical, K being a complex, bases b^*_p and b'_{p-1} can be found such that $[b_{p+1} : b^*_p]$ and $[b^*_p : b'_{p-1}]$ are both canonical.*

Repeated use of (4.5) gives

(4.6) *Bases for the groups of chains on a complex can be found such that the corresponding incidence matrices are all canonical.*

The convenience of this result justifies our unusual definition of a canonical matrix (cf. Lefschetz, l. c. [2], p. 27).

Suppose $b_n, \cdots, b_{p+1}, b_p, b_{p-1}, \cdots$ are bases taken in accord with (4.6). Let b'_p denote the first ϱ'_p (= the number of unit invariant factors of a pth incidence matrix of K) members of b_p, let b^2_p denote the next ϱ''_p (= $\varrho_p - \varrho'_p$ = the number of invariant factors >1) members of b_p, b^3_p

the next $\alpha_p - \varrho_p - \varrho_{p-1}$ members, b_p^4 the next ϱ''_{p-1} members, and b_p^5 the last ϱ'_{p-1} members. Then $[b_{p+1} : b_p]$ may be represented by

(4.7)

	b_p^1	b_p^2	b_p^3	b_p^4	b_p^5
b_{p+1}^1	0	0	0	0	0
b_{p+1}^2	0	0	0	0	0
b_{p+1}^3	0	0	0	0	0
b_{p+1}^4	0	t	0	0	0
b_{p+1}^5	1	0	0	0	0

,

where 0 represents a sub-matrix of zeros, 1 a square sub-matrix of zeros except for unit elements up the secondary diagonal, and t a square sub-matrix of zeros except for integers (invariant factors) >1 in ascending order up the secondary diagonal. Of course $[b_p : b_{p-1}]$ has a precisely similar form.

We see that b_p^1, b_p^2, b_p^3 jointly form a basis for Γ_p; b_p^1 and tb_p^2, where tb_p^2 represents the members of b_p^2 each multiplied by the corresponding invariant factor in t, jointly form a basis for F_p. F_p and Γ_p are clearly free groups. H_p has as basis the cycles of b_p^2 and b_p^3 together, or more exactly the co-sets of Γ_p with respect to F_p to which these cycles belong. The group having as basis the co-sets to which the members of b_p^2 belong is a finite group T_p, called the pth *torsion group*. The orders of the elements of the said basis of T_p are the invariant factors in t. These numbers, denoted by $\theta_i^p(K)$, are called pth *torsion coefficients* of K. The group having as basis the co-sets to which the members of b_p^3 belong is a free group B_p, called the pth *Betti group*.[11] The number of elements, $\alpha_p - \varrho_p - \varrho_{p-1}$, denoted by $R_p(K)$, in the said basis for B_p is called the pth *Betti number* of K. Clearly H_p is the direct sum of B_p and T_p. Hence the pth torsion coefficients and Betti number, characterizing H_p as they do, are termed pth *homology characters* of K.

A knowledge of the elementary divisors, i. e., the powers of primes occurring in the invariant factors >1, is equivalent to a knowledge of the invariant factors >1. Hence we shall call the elementary divisors of a pth incidence matrix pth *torsion divisors* of K.

For future reference we introduce some terms descriptive of complexes possessing no torsion coefficients and very simple Betti numbers. If all the Betti numbers are zero we shall call a complex *null-like*, since its homology characters are those of the null set. If all the Betti numbers

[11] Cf. p. 133, Hopf, H., *Eine Verallgemeinerung der Euler-Poincaréschen Formel.* Göttinger Nachr. (1928), p. 127-136.

are zero except the tth which is one, we shall call a complex *t-cell-like*, since it has the same homology characters as the t-cell. If all the Betti numbers are zero except the 0th and uth which are both one, u (>0) being the dimension of the complex, we shall call a complex *sphere-like*; but if the 0th and uth Betti numbers are arbitrary we shall term a complex *quasi-sphere-like*.

Dual and complement.

5. **Dual cells.** To each cell E_p, $p \leqq n$, let there correspond a cell E^{n-p} called its *dual*, or more precisely n-dual, such that

$$E^{n-q} \leqq E^{n-p} \text{ if, and only if, } E_p \leqq E_q, \tag{5.1}$$

$$[E^{n-p} : E^{n-p-1}] = [E_{p+1} : E_p]. \tag{5.2}$$

The set of cells $\{E_i^{n-p}\}$ dual to the cells $\{E_p^i\}$ of K we call the dual (more precisely n-dual) of K and denote by K^*. (Cf. Mayer's *reziproker, dualer Ring*, l. c. [9], p. 18.)

(5.3) *A necessary and sufficient condition that K^* be a complex is that K be a complex.*

By (4.4) a necessary and sufficient condition that K^* be a complex is that

$$[E_j^{n-p+1} : E_i^{n-p}]\,[E_i^{n-p} : E_h^{n-p-1}] = 0 \qquad \text{(all } n-p\text{)}, \tag{5.4}$$

which due to (5.2) is precisely equivalent to

$$[E_{p+1}^h : E_p^i]\,[E_p^i : E_{p-1}^j] = 0. \tag{5.5}$$

This gives the theorem since, of course, (5.5) is a necessary and sufficient condition that K be a complex.

The identification by (5.2) of the elements of the pth incidence matrix $[\{E_{p+1}^h\} : \{E_p^i\}]$ of K with the $(n-p-1)$st incidence matrix $[\{E^{n-p}\} : \{E^{n-p-1}\}]$ of K^* gives

$$\theta_i^p(K) = \theta_i^{n-p-1}(K^*). \tag{5.6}$$

Moreover

$$\begin{aligned} R_{n-p}(K^*) &= \alpha_{n-p}^* - \varrho_{n-p}^* - \varrho_{n-p-1}^* \\ &= \alpha_p - \varrho_{p-1} - \varrho_p. \end{aligned}$$

So

$$R_p(K) = R_{n-p}(K^*). \tag{5.7}$$

6. **Open and closed sub-sets.** The *star* $St(\{E_p\})$, (*closure*, $Cl(\{E_p\})$,) on K of a set $\{E_p\}$ of cells of K consists of all cells E_q (E_r) of K such that

$$E_p \leqq E_q \quad (E_r \leqq E_p). \tag{6.1}$$

If a set of cells of K is its own star (closure) on K we say that it is an *open* (*closed*) sub-set of K. The terms open, closed, closure bear close analogy to the same terms in point-set theory. A star is the counterpart of a neighborhood.

(6.2) *The complement of an open (closed) sub-set of K is a closed (open) sub-set of K.*

(6.3) *The logical sum and product of two open (closed) sub-sets of K are open (closed) sub-sets of K.*

(6.4) *The logical product of an open and a closed sub-set of K is a sub-set closed on the former and open on the latter.*

(6.5) *If $'K$ is an open (closed) sub-set of K, $'K^*$ is a closed (open) sub-set of K^*.*

These theorems, (6.2)–(6.5), are immediate consequences of the definitions on which they are based.

A complex, all of whose cells belong to K, is called a *sub-complex* of K. We prove:

(6.6) *An open (closed) sub-set of a complex K is a sub-complex of K.*

Since cells of course form bases the theorem is true, by (4.2), if for each E_{p+1} of the sub-set

$$F(F(E_{p+1})) = 0, \tag{6.7}$$

the boundary chains being on the sub-set. For a closed sub-set (6.7) follows from the fact that the boundary chain on the sub-set of a cell of the sub-set is the same as that on K. By passing to dual cells closed and open are interchanged, and so the theorem is true for an open sub-set.

(6.8) *The logical sum of two complexes each open (closed) with respect to that sum is a complex.*

The proof follows much the same lines as that of (6.6). For the closed case the boundary chain $F(E_{p+1})$ on the sum is the same as $F(E_{p+1})$ on the given complex that contains E_{p+1}, or the same as $F(E_{p+1})$ on either given complex if both contain it. Hence from the validity of (6.7) on one or other given complex we conclude the validity of (6.7) on the sum. The open case follows by taking duals.

We call the logical product of the star of E_p on K and the closure of E_q on K the *intercept*, $It(E_p, E_q)$, of E_p and E_q on K. If K is a complex $St(E_p)$ and $Cl(E_q)$ are, by (6.6), sub-complexes of K, and so by (6.4) and (6.6) $It(E_p, E_q)$ is a sub-complex also. Of course $It(E_p, E_q)$ is vacuous unless $E_p \leqq E_q$.

7. **Complementary open and closed sub-complexes.** Let our complex K be divided into complementary sub-complexes ${}^1K$ and ${}^2K$, closed and open, respectively. By adding a superscript 1 or 2 to the

are zero except the tth which is one, we shall call a complex *t-cell-like*, since it has the same homology characters as the t-cell. If all the Betti numbers are zero except the 0th and uth which are both one, u (>0) being the dimension of the complex, we shall call a complex *sphere-like*; but if the 0th and uth Betti numbers are arbitrary we shall term a complex *quasi-sphere-like*.

Dual and complement.

5. **Dual cells.** To each cell E_p, $p \leqq n$, let there correspond a cell E^{n-p} called its *dual*, or more precisely n-dual, such that

$$E^{n-q} \leqq E^{n-p} \text{ if, and only if, } E_p \leqq E_q, \tag{5.1}$$

$$[E^{n-p} : E^{n-p-1}] = [E_{p+1} : E_p]. \tag{5.2}$$

The set of cells $\{E_i^{n-p}\}$ dual to the cells $\{E_p^i\}$ of K we call the dual (more precisely n-dual) of K and denote by K^*. (Cf. Mayer's *reziproker, dualer Ring*, l. c. [9], p. 18.)

(5.3) *A necessary and sufficient condition that K^* be a complex is that K be a complex.*

By (4.4) a necessary and sufficient condition that K^* be a complex is that

$$[E_j^{n-p+1} : E_i^{n-p}]\,[E_i^{n-p} : E_h^{n-p-1}] = 0 \qquad \text{(all } n-p\text{)}, \tag{5.4}$$

which due to (5.2) is precisely equivalent to

$$[E_{p+1}^h : E_p^i]\,[E_p^i : E_{p-1}^j] = 0. \tag{5.5}$$

This gives the theorem since, of course, (5.5) is a necessary and sufficient condition that K be a complex.

The identification by (5.2) of the elements of the pth incidence matrix $[\{E_{p+1}^h\} : \{E_p^i\}]$ of K with the $(n-p-1)$st incidence matrix $[\{E^{n-p}\} : \{E^{n-p-1}\}]$ of K^* gives

$$\theta_i^p(K) = \theta_i^{n-p-1}(K^*). \tag{5.6}$$

Moreover

$$\begin{aligned} R_{n-p}(K^*) &= \alpha_{n-p}^* - \varrho_{n-p}^* - \varrho_{n-p-1}^* \\ &= \alpha_p - \varrho_{p-1} - \varrho_p. \end{aligned}$$

So

$$R_p(K) = R_{n-p}(K^*). \tag{5.7}$$

6. **Open and closed sub-sets.** The *star* $St(\{E_p\})$, (*closure*, $Cl(\{E_p\})$,) on K of a set $\{E_p\}$ of cells of K consists of all cells E_q (E_r) of K such that

$$E_p \leqq E_q \quad (E_r \leqq E_p). \tag{6.1}$$

If a set of cells of K is its own star (closure) on K we say that it is an *open* (*closed*) sub-set of K. The terms open, closed, closure bear close analogy to the same terms in point-set theory. A star is the counterpart of a neighborhood.

(6.2) *The complement of an open (closed) sub-set of K is a closed (open) sub-set of K.*

(6.3) *The logical sum and product of two open (closed) sub-sets of K are open (closed) sub-sets of K.*

(6.4) *The logical product of an open and a closed sub-set of K is a sub-set closed on the former and open on the latter.*

(6.5) *If $'K$ is an open (closed) sub-set of K, $'K^*$ is a closed (open) sub-set of K^*.*

These theorems, (6.2)–(6.5), are immediate consequences of the definitions on which they are based.

A complex, all of whose cells belong to K, is called a *sub-complex* of K. We prove:

(6.6) *An open (closed) sub-set of a complex K is a sub-complex of K.*

Since cells of course form bases the theorem is true, by (4.2), if for each E_{p+1} of the sub-set

$$F(F(E_{p+1})) = 0, \tag{6.7}$$

the boundary chains being on the sub-set. For a closed sub-set (6.7) follows from the fact that the boundary chain on the sub-set of a cell of the sub-set is the same as that on K. By passing to dual cells closed and open are interchanged, and so the theorem is true for an open sub-set.

(6.8) *The logical sum of two complexes each open (closed) with respect to that sum is a complex.*

The proof follows much the same lines as that of (6.6). For the closed case the boundary chain $F(E_{p+1})$ on the sum is the same as $F(E_{p+1})$ on the given complex that contains E_{p+1}, or the same as $F(E_{p+1})$ on either given complex if both contain it. Hence from the validity of (6.7) on one or other given complex we conclude the validity of (6.7) on the sum. The open case follows by taking duals.

We call the logical product of the star of E_p on K and the closure of E_q on K the *intercept*, $It(E_p, E_q)$, of E_p and E_q on K. If K is a complex $St(E_p)$ and $Cl(E_q)$ are, by (6.6), sub-complexes of K, and so by (6.4) and (6.6) $It(E_p, E_q)$ is a sub-complex also. Of course $It(E_p, E_q)$ is vacuous unless $E_p \leqq E_q$.

7. **Complementary open and closed sub-complexes.** Let our complex K be divided into complementary sub-complexes ${}^1K$ and ${}^2K$, closed and open, respectively. By adding a superscript 1 or 2 to the

items of notation we have built up for K we obtain items for ${}^1K$ or ${}^2K$; thus ${}^1E_p$ is a cell of ${}^1K$, ${}^2\varrho_p$ is the rank of a pth incidence matrix of ${}^2K$, etc. We adopt this convention as a permanent policy.

For each p let ${}^1b_p$ and ${}^2b_p$ be bases chosen in accord with (4.6). Then the resulting pth incidence matrix of K has the form:

(7.1)

	${}^1b_p^1$	${}^1b_p^2$	${}^1b_p^3$	${}^1b_p^4$	${}^1b_p^5$	${}^2b_p^1$	${}^2b_p^2$	${}^2b_p^3$	${}^2b_p^4$	${}^2b_p^5$
${}^1b_{p+1}^1$	0	0	0	0	0	0	0	0	0	0
${}^1b_{p+1}^2$	0	0	0	0	0	0	0	0	0	0
${}^1b_{p+1}^3$	0	0	0	0	0	0	0	0	0	0
${}^1b_{p+1}^4$	0	t	0	0	0	0	0	0	0	0
${}^1b_{p+1}^5$	1	0	0	0	0	0	0	0	0	0
${}^2b_{p+1}^1$	z	z	0	0	0	0	0	0	0	0
${}^2b_{p+1}^2$	z	x	0	0	0	0	0	0	0	0
${}^2b_{p+1}^3$	z	x	v	0	0	0	0	0	0	0
${}^2b_{p+1}^4$	z	x	x	x	z	0	t	0	0	0
${}^2b_{p+1}^5$	z	z	z	z	z	1	0	0	0	0

The 0's in the upper right "quarter-matrix", $[{}^1b_{p+1}:{}^2b_p]$, are due to ${}^1K$ being closed. The upper left and lower right "quarter-matrices", $[{}^1b_{p+1}:{}^1b_p]$ and $[{}^2b_{p+1}:{}^2b_p]$, are canonical by construction. The 0's in the lower left "quarter-matrix", $[{}^2b_{p+1}:{}^1b_p]$, are caused by the product condition (4.4) applied to (7.1) coupled in turn with the similar $(p+1)$st and $(p-1)$st incidence matrices. The v sub-matrix $[{}^2b_{p+1}^3:{}^1b_p^3]$ may be supposed canonical since we can make elementary changes in ${}^2b_{p+1}^3$ and ${}^1b_p^3$ without deranging anything else. The sub-matrices marked z may be reduced to zeros by using the unit invariant factors in the sub-matrices marked 1. In particular the sub-matrices $[{}^2b_{p+1}^1:{}^1b_p^2]$, $[{}^2b_{p+1}^4:{}^1b_p^5]$ become 0 through applying the product condition (4.4) after the reduction by the 1's has been accomplished in the $(p+1)$st and $(p-1)$st incidence matrices, respectively. The sub-matrices not marked z are undisturbed by the reduction of the z's.

From (7.1), or simple modifications thereof, inter-relations of K, ${}^1K$ and ${}^2K$ may be read off. For example, we see that

$$\varrho_p = {}^1\varrho_p + {}^2\varrho_p + {}^{1,2}\varrho_p, \tag{7.2}$$

where ${}^{1,2}\varrho_p$ denotes the rank of the v sub-matrix, $[{}^2b_{p+1}^3:{}^1b_p^3]$. Hence

$$R_p(K) = \{R_p({}^1K) - {}^{1,2}\varrho_p\} + \{R_p({}^2K) - {}^{1,2}\varrho_{p-1}\}, \tag{7.3}$$

the two terms in braces being each $\geqq 0$. (Cf. Lefschetz, l. c. [2], pp. 147–151).

Cycles on ${}^1K$, it should be remarked, are cycles on K but cycles on ${}^2K$ are not necessarily cycles on K. In general they are only cycles modulo ${}^1K$, i. e., neglecting the cells of ${}^1K$. Using the mod notation, ${}^2K = K \bmod {}^1K$, $R_p({}^2K) = R_p(K, {}^1K)$, etc. (Cf. Lefschetz, l. c.[2], p. 35).

If $R_p(K) = 0$, $n > p > 0$, (7.3) becomes

$$(7.4)\qquad \begin{cases} R_n(K) &= R_n({}^1K) + \{R_n({}^2K) - R_{n-1}({}^1K)\}, \\ R_p({}^2K) &= R_{p-1}({}^1K), \\ R_0(K) &= \{R_0({}^1K) - R_1({}^2K)\} + R_0({}^2K). \end{cases}$$

As for torsion nothing can be asserted in general save that any pth torsion divisor of K divides some pth torsion divisor of $[{}^2b^3_{p+1} : {}^1b^3_p]$. On the other hand any invariant factor of the v sub-matrix, $[{}^2b^3_{p+1} : {}^1b^3_p]$ divides some pth torsion coefficient of K. If however K is quasi-sphere-like we can show that

$$(7.5)\qquad \theta^p_i({}^2K) = \theta^{p-1}_i({}^1K) \qquad (n > p > 0).$$

Since $R_p(K) = 0$, $n > p > 0$, the v sub-matrix, $[{}^2b^3_{p+1} : {}^1b^3_p]$, must have maximum rank, all p, and be square $n > p+1 > 1$. Coupling this with the absence of torsion coefficients and the reducibility of the sub-matrices marked z in (7.1) it is evident that $[{}^2b^4_{p+1} : {}^1b^4_p]$, $n > p > 0$, must contain a minor of order ${}^2\varrho''_p$ (= order of $[{}^2b^4_{p+1} : {}^2b^2_p]$) with determinant prime to the determinant of $[{}^2b^4_{p+1} : {}^2b^2_p]$. Hence the number ${}^1\varrho''_{p-1}$ of columns of $[{}^2b^4_{p+1} : {}^1b^4_p] \geqq {}^2\varrho''_p$. A similar argument based on $[{}^2b^2_p : {}^1b^2_{p-1}]$, $n > p > 0$, gives ${}^2\varrho''_p \geqq {}^1\varrho''_{p-1}$. Therefore ${}^2\varrho''_p = {}^1\varrho''_{p-1}$ and the said minor is $[{}^2b^4_{p+1} : {}^1b^4_p]$ itself. Applying the product condition (4.4) we have

$$(7.6)\qquad [{}^2b^4_{p+1} : {}^1b^4_p][{}^1b^4_p : {}^1b^2_{p-1}] + [{}^2b^4_{p+1} : {}^2b^2_p][{}^2b^2_p : {}^1b^2_{p-1}] = 0,$$

from which we conclude, using the fact that the determinants of the matrices in (7.6) are prime in pairs, that

$$(7.7)\qquad [{}^1b^4_p : {}^1b^2_{p-1}] = [{}^2b^4_{p+1} : {}^2b^2_p]$$

since we know them both to be in canonical form. This proves (7.5) since (7.6) and (7.7) hold for $n > p > 0$.

Combining (7.4) and (7.5) we have

(7.8) *If ${}^1K$ and ${}^2K$ are complementary closed and open sub-complexes of a quasi-sphere-like complex K, the pth homology group ${}^2H_p$ is isomorphic with the $(p-1)$st homology group ${}^1H_{p-1}$, $n \geqq p \geqq 1$, except that the nth (0th) Betti group ${}^2B_n$ (${}^1B_0$) may have a basis exceeding that*

of the $(n-1)$*st* (1*st*) *Betti group* ${}^1B_{n-1}$ (${}^2B_1$) *by at most* $R_n(K)$ ($R_0(K)$) *members.* (Cf. Mayer, l. c. [9], pp. 221–24.)

We remark that the inter-relations of the complexes in (6.8) may be readily exhibited by putting the pth incidence matrix of the sum complex in suitable form suggested by (7.1). (Mayer, l. c. [9], pp. 31–42, derives inter-relations of this type.)

Let ${}^1K^*$ denote the set of cells of K^* dual to the cells of ${}^1K$. Then by applying (5.7) to ${}^1K$ and K, (7.4) becomes

$$(7.9)\quad \begin{cases} R_n(K) = R_0(K^*) = R_0({}^1K^*) + \{R_n({}^2K) - R_1({}^1K^*)\}, \\ R_p({}^2K) = R_{n-p-1}({}^1K^*) \qquad (n>p>1), \\ R_0(K) = R_n(K^*) = \{R_n({}^1K^*) - R_1({}^2K)\} + R_0({}^2K); \end{cases}$$

and applying (5.6) to ${}^1K$, (7.5) becomes

$$(7.10)\qquad \theta_i^p({}^2K) = \theta_i^{n-p}({}^1K^*).$$

These relations (cf. Lefschetz, l. c. [2], p. 144) provide a basis for duality theorems of the type suggested by the Alexander Duality Theorem[12] for a sub-(point)-set of an n-sphere. There are results, of course, formally similar to (7.9) and (7.10) when the change is to ${}^2K^*$ instead of ${}^1K^*$.

8. **Connectedness.** Two cells $E^{i_0}_{p_0}$, $E^{i_\nu}_{p_\nu}$ of K are said to be *connected on* K if cells $E^{i_1}_{p_1}, E^{i_2}_{p_2}, \cdots, E^{i_{\nu-1}}_{p_{\nu-1}}$ of K exist such that

$$E^{i_{\varkappa-1}}_{p_{\varkappa-1}} \prec E^{i_\varkappa}_{p_\varkappa} \quad \text{or} \quad E^{i_\varkappa}_{p_\varkappa} \prec E^{i_{\varkappa-1}}_{p_{\varkappa-1}} \qquad \varkappa = 1, 2, \cdots, \nu.$$

If the interpolated cells are of dimensions n and $(n-1)$ only, we term $E^{i_0}_{p_0}$, $E^{i_\nu}_{p_\nu}$ *strongly connected on* K. A set of cells is *connected* if every pair of cells of the set is connected on the set. It is readily seen that:

(8.1) *A necessary and sufficient condition that a set of cells be connected is that it contain no proper sub-set which is both open and closed.*

For convenience let us call a connected set K *strongly connected* if the star of each cell of K contains either one n-cell or several n-cells which are all strongly connected on the star. Then

(8.2) *A necessary and sufficient condition that a closed sub-set* L *of a strongly connected set* K *does not separate* K *is that the* n*-cells of* $K-L$ *be strongly connected on* $K-L$.

[N. B. L separates K if $K-L$ is not connected.]

[12] Alexander, J. W., *A proof and extension of the Jordan-Brouwer separation theorem.* Trans. Amer. Math. Soc., 23 (1922), pp. 333–349.

Since L is closed the star of a cell of $K-L$ is the same on $K-L$ as on K. Therefore $K-L$ is strongly connected if its n-cells are strongly connected. If the n-cells are not strongly connected we may divide them into complementary sets ${}^1\{E_n\}$ and ${}^2\{E_n\}$ such that no n-cell of one set is strongly connected on $K-L$ with an n-cell of the other. The closures of ${}^1\{E_n\}$ and ${}^2\{E_n\}$ on $K-L$ exhaust $K-L$, for the star of each cell contains an n-cell; and yet they have no cells in common, for a common cell would not have a strongly connected star. By (8.1), $K-L$ is not connected.

A basic n-cycle in which each n-cell of a complex K appears with a non-zero coefficient we call a *covering n-cycle* for K. If each coefficient is ± 1 the covering n-cycle is *simple*. For future reference we prove that:

(8.3) *If each $St(E_{n-1})$ of a complex K is n-cell-like and carries a basic n-cycle which contains with coefficient $+1$ every E_n for which $[E_n : E_{n-1}] \neq 0$, the sum of the n-cells of K is a simple covering n-cycle for K.*

The incidence matrix $[\{E_n\} : E_{n-1}]$ of $St(E_{n-1})$ must have rank 1, else $R_{n-1}(St(E_{n-1})) \neq 0$. Accordingly $\{E_n\}$ must consist of exactly two cells —say E_n', E_n''— else $R_n(St(E_{n-1})) \neq 1$. $[E_n' : E_{n-1}]$ and $[E_n'' : E_{n-1}]$ must be relatively prime, else $St(E_{n-1})$ has a torsion coefficient. Therefore a basic n-cycle on $St(E_{n-1})$ must be

$$\pm [E_n'' : E_{n-1}]\, E_n' \mp [E_n' : E_{n-1}]\, E_n'',$$

and so if one of $[E_n' : E_{n-1}]$, $[E_n'' : E_{n-1}] \neq 0$ the other by hypothesis $= \pm 1$. Since not both $= 0$, the rank being 1, $[\{E_n\} : E_{n-1}]$ has the form:

(8.4)

	E_{n-1}
E_n'	± 1
E_n''	∓ 1

.

This permits us to conclude that

$$C_n = \sum_{i=1}^{a_n} E_n^i \tag{8.5}$$

has no boundary on any $St(E_{n-1})$, and so is an n-cycle on K.

If (8.4) holds for each $St(E_{n-1})$ of K:

(8.6) *A necessary and sufficient condition that $R_n(K) = 1$ is that the n-cells of K be strongly connected on K.*

If the n-cells were not strongly connected C_n would split up into two or more distinct basic n-cycles, making $R_n(K) > 1$. On the other hand if the n-cells are strongly connected any n-cycle on K must be a multiple of C_n.

A complex K having a covering n-cycle, $R_n(K) = 1$, and each p-cell, $p < n$, on the boundary of at least one n-cell, is called an (orientable) *circuit.* If, under suitable orientation of the n-cells of K, (8.4) holds for each $St(E_{n-1})$, the circuit is *simple.* In the latter case the covering n-cycle is of course the simple one C_n. The first part of the argument under (8.6) shows that the n-cells of a circuit K are strongly connected on K. Accordingly K is connected, for by definition the rest of the cells are connected to the n-cells.

(8.7) *If a circuit L is a closed $(n-1)$-dimensional sub-complex of a strongly connected simple circuit K, a necessary and sufficient condition that L separate K is that the covering $(n-1)$-cycle on L bound (i. e. ~ 0) on K.*

Necessity: Let $K - L$ be divided into disconnected complementary sets ${}^1K$, ${}^2K$; let ${}^1C_n$ be the part of C_n lying on ${}^1K$. Then on ${}^1K + L$, which is a closed sub-set of K, $F({}^1C_n)$ is an $(n-1)$-cycle involving, with coefficients ± 1, $(n-1)$-cells which are on the boundary of just one n-cell of ${}^1K$. The ± 1 coefficients show $F({}^1C_n)$ to be basic. It lies on L since ${}^1C_n$ is a cycle on ${}^1K$. Therefore $F({}^1C_n) = \pm C_{n-1}$, the covering $(n-1)$-cycle of L. Incidentally this shows C_{n-1} to be simple.

Sufficiency: Let $F(C_n') = C_{n-1}$. By adding a multiple of C_n to C_n', if necessary, we can arrange that C_n' does not involve all the n-cells of K with non-zero coefficients. But any two n-cells strongly connected on $K - L$ must have the same coefficient in C_n'. Therefore the n-cells are not strongly connected on $K - L$, and so by (8.2) L separates K.

Cell Consolidation and Subdivision.

9. **Blocks.** Let K be divided into complementary sets ${}^1K$, ${}^2K$. If no cell of ${}^1K$ of dimension $\geqq t$ is on the boundary of a cell of ${}^2K$ and no cell of ${}^2K$ is on the boundary of a cell of ${}^1K$ of dimension $\leqq t$ we call ${}^2K$ a *t-block* of K. In other words a t-block of K is a sub-set closed modulo cells of dimensions $< t$ and open modulo cells of dimensions $> t$. It is the logical product of its star and its closure. Therefore by (6.4) and (6.6) it is a sub-complex of K if K is a complex.

The dual of a t-block of K is clearly an $(n-t)$-block of K^*. The logical product of a t-block of K with any sub-set of K is a t-block of the latter. The complement on K of a t-block is not in general a block. But when $t = n$ or 0, a t-block of K becomes precisely an open or closed sub-set of K and so the complement is also a block. A single t-cell is a t-block.

Let K be a complex having a t-block ${}^2K$ which is null-like except that $R_t({}^2K)$ may be arbitrary. Then K has as incidence matrices:

(9.1)

	$\{^1E_q\}$	$^2b^1_q$	$^2b^5_q$
$\{^1E_{q+1}\}$	x	z	z
$^2b^1_{q+1}$	0	0	0
$^2b^5_{q+1}$	0	1	0

	$\{^1E_t\}$	$^2b^1_t$	$^2b^3_t$	$^2b^5_t$
$\{^1E_{t+1}\}$	x	z	x	0
$^2b^1_{t+1}$	0	0	0	0
$^2b^5_{t+1}$	0	1	0	0

	$\{^1E_{t-1}\}$	$^2b^1_{t-1}$	$^2b^5_{t-1}$
$\{^1E_t\}$	x	0	0
$^2b^1_t$	0	0	0
$^2b^3_t$	x	0	0
$^2b^5_t$	z	1	0

	$\{^1E_r\}$	$^2b^1_{r-1}$	$^2b^5_{r-1}$
$\{^1E_r\}$	x	0	0
$^2b^1_r$	z	0	0
$^2b^5_r$	z	1	0

$(q > t > r).$

The bases for ^2K have been taken in accord with (4.6), as the canonical form of the sub-matrices $[^2b_{p+1} : {}^2b_p]$ indicates. The sub-matrices $[^2b_{q+1} : \{^1E_q\}]$, $q \geqq t$, $[\{^1E_r\} : {}^2b_{r-1}]$, $r \leqq t$, are 0 as a direct consequence of the definition of t-block. Application of the product condition (4.4) shows $[\{^1E_{t+1}\} : {}^2b^5_t]$ and $[^2b^1_t : \{^1E_{t-1}\}]$ to be 0. The unit invariant factors in the sub-matrices marked 1 may be used to reduce to zero the remaining non-zero elements in the same rows or columns. By doing this throughout the incidence matrices of all dimensions, every z in line (horizontally or vertically) with a 1 could be made 0. The only other sub-matrices affected by this reduction would be the remaining z's; they also would become 0, as shown by (4.4).

Let $^{1'}K$ denote a set of cells isomorphic with ^1K except possibly for boundary relations. Two sets of cells are *isomorphic* if between them there exists a one-to-one correspondence preserving dimensions, boundary relations, and incidence numbers. Let $^{2'}K$ denote a set of t-cells, $R_t(^2K)$ in number, such that

$$(9.2) \qquad \begin{cases} [\{^{1'}E_{t+1}\} : \{^{2'}E_t\}] = [\{^1E_{t+1}\} : {}^2b^3_t], \\ [\{^{2'}E_t\} : \{^{1'}E_{t-1}\}] = [^2b^3_t : \{^1E_{t-1}\}]. \end{cases}$$

Then $'K = {}^{1'}K + {}^{2'}K$ has as incidence matrices:

(9.3)

	$\{^{1'}E_q\}$
$\{^{1'}E_{q+1}\}$	x

	$\{^{1'}E_t\}$	$\{^{2'}E_t\}$
$\{^{1'}E_{t+1}\}$	x	x

	$\{^{1'}E_{t-1}\}$
$\{^{1'}E_t\}$	x
$\{^{2'}E_t\}$	x

	$\{^{1'}E_{r-1}\}$
$\{^{1'}E_r\}$	x

$(q > t > r).$

Comparison of (9.3) and (9.1)—the latter with its z's considered reduced to 0's—shows that the incidence matrices of $'K$ satisfy the product

A complex K having a covering n-cycle, $R_n(K) = 1$, and each p-cell, $p < n$, on the boundary of at least one n-cell, is called an (orientable) *circuit*. If, under suitable orientation of the n-cells of K, (8.4) holds for each $St(E_{n-1})$, the circuit is *simple*. In the latter case the covering n-cycle is of course the simple one C_n. The first part of the argument under (8.6) shows that the n-cells of a circuit K are strongly connected on K. Accordingly K is connected, for by definition the rest of the cells are connected to the n-cells.

(8.7) *If a circuit L is a closed $(n-1)$-dimensional sub-complex of a strongly connected simple circuit K, a necessary and sufficient condition that L separate K is that the covering $(n-1)$-cycle on L bound (i. e. ~ 0) on K.*

Necessity: Let $K - L$ be divided into disconnected complementary sets ${}^1K$, ${}^2K$; let ${}^1C_n$ be the part of C_n lying on ${}^1K$. Then on ${}^1K + L$, which is a closed sub-set of K, $F({}^1C_n)$ is an $(n-1)$-cycle involving, with coefficients ± 1, $(n-1)$-cells which are on the boundary of just one n-cell of ${}^1K$. The ± 1 coefficients show $F({}^1C_n)$ to be basic. It lies on L since ${}^1C_n$ is a cycle on ${}^1K$. Therefore $F({}^1C_n) = \pm C_{n-1}$, the covering $(n-1)$-cycle of L. Incidentally this shows C_{n-1} to be simple.

Sufficiency: Let $F(C_n') = C_{n-1}$. By adding a multiple of C_n to C_n', if necessary, we can arrange that C_n' does not involve all the n-cells of K with non-zero coefficients. But any two n-cells strongly connected on $K - L$ must have the same coefficient in C_n'. Therefore the n-cells are not strongly connected on $K - L$, and so by (8.2) L separates K.

Cell Consolidation and Subdivision.

9. **Blocks.** Let K be divided into complementary sets ${}^1K$, ${}^2K$. If no cell of ${}^1K$ of dimension $\geqq t$ is on the boundary of a cell of ${}^2K$ and no cell of ${}^2K$ is on the boundary of a cell of ${}^1K$ of dimension $\leqq t$ we call ${}^2K$ a *t-block* of K. In other words a t-block of K is a sub-set closed modulo cells of dimensions $< t$ and open modulo cells of dimensions $> t$. It is the logical product of its star and its closure. Therefore by (6.4) and (6.6) it is a sub-complex of K if K is a complex.

The dual of a t-block of K is clearly an $(n-t)$-block of K^*. The logical product of a t-block of K with any sub-set of K is a t-block of the latter. The complement on K of a t-block is not in general a block. But when $t = n$ or 0, a t-block of K becomes precisely an open or closed sub-set of K and so the complement is also a block. A single t-cell is a t-block.

Let K be a complex having a t-block ${}^2K$ which is null-like except that $R_t({}^2K)$ may be arbitrary. Then K has as incidence matrices:

(9.1)

	$\{^1E_q\}$	$^2b^1_q$	$^2b^5_q$
$\{^1E_{q+1}\}$	x	z	z
$^2b^1_{q+1}$	0	0	0
$^2b^5_{q+1}$	0	1	0

	$\{^1E_t\}$	$^2b^1_t$	$^2b^3_t$	$^2b^5_t$
$\{^1E_{t+1}\}$	x	z	x	0
$^2b^1_{t+1}$	0	0	0	0
$^2b^5_{t+1}$	0	1	0	0

$(q>t>r).$

	$\{^1E_{t-1}\}$	$^2b^1_{t-1}$	$^2b^5_{t-1}$
$\{^1E_t\}$	x	0	0
$^2b^1_t$	0	0	0
$^2b^3_t$	x	0	0
$^2b^5_t$	z	1	0

	$\{^1E_r\}$	$^2b^1_{r-1}$	$^2b^5_{r-1}$
$\{^1E_r\}$	x	0	0
$^2b^1_r$	z	0	0
$^2b^5_r$	z	1	0

The bases for ^2K have been taken in accord with (4.6), as the canonical form of the sub-matrices $[^2b_{p+1}:{}^2b_p]$ indicates. The sub-matrices $[^2b_{q+1}:\{^1E_q\}]$, $q \geqq t$, $[\{^1E_r\}:{}^2b_{r-1}]$, $r \leqq t$, are 0 as a direct consequence of the definition of t-block. Application of the product condition (4.4) shows $[\{^1E_{t+1}\}:{}^2b^5_t]$ and $[^2b^1_t:\{^1E_{t-1}\}]$ to be 0. The unit invariant factors in the sub-matrices marked 1 may be used to reduce to zero the remaining non-zero elements in the same rows or columns. By doing this throughout the incidence matrices of all dimensions, every z in line (horizontally or vertically) with a 1 could be made 0. The only other sub-matrices affected by this reduction would be the remaining z's; they also would become 0, as shown by (4.4).

Let $^{1'}K$ denote a set of cells isomorphic with ^1K except possibly for boundary relations. Two sets of cells are *isomorphic* if between them there exists a one-to-one correspondence preserving dimensions, boundary relations, and incidence numbers. Let $^{2'}K$ denote a set of t-cells, $R_t(^2K)$ in number, such that

$$(9.2)\qquad \begin{cases} [\{^{1'}E_{t+1}\}:\{^{2'}E_t\}] = [\{^1E_{t+1}\}:{}^2b^3_t], \\ [\{^{2'}E_t\}:\{^{1'}E_{t-1}\}] = [^2b^3_t:\{^1E_{t-1}\}]. \end{cases}$$

Then $'K = {}^{1'}K + {}^{2'}K$ has as incidence matrices:

(9.3)

	$\{^{1'}E_q\}$
$\{^{1'}E_{q+1}\}$	x

	$\{^{1'}E_t\}$	$\{^{2'}E_t\}$
$\{^{1'}E_{t+1}\}$	x	x

	$\{^{1'}E_{t-1}\}$
$\{^{1'}E_t\}$	x
$\{^{2'}E_t\}$	x

	$\{^{1'}E_{r-1}\}$
$\{^{1'}E_r\}$	x

$(q>t>r).$

Comparison of (9.3) and (9.1)—the latter with its z's considered reduced to 0's—shows that the incidence matrices of $'K$ satisfy the product

condition (4.4), since those of K do. Therefore $'K$ is a complex. It has the same torsion coefficients as K, for the invariant factors >1 of the corresponding matrices in (9.3) and (9.1) are the same. And the Betti numbers are the same, for clearly

$$\alpha_p - '\alpha_p = (\varrho_p - '\varrho_p) + (\varrho_{p-1} - '\varrho_{p-1}).$$

Hence

(9.4) *$'K$ is a complex with the same homology characters as K.*[13]

As a corollary we have that if ${}^0K + {}^2K$ (${}^0K \subset {}^1K$) is a sub-complex of K the corresponding set

(9.5) ${}^{0,}K + {}^{2,}K$ *is a sub-complex of $'K$ with the same homology characters as ${}^0K + {}^2K$.*

This is because the argument applied to K and $'K$ is equally applicable to ${}^0K + {}^2K$ and ${}^{0,}K + {}^{2,}K$.

If ${}^2K$ is a *nullblock*, i. e., a null-like block, $R_t({}^2K) = 0$ and so ${}^{2,}K$ is vacuous. Let ${}^{1,}K = {}^1K$. Then (9.4) asserts that

(9.6) *${}^1K$ is a complex with the same homology characters as K.*

Consequently nullblocks may be neglected in calculating homology characters.

If ${}^2K$ is a *t-cellblock*, i. e., a t-cell-like t-block, $R_t({}^2K) = 1$ and so ${}^{2,}K$ consists of a single t-cell.

10. **Welds.** To each cell E_p of K let there correspond a cell $P(E_p)$, not necessarily of dimension p, such that

$$P(E_p) \leqq P(E_q) \text{ if } E_p \prec E_q. \tag{10.1}$$

In some respects this correspondence is the analogue of a continuous transformation in point-set theory, e. g. from (10.1) it follows that:

(10.2) *The totality of cells of K corresponding to the cells of any open (closed) sub-set of $P(K)$ is open (closed) on K.*

It is evident that the product correspondence QP arising from $QP(E_p) = Q(P(E_p))$, where Q is a correspondence satisfying (10.1), is also a correspondence satisfying (10.1). A general correspondence P can be best handled if it can be resolved into the product of a number of elemental correspondences of some sort. We adopt as *elemental* a correspondence P having the property that to each cell of some sub-set ${}^1K$ of K there corresponds a unique cell of the same dimension but that to the cells of ${}^2K = K - {}^1K$ there corresponds a common cell, say E_t, i. e., $P({}^1E_p)$ is a p-cell, where $P({}^1E'_p) \neq P({}^1E''_p)$ if ${}^1E'_p \neq {}^1E''_p$, and $P({}^2K) = E_t$. In consequence of (10.1)

[13] This result embraces Mayer's *Operationen an Ringen, die die Homologiegruppen unverändert lassen*, l. c. [9], pp. 24-31.

$$(10.3)\qquad \begin{cases} P({}^1E_p) \prec E_t \text{ if } {}^1E_p \prec \text{some } {}^2E_q, \\ E_t \prec P({}^1E_s) \text{ if some } {}^2E_r \prec {}^1E_s. \end{cases}$$

This implies that no ${}^1E_p \prec {}^2E_q$, $p \geqq t$, and no ${}^2E_r \prec {}^1E_s$, $s \leqq t$. So by definition,

(10.4) *${}^2K$ is a t-block of K.*

From now on we confine ourselves to cases in which $P(K)$ is in some fashion superposed on K. Let

$$(10.5)\qquad P({}^1E_p) = {}^1E_p.$$

Then from (10.3) and (1.2) it follows that

$$(10.6)\qquad {}^1E_p \prec {}^1E_s \text{ if } {}^1E_p \prec \text{some } {}^2E_q \text{ and some } {}^2E_r \prec {}^1E_s.$$

If this is undesirably stringent let (10.5) be weakened in so far as it affects the set of cells of ${}^1K$ on the star or closure of ${}^2K$ on K. Labelling this set ${}^{1/2}K$ we merely require that ${}^{1/2}K$ and $P({}^{1/2}K)$ be isomorphic except for the discrepancies — $P({}^1E_p) \prec P({}^1E_s)$, ${}^1E_p \not\prec {}^1E_s$ — made necessary by the failure of (10.6). On the other hand let (10.3) be strengthened by the assumption that the boundary relations of E_t with cells of $P({}^1K)$ are just those it, (10.3), imposes. This strengthening amounts to replacing "if" in (10.3) by "if, and only if,". An immediate result of these assumptions is that:

(10.7) *To an open (closed) set of K containing ${}^2K$ there corresponds an open (closed) set of $P(K)$.*

Let K be a complex. The question then is — what conditions added to (10.5), or its weakened form, and the strengthened (10.3) ensure that our elemental correspondence preserves homology characters in the sense of (9.5) (${}^{a'}K$ being $P({}^aK)$). It is immediately evident that ${}^2K$ must be a t-cellblock, for when ${}^0K$ is vacuous ${}^2K$ must have the same homology characters as E_t. Conversely, if ${}^2K$ is a t-cellblock and ${}^2b_t^s$ can be chosen to satisfy (9.2), both (9.4) and (9.5) hold. In this event we call the correspondence P a *t-weld.*

The t-weld is an important operation. It combines the cells of a suitable sub-set ${}^2K$ of the complex K into a single cell without affecting essentially (i) the rest of K or (ii) the sub-sets of K containing ${}^2K$.[14] In (i) the reference is to (10.5) or its weakened form, and in (ii) to the strengthened (10.3) as well as to the results of (10.7), (9.4) and (9.5). The applicability of our welding operation is not as limited as might appear at first sight. Any ${}^2K$ which is a cellblock will do; we have only to create E_t and

[14] The idea, due to Alexander, of forming a cell-like set of cells into a cell, took form in the *Zelle* of van Kampen, l. c. [5], and the *combinatorial cell* of Lefschetz, l. c. [2], pp. 105–6. Both of these "cells" are t-cell-like, t-dimensional sets of cells, corresponding to the type of t-cellblock described in (11.8).

$P(^{1/2}K)$ if necessary, endowing them with the requisite boundary relations and incidence numbers—those of E_t determined by the strengthened (10.3) and (9.2), and those of $P(^{1/2}K)$ by the weakened (10.5).

Fig. I provides some simple geometrical illustrations of welds. From the complex a the complex b is obtained by a 0-weld which replaces the triangle, its sides and vertices by a single 0-cell. The complex d is

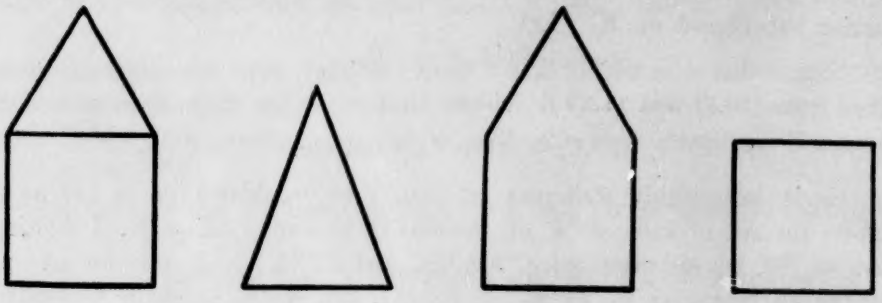


Fig. I.

obtained from a by a 1-weld which replaces the triangle, its sides and the one vertex not in common with the square by a single 1-cell. The complex c is obtained from a by a 2-weld which replaces the interiors of the triangle and square and the common side by a single 2-cell.

11. **Subdivision.** Consider the complex K to be divided into a set of complementary sub-sets. Let $'\alpha_p$ denote the number of sub-sets of dimension p; if $'\alpha_p > 0$, let $K_p^1, K_p^2, \cdots, K_p^{'\alpha_p}$ denote these sub-sets. The division of K into sub-sets is to be such that each K_t^k is open on $\{K_p\}_{p\leq t}$. This makes $\{K_p\}_{p\leq t}$ closed on K. So applying (6.6), it follows that $\{K_p\}_{p\leq t}$ is a sub-complex of K, and K_t^k a sub-complex of $\{K_p\}_{p\leq t}$. Let each K_t^k be t-cell-like. Then K has as pth incidence matrix:

(11.1)

	${}^1_pb^3_p$	${}^1_pb^5_p$	$\cdots$	${}^{'\alpha_p}_pb^3_p$	${}^{'\alpha_p}_pb^5_p$	${}^1_{p+1}b^1_p$	${}^1_{p+1}b^5_p$	$\cdots$	${}^{'\alpha_{p+1}}_{p+1}b^1_p$	${}^{'\alpha_{p+1}}_{p+1}b^5_p$	$\cdots$	${}^{j_1}_qb^1_p$	${}^{j_1}_qb^5_p$	$\cdots$
${}^1_{p+1}b^3_{p+1}$	x	0	$\cdots$	x	0	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$
${}^1_{p+1}b^5_{p+1}$	z	z	$\cdots$	z	z	1	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	
${}^{'\alpha_{p+1}}_{p+1}b^3_{p+1}$	x	0	$\cdots$	x	0	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$
${}^{'\alpha_{p+1}}_{p+1}b^5_{p+1}$	z	z	$\cdots$	z	z	0	0	$\cdots$	1	0	$\cdots$	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	
${}^{j_1}_qb^1_{p+1}$	z	z	$\cdots$	z	z	z	z	$\cdots$	z	z	$\cdots$	0	0	$\cdots$
${}^{j_1}_qb^5_{p+1}$	z	z	$\cdots$	z	z	z	z	$\cdots$	z	z	$\cdots$	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	

As the canonical form of the sub-matrices $[{}^k_t b_{p+1} : {}^k_t b_p]$ indicates, each basis ${}^k_t b_p$ for K^k_t is chosen in accordance with (4.6). Because each K_s is open on $\{K_r\}_{r\geq s}$, the sub-matrices $[{}_r b_{p+1} : {}_s b_p]$, $r \leqq s$, are 0. The product condition (4.4) shows that the sub-matrices $[{}_{p+1}b^3_{p+1} : {}_p b^5_p]$ are 0. The unit invariant factors in $[{}^k_t b^5_{p+1} : {}^k_t b^1_p]$ can be used to reduce to zero the remaining non-zero elements in the same row. By doing this systematically for each ${}^h_t K$, taking $p = t-1, t-2, \cdots, 0$ in turn, all sub-matrices marked z, horizontally in line with a sub-matrix marked 1, could be made 0 throughout the (11.1) matrices of all dimensions. The only other sub-matrices affected by this reduction would be the remaining z's; they would also become 0, as application of (4.4) goes to show.

To each K^k_t let there correspond a unique t-cell ${}'E^k_t$ such that

$$[{}'E^j_{t+1} : {}'E^k_t] = [{}^j_{t+1} b^3_{t+1} : {}^k_t b^3_t]. \tag{11.2}$$

Then the set ${}'K = \{{}'E^k_t\}$ has a pth incidence matrix:

$$\begin{array}{c|ccc} & {}'E^1_p & \cdots\cdots & {}'E^{{}'\alpha_p}_p \\ \hline {}'E^1_{p+1} & x & \cdots\cdots & x \\ \vdots & \vdots & & \vdots \\ {}'E^{{}'\alpha_{p+1}}_{p+1} & x & \cdots\cdots & x \end{array} \tag{11.3}$$

Since the (11.1) matrices of K satisfy (4.4), so do the (11.3) ones of ${}'K$. Therefore ${}'K$ is a complex. Comparison of (11.1) and (11.3) shows that

$$\alpha_p - {}'\alpha_p = (\varrho_p - {}'\varrho_p) + (\varrho_{p-1} - {}'\varrho_{p-1}).$$

This means that $R_p({}'K) = R_p(K)$. The invariant factors of (11.3) and (11.1) are the same; i. e., ${}'K$ and K have the same torsion coefficients. Hence

(11.4) *${}'K$ is a complex with the same homology characters as K.*

As an immediate corollary we have that:

(11.5) *To any sub-complex of K formed from some of the sub-sets K^k_t there corresponds a sub-complex of ${}'K$ with the same homology characters.*

This is because the sub-complex of K and the corresponding sub-set of ${}'K$ are related just as K and ${}'K$ are.

Now suppose the correspondence between K and ${}'K$, in addition to the properties it has already attained in this section, is a correspondence P—i. e., $P^{-1}({}'E^h_t) = K^k_t$—satisfying (10.1) and also the condition

$$E_p \prec E_u \quad \text{if} \quad P(E_p) \prec E_u \tag{11.6}$$

(E_u is any cell, not necessarily of ${}'K = P(K)$). We assume that the boundary relations between cells of $P(K)$ are just those imposed by (10.1),

and that the boundary relations between cells of K and cells of $P(K)$ are just those imposed by (11.6) when E_u is a cell of $P(K)$. Under these circumstances we define K to be a *subdivision* of $'K = P(K)$.

Let $'K$ be a subdivision of $''K = Q('K)$. The correspondence $''K = QP(K)$ is then one satisfying (10.1) and (11.6). We extend to it the assumptions set down under (11.6). The sub-set corresponding to a cell $''E_t$ of $''K$ is clearly t-dimensional and open on K modulo cells corresponding to cells of $''K$ of dimensions $> t$. By (11.5) $P^{-1}Q^{-1}(''E_t)$ is t-cell-like. A suitably chosen basic t-cycle on it will satisfy the same conditions of the (11.2) sort as will the basic t-cycle on $Q^{-1}(''E_t)$. So we conclude that K is a subdivision of $''K = QP(K)$.

Let

$$ {}^k_tK = \{'E_q\}_{q>t} + \{'E_t^i\}^{i<k} + \{K_t^j\}^{j\geq k} + \{K_r\}_{r<t}, $$

and let

$$ \begin{aligned} P_t^k({}^k_tK) &= {}^k_tK - K_t^k + 'E_t^k \\ &= {}^{k+1}_tK \quad (\text{or } {}^1_{t-1}K \text{ if } k = '\alpha_t). \end{aligned} $$

Conditions (10.1) and (11.6), as interpreted in accordance with the assumptions set down under (11.6), transmit themselves to each P_t^k, thereby ensuring that P_t^k satisfy the strengthened (10.3). By construction P_t^k is elemental and satisfies (10.5). Condition (9.2) is compatible with (11.2). Each P_t^k may accordingly be regarded as a t-weld, and so P may be resolved into a sequence of t-welds:

$$ (11.7) \qquad P_n^1,\ P_n^2,\ \cdots,\ P_t^k,\ \cdots,\ P_0^{'\alpha_0}. $$

Moreover, ${}^h_tK$ is a subdivision of $P_t^k({}^k_tK)$.

Returning temporarily to the notation of § 10, let P be a t-weld for which (10.5) holds in full. By comparing the properties of this t-weld with the definition of subdivision we find that:

(11.8) *In order that K be a subdivision of $P(K)$ it is necessary and sufficient that every ${}^2E_q$ be on the boundary of each cell on whose boundary $P({}^2K)$ lies and that ${}^2K$ be t-dimensional.*

The first condition arises of course in (11.6). The second is in general redundant. This is because $P({}^2K)$ will usually be on the boundary of some $(t+1)$-cell, thus preventing ${}^2K$ from having cells of dimension $> t$. But ${}^2K$, being t-cell-like, must contain t-cells, and so is t-dimensional.

Multiplication.

12. **Rules of multiplication.** We introduce an operation by which an (ordered) pair of (oriented) cells of dimensions p and q may determine

a third (oriented) cell of dimension $p+q+w$ (w an integer). This operation we call multiplication, or more properly, *w-multiplication*. Put in symbols the w-product of two cells takes the form

$$E_p \overset{w}{\times} E_q = \pm E_{p+q+w} \quad \text{or} \quad 0. \tag{12.1}$$

The ambiguity of sign arises from the fact that the orientation determined by the multiplication may be either the same as or opposite to the chosen orientation of the $(p+q+w)$-cell. The "or 0" alternative formally expresses the possibility that the $(p+q+w)$-cell may fail to exist.

We postulate that:

$$(s_i E_p^i) \overset{w}{\times} (t_j E_q^j) = s_i t_j (E_p^i \overset{w}{\times} E_q^j), \tag{12.2}$$

$$E_q \overset{w}{\times} E_p = (-1)^{(p+w)(q+w)} E_p \overset{w}{\times} E_q, \tag{12.3}$$

$$(E_p \overset{w}{\times} E_q) \overset{w}{\times} E_r = E_p \overset{w}{\times} (E_q \overset{w}{\times} E_r), \tag{12.4}$$

$$E_p \overset{w}{\times} E_r \prec E_q \overset{w}{\times} E_r \text{ if, and only if, } E_p \prec E_q, \tag{12.5}$$

$$[(E_{p+1} \overset{w}{\times} E_q) : (E_p \overset{w}{\times} E_q)] = [E_{p+1} : E_p]. \tag{12.6}$$

Of course it is understood that (12.5) and (12.6) are without meaning unless the w-products appearing therein are non-vacuous. From (12.3) and (12.6) follows

$$[(E_p \overset{w}{\times} E_{q+1}) : (E_p \overset{w}{\times} E_q)] = (-1)^{p+w} [E_{q+1} : E_q]. \tag{12.7}$$

The w-product

$$^1K \overset{w}{\times} {}^2K \tag{12.8}$$

of two sets ^1K, ^2K is defined to be the set of all cells obtained by multiplying in turn each cell of ^1K by each cell of ^2K. Making the assumption that the cells of the set (10.8) are all non-vacuous and distinct, it is easily calculated from (10.6) and (12.7) that

$$F(^1C_p \overset{w}{\times} {}^2C_q) = F(^1C_p) \overset{w}{\times} {}^2C_q + (-1)^{p+w} {}^1C_p \overset{w}{\times} F(^2C_q). \tag{12.9}$$

Necessary and sufficient conditions that

$$F(F(^1C_p \overset{w}{\times} {}^2C_q)) = 0 \tag{12.10}$$

are that $F(F(^1C_p)) = 0$ and $F(F(^2C_q)) = 0$, i. e., that ^1K, ^2K are complexes. Since any chain on the set (12.8) can be expressed as a sum of w-products of chains on ^1K and ^2K, (12.10) is a necessary and sufficient condition that the set (12.8) be a complex.

In §§ 13, 14, 18, three distinct types of w-products are considered. They are, respectively, the *product* ($w = 0$), the *intersection* ($w = -n$), and

the *join* ($w = 1$). In fact, our main purpose in introducing w-multiplication is to unify as far as possible these three concepts. (The relationship of products and joins is no new realization; e. g., cf. Lefschetz, l. c. [2], p. 110).

13. **Products.** A w-product (12.8), consisting entirely of distinct non-vacuous cells, determined by sets ${}^1K$, ${}^2K$ which are complexes, is by (11.10) a complex. Its homology characters are completely determined by those of ${}^1K$ and ${}^2K$. In verifying this let us simplify our notation somewhat by using ordinary products: then $w = 0$, and the operation symbol is a plain $\times$. (Cf. Lefschetz, l. c. [2], p. 220ff., for the point-theoretic meaning of products.)

For each dimension choose bases ${}^1b_p$, ${}^2b_q$ for ${}^1K$, ${}^2K$ in accord with (4.6). Let ${}^1b_p \times {}^2b_q$, $p+q=r$, denote the set of r-chains on ${}^1K \times {}^2K$ obtained by multiplying in turn each p-chain of ${}^1b_p$ by each q-chain of ${}^2b_q$; then $\{{}^1b_p \times {}^2b_q\}_{p+q=r}$ constitutes a basis for the r-chains on ${}^1K \times {}^2K$. The rth incidence matrix $[\{{}^1b_p \times {}^2b_q\}_{p+q=r+1} : \{{}^1b_p \times {}^2b_q\}_{p+q=r}]$ contains at most two non-zero elements in each row and column. By simple changes we may arrange to have at most one non-zero element in each row and column.

Let ${}^1C_{p+1}$, ${}^1C_p$, ${}^2C_{q+1}$, ${}^2C_q$, be members of ${}^1b_{p+1}$, ${}^1b_p$, ${}^2b_{q+1}$, ${}^2b_q$, respectively, such that $F({}^1C_{p+1}) = \lambda\, {}^1C_p$ and $F({}^2C_{q+1}) = \mu\, {}^2C_q$; then by (12.9)

$$(13.1)\qquad \begin{cases} F({}^1C_{p+1} \times {}^2C_{q+1}) = \lambda\, {}^1C_p \times {}^2C_{q+1} + (-1)^{p+1}\, \mu\, {}^1C_{p+1} \times {}^2C_q, \\ F({}^1C_p \times {}^2C_{q+1}) = (-1)^p\, \mu\, {}^1C_p \times {}^2C_q, \\ F({}^1C_{p+1} \times {}^2C_q) = \lambda\, {}^1C_p \times {}^2C_q, \\ F({}^1C_p \times {}^2C_q) = 0. \end{cases}$$

By elementary changes of basis ${}^1C_p \times {}^2C_{q+1}$ and ${}^1C_{p+1} \times {}^2C_q$ may be replaced by chains C'_{p+q+1} and C''_{p+q+1} such that

$$(13.2)\qquad \begin{cases} F({}^1C_{p+1} \times {}^2C_{q+1}) = (\lambda, \mu)\, C'_{p+q+1}, & ((\lambda, \mu) = \text{G. C. D. of } \lambda, \mu) \\ F(C'_{p+q+1}) = 0, \\ F(C''_{p+q+1}) = (\lambda, \mu)\, {}^1C_p \times {}^2C_q, \\ F({}^1C_p \times {}^2C_q) = 0. \end{cases}$$

After effecting all possible changes of this sort we get incidence matrices with at most one non-zero element in each row and column. From these reduced matrices the homology characters of ${}^1K \times {}^2K$ may be computed.

The all-zero rows in our reduced $(r-1)$st incidence matrix corresponding to all-zero columns in the rth are precisely those determined by $\{{}^1b^3_p \times {}^2b^3_q\}_{p+q=r}$. Their number is the rth Betti number of ${}^1K \times {}^2K$; i. e.,

$$(13.3)\qquad R_r({}^1K \times {}^2K) = \sum_{p+q=r} R_p({}^1K)\, R_q({}^2K).$$

If ${}^1K$ and ${}^2K$ are without torsion coefficients each $\lambda, \mu, (\lambda, \mu) = 1$, and so
(13.4) *${}^1K \times {}^2K$ has no torsion coefficients if ${}^1K$, ${}^2K$ have none.*
(Lefschetz, l. c. [2], pp. 225–230, has (12.3) and (12.4).)

An actual formula for the torsion coefficients of ${}^1K \times {}^2K$ seems out of the question, but from our reduced matrices may be obtained a formula for the number ${}^{1\times 2}n_r(\pi^u)$ of rth torsion divisors of ${}^1K \times {}^2K$ having the value π^u (π a prime, u an exponent); it is

$$
\begin{aligned}
{}^{1\times 2}n_r(\pi^u) &= \sum_{p+q=r} \{{}^1n_p(\pi^u)\, R_q({}^2K) + R_p({}^1K)\, {}^2n_q(\pi^u)\} \\
&+ \sum_{p+q=r,\, r-1} \{{}^1n_p(\pi^u)\, {}^2n_q(\pi^u) + \sum_{v>u} ({}^1n_p(\pi^u)\, {}^2n_q(\pi^v) + {}^1n_p(\pi^v)\, {}^2n_q(\pi^u))\},
\end{aligned} \tag{13.5}
$$

where of course ${}^1n_p(\pi^u)$ is the number of pth torsion divisors of ${}^1K$ which $= \pi^u$, etc.

To have results for the general w-product we need only put in w wherever necessary in (13.3)–(13.5); thus (13.3) becomes

$$
R_r({}^1K \overset{w}{\times} {}^2K) = \sum_{p+q+w=r} R_p({}^1K)\, R_q({}^2K). \tag{13.6}
$$

Criteria for the absolute manifold.

14. **Intersections.** By the intersection (in n dimensions) of E_q and E_p we mean the particular w-product (12.1) obtained by setting $w = -n$. In conformity with established notation this $(-n)$-product is written $E_q \cdot E_p$ (cf. Lefschetz, l. c. [2], p. 165). We postulate that

$$
\left.
\begin{aligned}
E_q \cdot E_r &\neq 0 \qquad (14.1) \\
E_q \cdot E_p &\prec E_r \qquad (14.2)
\end{aligned}
\right\} \text{ if } E_q \cdot E_p \neq 0 \text{ and } E_p \prec E_r.
$$

The intersection ${}^2K \cdot {}^1K$ of two complexes ${}^2K$, ${}^1K$ consists in general of both vacuous and non-vacuous cells, the latter of which we assume to be distinct among themselves. However, due to (14.1), the presence of vacuous cells does not prevent (12.9) and (12.10) from carrying over as

$$
F({}^2C_q \cdot {}^1C_p) = {}^2C_q \cdot F({}^1C_p) + (-1)^{p-n} F({}^2C_q) \cdot {}^1C_p, \tag{14.3}
$$

$$
F(F({}^2C_q \cdot {}^1C_p)) = 0. \tag{14.4}
$$

From (14.4) we conclude that ${}^2K \cdot {}^1K$ is a complex.

The intuitive background of intersections is readily illustrated by a simple geometrical example. Taking $n = 2$, let ${}^1K$ and ${}^2K$ be aggregates of polygonal interiors (2-cells), edges (1-cells), and vertices (0-cells) which overlap in such a fashion that no vertex of the one aggregate lies

on a vertex or edge of the other. Then ${}^2K \cdot {}^1K$ is the "fragmentary" aggregate contained in the common part of ${}^1K$ and ${}^2K$. In Fig. II appears an example of this sort. The interiors, sides and vertices of the four triangles form ${}^1K$, say, while the interiors, sides and vertices of the nine squares form ${}^2K$. ${}^2K \cdot {}^1K$ consists of the interiors, sides and vertices of the variety of small triangles, squares, quadrilaterals and pentagons, into which ${}^1K$ and ${}^2K$ decompose their common part.

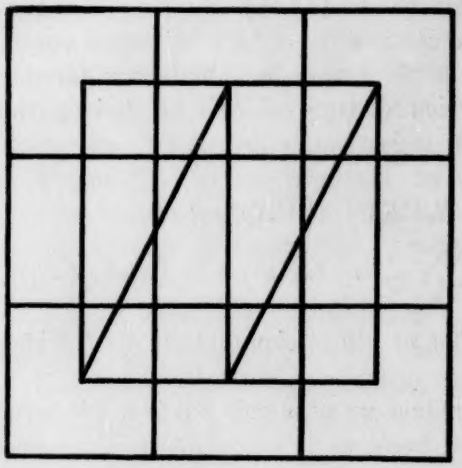


Fig. II.

In considering an intersection complex ${}^2K \cdot {}^1K$ it is natural to take the following for granted: that ${}^1K \cdot {}^1K = 0 = {}^2K \cdot {}^2K$; that no ${}^1E_p \prec$ some ${}^2E_q$, or vice versa; that the boundary relations of cells of ${}^2K \cdot {}^1K$ are just those imposed by (12.5); and that the boundary relations of cells of ${}^2K \cdot {}^1K$ with cells of ${}^1K$ and ${}^2K$ are just those imposed by (14.2). Then

(14.5) ${}^2E_q \cdot {}^1E_p \leqq {}^2E_s \cdot {}^1E_r$ if, and only if, ${}^1E_p \leqq {}^1E_r$ and ${}^2E_q \leqq {}^2E_s$, and

(14.6) ${}^2E_q \cdot {}^1E_p \prec {}^1E_r$ or ${}^2E_s$ if, and only if, ${}^1E_p \prec {}^1E_r$ or ${}^2E_q \prec {}^2E_s$ respectively.

Because of (14.5) the sub-set ${}^2K \cdot {}^1E_t$ of ${}^2K \cdot {}^1K$ is open on $\{{}^2K \cdot {}^1E_p\}_{p \leqq t}$. So, if each ${}^2K \cdot {}^1E_t$ is t-dimensional and t-cell-like with a basic t-cycle ${}^2_tC_n \cdot {}^1E_t$ such that

(14.7) $$[{}^2_{t+1}C_n \cdot {}^1E_{t+1} : {}^2_tC_n \cdot {}^1E_t] = [{}^1E_{t+1} : {}^1E_t],$$

theorem (11.4) applies to $K = {}^2K \cdot {}^1K$ and $'K = {}^1K$. We investigate this.

Let ${}^1E_p({}^2K)$ denote the set of cells of ${}^2K$ having non-vacuous intersections with ${}^1E_p$. Due to (14.1), ${}^1E_p({}^2K)$ is open on ${}^2K$ and ${}^1E_p({}^2K) \subset {}^1E_r({}^2K)$ if ${}^1E_p \prec {}^1E_r$. In order that ${}^2K \cdot {}^1E_t$ be t-dimensional and t-cell-like with ${}^2_tC_n \cdot {}^1E_t$ as basic t-cycle it is necessary and sufficient that ${}^1E_t({}^2K)$ be n-dimensional and n-cell-like with ${}^2_tC_n$ as basic n-cycle. On ${}^2K \cdot {}^1K$ the basic $(t+1)$-cycle ${}_{t+1}{}^2_tC_n \cdot {}^1E_{t+1}$ of ${}^2K \cdot {}^1E_{t+1}$ has the boundary

$$F({}_{t+1}{}^2_tC_n \cdot {}^1E_{t+1}) = [{}^1E_{t+1} : {}^1E_t]_{t+1}{}^2_tC_n \cdot {}^1E_t.$$

So in order that (14.7) be satisfied it is necessary and sufficient that

(14.8) $${}_{t+1}{}^2_tC_n \cdot {}^1E_t = {}^2_tC_n \cdot {}^1E_t \quad \text{or} \quad [{}^1E_{t+1} : {}^1E_t] = 0.$$

Of course there is a precisely similar investigation of the correspondence between the sub-sets ${}^2E_q \cdot {}^1K$ of ${}^2K \cdot {}^1K$ and the cells ${}^2E_q$ of ${}^2K$. In place of (14.7) we should have

$$[{}^2E_{t+1} \cdot {}^1_{t+1}C_n : {}^2E_t \cdot {}^1_tC_n] = [{}^2E_{t+1} : {}^2E_t], \tag{14.7'}$$

and in place of (14.8)

$${}^2E_t \cdot {}^1_{t+1}C_n = {}^2E_t \cdot {}^1_tC_n \quad \text{or} \quad [{}^2E_{t+1} : {}^2E_t] = 0. \tag{14.8'}$$

Because of (14.5), (14.1) and (14.6) the correspondences

$$P({}^2E_q \cdot {}^1E_p) = {}^1E_p, \qquad P({}^2E_q \cdot {}^1E_p) = {}^2E_q$$

satisfy (10.1) and (11.6), as interpreted in accordance with the assumptions set down under (11.6). Summarizing, we have:

(14.9) *In order that the intersection* ${}^2K \cdot {}^1K$ *of the complexes* ${}^2K$, ${}^1K$ *be a subdivision of* ${}^aK$, $a = 1$ *or* 2, *it is necessary and sufficient that each* ${}^aE_t({}^bK)$, $b = 2$ *or* 1, *be* n*-dimensional and* n*-cell-like with a basic* n*-cycle* ${}^b_tC_n$ *satisfying* (14.8) *or* (14.8').

In Figure II ${}^2K \cdot {}^1K$ is a subdivision of ${}^1K$ but not of ${}^2K$.

Condition (14.8) or (14.8') is met very satisfactorily if ${}^bK$ has a simple covering n-cycle ${}^bC_n$, for then the part of ${}^bC_n$ on ${}^aE_t({}^bK)$ constitutes a suitable ${}^b_tC_n$.

15. **Star and closure uniformity.** Let us consider the intersection complex $K \cdot K^*$ of a complex K and its dual K^*. We assume that

$$E^i_p \cdot E^{n-p}_j \neq 0 \quad \text{if, and only if, } i = j. \tag{15.1}$$

Then, in accordance with (14.1),

$$E_q \cdot E^{n-p} \neq 0 \quad \text{if, and only if, } E_p \leqq E_q. \tag{15.2}$$

This causes (14.5) to become

$$E_q \cdot E^{n-p} \leqq E_s \cdot E^{n-r} \quad \text{if, and only if, } E_r \leqq E_p \leqq E_q \leqq E_s. \tag{15.3}$$

We apply Theorem (14.9) to $K \cdot K^*$. By (15.2) the sub-set $E^{n-p}(K)$ of K is the star, $St(E_p)$, of E_p on K; and $E_q(K^*)$ the star, $St(E^{n-q})$, of E^{n-q} on K^*. Suppose $K \cdot K^*$ is a sub-division of K^* (or K). Then each $St(E_p)$ ($St(E^{n-q})$) must be n-cell-like with basic n-cycles satisfying (14.8) (or (14.8')). The star of an n-cell is, of course, just itself. By changing the orientation of certain n-cells, if necessary, we arrange that each appears with coefficient $+1$ in the basic n-cycle it bears. Then the sum of the n-cells of K (K^*) is a simple covering n-cycle, for Theorem (8.3) is applicable.

If every $St(E_p)$ of K is n-cell-like and if K carries a simple covering n-cycle we describe the complex K as having *star uniformity.* This use of the word "uniformity" has the justification—that, if the stars of the cells of K have uniform, i. e., the same, homology characters, those characters must be the characters of an n-cell, for the star of an n-cell is the n-cell itself. If K^* has star uniformity we describe K as having *closure uniformity.* In terms of these definitions the application of Theorem (14.9) to $K \cdot K^*$ finds expression in the theorems:

(15.4) *A necessary and sufficient condition that $K \cdot K^*$ be a subdivision of K^* is that K have star uniformity.*

(15.5) *A necessary and sufficient condition that $K \cdot K^*$ be a subdivision of K is that K have closure uniformity.*

If K has both star and closure uniformity $K \cdot K^*$ is a common subdivision of K and K^*. These, therefore, have the same homology characters—which, due to (5.6) and (5.7), must satisfy the duality relations:

$$\theta_i^p(K) = \theta_i^{n-p-1}(K), \tag{15.6}$$

$$R_p(K) = R_{n-p}(K). \tag{15.7}$$

16. Intercept uniformity. Let us examine $K \cdot K^*$ for star and closure uniformity.

In accordance with (15.3) the star of a (non-vacuous) cell of $K \cdot K^*$ may be expressed as a wholly non-vacuous intersection complex as follows:

$$\begin{cases} St(E_q \cdot E^{n-p}) = St(E_q) \cdot St(E^{n-p}), \\ St(E_n \cdot E^{n-p}) = E_n \cdot St(E^{n-p}), \\ St(E_q \cdot E^n) = St(E_q) \cdot E^n. \end{cases} \tag{16.1}$$

Hence, applying the $w = -n$ counterparts of (13.3) and (13.4) to these intersection complexes, we have that the star of each cell of $K \cdot K^*$ is n-cell-like if the star of each cell of K and K^* is n-cell-like, and conversely. Moreover, if K and K^* have C_n and C^n, respectively, as simple covering n-cycles, $K \cdot K^*$ has $C_n \cdot C^n$ as a simple covering n-cycle, and conversely. Therefore,

(16.2) *A necessary and sufficient condition that $K \cdot K^*$ have star uniformity (the simple covering n-cycle being of the form $C_n \cdot C^n$) is that K have star and closure uniformity.*

As a first step in testing $K \cdot K^*$ for closure uniformity we notice that a simple covering n-cycle on $K \cdot K^*$ is obtained by assigning the dual of each 0-cell $E_p \cdot E^{n-p}$ the coefficient $(-1)^{p(p+1)/2}$, for by (14.3), (15.2) and (5.2)

$$F(E_{p+1} \cdot E^{n-p}) = [E_{p+1} : E_p]\,(E_{p+1} \cdot E^{n-p-1} + (-1)^p E_p \cdot E^{n-p}). \tag{16.3}$$

It remains to find under what conditions the closure of each cell of $K \cdot K^*$ is 0-cell-like. Let $E_r, E_{r_1}^{i_1}, E_{r_2}^{i_2}, \cdots, E_{r_\nu}^{i_\nu}, E_s$ $(r < r_1 \leqq r_2 \leqq \cdots \leqq r_\nu < s)$ denote the cells of the intercept, $It(E_r, E_s)$, of E_r and E_s on K. (The definition of intercept was given at the end of § 6). Then, in accordance with (15.3), $Cl(E_s \cdot E^{n-r})$ may be expressed as

$$\begin{aligned} &It(E_r, E_s) \cdot E^{n-r} + It(E_{r_1}^{i_1}, E_s) \cdot E_{i_1}^{n-r_1} + It(E_{r_2}^{i_2}, E_s) \cdot E_{i_2}^{n-r_2} + \cdots \\ &\cdots + It(E_{r_\nu}^{i_\nu}, E_s) \cdot E_{i_\nu}^{n-r_\nu} + E_s \cdot E^{n-s}. \end{aligned} \tag{16.4}$$

The terms in (16.4) are so arranged that each is an open (closed) sub-set of $Cl(E_s \cdot E^{n-r})$ modulo the totality of preceding (succeeding) terms. Hence if every $It(E_p, E_q)$ is null-like we may suppress (cf. (9.6)) the terms of (16.4) in turn from the beginning until only $E_s \cdot E^{n-s}$ is left, thus showing that $Cl(E_s \cdot E^{n-r})$ is 0-cell-like.

Let us try to reverse this last argument. We accordingly suppose that the closure of each cell of $K \cdot K^*$ is 0-cell-like. In (16.3)

$$[E_{p+1} : E_p] = \pm 1 \quad \text{if} \quad E_p < E_{p+1}, \tag{16.5}$$

else $Cl(E_{p+1} \cdot E^{n-p})$ has a torsion coefficient. This ensures that any 0-cell on $Cl(E_s \cdot E^{n-r})$ may be taken as its basic non-bounding 0-cycle. Consequently the removal of $E_s \cdot E^{n-s}$ leaves (16.4) null-like. If $s = r+1$, this means that $It(E_r, E_s)$ is null-like. If $s > r+1$ we assume each $It(E_p, E_q)$ null-like for $q - p < s - r$. Then knocking terms off the end of (16.4) until only $It(E_r, E_s) \cdot E^{n-r}$ is left, it results that $It(E_r, E_s)$ is null-like. Thus we conclude by induction that each intercept of K is null-like.

We describe the complex K as having *intercept uniformity* if every intercept of K is null-like. This use of the term "uniformity" is justified by the fact that, if the intercepts of pairs of cells of K have uniform (i. e., the same) homology characters, those characters must belong to a null-like set because the intercept of a pair of cells $E_p \not\leqq E_q$ is vacuous. Of course, K^* has intercept uniformity if K has, and conversely. In terms of intercept uniformity the results of the above discussion are embodied in the theorem:

(16.6) *A necessary and sufficient condition that $K \cdot K^*$ have closure uniformity is that K have intercept uniformity.*

The investigation is rendered complete by the theorem:

(16.7) *A necessary and sufficient condition that $K \cdot K^*$ have intercept uniformity is that K have intercept uniformity.*

The proof is immediate, for by (15.3) the (non-vacuous) intercept of a pair of cells of $K \cdot K^*$ may be expressed as a wholly non-vacuous intersection complex as follows:

$$(16.8)\quad \begin{cases} It(E_q \cdot E^{n-p},\ E_s \cdot E^{n-r}) = It(E_q, E_s) \cdot It(E^{n-p}, E^{n-r}), \\ It(E_q \cdot E^{n-p},\ E_q \cdot E^{n-r}) = E_q \cdot It(E^{n-p}, E^{n-r}), \\ It(E_q \cdot E^{n-p},\ E_s \cdot E^{n-p}) = It(E_q, E_s) \cdot E^{n-p}. \end{cases}$$

17. **The absolute manifold.** If K has star, closure and intercept uniformity, so has $K \cdot K^*$, as shown by (16.2), (16.6) and (16.7); in addition, as shown by (15.5), $K \cdot K^*$ is a subdivision of K, which entails that it have the same homology characters as K. We summarize this in the statement:

(17.1) *A complex K with star, closure and intercept uniformity is essentially unaffected by intersection with its dual.*

To such a complex K we give the name *n*-dimensional (orientable) *absolute manifold* and the symbol M.

Let $M^{(\nu+1)} = M^{(\nu)} \cdot M^{(\nu)*}$ $(M^{(0)} \equiv M)$. Then in consequence of (17.1) each $M^{(\nu+1)}$ is an absolute manifold with the same homology characters as M. Each $M^{(\nu+1)}$ is a subdivision of $M^{(\nu)}$ and so (§ 11) a subdivision of all $M^{(\lambda)}$ $(\lambda \leqq \nu)$ — in particular, of M.

(17.2) *The product ${}^1M \times {}^2M$ of two absolute manifolds is an absolute manifold.* This follows from applying (13.3) and (13.4) to $St({}^1E_p \times {}^2E_q)$, $Cl({}^1E_p \times {}^2E_q)$, $It({}^1E_p \times {}^2E_q,\ {}^1E_r \times {}^2E_s)$ expressed as $St({}^1E_p) \times St({}^2E_q)$, etc. (It is assumed that the boundary relations between cells of ${}^1M \times {}^2M$ are just those imposed by (12.5).) In this connection it is perhaps interesting to notice that $M^{(1)*}$ can be thought of as the closure on $M^* \times M$ of the set $\{E^{n-p} \times E_p\}$ of products of dual cells. The results of §§ 15, 16 could have been developed quite as well in terms of this "sub-product" complex as in terms of the intersection complex.

On M (16.5) holds, else $It(E_p, E_{p+1})$ is not null-like. In conjunction with (1.3) this requires that for any pair of cells of M of consecutive dimensions

$$(17.3)\quad \begin{cases} [E_{p+1} : E_p] = \pm 1 \text{ or } 0,^{15} \\ E_p \prec E_{p+1} \text{ if, and only if, } [E_{p+1} : E_p] \neq 0. \end{cases}$$

If $E_p \prec E_q$, $q - p > 1$, there is at least one E_{p+1} such that $E_p \prec E_{p+1} \prec E_q$, else $R_p(It(E_p, E_q)) = 1$. Repetition of this argument leads to the result that

$$(17.4)\qquad E_p \prec E_{p+1} \prec E_{p+2} \prec \cdots \prec E_{q-1} \prec E_q \text{ if } E_p \prec E_q.$$

[15] de Rham makes this a basic assumption in his treatment of (absolute) manifolds, l. c. [8], p. 119.

Taken together, (17.3) and (17.4) show that the boundary relations between cells of M may be completely deduced from the incidence numbers.

(17.5) *The sub-complex* $Bd(E_t) = Cl(E_t) - E_t$ *of* M *is a sphere-like* $(t-1)$-*dimensional absolute manifold.*[16]

In consequence of (17.3) $F(E_t)$ provides a simple covering $(t-1)$-cycle for $Bd(E_t)$. The star on $Bd(E_t)$ of a cell E_p is just $It(E_p, E_t) - E_t$ and so is $(t-1)$-cell-like. Therefore $Bd(E_t)$ has star uniformity. On $Bd(E_t)$ $Cl(E_p)$ and $It(E_p, E_q)$ are the same as on M. And since the $(t-1)$-dimensional dual of $Bd(E_t)$ is isomorphic, except for dimension, to $St(E^{n-t}) - E^{n-t}$ it has a simple covering $(t-1)$-cycle. Therefore $Bd(E_t)$ has closure and intercept uniformity. Since $Cl(E_t)$ is 0-cell-like, $Bd(E_t)$ is sphere-like.

By (8.6) the n-cells of each $St(E_p)$ of M are strongly connected on the star. This means that for M connectedness and strong connectedness are equivalent. So on account of (8.6),

(17.6) *A necessary and sufficient condition that* M *be connected is that* $R_n(M) = 1$.

Consequently a connected absolute manifold is a simple circuit.

Let $\tilde{M}^*$ denote the result of re-orienting the cells of M^* by the rule $\tilde{E}^{n-p} = (-1)^{p(p+1)/2} E^{n-p}$. Then on $M \cdot \tilde{M}^*$ (16.3) becomes

$$F(E_{p+1} \cdot \tilde{E}^{n-p}) = (-1)^p [E_{p+1} : E_p] (E_p \cdot \tilde{E}^{n-p} - E_{p+1} \cdot \tilde{E}^{n-p-1}).$$

Using (17.4) we have

$$F\left(\sum_{\nu=0}^{q-p-1} (-1)^{p+\nu} [E_{p+\nu+1} : E_{p+\nu}] E_{p+\nu+1} \cdot \tilde{E}^{n-p-\nu}\right) = E_p \cdot \tilde{E}^{n-p} - E_q \cdot \tilde{E}^{n-q}.$$

Hence

$$E_p \cdot \tilde{E}^{n-p} \sim E_q \cdot \tilde{E}^{n-q} \quad \text{on} \quad M \cdot \tilde{M}^*, \tag{17.7}$$

if $E_p \prec E_q$ and so also if E_p and E_q are connected on M.

Suppose M connected. Let γ_0 denote a 0-cycle on $M \cdot \tilde{M}^*$ and $Ki(\gamma_0)$ the sum of the coefficients expressing γ_0 in terms of the 0-cells of $M \cdot \tilde{M}^*$. Then due to (17.7) we have

$$\gamma_0 \sim Ki(\gamma_0) E_0,$$

where E_0 is an arbitrary 0-cell of $M \cdot \tilde{M}^*$. The number $Ki(\gamma_0)$ is called the *Kronecker index* of γ_0. In terms of it (15.1) takes the form

$$Ki(E_p^i \cdot \tilde{E}_j^{n-p}) = \delta_j^i,$$

and so

$$Ki(s_i E_p^i \cdot t^j \tilde{E}_j^{n-p}) = s_i t^i. \tag{17.8}$$

[16] In Lefschetz' definition of manifold, l. c. [2], p. 120, this appears as one of the conditions.

Hence, if $\{C_p^i\}$ and $\{\tilde{C}_i^{n-p}\}$ are bases such that

$$[C_p^i/E_p^h] = [\tilde{E}_h^{n-p}/\tilde{C}_i^{n-p}]$$

it follows that

$$Ki(C_p^i \cdot \tilde{C}_j^{n-p}) = [C_p^i/E_p^h]\,[\tilde{C}_j^{n-p}/\tilde{E}_h^{n-p}] = \delta_j^i. \tag{17.9}$$

Let C_p, $0<p<n$, denote a p-cycle on an absolute manifold M which is sphere-like. Then there is a chain C'_{p+1} such that $F(C'_{p+1}) = C_p$. Let $\tilde{C}^{n-p}$ denote an $(n-p)$-cycle on $\tilde{M}^*$; then

$$F(C'_{p+1} \cdot \tilde{C}^{n-p}) = (-1)^p\, C_p \cdot \tilde{C}^{n-p}$$

and so $C_p \cdot \tilde{C}^{n-p} \sim 0$ on $M \cdot \tilde{M}^*$, i. e., $Ki(C_p \cdot \tilde{C}^{n-p}) = 0$. Let $\tilde{C}^{n-p-1}$ denote an $(n-p-1)$-cycle on $\tilde{M}^*$; then $Ki(C_p \cdot \tilde{C}'^{n-p})$ is the same for each $\tilde{C}'^{n-p}$ such that $F(\tilde{C}'^{n-p}) = \tilde{C}^{n-p-1}$, because any two $\tilde{C}'^{n-p}$'s will differ by an $(n-p)$-cycle $\tilde{C}^{n-p}$. The common value $Ki(C_p \cdot \tilde{C}'^{n-p})$ is called the *looping* (*linking*) *coefficient* of C_p and $\tilde{C}^{n-p-1}$, and denoted by $Lc(C_p, \tilde{C}^{n-p-1})$. It is easily verified that

$$Lc(C_p, \tilde{C}^{n-p-1}) = (-1)^{p-1}\, Ki(C'_{p+1} \cdot \tilde{C}^{n-p-1}).$$

Joins and attendant considerations.

18. **Joins.** By setting $w = 1$ in (12.1) the join of E_p and E_q is obtained. We write this join simply $E_p\,E_q$. The cells E_p, E_q are termed *components* of the join.

The treatment of joins is facilitated by the introduction of an ideal cell E_{-1} playing the rôle of the unit element in the 1-multiplication.[17] This cell is such that, for every E_p,

$$E_p\,E_{-1} = E_p \quad \text{and} \quad E_{-1} \leqq E_p, \tag{18.1}$$

and, for every E_0, suitably oriented,

$$[E_0 : E_{-1}] = 1. \tag{18.2}$$

This suitable orientation will be assumed whenever (18.2) is of consequence.

If E_{-1} be added to an arbitrary complex it is by no means certain that the resulting set of cells will be a complex. In case it is we say the primitive complex is *augmentable* and call the resulting complex its *augmented* complex. The only change in homology characters is a diminution by unity of the 0th Betti number. Clearly

[17] Cf. the symbolic 1 of Alexander, J. W., *The combinatorial theory of complexes.* Ann. Math. 31 (1930), p. 294.

(18.3) *A necessary and sufficient condition that a complex K be augmentable is that K^* have a simple covering n-cycle.*

The covering n-cycle constitutes a boundary chain for an ideal $(n+1)$-cell dual to E_{-1}. The suitable orientation associated with (18.2) corresponds to the orientation of the n-cells of K^* which causes the simple covering n-cycle to contain each n-cell of K^* with coefficient $+1$.

We assume that every cell E_p has, neglecting the order of 1-multiplication, a unique decomposition

$$E_p = \pm E_{p_0} E_{p_1} \cdots E_{p_x} \tag{18.4}$$

into prime components, i. e., into components having no other components than themselves and E_{-1}. Another assumption made concerning joins is that

$$E_p E_p = 0 \qquad (p \geqq 0). \tag{18.5}$$

A p-cell is called a *p-simplex* if all its prime components are 0-cells, or vertices — to give them a more descriptive name. The join of two vertices A, B is by (12.3) "skew-commutative", i. e.,

$$AB = -BA, \tag{18.6}$$

a rule valid, because of (18.5), even when $A = B$. Under suitable ordering of its vertices a p-simplex σ_p is expressible as

$$A^0 A^1 \cdots A^p.$$

Any r-component of σ_p must on account of (18.5) be an r-simplex

$$\sigma_r = A^{i_0} A^{i_1} \cdots A^{i_r},$$

where $(i_0\, i_1 \cdots i_r)$ is a combination of $(0\,1 \cdots p)$. If $r = p-1$, we have by (12.6), (18.1), (18.2) and (18.6) that

$$[\sigma_p : \sigma_{p-1}] = \pm 1 \tag{18.7}$$

according as $(i_p\, i_0 \cdots i_{p-1})$ is an even or odd permutation of $(0\,1 \cdots p)$. Hence $\sigma_{p-1} \prec \sigma_p$. By applying this argument successively to the intermediate components

$$A^{i_0} A^{i_1} \cdots A^{i_r} A^{i_{r+1}} \cdots A^{i_s} \qquad (r < s < p)$$

we conclude that the component $\sigma_r \prec \sigma_p$. In other words every component of a simplex other than itself is a simplex on its boundary.

The set of simplexes consisting of the components of a p-simplex, including the p-simplex itself but excluding E_{-1}, is called the *closed* p-simplex.

(18.8) *A closed p-simplex is a 0-cell-like augmentable complex.*

Each vertex is an augmentable complex. By taking the iterated join of the augmented vertices we obtain precisely the closed p-simplex plus E_{-1}. Theorems (13.5), (13.6) show that this complex like the augmented vertices of which it is composed is null-like. Removal of E_{-1} completes the argument.

(18.9) *The star of a component of a p-simplex on the closed p-simplex is null-like.*

Let $\sigma_p = \sigma_r \sigma_{p-r-1}$. The star of σ_r on the closed simplex $\bar{\sigma}_p$ is $\sigma_r {}^+\bar{\sigma}_{p-r-1}$ where ${}^+\bar{\sigma}_{p-r-1}$ denotes the augmented closed simplex obtained from σ_{p-r-1}. Hence the theorem.

19. **The simplicial complex.** In the usual sense of the term a *simplicial complex* π is a closed sub-complex of a closed simplex $\bar{\sigma}_N$. We prefer, however, the designation *closed simplicial complex* for such a π and reserve the above unqualified term for any sub-complex of a $\bar{\sigma}_N$.

(19.1) *A closed simplicial complex π is augmentable.*

For ${}^+\pi$ is closed on ${}^+\bar{\sigma}_N$ and so is a complex.

A necessary and sufficient condition that a set K of cells, including E_{-1}, be an augmented closed simplicial complex is that to every pair of cells $E_p \prec E_q$ of K there correspond a cell E_{q-p-1} of K such that

$$E_p E_{q-p-1} = \pm E_q \tag{19.2}$$

and (at least) one cell E_r of K, for each $p < r < q$, such that

$$E_p \prec E_r \prec E_q. \tag{19.3}$$

Necessity follows directly from the properties of closed simplexes recorded in the preceding section. To prove sufficiency let us note that since $E_{-1} \prec E_q$ there is, by (19.3), at least one 0-cell E_0 of $K \prec E_q$ such that, by (19.2),

$$E_0 E_{q-1} = \pm E_q. \tag{19.4}$$

Since due to (12.6) and (18.2) $[E_q : E_{q-1}] = \pm 1$,

$$E_{q-1} \prec E_q. \tag{19.5}$$

In turn there is a 0-cell $E_0' \prec E_{q-1}$, and so $\prec E_q$, such that

$$E_0 E_0' E_{q-2} = \pm E_q.$$

Finally we get

$$E_0 E_0' E_0'' \cdots E_0^{(q)} = \pm E_q,$$

where E_0, E_0', etc., $\prec E_q$, which fact taken with (18.4) and (19.2) implies that each $(q-1)$-component of E_q belongs to K. Since E_q is an arbitrary cell of K the argument is complete.

By (18.8) the closure of each cell of a closed simplicial complex π is 0-cell-like. Theorems (19.1) and (18.3) unite to show that π^* has a simple covering cycle. Hence

(19.6) *A closed simplicial complex has closure uniformity.*

In consequence of (18.9)

(19.7) *A closed simplicial complex has intercept uniformity.*

So

(19.8) *A necessary and sufficient condition that a closed simplicial complex be an absolute manifold is that it have star uniformity.*

20. **Focal subdivision.** On a complex ${}^1K+E_t$ let $Bd(E_t)=Cl(E_t)-E_t$ and, supposing $Bd(E_t)$ augmentable, let

$$(20.1) \qquad {}^2K = E_0 {}^+Bd(E_t) \quad \text{or} \quad {}^+Bd(E_t)E_0.$$

Besides the boundary and incidence relations rendered necessary by the fact that ${}^2K$ is a join we suppose that

$$(20.2) \qquad {}^2E_r \prec E_u \quad \text{if} \quad E_t \prec E_u,$$

$$(20.3) \qquad [{}^1E_{t+1} : {}^2E_t] = \varepsilon [{}^1E_{t+1} : E_t][E_t : {}^1E_{t-1}],$$

where $\varepsilon\, {}^2E_t = E_0\, {}^1E_{t-1}$ or $-(-1)^t\, {}^1E_{t-1}E_0$.

On ${}^1K+{}^2K$, $F(F({}^2E_{p+1}))$ is the same as on $Cl({}^2K)$ and so is 0 since $Cl({}^2K)$, being the join of ${}^+E_0$ and $Bd(E_t)$, is a complex. On ${}^1K+{}^2K$ $F(F({}^1E_{p+1}))$ is the same as on ${}^1K+E_t$, except possibly for $p=t$ or $t+1$. On ${}^1K+E_t$ let

$$F({}^1E_{t+1}) = {}^1C_t + [{}^1E_{t+1} : E_t]\, E_t;$$

then

$$F({}^1C_t) + [{}^1E_{t+1} : E_t]\, F(E_t) = 0$$

On ${}^1K+{}^2K$, due to (20.3),

$$F({}^1E_{t+1}) = {}^1C_t + [{}^1E_{t+1} : E_t]\, E_0\, F(E_t) \quad \text{or} \quad {}^1C_t + (-1)^t [{}^1E_{t+1} : E_t]\, F(E_t)\, E_0.$$

Hence

$$F(F({}^1E_{t+1})) = F({}^1C_t) + [{}^1E_{t+1} : E_t]\, F(E_t) = 0.$$

On ${}^1K+{}^2K$, $F(F({}^1E_{t+2}))$ is also 0 because the coefficient of a ${}^2E_t$ is $\varepsilon [F({}^1E_{t+2}) : E_t][E_t : {}^1E_{t-1}]$, which is 0 since $[F({}^1E_{t+2}) : E_t] = 0$. This completes the argument that ${}^1K+{}^2K$ is a complex.

We assume that the boundary relations on ${}^2K$ and between cells of ${}^2K$ and cells of ${}^1K$ are just those imposed by (20.1) and (20.2). Then ${}^2K$ is clearly a t-dimensional t-block on ${}^1K+{}^2K$. So in accordance with (11.8) ${}^1K+{}^2K$ is a subdivision of ${}^1K+E_t$ if ${}^1K+E_t$ is a t-weld of ${}^1K+{}^2K$. The latter requirement is met if ${}^2K$ is t-cell-like with a basic t-cycle ${}^2b_t^2$ satisfying (9.2)—which in this case has the form

$$\begin{cases} [{}^1E_{t+1} : {}^2b_t^3] = [{}^1E_{t+1} : E_t], \\ [{}^2b_t^3 : {}^1E_{t-1}] = [E_t : {}_1E_{t-1}]. \end{cases}$$

These incidence relations are shown by (20.3) and (20.1) to be satisfied if, and only if,

$$^2b_t^3 = E_0 F(E_t) \quad \text{or} \quad (-1)^t F(E_t) E_0,$$

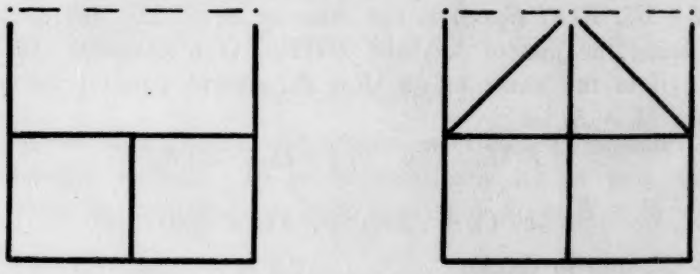
which of course carries the implication that $F(E_t)$ is a basic $(t-1)$-cycle. To require ${}^2K$ to be t-cell-like is to require ${}^+Bd(E_t)$ to be $(t-1)$-cell-like, or—$F(E_t)$ being basic—to require ${}^+Cl(E_t)$ to be null-like, i. e., to require $Cl(E_t)$ to be 0-cell-like and augmentable. Hence

(20.4) *In order that* ${}^1K+{}^2K$ *be a subdivision of the complex* ${}^1K+E_t$ *it is necessary and sufficient that* $Cl(E_t)$ *be* 0*-cell-like and augmentable.*

Intuitively, in the case of a geometrical 2-complex consisting of polygonal interiors, edges, and vertices, this subdivision may be thought of as adding a vertex at the centre of one polygon and joining it by segments to the vertices of that polygon.

By replacing ${}^+Bd(E_t)$ by $Bd(E_t)$ itself in the above discussion we get the theorem:

(20.5) *In order that* ${}^1K+{}^2K$ *be a subdivision of* ${}^1K+E_t$ *it is necessary and sufficient that* $Cl(E_t)$ *be null-like.*



(a) Fig. III. (b)

Fig. III illustrates this type of subdivision. (The edges and vertices indicated by the broken lines are not part of the 2-complex considered.)

We call a subdivision complying with (20.4) or (20.5) a *focal subdivision* —the 0-cell E_0 being the "focus" of the subdivision. A cell E_t having ${}^{(+)}Cl(E_t)$ null-like we term *focally subdivisible* ($^{(+)}$ means augmented if possible). To distinguish between the focal subdivision corresponding to (20.4) and that corresponding to (20.5) the former may be called *closed* and the latter *open*.[18]

[18] For a closed simplicial complex closed focal subdivision differs from the *elementary subdivision* introduced by Alexander, l. c. [4], p. 303, in that E_t alone is subdivided—not its whole star, as is essential if the new cells are to be simplexes. Closed focal subdivision is like the *subdivision pyramidale* used by de Rham, l. c. [3], p. 130.

Focal subdivision is transitive in that:

(20.6) *The new cells introduced by focal subdivision are themselves focally subdivisible.*

For $^{(+)}Cl(E_0\,{}^1E_p)$ is null-like, being precisely the join of $^+E_0$ with $^{(+)}Cl({}^1E_p)$. (Since here orientation is not of consequence we do need to add "or $^{(+)}Cl({}^1E_p\,E_0)$".)

(20.7) *Intercept uniformity is unaffected by focal subdivision.*

Suppose ${}^1E_p \prec E_t \prec {}^1E_r$. In $It({}^1E_p, {}^1E_r)$, $It({}^1E_p, E_t)$ is replaced by the join of $^+E_0$ and $It({}^1E_p, E_t) - E_t$, i. e., a nullblock is replaced by a nullblock. Hence $It({}^1E_p, {}^1E_r)$ remains null-like.

Consider $It(E_0\,{}^1E_p, {}^1E_r)$. It can be thought of as arising from $It(E_t, {}^1E_r)$ by replacing E_t by the join of E_0 with $It({}^1E_p, E_t) - E_t$. The latter plays the rôle of a subdivision of E_t. Hence $It(E_0\,{}^1E_p, {}^1E_r)$ like $It(E_t, {}^1E_r)$ is null-like.

Now suppose ${}^1E_p \prec {}^1E_q \prec E_t$. $It(E_0\,{}^1E_p, E_0\,{}^1E_q)$ is the join of E_0 with $It({}^1E_p, {}^1E_q)$, and so is null-like. $It({}^1E_p, E_0\,{}^1E_q)$ consists of $It({}^1E_p, {}^1E_q) + It(E_0\,{}^1E_p, E_0\,{}^1E_q)$ and so is null-like. The argument is complete.

By considering $St({}^1E_p)$ or $St(E_0\,{}^1E_p)$ instead of $It({}^1E_p, {}^1E_r)$ or $It(E_0\,{}^1E_p, {}^1E_r)$, respectively, we have that:

(20.8) *Assuming intercept uniformity, the stars of cells on* ${}^1K + {}^2K$ *have the same homology characters as the corresponding stars on* ${}^1K + E_t$. [N. B. To $St(E_0\,{}^1E_p)$ corresponds $St(E_t)$.]

On the other hand suppose $St(E_0\,{}^1E_p)$ has the same homology characters as $St(E_t)$. Then certainly $It({}^1E_p, E_t)$ must be $(t-1)$-cell-like, and so $It({}^1E_p, E_t)$ must be null-like unless $F(E_t)$ on $It({}^1E_p, E_t)$ fails to be basic. The latter contingency cannot occur if not all $[{}^1E_{t+1} : E_t] = 0$, i. e., if $R_t(St(E_t)) = 0$, else by (20.3) $St(E_0\,{}^1E_p)$ will not have the same t-dimensional homology characters as $St(E_t)$. Hence

(20.9) *If* $St(E_0\,{}^1E_p)$ *has the same homology characters as* $St(E_t)$ *and* $R_t(St(E_t)) = 0$, $It({}^1E_p, E_t)$ *is null-like.*

21. **Derivability.** A complex all of whose cells are focally subdivisible we call *derivable*. In consequence of (20.6)

(21.1) *Derivability is unaffected by focal subdivision.*

If a cell of a derivable complex K is focally subdivisible in the closed sense so must be all cells of K on its boundary. Accordingly the cells of K focally subdivisible in the closed sense form a closed sub-complex of K. This sub-complex is augmentable; just take the logical sum of the augmented closures of all its cells and apply (6.8). The complementary open sub-complex of cells focally subdivisible in the open sense we call the *fringe* of K. and denote by J.

The closure of every cell of a derivable K contains cells of all lower dimensions, else there would be an E_p, $p>0$, with $R_p(Cl(E_p)) = 1$. So, since every 0-cell of K belongs to $K-J$,

$$St(K-J) = K. \tag{21.2}$$

The closure on K of each cell of J must contain a 1-cell of J, else the closure would be augmentable, i. e.,

$$St(\{E_1\}_J) = J. \tag{21.3}$$

Due to (18.3), since $K-J$ is augmentable,

(21.4) *K^*-J^* has a simple covering n-cycle.*

The condition that the closure of a 1-cell E_1 of J be null-like implies that on the boundary of E_1 there is a single E_0 and that $[E_1 : E_0] = \pm 1$. Applied to K^* this shows that

(21.5) *J^* has a simple covering $(n-1)$-cycle,*

viz., the boundary chain on J^* of the simple covering n-cycle on K^*-J^*.

On a derivable complex K first subdivide focally all the 1-cells, then all the 2-cells, etc., and finally all the n-cells, multiplying each time on the right by the new 0-cell introduced as a focus. The resulting subdivision of K is called the *regular* (or barycentric) subdivision of K. Let $A^i_{(t)}$ denote the focus introduced in subdividing E^i_t, and for uniformity let $E^i_0 = A^i_{(0)}$. Then every p-cell of the regular subdivision is a simplex,

$$A^{i_0}_{(t_0)} A^{i_1}_{(t_1)} \cdots A^{i_p}_{(t_p)} \quad \text{corresponding to} \quad E^{i_0}_{t_0} \prec E^{i_1}_{t_1} \prec \cdots \prec E^{i_p}_{t_p}, \tag{21.6}$$

where $E^{i_0}_{t_0} \subset K-J$. Let K' denote the closed simplicial complex consisting of all simplexes (21.6) with the restriction $E^{i_0}_{t_0} \subset K-J$ removed. We call K' the (first) *derived complex* of K (cf. Alexander, l. c. [4], p. 303). Let $J^{*\prime}$ denote the closed simplicial complex consisting of all simplexes (21.6) such that $E^{i_0}_{t_0} \subset J$. Then the regular subdivision of K is precisely the simplicial complex $K'-J^{*\prime}$.

Suppose K^* is derivable with L^* as its fringe. In forming its regular subdivision we multiply each time on the left by the new focal vertex. We assume that the focus introduced in subdividing E^{n-t}_i is $A^i_{(t)}$, and that $E^{n-n}_i = A^i_{(n)}$. Let L' denote the closed simplicial complex consisting of the simplexes (21.6) such that $E^{i_p}_{t_p} \subset L$. Then the regular subdivision of K^* is precisely the simplicial complex $K'-L'$.

Suppose $K-L$, where L is any closed $(n-1)$-dimensional sub-complex of K, has a simple covering n-cycle. Let $A_{(p)}[^{(+)}_{\cdots}]$ denote the set, aug-

mented if possible, of all simplexes of $K'-L'$ having $A_{(p)}$ as first vertex. Then

(21.7) *Subject to* (17.3), *a sufficient condition that* K^* *be derivable with fringe* L^* *is that each* $A_{(p)}[\overset{(+)}{\cdots}]$ *be* $(n-p)$*-cell-like.*[19]

The $[\overset{(+)}{\cdots}]$ of each $A_{(n-1)}[\overset{(+)}{\cdots}]$ consists of a single vertex or two vertices plus E_{-1}; so, using (17.3) and the fact that $K-L$ has a simple covering n-cycle, each E^1 is focally subdivisible with $A_{(n-1)}[\overset{(+)}{\cdots}]$ as subdivision. Suppose each E^{n-q} subdivisible with $A_{(q)}[\overset{(+)}{\cdots}]$ as subdivision, $q > p$. Then the $[\cdots]$ of $A_{(p)}[\overset{(+)}{\cdots}]$ is a regular subdivision of $Bd(E^{n-p}) = Cl(E^{n-p}) - E^{n-p}$. Consequently ${}^{(+)}Bd(E^{n-p})$ is $(n-p-1)$-cell-like, and so due to (17.3) ${}^{(+)}Cl(E^{n-p})$ is null-like.

If K and K^* are both derivable it follows from (5.6) and (5.7) that between their regular subdivisions there are the duality relations

$$\theta_i^p(K'-J^{*\prime}) = \theta^{n-p-1}_i(K'-L'), \tag{21.8}$$

$$R_p(K'-J^{*\prime}) = R_{n-p}(K'-L'). \tag{21.9}$$

Relative manifold criteria.

22. Relative star and closure uniformity. In consequence of (18.8), (19.1) and (19.7) K' is derivable, augmentable, and has intercept uniformity. What can be said concerning the derivability of K'^*?

Suppose K derivable. On the boundary of a 1-cell $E_1 \subset J$ there is just a single 0-cell, say E_0, since $Cl(E_1)$ is null-like by (20.5). Consequently the star on K' of an $(n-1)$-simplex $A_{(1)}A_{(2)}\cdots A_{(n)}$ of $J^{*\prime}$ is ${}^{+}A_{(0)}A_{(1)}A_{(2)}\cdots A_{(n)}$, and so is null-like. Similarly, supposing K^* derivable, the star on K' of an $(n-1)$-simplex $A_{(0)}A_{(1)}\cdots A_{(n-1)}$ of L' is null-like. In view of (21.3) these facts imply that if K'^* is derivable $J^{*\prime *}$ and L'^* belong to its fringe. We assume that the fringe of K'^* is precisely $(J^{*\prime} \dot{+} L')^*$.

Let $A_{(p)}\{\cdot\overset{+}{\cdot}\cdot\}A_{(q)}$ denote the set of all simplexes (21.6) of K' having $A_{(p)}$ and $A_{(q)}$ as first and last vertices, respectively; this set includes the 1-simplex $A_{(p)}A_{(q)}$ — hence the $+$ in $\{\cdot\overset{+}{\cdot}\cdot\}$. (In case $q = p+1$, $\{\cdot\overset{+}{\cdot}\cdot\} = E_{-1}$.) In similar fashion let $A_{(p)}\{\cdot\overset{+}{\cdot}\cdot\}$ and $\{\cdot\overset{+}{\cdot}\cdot\}A_{(q)}$ denote the sets of all simplexes of K' having respectively $A_{(p)}$ as first vertex and $A_{(q)}$ as last vertex. Then

(22.1) *In order that* K'^* *be derivable with* $(J^{*\prime} \dot{+} L')^*$ *as fringe it is necessary and sufficient that each* $A_{(p)}\{\cdot\overset{+}{\cdot}\cdot\}A_{(q)}$ *be* $(q-p)$*-cell-like, that each* $A_{(p)}\{\cdot\overset{+}{\cdot}\cdot\}$ *be* $(n-p)$*-cell-like or null-like according as* $E_p \subset K-L$ *or* L, *and that each* $\{\cdot\overset{+}{\cdot}\cdot\}A_{(q)}$ *be* q*-cell-like or null-like according as* $E_q \subset K-J$ *or* J.

[19] This theorem can be used as a basis for proving Property I, Lefschetz, l. c. [2], p. 122.

Necessity follows from the cell- or null-like nature of the following stars:

$$A_{(0)}A_{(1)}\cdots A_{(p)}\{\overset{+}{\cdots}\}A_{(q)}A_{(q+1)}\cdots A_{(n)},$$
$$A_{(0)}A_{(1)}\cdots A_{(p)}\{\overset{+}{\cdots}\},$$
$$\{\overset{+}{\cdots}\}A_{(q)}A_{(q+1)}\cdots A_{(n)}.$$

Sufficiency follows from the fact that the star of any t-simplex $A_{(p_0)}A_{(p_1)}\cdots A_{(p_t)}$ can be written:

$$\{\overset{+}{\cdots}\}A_{(p_0)}\{\overset{+}{\cdots}\}A_{(p_1)}\{\overset{+}{\cdots}\}\quad\cdots\quad\{\overset{+}{\cdots}\}A_{(p_t)}\{\overset{+}{\cdots}\}.$$

(22.2) *A sufficient — and, subject to* (17.3), *necessary — condition that each* $A_{(p)}\{\overset{+}{\cdots}\}A_{(q)}$ *be* $(q-p)$*-cell-like is that* K *have intercept uniformity.* Sufficiency is a consequence of the fact that $A_{(p)}\{\overset{+}{\cdots}\}A_{(q)}$ "divided" by $A_{(p)}$ may be regarded as the regular subdivision of the $(q-p-1)$-cell of a complex isomorphic with $It(E_p, E_q)-E_p$ except that the dimensions of its cells are uniformly $(p+1)$ less than those of the latter. For necessity we reverse the argument, using induction on $q-p$. But to start the induction each $It(E_p, E_{p+1})$ must be null-like, i. e., (17.3) must be valid. Suppose all intercepts $It(E_p, E_q)$ proved null-like for $q-p<s-r$. Then the $\{\cdots\}$ of $A_{(r)}\{\overset{+}{\cdots}\}A_{(s)}$ may be regarded as a regular subdivision of a complex isomorphic, except for dimensions, to $It(E_r, E_s)-E_r-E_s$, thereby showing the latter to be sphere-like. On $It(E_r, E_s)$, in view of (17.3), $F(E_s)$ is basic and $E_r\sim 0$; therefore $It(E_r, E_s)$ is null-like.

Let $Cl(E_p, J)$ denote the set of cells obtained by dropping from $Cl(E_p)$ the cells, if any, it has in common with J. Then

(22.3) *A necessary and sufficient condition that each* $\{\overset{+}{\cdots}\}A_{(p)}$, $E_p\subset J$, *be null-like is that each corresponding* $Cl(E_p, J)$ *be* 0*-cell-like.*

The theorem clearly holds for $p=1$. Suppose it holds for $p<q$. From $\{\overset{+}{\cdots}\}A_{(q)}$, $E_{(q)}\subset J$, we may remove the null-like sets without affecting homology characters; the part left is ${}^{+}\Omega A_{(q)}$ where Ω is the regular subdivision of $Cl(E_q, J)$. Hence $\{\overset{+}{\cdots}\}A_{(q)}$ is null-like if, and only if, $Cl(E_q, J)$ is 0-cell-like.

Similarly, letting $St(E_p, L)$ denote the set of cells of $St(E_p)$ having no cells in common with L, we have:

(22.4) *A necessary and sufficient condition that each* $A_{(p)}\{\overset{+}{\cdots}\}$, $E_p\subset L$, *be null-like is that each corresponding* $St(E_p, L)$ *be* n*-cell-like.*

From the derivability in the closed sense of $K-J$ and K^*-L^* it follows respectively that $Cl(E_q, J)=Cl(E_q)$, $E_q\subset K-J$, is 0-cell-like and that $St(E_p, L)=St(E_p)$, $E_p\subset K-L$, is n-cell-like. Combining these facts with (21.4), (22.3), and (22.4) we are led to definitions of relative star and closure uniformity. We say that K has *star uniformity*

mod L if each $St(E_p, L)$ is n-cell-like and if $K-L$ has a simple covering n-cycle; we say K has *closure uniformity mod* J if each $Cl(E_p, J)$ is 0-cell-like and if $K-J$ is augmentable, i. e., if K^* has star uniformity mod J^*. When L or J is vacuous, the corresponding relative uniformity reduces of course to an ordinary uniformity of § 15.

Suppose $K-L$ and L have simple covering cycles, that on L the boundary of that on $K-L$. Let E_p be a cell of L. Then from two of—$St(E_p)$ null-like, $St(E_p, L)$ n-cell-like, $St(E_p)-St(E_p, L)(=St(E_p)$ on $L)$ $(n-1)$-cell-like—follows the third. Hence, using (21.4) and (21.5) applied to $K-L$ and L, we conclude that

(22.5) *Of the statements:*

(a) K^* *is derivable with* L^* *as fringe,*

(b) K *has star uniformity mod* L, C_n *being the simple covering* n*-cycle on* $K-L$,

(c) L *has star uniformity with* $F(C_n)$ *as its simple covering* $(n-1)$*-cycle.*

any two imply the third.

In case (b) and (c) hold for a complex K we say that K has *star uniformity bounded by* L. If K^* has star uniformity bounded by J^* we say K has *closure uniformity fringed by* J. Then by the above work:

(22.6) *In order that* K, K^*, K'^* *be derivable with* J, L^*, $(J^{*\prime}+L')^*$, *respectively, as fringes it is sufficient—and, subject to* (17.3), *necessary—that* K *have star uniformity bounded by* L, *closure uniformity fringed by* J, *and intercept uniformity.*

Under suitable modification the results of § 15 apply to relative star and closure uniformity. Thus,

(22.7) *A necessary and sufficient condition that* $(K-L)\cdot K^*$ *be a subdivision of* K^* *is that* K *have star uniformity mod* L.

And of course there is a theorem corresponding to (15.5) as this does to (15.4). If K has star uniformity mod L and closure uniformity mod J, $(K-L)\cdot(K-J)^*$ is a common subdivision of $K-L$ and K^*-J^*, and so

$$\theta_i^p(K-L) = \theta_i^{n-p-1}(K-J), \tag{22.8}$$

$$R_p(K-L) = R_{n-p}(K-J). \tag{22.9}$$

(These duality relations form a purely combinatorial counterpart of—or, perhaps we should say, basis for—the Fundamental Duality Theorem of Lefschetz, l. c. [3], p. 142.)

23. **Manifold with regular boundary.** Let $'K$ denote any complex obtained from the complex K by one or more focal subdivisions. From (20.8), (20.9) and (21.1) it follows that:

(23.1) *In order that every cell of K, K^*, $'K$, $'K^*$ be focally subdivisible it is necessary and sufficient that K and K^* be derivable and that K have intercept uniformity.*

Suppose K, in addition to satisfying the conditions in (23.1), is augmentable. Then K will have no fringe, but K^* of course may have a fringe L^*. The regular subdivision of K will be exactly the derived complex K'. Hence K' constitutes a $'K$ of Theorem (23.1), and so K'^* is derivable with L'^* (cf. (20.8)) as fringe. Applying (22.6) we see that K has star uniformity bounded by L and closure uniformity, besides the intercept uniformity hypothesized. The star uniformity bounded by L implies, by definition, that L has star uniformity. Being part of K, L has closure and intercept uniformity; therefore L is an absolute manifold. Due to K's closure and intercept uniformity the results (17.3), (17.4), and (17.5) apply to K. A complex K such as just described—i. e., satisfying (23.1), and augmentable—we call an n-dimensional (orientable) *manifold with regular boundary* L.[20] In case L is vacuous, K becomes, of course, an absolute manifold. If K^* is a manifold with regular boundary J^* we say that K is a *manifold with regular fringe* J.

Suitably modified, Theorem (16.2) carries over for the types of star and closure uniformity introduced in § 22. Hence if K is a manifold with regular boundary L, $K \cdot K^*$ has star uniformity bounded by $L \cdot K^*$. In consequence of (16.6) and (16.7) $K \cdot K^*$ has closure and intercept uniformity. Therefore $K \cdot K^*$ is a manifold with regular boundary $L \cdot K^*$. By the counterpart of (22.7) $K \cdot K^*$ and $L \cdot K^*$ are (since $J = 0$) subdivisions of K and L. We express these facts in the statement:

(23.2) *A manifold with regular boundary is essentially unaffected by intersection with its dual.*

This reduces to (17.1) when L is vacuous.

On account of (19.6), (19.7) and (20.8) K' is a manifold with regular boundary L' if K is a manifold with regular boundary L. L' is a regular subdivision of L just as K' is of K. We express these facts by the statement:

(23.3) *A manifold with regular boundary is essentially unaffected by regular subdivision.*

This includes as a special case, viz., for $L = 0$, a similar statement concerning an absolute manifold.

Let us now consider a complex K with star uniformity bounded by L, closure uniformity fringed by J, and intercept uniformity—i. e., a K meeting the conditions of (22.6). The following facts follow immediately from definitions or previous theorems:

[20] Using (17.5) and (21.7) it is not difficult to show that this definition agrees with that of Lefschetz, l. c. [2], pp. 119–20.

(23.4) *L is an $(n-1)$-dimensional manifold with regular fringe $L \cap J$.*
(23.5) *J^* is an $(n-1)$-dimensional manifold with regular fringe $L^* \cap J^*$.*
[N. B. $L \cap J$ denotes the set of cells common to L and J.]
(23.6) *$K \cdot K^*$ is a manifold with regular boundary $(K \cdot J^* \dot{+} L \cdot K^*)$.*
(23.7) *K' is a manifold with regular boundary $(L' \dot{+} J^{*\prime})$.*

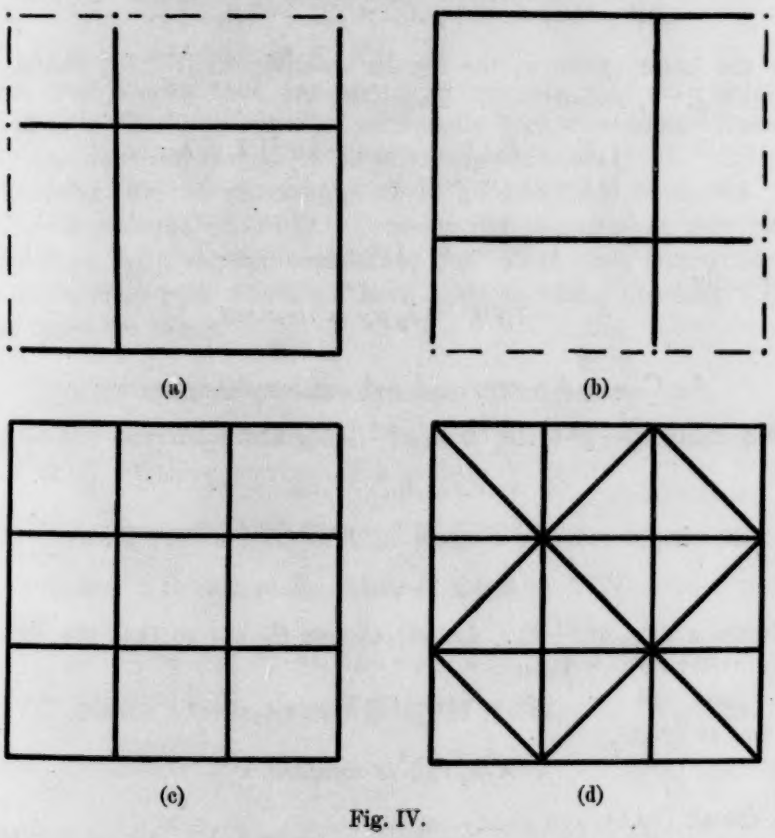


Fig. IV.

Fig. IV illustrates: (a) a simple 2-dimensional geometric complex K with star uniformity bounded by L, closure uniformity fringed by J, and intercept uniformity; (b) the dual complex K^*; (c) the intersection complex $K \cdot K^*$; (d) the derived complex K'. L consists of the four 1-cells and three 0-cells at the bottom and right of (a), and J of the three 2-cells and four 1-cells immediately adjacent to the broken lines and dots which limit the figure without belonging to it. Thus L and J have two common 1-cells in this case.

24. **Realization.** Let K be a complex with star uniformity bounded by L, closure uniformity fringed by J, and intercept uniformity. We identify the sets of simplexes $A_{(p)}\{\cdot \dot{+} \cdot\} A_{(q)}$ of K' with the cells $E_q \cdot E^{n-p}$

of $K \cdot K^*$ with a view to applying Theorem (11.4). By (22.2) intercept uniformity ensures that each $A_{(p)}\{\overset{+}{\cdot\cdot\cdot}\}A_{(q)}$ is $(q-p)$-cell-like. Our main task is to show that (11.2) can be satisfied.

Let

$$\text{(24.1)} \qquad \begin{cases} C_{p-1}A_{(p)} &= \lambda_p A_{(0)}A_{(1)}\cdots A_{(p)} + \cdots \\ A_{(p)}C_{n-p-1} &= \mu_p A_{(p)}A_{(p+1)}\cdots A_{(n)} + \cdots \end{cases}$$

denote the basic cycles on the regular subdivisions $[\overset{(+)}{\cdot\cdot\cdot}]A_{(p)}$ and $A_{(p)}[\overset{(+)}{\cdot\cdot\cdot}]$ of E_p and E^{n-p}, respectively. Then

$$\text{(24.2)} \qquad \begin{cases} [E_{p+1}:E_p] &= (-1)^{p+1}\lambda_{p+1}\lambda_p, \\ [E^{n-p}:E^{n-p-1}] &= \mu_p\mu_{p+1}, \end{cases}$$

and so

$$\lambda_p\mu_p = (-1)^{p+1}\lambda_{p+1}\mu_{p+1}.$$

Consequently

$$\text{(24.3)} \qquad (-1)^{p(p+1)/2}\lambda_p\mu_p \text{ is constant.}$$

Let

$$\text{(24.4)} \qquad A_{(p)}C_{q-p-2}A_{(q)} = \gamma_{p,q}A_{(p)}A_{(p+1)}\cdots A_{(q-1)}A_{(q)} + \cdots$$

denote a basic $(q-p)$-cycle on $A_{(p)}\{\overset{+}{\cdot\cdot\cdot}\}A_{(q)}$ and let

$$C_n = \varepsilon A_{(0)}A_{(1)}\cdots A_{(n)} + \cdots$$

denote the simple covering n-cycle on $K'-(L'\dot{+}J^{*\prime})$. Then

$$C_{p-1}A_{(p)}C_{q-p-2}A_{(q)}C_{n-q-1}$$

constitutes a part of $\pm C_n$. Let us choose C_{q-p-2} so that the sign is $(-1)^{[p(p+1)+q(q+1)]/2}$. Then

$$(-1)^{[p(p+1)+q(q+1)]/2}\lambda_p\mu_q\nu_{p,q} = \varepsilon,$$

i. e., due to (24.3),

$$\text{(24.5)} \qquad \lambda_q\mu_p\nu_{p,q} \text{ is constant.}$$

From (24.4), (24.5), and (24.2) we have

$$\begin{aligned} &[A_{(p)}C_{q-p-2}A_{(q)} : A_{(p+1)}C_{q-(p+1)-2}A_{(q)}] \\ &\quad = \nu_{p,q}\nu_{p+1,q} \\ &\quad = \mu_p\mu_{p+1} \\ &\quad = [E^{n-p}:E^{n-p-1}] \\ &\quad = [E_q\cdot E^{n-p} : E_q\cdot E^{n-p-1}], \end{aligned}$$

and

$$\begin{aligned} &[A_{(p)}C_{q-p-2}A_{(q)} : A_{(p)}C_{q-1-p-2}A_{(q-1)}] \\ &\quad = (-1)^{q-p}\nu_{p,q}\nu_{p,q-1} \\ &\quad = (-1)^{q-p}\lambda_q\lambda_{q-1} \\ &\quad = (-1)^p[E_q:E_{q-1}] \\ &\quad = [E_q\cdot E^{n-p} : E_{q-1}\cdot E^{n-p}]. \end{aligned}$$

Consequently (11.2) is satisfied, and so Theorem (11.4) applies to K' and $K \cdot K^*$. By agreeing that

$$A_{(p)} A_{(p+1)} \cdots A_{(q)} \prec E_u \quad \text{if} \quad E_q \cdot E^{n-p} \prec E_u, \tag{24.6}$$

where E_u is any cell, not necessarily of K, we may go further and assert that

(24.7) *K' is a subdivision of $K \cdot K^*$.*

It is well known that any simplicial complex may be geometrically realized in a Euclidean space of sufficiently high dimension. Hence

(24.8) *Any derivable complex is geometrically realizable.*

In particular, the above complex K is geometrically realizable. So is $K \cdot K^*$, on account of (24.7). Consequently the absolute manifolds and the manifolds with regular boundaries, their duals, their intersections with their duals, etc., with which we have dealt so much, are fully realizable in the geometric sense.

Correspondences and fixed cells.

25. **Chain correspondences.** To each p-cell ${}^aE_p^i$ of the complex ${}^aK$ ($a = 1$ or 2) let there correspond a p-chain

$$T^b({}^aE_p^i) = [{}^aE_p^i : {}^bE_p^j]\, {}^bE_p^j \tag{25.1}$$

on the complex ${}^bK$ ($b = 2$ or 1). Then to the chain $t_i\, {}^aE_p^i$ there corresponds the chain

$$T^b(t_i\, {}^aE_p^i) = t_i [{}^aE_p^i : {}^bE_p^j]\, {}^bE_p^j = t_i\, T^b({}^aE_p^i).$$

Let $\{{}^aC_p^h\}$ denote a basis for the group of p-chains on ${}^aK$. Then

$$T^b({}^aC_p^h) = [{}^aC_p^h : {}^bC_p^k]\, {}^bC_p^k \tag{25.2}$$

where

$$[{}^aC_p^h : {}^bC_p^k] = [{}^aC_p^h / {}^aE_p^i]\, [{}^aE_p^i : {}^bE_p^j]\, [{}^bE_p^j / {}^bC_p^k]. \tag{25.3}$$

(N. B. ${}^aC_p^h = [{}^aC_p^h / {}^aE_p^i]\, {}^aE_p^i, \qquad {}^bE_p^j = [{}^bE_p^j / {}^bC_p^k]\, {}^bC_p^k$.)

In group terminology (25.1) establishes a homomorphism between the group of p-chains on ${}^aK$ and a sub-group (not necessarily proper, of course) of p-chains on ${}^bK$. To ensure that a homomorphism is induced between the corresponding homology groups we postulate that

$$T^b(F({}^aC_{p+1})) = F(T^b({}^aC_{p+1})) \tag{25.4}$$

for an arbitrary chain ${}^aC_{p+1}$; or symbolically $T^bF = FT^b$, i. e., the operations F and T^b are commutative. This amounts to the requirement that

$$T^b(F({}^aC^g_{p+1})) = F(T^b({}^aC^g_{p+1})),$$

i. e., that

$$[{}^aC^g_{p+1} : {}^aC^h_p]\,[{}^aC^h_p : {}^bC^k_p] = [{}^aC^g_{p+1} : {}^bC^f_{p+1}]\,[{}^bC^f_{p+1} : {}^bC^k_p]. \tag{25.5}$$

If desired, (25.1) can be regarded as also establishing a chain correspondence between ${}^bK^*$ and ${}^aK^*$, viz.,

$$T^a({}^bE^{n-p}_j) = [{}^bE^{n-p}_j : {}^aE^{n-p}_i]\,{}^aE^{n-p}_i, \tag{25.6}$$

where

$$[{}^bE^{n-p}_j : {}^aE^{n-p}_i] = [{}^aE^i_p : {}^bE^j_p].$$

If the bases $\{{}^aC^h_p\}$ are canonical throughout, (25.5) takes the schematic form

$$\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ s & 0 & 0 \end{Vmatrix} \begin{Vmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{Vmatrix} = \begin{Vmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{Vmatrix} \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ s & 0 & 0 \end{Vmatrix}. \tag{25.7}$$

The sub-matrices marked s consist of zeros except for elements ${}^a\zeta^u_p$, $u = 1, 2, \cdots, {}^a\varrho_p$, up the secondary diagonal. The sub-matrices of $[\{{}^aC^h_p\} : \{{}^bC^k_p\}]$, $[\{{}^aC^g_{p+1}\} : \{{}^bC^f_{p+1}\}]$ marked 0 consist of zeros because of equations (25.5). It is to be observed that

$${}^a\zeta^u_p\,[{}^aC^u_p : {}^bC^v_p] = [{}^aC^{{}^a\alpha_{p+1}-u+1}_{p+1} : {}^bC^{{}^b\alpha_{p+1}-v+1}_p]\,{}^b\zeta^v_p. \tag{25.8}$$

Now consider the correspondence

$$T^a({}^aC^h_p) = [{}^aC^h_p : {}^aC^i_p]\,{}^aC^i_p \tag{25.9}$$

where

$$T^a({}^aC^h_p) = T^a(T^b({}^aC^h_p)),$$

and so

$$[{}^aC^h_p : {}^aC^i_p] = [{}^aC^h_p : {}^bC^k_p]\,[{}^bC^k_p : {}^aC^i_p]. \tag{25.10}$$

Since

$$[{}^bC^f_{p+1} : {}^bC^k_p]\,[{}^bC^k_p : {}^aC^i_p] = [{}^bC^f_{p+1} : {}^aC^e_{p+1}]\,[{}^aC^e_{p+1} : {}^aC^i_p]$$

we have by multiplying (25.5) by $[{}^bC^k_p : {}^aC^i_p]$ that

$$[{}^aC^g_{p+1} : {}^aC^h_p]\,[{}^aC^h_p : {}^aC^i_p] = [{}^aC^g_{p+1} : {}^aC^e_{p+1}]\,[{}^aC^e_{p+1} : {}^aC^i_p].$$

This shows that (25.9) establishes a homomorphism extending to the homology groups of ${}^aK$.

In accordance with the last paragraph we may allow a and b to take the values 1, 2 independently in formulas (25.1)–(25.8).

26. Weld correspondences. Consider a complex $K = {}^1K + {}^2K$ in which ${}^2K$ is a t-cellblock. Let K become $'K$ by the welding of ${}^2K$ into

a single cell $E_t = {}^2{}'E_t$. For convenience let us take the bases appearing in (9.1) with the slight modification that

$$\{{}^1E_p\}' = \{{}^1E_p\} + {}^2b^3_p$$

(${}^2b^3_p$ is of course vacuous unless $p = t$) and let us agree that ${}^2b'^1_{p-1}$ denotes a sub-basis obtained from ${}^2b^1_{p-1}$ by just reversing the order of its elements.

With the weld we associate a chain correspondence between $'K$ and K such that:

$$[\{'E_p\} : \{{}^1E_p\}'] = [\{{}^1E_p\}'/\{{}^1E_p\}'], \tag{26.1}$$

$$[\{'E_p\} : {}^2b^1_p] = 0, \tag{26.2}$$

$$[\{'E_p\} : {}^2b^5_p] = -[\{{}^1E_p\}' : {}^2b'^1_{p-1}]. \tag{26.3}$$

($[\{{}^1E_p\}'/\{{}^1E_p\}']$ is of course a unit matrix; by (9.1) $[\{{}^1E_p\}' : {}^2b'^1_{p-1}] = 0$, $p \leqq t$.) This correspondence satisfies (25.5) since (26.3) implies

$$[\{'E_p\} : \{{}^1E_p\}'][\{{}^1E_p\}' : {}^2b_{p-1}] + [\{'E_p\} : {}^2b^5_p][{}^2b^5_p : {}^2b^1_{p-1}] = 0$$

and since

$$[\{'E_{p+1}\} : \{'E_p\}][\{'E_p\} : {}^2b^5_p] = [\{'E_{p+1}\} : \{{}^1E_{p+1}\}'][\{{}^1E_{p+1}\}' : {}^2b^5_p]$$

in consequence of (26.1), (26.2) and the boundary condition

$$[\{{}^1E_{p+1}\}' : \{{}^1E_p\}'][\{{}^1E_p\}' : {}^2b^1_{p-1}] = [\{{}^1E_{p+1}\}' : {}^2b^5_p][{}^2b^5_p : {}^2b^1_p].$$

Considerations of duality suggested by (25.6) and the fact that the dual of a t-cellblock is an $(n - t)$-cellblock indicate the existence of a correspondence converse to that in the last paragraph. It is defined by:

$$[\{{}^1E_p\}' : \{'E_p\}] = [\{{}^1E_p\}'/\{{}^1E_p\}'], \tag{26.4}$$

$$[{}^2b^1_p : \{'E_p\}] = -[{}^2b'^5_{p+1} : \{{}^1E_p\}'], \tag{26.5}$$

$$[{}^2b^5_p : \{'E_p\}] = 0. \tag{26.6}$$

From the two above correspondences taken in order there results a correspondence on $'K$. It is the identity correspondence

$$[\{'E_p\} : \{'E_p\}] = [\{'E_p\}/\{'E_p\}]. \tag{26.7}$$

This verifies the fact that the corresponding homology groups of K and $'K$ are isomorphic.

Now let K and $'K$ have the significance attached to them in § 11. The correspondence between $'K$ and K defined by

$$['E^i_p : {}^j_p b^3_p] = \delta^i_j \tag{26.8}$$

clearly satisfies (25.5). A converse correspondence—which would follow (26.8) to produce an identity correspondence on $'K$ analogous to (26.7)—does not have a simple direct definition. But the existence of such a correspondence is guaranteed by the fact that the passage from K to $'K$ may be achieved by a succession of welds.

27. **Fixed cells.** Let us consider the correspondence on a complex K determined by

$$T(E_p^i) = [E_p^i : E_p^j]\, E_p^j. \tag{27.1}$$

The formulas (25.2)–(25.8) carry over with a and b dropped. A particular consequence of (25.8) is that

$$[C_p^u : C_p^u] = [C_{p+1}^{\alpha_{p+1}-u+1} : C_{p+1}^{\alpha_{p+1}-u+1}] \qquad (u \text{ not summed}).$$

Hence

$$\sum_p (-1)^p [C_p^h : C_p^h] = \sum_p (-1)^p [C_p^d : C_p^d]$$

where $d = \varrho_p + 1, \varrho_p + 2, \cdots, \alpha_p - \varrho_{p-1}$, or, since by (25.3)

$$[C_p^h : C_p^h] = [E_p^i : E_p^i],$$

we have that

$$\sum_p (-1)^p [E_p^i : E_p^i] = \sum_p (-1)^p [C_p^d : C_p^d]. \tag{27.2}$$

In case (27.1) is the identity correspondence

$$T(E_p^i) = E_p^i$$

it is clear that

$$[E_p^i : E_p^j] = \delta_j^i = [C_p^i : C_p^j]$$

and that (25.5) is satisfied. Formula (27.2) becomes the Euler-Poincaré formula

$$\sum_p (-1)^p \alpha_p = \sum_p (-1)^p R_p(K). \tag{27.3}$$

Consequently Hopf has aptly termed (27.2) a generalized Euler-Poincaré formula (cf. Hopf, l. c. [11], p. 136).

For a given value of i $[E_p^i : E_p^i]$ is the algebraic number of times the cell E_p^i corresponds to itself under (27.1). We call the sum $[E_p^i : E_p^i]$ the algebraic number of fixed p-cells of the correspondence, and

$$\sum_p (-1)^p [E_p^i : E_p^i]$$

the (algebraic) number of *fixed cells.*

Suppose the correspondence (27.1) has the property that

$$T(C_p^\lambda) \sim C_p^\lambda.$$

We call such a correspondence a *combinatorial deformation* Due to (27.2) it is seen that

(27.4) *The number of fixed cells of a combinatorial deformation is the Euler-Poincaré number* $\sum(-1)^p R_p(K)$.

It is also immediate that

(27.5) *On a complex all of whose Betti numbers are zero the number of fixed cells of any correspondence is zero.*

Let K be an augmented subdivision of $'K$, itself augmented and derivable. Consider any chain correspondence between K and $'K$ having the property, that to a chain on ${}^i_pK$ (notation of § 11) there corresponds a chain on $Cl('E^i_p)$; E_{-1} corresponds to itself. From (26.8) followed by this correspondence there results a correspondence T on $'K$.

(27.6) *T is an identity correspondence.*

The theorem is true for dimension -1. Suppose it holds for dimensions $< t$. Under T chains on a $Cl('E_t)$ go over into chains on the same closure. So by (27.5) the number of fixed cells on the nullblock $Cl('E_t)$ is zero. But all cells on the boundary of $'E_t$ are by supposition fixed. Therefore $'E_t$ must itself be fixed, and the induction is complete. The theorem easily extends to the case of a derivable $'K$ having a fringe.

Let K and $'K$ be arbitrary closed augmented simplicial complexes. To each vertex A^i of K let there correspond a vertex $T'(A^i)$ of $'K$ such that if

$$E_p = A^{i_0} A^{i_1} \cdots A^{i_p}$$

is a simplex of K, then

$$T'(A^{i_0}),\ T'(A^{i_1}),\ \cdots,\ T'(A^{i_p})$$

are vertices of some simplex of $'K$. Let

(27.7) $$T'(E_p) = T'(A^{i_0})\, T'(A^{i_1}) \cdots T'(A^{i_p}).$$

Consequently to E_p there corresponds 0 or $\pm$ some $'E_p$. Of course

$$T'(E_{-1}) = E_{-1}.$$

Clearly (27.7) satisfies condition (25.5).

Now suppose K is a simplicial subdivision of $'K$ and that $T'(A)$ is a vertex of $'E^i_p$ if A is a vertex on ${}^i_pK$. Theorem (27.6) applies. The result for $'K$ a closed n-simplex yields the well-known lemma[21] used in proving the topological invariance of dimensionality, viz.,

[21] Cf. e. g. Sperner, E., *Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes.* Hamburger Abhand. 6 (1928), pp. 265-272.

(27.8) *The number of n-simplexes of the subdivision going over into the main n-simplex with orientation retained exceeds by unity the number going over with orientation reversed.*

28. **Correspondences and products.** In the study of correspondences products have played an important part (cf. Lefschetz, l. c. [2], Chap. VI). It is the object of this section to expose the combinatorial formalities linking correspondences with products.

Consider the product complex ${}^1K^* \times {}^2K$ and also its dual. The dual of ${}^1E^{n-p} \times {}^2E_q$ may be written in conformity with (5.2) as

$$(-1)^{nq}\, {}^1E_p \times {}^2E^{n-q}. \tag{28.1}$$

Accordingly the dual of ${}^1K^* \times {}^2K$ may conveniently be written as ${}^1K \times {}^2K^*$, and the intersection complex as $({}^1K^* \times {}^2K) \cdot ({}^1K \times {}^2K^*)$. But the latter may also be considered as the product $({}^1K^* \cdot {}^1K) \times ({}^2K \cdot {}^2K^*)$ of the two intersection complexes, since computation shows that as far as incidence numbers are concerned

$$({}^1E^{n-p} \cdot {}^1E_r) \times ({}^2E_q \cdot {}^2E^{n-s}) = (-1)^{ps}\, ({}^1E^{n-p} \times {}^2E_q) \cdot ({}^1E_r \times {}^2E^{n-s}). \tag{28.2}$$

In what follows we shall assume the existence of Kronecker indices whenever there is an occasion for them. (As a matter of fact (17.3) together with a convention as to connected pieces yields a Kronecker index on $K \cdot K^*$.)

On ${}^1K^* \times {}^2K$

$$\sum_p (-1)^{(n-p)(n-p+1)/2}\, [{}^1E_p^i : {}^2E_p^j]\, {}^1E_i^{n-p} \times {}^2E_p^j \quad \textit{is an n-cycle} \tag{28.3}$$

in consequence of (25.5). Similarly

$$\sum_p (-1)^{p(p+1)/2}\, [{}^2E_p^j : {}^1E_p^i]\, {}^1E_p^i \times {}^2E_j^{n-p} \quad \textit{is an n-cycle} \tag{28.4}$$

on ${}^1K \times {}^2K^*$. Since

$$Ki\,(({}^1E^{n-p} \times {}^2E_p) \cdot ((-1)^{np}\, {}^1E_p \times {}^2E^{n-p})) = (-1)^{n(n+1)/2}$$

we have that the Kronecker index of the n-cycles in (28.3) and (28.4) is

$$\sum_p (-1)^p\, [{}^1E_p^i : {}^2E_p^j]\, [{}^2E_p^j : {}^1E_p^i], \tag{28.5}$$

which is precisely the common number of fixed cells of the correspondences (cf. (25.10)) defined on ${}^1K$ and ${}^2K$ by

$$[{}^1E_p^i : {}^1E_p^h], \qquad [{}^2E_p^j : {}^2E_p^k].$$

Consider the intersection of

$${}^1C^n \times {}^2C_q \qquad ({}^1C^n = \text{a covering } n\text{-cycle};\ {}^2C_q = t_k\, {}^2E_q^k)$$

with the n-cycle of (28.4). In accordance with (28.2) it may be written

$$\sum (-1)^{p(p+1)/2} [{}^2E_p^j : {}^1E_p^i] ({}^1C^n \cdot {}^1E_p^i) \times ({}^2C_q \cdot {}^2E_j^{n-p}). \tag{28.6}$$

For $p > q$, $({}^2C_q \cdot {}^2E_j^{n-p})$ is vacuous; for $p < q$, $({}^1C^n \cdot {}^1E_p^i)$ has dimension $< q$. So take $p = q$, suppress ${}^1C^n$, and replace $({}^2C_q \cdot {}^2E_j^{n-q})$ by its Kronecker index $(-1)^{q(q+1)/2} t_j$. The result is the chain

$$T^1({}^2C_q) = t_j [{}^2E_q^j : {}^1E_q^i] {}^1E_q^i.$$

Thus a definite process of intersection and "projection" gives a product interpretation of the correspondence T^1.

29. **Remarks.** Our first remark bears on the problem[22] of classifying sets of cells which are isomorphic except for incidence numbers. Let $\{E_0^i\}$ $(i = 0, 1, \cdots, n)$ be a set of 0-cells each of which has on its boundary a common (-1)-cell E'_{-1}. An augmented closed simplex ${}^+\bar{\sigma}_n$ and the iterated join

$$\prod_0^n (E_0^i + E'_{-1}) \tag{29.1}$$

are clearly isomorphic except for incidence numbers. By (12.10) the join (29.1) is a complex no matter what integral values

$$[E_0^i : E'_{-1}] \tag{29.2}$$

may have. Unless all the numbers (29.2) are zero the Betti numbers of (29.1) are all zero in consequence of (13.6), and so are the same as those of ${}^+\bar{\sigma}_n$. If the numbers (29.2) are ($\neq 0$ and) relatively prime (13.5) shows that (29.1) like ${}^+\bar{\sigma}_n$ has no torsion coefficients.

Our discussion of the groups associated with a complex (cf. § 4) would be incomplete without some mention of "chains with rational coefficients" and "chains modulo m".

Instead of G_p consider the group G_p^R of all chains

$$t_i E_p^i \qquad (i = 1, 2, \cdots, \alpha_p) \tag{29.3}$$

with rational coefficients t_i. Any set $\{C_p^i\}$ of these chains such that the determinant $|[C_p^i / E_p^j]| \neq 0$ constitutes a basis for G_p^R. Bases of this sort can be chosen, for each p, so that $[\{C_{p+1}^h\} : \{C_p^i\}]$ consists of zeros except

[22] Suggested by Professor Lefschetz.

for ϱ_p 1's up the secondary diagonal. The Betti group B_p is replaced by a group B_p^R related to B_p just as G_p^R is related to G_p. The torsion group T_p falls out entirely.

For a discussion of the group $G_p^{(m)}$—of all chains (29.3) with t_i integers reduced modulo m—the reader is referred to a note[23] dealing especially with this topic.

If it would be worth while recourse might be had to general ring and ideal theory for the coefficients t_i, in the forms (29.3). In the present paper the disadvantages of such an abstraction seemed to outweigh the advantages.

Consider a chain correspondence between a complex K and a complex $\{E_p'\}$. We assume that there is just one E_p' of each dimension, and that each $[E_{p-1}' : E_p'] = 0$. The correspondence can accordingly be expressed as

$$T(C_p) = f(C_p)\, E_p' \tag{29.4}$$

where $f(C_p)$ is a linear (numerical) function of the elements of the group of p-chains on K. As a consequence of (25.4) we have that

$$f(C_p) = 0 \quad \text{if} \quad C_p \sim 0.$$

The function $f(C_p)$ is therefore analogous to an integral of total differentials on an analytic manifold. $Ki(C_p \cdot C^{n-p})$, for C^{n-p} a fixed cycle, is such a function.

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[23] Tucker, A. W., *Modular homology characters.* Proc. Nat. Acad. 18 (1932), pp. 467–471.

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PRINCETON UNIVERSITY.

REGULAR FAMILIES OF CURVES.[1]

By Hassler Whitney.[2]

1. **Introduction.** The topological properties of families of curves, occurring as solutions of differential equations, were first studied extensively by Poincaré. Other authors have continued these investigations, building up a considerable theory of the behavior of such curves, as determined by the differential equations considered.[3] These equations usually are, or may be, taken in the form

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n);$$

the solutions may then be considered as the paths of particles in n-space, t being the time.

In this paper we study families of curves, given abstractly. The curves are required to satisfy a certain condition, the "regularity condition" (§ 6); it is certainly satisfied by solutions of differential equations. The main object of the paper is first, to prove the existence of cross-sections of these curves (§ 17), and secondly, to show how a function $q = f(p, t)$ can be defined (under slight further restrictions), which may be interpreted as defining a motion of particles along the curves of the family (§ 27).

2. **Summary of results.** Part I is devoted to a function of point sets μ; it is this function which makes possible the proofs of the main theorems.

In Part II regular families of curves are defined, and necessary and sufficient conditions that a family be regular are given. A definition of the distance between two arcs due to Fréchet[4] is fundamental. The "motions" are defined, and the orientability of a regular family is discussed. A function $q = g(p, t)$ is introduced; corresponding to a point p on a curve C and a number t, there is a point q, which moves away from p as t increases. By means of this function, a set of arcs lying between two cross-sections can be mapped topologically into a family of straight

[1] Received October 26, 1932.—Presented to the American Mathematical Society, March 25, 1932. An outline of the paper will be found in the Proceedings of the National Academy of Sciences, vol. 18 (1932), pp. 275-278 and 340-342.

[2] National Research Fellow.

[3] See for instance L. Bieberbach, Differentialgleichungen, Berlin, 1930, Part I, Ch. IV.

[4] M. Fréchet, Sur quelques points du calcul fonctionnel; Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), pp. 1-74.

lines. A general definition of cross-sections is given, their existence is proved, and a few of their properties are discussed.

Suppose we have a steady flow of fluid through space. The path of each moving particle is a curve C; these curves are fixed in position. There may also be invariant points. Given a position p of a particle and a length of time t, we may represent by $q = f(p, t)$ the position of the particle after the time t has elapsed. The set of positions the particle moves over during this time defines the motion (pq). In Part III we show under what conditions such a flow may be defined over a given family of curves and invariant points.

A family of paths is first defined. Tubes are introduced, and it is shown that the ends of a tube are cross-sections, and conversely, a tube can be formed from a cross-section. A sequence of tubes can be found covering the curves of the family, and having certain properties. Using the function g of Part II, we define the function f over each tube in turn, till it is defined over the whole set of curves. Letting f be constant at the invariant points completes the definition of the flow. A function $\tau(pq)$ is also defined for each motion (pq); it may be interpreted as the time taken by a particle in travelling from p to q. We prove finally that any two flows defined over a family of paths are equivalent in a certain general sense.[5]

I. The function of sets μ.

3. Definition and elementary properties of μ. Let R be a separable metric space,[6] and let $a_1, a_2, \cdots$ be a sequence of points dense in R. Corresponding to these points we define functions $f_1(p), f_2(p), \cdots$ by the equations

$$f_i(p) = \frac{1}{1+\varrho(a_i, p)} \qquad (i = 1, 2, \cdots). \tag{3.1}$$

We note the following inequalities:

$$\begin{gathered} 0 < f_i(p) \leqq 1; \\ |f_i(p) - f_i(q)| \leqq |\varrho(a_i, p) - \varrho(a_i, q)| \leqq \varrho(p, q). \end{gathered} \tag{3.2}$$

[5] We make use of the following notation:

$\varrho(p, q)$ = distance between the points (or point sets) p, q.
$\delta(A)$ = diameter of the point set A.
$V_\varepsilon(A)$ = all points p such that $\varrho(p, A) < \varepsilon$.
$W_\varepsilon(A)$ = all points p such that $\varrho(p, A) \leqq \varepsilon$.

We say a point p (or point set P) lies within, or interior to, the set A if p is (each point of P is) an inner point of A. P lies within an arc if it lies in the arc minus its end points. $A \subset A'$ signifies that each point of A is a point of A'.

[6] See for instance Hausdorff, Mengenlehre, Berlin, 1927, Ch. VI.

Let A be any non-void point set in R. For a given i, consider the various values of $f_i(p)$ for $p \subset A$; these values have a least upper bound U. B. and a greatest lower bound L. B. We put

$$(3.3) \quad \mu_i(A) = \underset{p \subset A}{\text{U. B.}} f_i(p) - \underset{p \subset A}{\text{L. B.}} f_i(p) = \underset{p, q \subset A}{\text{U. B.}} |f_i(p) - f_i(q)|,$$

and

$$(3.4) \quad \mu(A) = \sum_{i=1}^{\infty} \frac{\mu_i(A)}{2^i}.$$

Using (3.2), we find

$$(3.5) \quad \mu_i(A) \leqq \delta(A), \quad \mu(A) \leqq \delta(A), \quad \mu(A) \leqq 1.$$

In the following theorems, we give some elementary properties of the function μ.

THEOREM 3A. *If A contains but a single point, then $\mu(A) = 0$.*

For each $\mu_i(A) = 0$.

THEOREM 3B. *If $A \subset A'$, then $\mu(A) \leqq \mu(A')$.*

THEOREM 3C. *If $\bar{A} = A$ plus its limit points, then $\mu(\bar{A}) = \mu(A)$.*

For this reason, the function μ is of interest mainly in dealing with closed sets.

THEOREM 3D. *If $A \subset W_\varepsilon(A')$, then*

$$\mu(A) < \mu(A') + 2\varepsilon.$$

Consider a fixed i, and take any point $p \subset A$. There is a corresponding point $p' \subset A'$ for which $\varrho(p, p') \leqq \varepsilon + \beta$, where $\beta > 0$ is so small that for some $\alpha > 0$, $|f_i(p) - f_i(p')| < \varepsilon - \alpha$. Therefore

$$\underset{p \subset A}{\text{U. B.}} f_i(p) \leqq \underset{p' \subset A'}{\text{U. B.}} f_i(p') + \varepsilon - \alpha, \qquad \underset{p \subset A}{\text{L. B.}} f_i(p) \geqq \underset{p' \subset A'}{\text{L. B.}} f_i(p') - \varepsilon + \alpha,$$

and thus $\mu_i(A) < \mu_i(A') + 2\varepsilon$. Dividing by 2^i and summing over i gives the theorem.

THEOREM 3E. *If $A \subset A'$ and A' contains a point p at a positive distance d from A, then*

$$\mu(A) < \mu(A').$$

This is true, in particular, if A is closed and $A \subset A'$, $A \neq A'$.

Of course $\mu_i(A) \leqq \mu_i(A')$ for each i; the theorem will follow when we have found an i such that $\mu_i(A) < \mu_i(A')$. As the sequence $a_1, a_2, \cdots$ is dense in R, there is a point a_i such that $\varrho(a_i, p) < d/3$; then $f_i(p) > 1/(1 + d/3)$. Now $\varrho(a_i, A) > 2d/3$, hence $f_i(q) < 1/(1 + 2d/3)$ for $q \subset A$. Hence $\underset{q \subset A}{\text{U. B.}} f_i(q) < \underset{p' \subset A'}{\text{U. B.}} f_i(p')$, and thus $\mu_i(A) < \mu_i(A')$.

We can state the last two theorems in the form: $\mu(A)$ *is a continuous and increasing function of* A.

THEOREM 3F. *If* $A \cdot A' \neq 0$, *then*

$$\mu(A+A') \leqq \mu(A)+\mu(A').$$

The proof is simple.[7]

4. **Comparison of the function μ with other functions.** Functions obeying both Theorems 3D and 3E do not seem to have been used before. We mention here some functions which obey these theorems partially.

(a) The function $\delta(A)$ obeys Theorem 3D but not Theorem 3E.

(b) Let R^n be Euclidean n-space, and let $m_n(A)$ be the n-dimensional Lebesgue measure of a set A. Theorem 3D, in a weakened form, is obeyed by this function. Theorem 3E will be obeyed if, for instance, we restrict ourselves to the closures of open sets.

(c) The capacity of a set, say in three dimensions, obeys Theorem 3D in a weakened form, but not Theorem 3E: The capacity of a sphere is the same as the capacity of its surface.

(d) Let R be a separable space, defined by the sequence of neighborhoods $U_1, U_2, \cdots$. (If R is metric and the points $a_1, a_2, \cdots$ are dense in R, we can take for these neighborhoods open spheres of rational diameters about these points.) For any set A, put $\alpha_i(A) = 1$ or 0 according as $U_i \cdot A \neq 0$ or $= 0$; put $\nu(A) = \sum_1^\infty \alpha_i/2^i$. This function satisfies Theorem 3E, as stated for closed sets, but does not satisfy Theorem 3D. It has the remarkable property that any two distinct closed sets have distinct numbers ν.

5. **The function μ'.** Let R be a Euclidean space, or more generally, a metric space in which every bounded set is totally bounded.[8] Take any

[7] As a simple application of the function μ, we prove *Brouwer's Reduction Theorem.* We consider only *closed* subsets of a separable metric space R. If a (closed) set A has the property P, we write: $P(A)$. Assume of P that if $A_0 \supset A_1 \supset \cdots \supset A$ (A is the limiting set of the sequence), and $P(A_i)$, each i, then $P(A)$. *If* $P(B_0)$, *then there is a set* $B \subset B_0$ *such that* (1) $P(B)$, *and* (2) *if* $B' \subset B$ *and* $P(B')$, *then* $B' = B$.

Say $\mu(B_0) = \mu_0$. Let μ_0' be the lower bound of numbers $\mu(B')$ for $B' \subset B_0$ and $P(B')$, and take $B_1 \subset B_0$ so that $P(B_1)$ and $\mu(B_1) = \mu_1 < \mu_0' + 1$. In general, let μ_i' be the lower bound of numbers $\mu(B')$ for $B' \subset B_i$ and $P(B')$, and take $B_{i+1} \subset B_i$ so that $P(B_{i+1})$ and $\mu(B_{i+1}) = \mu_{i+1} < \mu_i' + 1/i$. Then $B_0 \supset B_1 \supset \cdots \supset B$, and $P(B)$. As $B_{i+1} \subset B_i$, $\mu_{i+1} \leqq \mu_i$. As any set in B_{i+1} lies also in B_i, $\mu_{i+1}' \geqq \mu_i'$. As $\mu_{i+1} \leqq \mu_i' + 1/i$, $\lim \mu_i = \lim \mu_i' =$ a number μ. As $B \subset B_i$, $\mu(B) \leqq \mu$ (in fact, $\mu(B) = \mu$). Suppose $B' \subset B$, $B' \neq B$ and $P(B')$. By Theorem 3E, $\mu(B') < \mu(B)$; hence, for some k, $\mu(B') < \mu_k'$. But this contradicts the definition of μ_k'.

As we used only Theorem 3E, not Theorem 3D, the function ν (§ 4) could be used in place of the function μ in this proof.

[8] See Hausdorff, loc. cit., p. 108.

bounded set A in R, and a fixed i. If $p_1, p_2, \cdots, p_i$ are points of A, there is associated with this set of points P_i a number $d(P_i) = \min \varrho(p_k, p_l)$ (which is 0 if two of the points coincide). Let $\mu_i'(A)$ be the upper bound of the numbers $d(P_i)$ for such sets P_i in A, and put

$$\mu'(A) = \sum_2^\infty \frac{\mu_i'(A)}{2^{i-1}}. \tag{5.1}$$

This function (for bounded sets) is easily seen to obey both Theorems 3 D and 3 E.[9]

In a general separable space, Theorem 3 E will not be satisfied by this function. For set $A = p_1 + p_2 + \cdots$, each p_i being that point in Hilbert space which has the ith coördinate equal to 1, the other coördinates being 0. If A' is any infinite subset of these points, $\mu'(A') = \mu'(A) = \sqrt{2}$.

A compensating advantage in the function μ' is the following. If A and A' are any two congruent sets, then $\mu'(A') = \mu'(A)$. If A' is similar to A (in the common geometric sense), each distance in A' being k times the corresponding distance in A, then $\mu'(A') = k\,\mu'(A)$.

II. Regular families of curves.

6. **Definitions.** Consider two arcs pq, $p'q'$. Let H be a homeomorphism between these arcs, in which p and q correspond to p' and q' respectively. Let $d(H)$ be the upper bound of all numbers $\varrho(r, r')$ for corresponding points r, r' in this homeomorphism, and let

$$\sigma(pq, p'q') = \sigma(p'q', pq) = \sigma(qp, q'p') \tag{6.1}$$

be the lower bound of the numbers $d(H)$ for all such homeomorphisms between the two arcs. This number we call the *span* between the two (oriented) arcs (écart in the sense of Fréchet).[10] Some simple properties of the span are given in the following theorems.

THEOREM 6 A. *If pqr and $p'q'r'$ are arcs and $\sigma(pq, p'q') < \varepsilon$, $\sigma(qr, q'r') < \varepsilon$, then $\sigma(pqr, p'q'r') < \varepsilon$.*

THEOREM 6 B. *If $\sigma(pr, p'r') < \varepsilon$ and q lies within pr, then there is a point q' within $p'r'$ such that $\sigma(pq, p'q') < \varepsilon$ and $\sigma(qr, q'r') < \varepsilon$.*

THEOREM 6 C. *For any three arcs pq, $p'q'$, $p''q''$,*

$$\sigma(pq, p''q'') \leqq \sigma(pq, p'q') + \sigma(p'q', p''q''). \tag{6.2}$$

We shall use the word *curve* in this part as the topological image of a line segment with or without either end point, or of a circle. Consider

[9] See the first Proceedings note referred to above.

[10] In Fréchet's definition, all homeomorphisms are allowed. In terms of σ, his écart is the lesser of the two numbers $\sigma(pq, p'q')$, $\sigma(pq, q'p')$.

a set of curves, the points of which form a separable metric space. We shall say these curves form a *regular family* F if no two of them intersect, and if the following condition is satisfied: Given any arc pq of a curve C of the family and any $\varepsilon > 0$, there is a $\delta > 0$ such that if p' lies on C' and $\varrho(p, p') < \delta$, then there is an arc $p'q'$ of C' such that

$$\sigma(pq, p'q') < \varepsilon.$$

It is easily shown that the condition that no two of the curves intersect may be replaced by the following: No two of the curves differ, as point sets, by at most two points.

Any arc lying on a curve of the regular family we shall call an *arc* of the regular family.

7. **Conditions under which a family of curves is regular.** We prove here

THEOREM 7A. *A necessary and sufficient condition that a family of non-intersecting curves be regular is that each point p of any curve C be on an arc pq of C (where q can be taken on either side of p if p is not an end point of C), with the following property. For every $\varepsilon > 0$ there is a $\delta > 0$ such that if p' lies on C' and $\varrho(p, p') < \delta$, then there is an arc $p'q'$ of C' such that*

(1) *$p'q' \subset V_\varepsilon(pq)$ and $q' \subset V_\varepsilon(q)$, and*

(2) *if r' and s' are on $p'q'$ and $\varrho(r', s') < \delta$, then the arc $r's'$ is of diameter $< \varepsilon$.*

We prove first the necessity of the condition. Given a point p of C, take any arc pq of C and any $\varepsilon > 0$. Divide pq into arcs of diameter $< \varepsilon/4$, and take η less than one third the distance between any two of these arcs without common points. Take $\delta < \eta$ so that if $p' \subset V_\delta(p)$, then there is an arc $p'q'$ with $\sigma(pq, p'q') < \eta$. (1) is certainly satisfied. To prove (2), take r' and s' on $p'q'$ with $\varrho(r', s') < \delta$; let r and s be corresponding points of pq so that $\sigma(pr, p'r')$, $\sigma(rs, r's')$ and $\sigma(sq, s'q')$ are $< \eta$. Then $\varrho(r', r) < \eta$ and $\varrho(s', s) < \eta$, hence $\varrho(r, s) < 3\eta$. Consequently r and s are either on the same or adjacent arcs of our subdivision of pq, and $\delta(rs) < \varepsilon/2$. But $\sigma(rs, r's') < \eta < \varepsilon/4$, and hence $\delta(r's') < \varepsilon$, as required.

We turn now to the sufficiency of the condition. The theorem gives us certain arcs pq with q on either side of p; we shall show that the regularity condition is satisfied, first, for these special arcs pq, then, for any arc pq. Divide $pq = p_0 p_m$ into arcs $p_0p_1, \cdots, p_{m-1}p_m$, each of diameter $< \varepsilon/6$, and let ε_1 be less than half the distance between any two of the points $p_0, \cdots, p_m$. Take a number δ_1 corresponding to ε_1, as given in the theorem. From each point p_i ($i \neq 0, m$) follow C towards

p_0 and towards p_m to the first points r_i and s_i respectively not in $V_{\delta_1/4}(p_i)$, and define the arcs $C_i^- = p_0 r_i$, $C_i = r_i s_i$, $C_i^+ = s_i p_m$. Take $\varepsilon_2 < \delta_1/4$ and $< \varrho(C_i^-, C_i^+)/2$ $(i = 1, \cdots, m-1)$. Take δ corresponding to ε_2 as given in the theorem.

If p_0' is any point in $V_\delta(p_0)$, there is an arc $p_0' p_m'$ satisfying (1) and (2), with ε replaced by ε_2; we shall show that $\sigma(p_0 p_m, p_0' p_m') < \varepsilon$. We note first that $p_0' p_m'$ has points in $V_{\delta_1/2}(p_i)$, $i = 1, \cdots, m-1$. For $p_0' \subset V_\delta(p_0) \subset V_{\varepsilon_2}(C_i^-)$, $p_m' \subset V_{\varepsilon_2}(p_m) \subset V_{\varepsilon_2}(C_i^+)$, $\varrho(C_i^-, C_i^+) > 2\varepsilon_2$, and $p_0' p_m'$ is connected; hence there is a point r' of $p_0' p_m'$ not in $V_{\varepsilon_2}(C_i^- + C_i^+)$. But $p_0' p_m' \subset V_{\varepsilon_2}(p_0 p_m)$, and thus $r' \subset V_{\varepsilon_2}(C_i) \subset V_{\delta_1/2}(p_i)$.

Let p_i' be a point of $p_0' p_m'$ in $V_{\delta_1/2}(p_i)$, $i = 1, \cdots, m-1$. These points are in the order $p_0', p_1', \cdots, p_{m-1}', p_m'$ (they are of course distinct). For if they are not, then there is a point p_j' lying between p_0' and p_i', $j > i$. We see then just as above that $p_0' p_j'$ contains a point $r' \subset V_{\delta_1/2}(p_i)$ (evidently $p_j' \subset V_{\varepsilon_2}(C_i^+)$); hence $\varrho(r', p_i') < \delta_1$. But also $\varrho(p_i', p_i) < \delta_1/2$, $\varrho(p_j', p_j) < \delta_1/2$, $\varrho(p_i, p_j) > 2\varepsilon_1$, and $\delta_1 < \varepsilon_1$; hence $\varrho(p_i', p_j') > \varepsilon_1$ and $\delta(r' p_j' p_i') > \varepsilon_1$, contradicting our choice of δ_1.

Establish a homeomorphism between each arc $p_i p_{i+1}$ and the corresponding arc $p_i' p_{i+1}'$; this gives a homeomorphism between $p_0 p_m$ and $p_0' p_m'$. If we show that $\delta(p_i p_{i+1} + p_i' p_{i+1}') < \varepsilon$, it will follow that $\sigma(p_i p_{i+1}, p_i' p_{i+1}') < \varepsilon$, and hence $\sigma(p_0 p_m, p_0' p_m') < \varepsilon$, as required. If we put $r_{-1} = r_0 = p_0$ and $s_m = s_{m+1} = p_m$ and note that $r_{i-1} s_{i+2} \subset V_{\varepsilon/6+\delta_1}(p_i p_{i+1})$, we see it is sufficient to show that $p_i' p_{i+1}' \subset V_{\varepsilon_2}(r_{i-1} s_{i+2})$; for then $p_i' p_{i+1}' \subset V_{\varepsilon/3}(p_i p_{i+1})$, and thus $\delta(p_i p_{i+1} + p_i' p_{i+1}') < \varepsilon$. If this inclusion is false, then there is a point s' of $p_i' p_{i+1}'$ in $V_{\varepsilon_2}(C_{i-1}^-)$ or $V_{\varepsilon_2}(C_{i+2}^+)$, say the former. Just as before, we see then that $s' p_m'$ contains a point r' in $V_{\delta_1/2}(p_{i-1})$. It follows that $\varrho(r', p_{i-1}') < \delta_1$ and $\delta(r' s' p_{i-1}') > \varepsilon_1$, which cannot be. This completes the proof for the special arcs pq.

Consider now any arc pq. We saw above that there is an arc pq_0 satisfying the required condition, with q_0 lying on the same side of p as q. If there is such an arc with q_0 lying beyond q, then it follows from Theorem 6B that pq has the required property. If not, then consider all such arcs pq_0, and let q_1 be the bound of these points q_0 on C; q_1 lies on pq. There is an arc $q_1 q_2$ with the required property, q_2 lying beyond q_1 from p.

Given an $\varepsilon > 0$, take $\varepsilon_1 < \varepsilon$, $< \varrho(p, q_1 q_2)/2$ and $< \varrho(q_2, pq_1)/2$, and take $\eta < \varepsilon_1/2$ so that if $q_1' \subset V_\eta(q_1)$, then there is an arc $q_1' q_2'$ with $\sigma(q_1 q_2, q_1' q_2') < \varepsilon_1$. There is a point $q_0 \neq q_1$ on pq_1 such that $\delta(q_0 q_1) < \eta/3$, and pq_0 has the required property. Take δ so that for any $p' \subset V_\delta(p)$, there is an arc $p' q_0'$ with $\sigma(pq_0, p' q_0') < \eta/3$. Take a point q_1' beyond q_0' from p' so that $\delta(q_0' q_1')$

$<\eta/3$; then $\sigma(pq_1, p'q_1')<\eta$. Take an arc $q_1'q_2'$ with $\sigma(q_1q_2, q_1'q_2')<\varepsilon_1$. On account of our choice of ε_1, q_2' lies beyond q_0' from p'. Hence for any $p'\subset V_\delta(p)$ there is an arc $p'q_2'$ with $\sigma(pq_2, p'q_2')<\varepsilon$, and pq_2 has the required property. But this contradicts the definition of q_1. The theorem is now completely proved.

8. **Examples of regular families.**

(a) In the xy-plane, the parabolas $y=a(1+x^2)$, one curve for each value of a, form a regular family. We could also restrict a to any set of values we like, say to the values $0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \cdots$.

(b) Using polar coördinates, the curves

$$r=1, \quad \text{and} \quad r=1-\frac{1}{\theta+\gamma} \qquad (\theta>1-\gamma),$$

one curve for each γ, $0\leqq\gamma<2\pi$, form a regular family. We might replace the curve $r=1$ by a finite number of curves, formed by dropping out points or arcs from the curve.

(c) One curve C is an open segment in the plane, and each other curve consists of those points in the plane at a fixed distance from C.

(d) Let F be a set of curves forming a separable metric space, and let each curve be an arc. Suppose (1) for each point p on a curve C of F and each $\varepsilon>0$ there is a $\delta>0$ such that if p' is on a curve C' of F and $\rho(p,p')<\delta$, then $C\subset V_\varepsilon(C')$ and $C'\subset V_\varepsilon(C)$,[11] and (2) the curves of F are equicontinuous.[12] Then F is a regular family.

To show this, we show that the conditions of Theorem 7A are satisfied; it is evident that no two of the curves can intersect. Let pq be any curve of F, and take any point p_1 interior to pq. We shall show that p_1q has the required property. (The case $p_1=p$ or q admits a simpler proof.) Given an $\varepsilon>0$, take $p_2\subset pp_1$ so that $\delta(p_2p_1)<\varepsilon/2$, and put $2\varepsilon_1=\rho(p_1, pp_2)$. Take ε_2 so that if p' and r' are on some C' and $\rho(p',r')<3\varepsilon_2$, then $\delta(p'r')<\varepsilon_1$. Take $r_1\subset p_2p_1$ and r_2 in p_1q so that $r_1p_1r_2\subset V_{\varepsilon_2}(p_1)$. Put $2\varepsilon_3=\rho(pr_1, r_2q)$, and take δ so that if $p'\subset C'\cdot V_\delta(p_1)$, then $C\subset V_{\varepsilon_3}(C')$ and $C'\subset V_{\varepsilon_3}(C)$. Take any such p'; then there is a point $q'\subset C'\cdot V_{\varepsilon_3}(q)$. We shall show that condition (1) of Theorem 7A is satisfied; (2) holds automatically.

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[11] We may express this by saying that the curves of F form a continuous (i. e. both lower and upper semicontinuous) collection.

[12] That is, for every $\varepsilon>0$ there is a $\delta>0$ such that if p and q are end points of an arc pq of any curve C of F and $\rho(p,q)<\delta$, then $\delta(pq)<\varepsilon$; in other words, the curves of F are uniformly locally connected.

It is of course sufficient to assume that the curves are equicontinuous in the neighborhood of each curve of the family.

We need merely show that $p'q' \subset V_\varepsilon(p_1 q)$, or, $p'q' \subset V_{\varepsilon/2}(p_2 q)$. As $C' \subset V_{\varepsilon_3}(C)$, it suffices to show that $p'q' \cdot V_{\varepsilon_3}(pp_2) = 0$. Suppose $s' \subset p'q' \cdot V_{\varepsilon_3}(pp_2)$. Then as $C' \subset V_{\varepsilon_3}(C)$ and $\varrho(pr_1, r_2 q) = 2\varepsilon_3$, there is a point $r' \subset s'q' \cdot V_{\varepsilon_3}(r_1 p_1 r_2) \subset V_{2\varepsilon_3}(p_1)$. But then $\varrho(p', r') < 3\varepsilon_2$, and also $\varrho(p', s') \geqq \varepsilon_1$ and thus $\delta(p'r') \geqq \varepsilon_1$, contradicting our choice of ε_2.

In regard to this example, see § 14.

(e) Let C be a simple closed curve in the plane, containing inside it the point a. Let every point inside C except for a lie on a simple closed curve containing a inside it. If no two of these curves intersect, and they are equicontinuous, then they form a regular family. The proof is not difficult. See also § 20.

9. **The neighborhood of an arc.** Take an arc pq of the regular family F and a $\lambda > 0$. The λ-neighborhood of pq, $N_\lambda(pq)$, consists of all arcs $p'q'$ of F such that $\sigma(pq, p'q') < \lambda$. $N_\lambda(pq)$ contains, in particular, arcs contained in and arcs containing pq.

THEOREM 9A. *The set of points on $N_\lambda(pq)$ is open.*

For suppose $r' \subset p'q'$, $\sigma(pq, p'q') = \lambda' < \lambda$. Take δ small enough so that for every $r^* \subset V_\delta(r')$ there are arcs r^*p^* (if $r' \neq p'$) and r^*q^* (if $r' \neq q'$) with $\sigma(r'p', r^*p^*) < \lambda - \lambda'$ and $\sigma(r'q', r^*q^*) < \lambda - \lambda'$, and so that p^* and q^* are on opposite sides of r^*. Then $\sigma(p'q', p^*q^*) < \lambda - \lambda'$ and hence $\sigma(pq, p^*q^*) < \lambda$; thus every r^* in $V_\delta(r')$ is on an arc of $N_\lambda(pq)$, and this set of points is open.

THEOREM 9B. *If pq is interior to a curve C of F, then for each $\varepsilon > 0$ there is a $\lambda > 0$ such that if the arcs $p'q'$ and $p''q''$ are both in $N_\lambda(pq)$ and they have common points, then they differ at most by a point set lying in $V_\varepsilon(p) + V_\varepsilon(q)$.*

Let pq be contained within an arc $p_0 q_0$ and contain within it an arc $p_1 q_1$ so that $p_0 p p_1 \subset V_{\varepsilon/2}(p)$ and $q_1 q q_0 \subset V_{\varepsilon/2}(q)$. Take $2\lambda_0 < \varepsilon$ and less than any of the distances $\varrho(p_0, pq)$, $\varrho(q_0, pq)$, $\varrho(p, p_1 q_0)$, $\varrho(q, p_0 q_1)$, and take $\lambda < \lambda_0$ so that all points of $N_\lambda(pq)$ are on arcs of $N_{\lambda_0}(p_0 q_0)$. Suppose now $p'q'$ and $p''q''$ are in $N_\lambda(pq)$ and have the common point r'. Let $p_0'q_0'$ be an arc of $N_{\lambda_0}(p_0 q_0)$ containing r', and take p_1' and q_1' on $p_0'q_0'$ so $\sigma(p_0 p_1, p_0' p_1')$, $\sigma(p_1 q_1, p_1' q_1')$ and $\sigma(q_1 q_0, q_1' q_0')$ are all $< \lambda_0$. p_0' and q_0' are not on $p'q'$, hence $p'q' \subset p_0'q_0'$. $p_1'q_0' \subset V_{\lambda_0}(p_1 q_0)$, hence $p_1'q_0' \cdot V_{\lambda_0}(p) = 0$, and thus $p' \subset p_0' p_1'$; similarly $q' \subset q_1' q_0'$. In the same way we see that $p_1'q_1' \subset p''q'' \subset p_0'q_0'$. Therefore $p'q'$ and $p''q''$ differ at most by subsets of $p_0'p_1'$ and $q_1'q_0'$, which lie in $V_\varepsilon(p) + V_\varepsilon(q)$.

10. **Orientability of a regular family.** Suppose a positive direction is assigned to each curve of a regular family F. An arc pq of F is now *positively oriented* if q lies in the positive direction from p along pq; otherwise it is *negatively oriented*. pq and qp have then the opposite orientation.

We shall say the curves of F are *similarly oriented* in the neighborhood of the arc pq of F if the following condition is satisfied. If $p_1q_1, p_2q_2, \cdots$ is a sequence of oriented arcs of F approaching pq: $\sigma(pq, p_iq_i) \to 0$, then for every i greater than some N, pq and p_iq_i are similarly oriented. If it is possible to assign a direction to each curve of F so that the curves are similarly oriented in the neighborhood of each arc of F, we shall say F is *orientable*. We always assume hereafter that the curves of a regular orientable family are properly oriented.

Each regular family described in § 8 is orientable except that in (c).

11. **The motions of a regular oriented family.** Let C be a curve of the regular oriented family F. Let K be a correspondence between the numbers x, $0 \leqq x \leqq 1$, and points p_x of C. If this correspondence maps $(0, 1)$ continuously onto C, and 0 goes into p_0 and 1 into p_1, we say K defines the *motion* (p_0p_1). We shall say two correspondences K_0 and K_1 define the same motion (p_0p_1) if one is deformable into the other; that is, there is a correspondence K_y for each y, $0 \leqq y \leqq 1$, such that if p_{xy} denotes the point corresponding to x in K_y, then $p_{0y} = p_0$, $p_{1y} = p_1$, and p_{xy} is continuous in both variables x and y.

A *null motion* (pp) is determined by letting all of $(0, 1)$ correspond to a single point p.

Suppose the curve C of F is the image of a line segment and p and q are distinct points of C. Then if we put the arc pq into $1-1$ continuous correspondence with $(0, 1)$, this defines a motion (pq); evidently there is no motion $(pq)'$ different from this one. In this case the motion (pq) is uniquely determined by its end points p and q. However, if C is a simple closed curve, and we follow the points p_x as x increases from 0 to 1 in the motion (pq), we may pass around C a number of times, and thus find different motions (pq). In any case, we may define any non-null motion (pq) as follows. Let $pp_1, p_1p_2, \cdots, p_{m-1}q$ be a set of arcs of C, all positively or all negatively oriented. Putting $p = p_0$ and $q = p_m$ and putting each p_ip_{i+1} in $1-1$ continuous correspondence with the interval $(i/m, (i+1)/m)$ gives a correspondence between $(0, 1)$ and points of C, defining the motion (pq).

A non-null motion (pq) is *positively* or *negatively oriented* according as the above arcs defining (pq) are positively or negatively oriented.

If F is a family of paths and the functions τ and f are defined as in § 27, then we may define the motion (pq) uniquely by means of the following correspondence:

$$p_x = f[p, x\tau(pq)] \qquad (0 \leqq x \leqq 1). \tag{11.1}$$

Let (pq) and (qr) be two motions on a curve C, the first ending, the second starting, at the point q. Define these in terms of the intervals

$(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ respectively. This gives a correspondence between $(0, 1)$ and points of C, defining the motion $(pr) = (pq) + (qr)$, the *sum* of (pq) and (qr). (qp) is the *negative* of (pq) if $(pq) + (qp)$ is the null motion.

12. **The space of motions.** Given two motions (pq) and $(p'q')$ on C and C' respectively, we can define the *span* $\sigma[(pq), (p'q')]$ between them just as we defined the span between arcs: Let K be any correspondence defining (pq), and K', one defining $(p'q')$; this induces a correspondence K^* between points of C and points of C'. $d(K^*)$ is the upper bound of $\varrho(r, r')$ for corresponding points in K^*, and $\sigma[(pq), (p'q')]$ is the lower bound of $d(K^*)$ for such correspondences K^*.

We may let the motions of F be points of a new space $\mathfrak{M}$, letting the distance between two of these points be the span between the two motions. $\mathfrak{M}$ is metric and separable. A sequence of motions dense in $\mathfrak{M}$ may be determined as follows. Let $a_1, a_2, \cdots$ be a sequence of points dense in R. For each $i, j > 0$ choose a point a_{ij} on the curve C_i through a_i so that $\varrho(a_j, a_{ij}) < \varrho(a_j, C_i) + 1/j$; then the points a_{ij} are dense on C_i. For any two points a_{ij} and $a_{ij'}$ on C_i, there is either one or a denumerable number of motions $(a_{ij}\, a_{ij'})$; the set S of all such motions is dense in $\mathfrak{M}$. For given an $\varepsilon > 0$ and a motion $(p_0\, p_m)$ of $\mathfrak{M}$, consider a set of arcs $p_0\, p_1, \cdots, p_{m-1}\, p_m$ defining $p_0\, p_m$ (see above). If a_{i_0} is close enough to p_0, there is an arc $a_{i_0}\, a_{i_1}$ close to $p_0\, p_1$, next an arc $a_{i_1}\, a_{i_2}$ close to $p_1\, p_2$, etc. This set of arcs determines a motion $(a_{i_0}\, a_{i_m})$ within ε of $p_0\, p_m$, as required.

13. **The function g.** Let p be a point on a curve C of a regular family, and consider a variable point q on C. For each arc pq there is a number $\mu(pq)$ (see Part I); $\mu(pq)$ increases continuously from zero as q moves away from p. Given a number α, if there is a point q such that $\mu(pq) = \alpha$, we shall say $q = g(p, \alpha)$:

$$q = g(p, \alpha) \text{ equivalent to } \mu(pq) = \alpha. \tag{13.1}$$

g is in general two-valued: if p is interior to C and α is small enough, there will be two corresponding points q; but if α is large enough, there will be one or no corresponding points.

The importance of the function g lies in its continuity:

THEOREM 13A. *In a regular family of curves, $g(p, \alpha)$ is a continuous function of both variables.*

By this we mean that if the arc pq is interior to the curve C and $q = g(p, \alpha)$, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $p' \subset V_\delta(p)$ and $|\alpha' - \alpha| < \delta$, then there is a point $q' = g(p', \alpha') \subset V_\varepsilon(q)$.

To prove this, take first $\eta < \varepsilon/2$ so small that there are points

$$q_1 = g(p, \alpha - 3\eta), \qquad q_2 = g(p, \alpha + 3\eta)$$

on the same side of p as q, and so that the arc $q_1 q q_2 \subset V_{\varepsilon/2}(q)$. Take $\delta < \eta$ so small that any point $p' \subset V_\delta(p)$ is an end point of an arc $p' q_2'$ with $\sigma(pq_2, p' q_2') < \eta$. Consider a homeomorphism between pq_2 and $p'q_2'$ in which corresponding points are distant $< \eta$; let q_1' be the point corresponding to q_1.

We shall now show that

$$\mu(p'q_1') < \alpha - \eta, \qquad \mu(p'q_2') > \alpha + \eta. \tag{13.2}$$

To prove the first inequality, every point of $p'q_1'$ being within η of the corresponding point of pq_1, $p'q_1' \subset V_\eta(pq_1)$; hence, by Theorem 3D, $\mu(p'q_1') < \mu(pq_1) + 2\eta = \alpha - \eta$. Also $pq_2 \subset V_\eta(p'q_2')$, and thus the second inequality follows.

Now take any such point $p' \subset V_\delta(p)$ and any α', $|\alpha' - \alpha| < \delta < \eta$. From the above inequalities we see that there is a point $q' = g(p', \alpha')$ in the arc $q_1' q_2'$. But the correspondence between pq_2 and $p'q_2'$ shows that $q_1' q_2' \subset V_\eta(q_1 q_2) \subset V_\varepsilon(q)$; hence $q' \subset V_\varepsilon(q)$.

THEOREM 13B. *g is uniformly continuous over any compact and closed set of points p and bounded closed range of values α.*

This is an immediate consequence of the Borel Theorem.

14. **An application.** Assume that the regular family in (d), § 8 is orientable. Then each curve of F has a first point p_0 and a last point p_1. Let S_0 be the set of all first points, and S_1, the set of all last points. It is easily seen that S_0 and S_1 are closed, and are homeomorphic. We prove now

THEOREM 14A. *F can be mapped in Euclidean or Hilbert space so as to become a family of straight line segments.*

First map S_0 into a subset of such a space in which the first coördinate $x = 0$. Let p be any point of F, on the arc $p_0 p_1$ of F, and say p_0 is mapped into the point p_0'; then map p into the point whose first coördinate $x = \mu(p_0 p)/\mu(p_0 p_1)$, and whose other coördinates agree with those of p_0'. By Theorem 3D, we see that $\mu(p_0 p_1)$ is a continuous function of p_0; hence, using Theorem 13A, we see that the points of F are mapped topologically into space as required.

A special case of this theorem has been proved by R. L. Moore.[13]

15. **The function g'.** Let F be a regular oriented family. Take any point p of a curve C of F, and any number $\alpha > 0$. If q is such a point of C that $\mu(pq) = \alpha$, put $q = g'(p, \alpha)$ or $q = g'(p, -\alpha)$ according as q is in the positive or negative direction from p along pq.

[13] R. L. Moore, Fundamenta Mathematicae, vol. 4 (1923), pp. 106–117, Theorem I. See also his Colloquium Lectures, Foundations of point set Theory, p. 397. Note that it is fairly easy to show directly that the above mapping is continuous, without showing that the family is regular.

g' is single-valued, and is continuous in the same sense that g was.

THEOREM 15A. *In a regular oriented family, if* $p_i \to p$, $\alpha_i \to \alpha \neq 0$, *and* $q_i = g'(p_i, \alpha_i)$, $q = g'(p, \alpha)$, *then* $p_i q_i \to pq$.

This is a strengthening of Theorem 13A. Given an $\varepsilon > 0$, choose $\delta > 0$ as in the proof of that theorem, and using the same notation, note that $\sigma(pq_1, p'q_1') < \eta$. Take any correspondence between $q_1 q$ and $q_1' q'$; as $q_1' q' \subset q_1' q_2' \subset V_\varepsilon(q)$, we see thus that $\sigma(q_1 q, q_1' q') < \varepsilon$, and hence $\sigma(pq, p'q') < \varepsilon$. From this we see that if $q_i^* = g(p_i, \alpha_i)$ (using the notation of the present theorem) is taken on the right side of p_i, then $p_i q_i^* \to pq$. But as the family is oriented, pq and $p_i q_i^*$ have the same orientation for $i >$ some N; hence $q_i = q_i^*$ for $i > N$, and $p_i q_i \to pq$.

16. **Cross-sections.** Let p be a point interior to a curve C of the regular family F. A set of points S forms a *cross-section through* p if the following conditions are satisfied:

(1) Each point p' of S lies within an arc $q_0' q_1'$ of F such that for some $\lambda' > 0$, each arc of $N_{\lambda'}(q_0' q_1')$ (see § 9) contains at most one point of S.

(2) p lies within an arbitrarily small arc $q_0 q_1$ of F such that for some $\lambda > 0$, each arc of $N_\lambda(q_0 q_1)$ contains exactly one point of S (hence $p \subset S$).

(3) S is closed.

S is a cross-section if it is a cross-section through some point. We might call those points of S satisfying (2) *inner points* of the cross-section.

It would be reasonable to require further that all points of S be inner points or limit points of inner points; also if R is locally connected, we might demand that S be connected.

Example. Let F be a regular family of curves filling a region of the plane (or of n-space); assume the curves have directional derivatives which are continuous in the region. At the point p within the arc $q_0 q_1$ erect a small line segment (or portion of an $(n-1)$-plane) perpendicular to the direction of $q_0 q_1$ at p; this forms a cross-section through p.

17. **Existence of cross-sections.** We prove here

THEOREM 17A. *Through any point* p *interior to a curve* C *of the regular family* F *there is a cross-section* S.

Let $q_0 q_1$ be an arc interior to C and containing p within it. Take $4\sigma < \varrho(q_0, pq_1)$ and $< \varrho(q_1, pq_0)$, and put $U = W_{2\sigma}(q_0)$. For any superscript $*$, if the curve C^* of F contains a point p^* near p, let q_ζ^* ($\zeta > 0$) be the first point of C^* in $W_\zeta(q_0)$ (if it exists), following C^* from p^* towards U; q_ζ is that point on C. Take η so small that the arc $q_{2\eta} q_0 \subset V_\sigma(q_0)$. Say $\mu(U + pq_\eta) = \alpha$; we shall show that on each arc near $q_0 q_1$ there is a point p' for which $\mu(U + p'q_\eta') = \alpha$, and that these points form a cross-section S.

Take $\lambda<\eta$ small enough so that any two arcs of $N_\lambda(q_0 q_1)$ with common points differ at most by a point set in $V_\eta(q_0)+V_\sigma(q_1)$ (see Theorem 9B). Then we can assign a positive direction to each arc $q_0' q_1'$ of $N_\lambda(q_0 q_1)$ so that $q_0'(\subset V_\eta(q_0))$ lies in the positive direction from q_1'.

Take $\beta<\eta/2$ so small that if p_1 and p_2 are the points of $q_0 q_1$ for which

$$\mu(U+p_1 q_\eta) = \alpha-2\beta, \quad \mu(U+p_2 q_\eta) = \alpha+2\beta, \tag{17.1}$$

then $p_1 p p_2 \subset V_{\varepsilon/2}(p)$, where ε is some number $<\sigma$. If r^* is any point near p on an arc of $N_\lambda(q_0 q_1)$, put

$$r^*(t) = g'(r^*, t) \qquad (t \geqq 0),$$

in the positive direction from r^*, as defined above. Put also

$$t_{k\zeta} = \mu(p_k q_\zeta) \qquad (k=1,2;\ \zeta \geqq 0).$$

Then in particular,

$$p_k(t_{k\zeta}) = g'[p_k, \mu(p_k q_\zeta)] = q_\zeta.$$

By the uniform continuity of the function g', we can take $\gamma_1<\varepsilon/2$ so that if $p_1' \subset V_{\gamma_1}(p_1)$, then

$$\varrho[p_1(t), p_1'(t)]<\beta \qquad (0 \leqq t \leqq t_{10}). \tag{17.2}$$

We shall show that $U+p_1' q_\eta' \subset V_\beta(U+p_1 q_\eta)$; it will follow that

$$\mu(U+p_1' q_\eta')<\mu(U+p_1 q_\eta)+2\beta = \alpha. \tag{17.3}$$

It is sufficient to show that $p_1' q_\eta' \subset V_\beta(U+p_1 q_\eta)$. Putting $t=t_{10}$ in (17.2), we see that $p_1'(t_{10}) \subset V_\beta(q_0)$; hence q_η' lies between p_1' and $p_1'(t_{10})$, and $p_1' q_\eta' \subset p_1' p_1'(t_{10})$. By (17.2), $p_1' p_1'(t_{1\eta}) \subset V_\beta(p_1 q_\eta)$, and by the choice of η, $p_1'(t_{1\eta})\, p_1'(t_{10}) \subset V_\beta(q_\eta q_0) \subset V_\beta(q_{2\eta} q_0) \subset U$; thus (17.3) holds.

Take $\gamma_2<\varepsilon/2$ so that if $p_2' \subset V_{\gamma_2}(p_2)$, then (17.2) holds with the subscript 1 replaced by 2. We shall show that $U+p_2 q_\eta \subset V_\beta(U+p_2' q_\eta')$; then

$$\mu(U+p_2' q_\eta')>\mu(U+p_2 q_\eta)-2\beta = \alpha. \tag{17.4}$$

As $p_2 q_\eta \subset p_2 q_0$ and $q_{2\eta} q_0 \subset U$, it is sufficient to show that $p_2 q_{2\eta} \subset V_\beta(U+p_2' q_\eta')$. By the choice of γ_2, $p_2' p_2'(t_{2,2\eta}) \subset V_\beta(p_2 q_{2\eta})$; thus, as $p_2 q_{2\eta} \cdot V_{2\eta}(q_0) = 0$ and $\beta<\eta$, $p_2' p_2'(t_{2,2\eta}) \cdot W_\eta(q_0) = 0$. Therefore the point q_η' lies beyond $p_2'(t_{2,2\eta})$. Hence $p_2 q_{2\eta} \subset V_\beta[p_2' p_2'(t_{2,2\eta})] \subset V_\beta(p_2' q_\eta')$, and (17.4) is proved.

Take $\gamma<\gamma_1$, γ_2 and λ, and take δ so small that any point of $W_\delta(p)$ lies on an arc $q_0' q_1'$ of $N_\gamma(q_0 q_1)$; this arc contains points p_1' and p_2' so that $\sigma(q_0 p_1, q_0' p_1')$, $\sigma(p_1 p_2, p_1' p_2')$ and $\sigma(p_2 q_1, p_2' q_1')$ are $<\gamma$. Then $p_1' \subset V_{\gamma_1}(p_1)$ and $p_2' \subset V_{\gamma_2}(p_2)$, and on account of (17.3) and (17.4) there will be a point p' of $p_1' p_2'$ for which

$$\mu(U+p' q_\eta') = \alpha. \tag{17.5}$$

We shall show that the set S of all these points forms a cross-section through p.

We show first that (1) holds. Take any arc A' of $N_\lambda(q_0 q_1)$ containing a point p'' of S. p'' was determined as follows: A point r'' was taken in $W_\vartheta(p)$, on the arc A'' say of $N_\lambda(q_0 q_1)$, this determined a point q''_η, and this determined p''. A' and A'' have p'' in common; hence they differ at most by a point set in $V_\eta(q_0)+V_\sigma(q_1)$, because of the choice of λ. Hence $A'\cdot W_\vartheta(p)\neq 0$, and no matter what point r' is taken in $A'\cdot W_\vartheta(p)$, determining q'_η and $p'\subset S$, $q'_\eta = q''_\eta$ and $p' = p''$. Thus p' is the only point of A' on S, from which (1) follows immediately.

For some $\lambda_0>0$, each arc of $N_{\lambda_0}(q_0 q_1)$ cuts $W_\vartheta(p)$ and therefore contains a point of S. This evidently holds also for any smaller arc $r_0 r_1$ containing p (as S is closed); hence (2) holds.

We must prove still that S is closed. Let $p^1, p^2, \cdots$, be a sequence of points in S, lying on arcs $A^1, A^2, \cdots$, of $N_\gamma(q_0 q_1)$. These arcs all cut the closed set $W_\vartheta(p)$, hence if the sequence has a limit point, it lies on an arc A^* of $N_\gamma(q_0 q_1)$ cutting $W_\vartheta(p)$, and A^* contains a point p^* of S. Assume now that p^* exists. We shall show that for an arbitrary $\varepsilon^*>0$ there is a $\delta^*>0$ such that the points of S on all arcs of $N_\gamma(q_0 q_1)$ cutting $V_{\vartheta^*}(p^*)$ lie in $V_{\varepsilon^*}(p^*)$; it follows then that $p^i\to p^*$. If $p^*=p$, this follows from what we saw above. For replacing the ε used above by an arbitrary ε and choosing γ_1, γ_2 and δ accordingly, we have $\sigma(p_2 p_1, p'_2 p'_1)<\gamma<\varepsilon/2$, hence $p'\subset V_{\varepsilon/2}(p_1 p_2)\subset V_\varepsilon(p)$.

Suppose now $p^*\neq p$. Given an $\varepsilon^*>0$, take $\beta^*\leqq\beta$ and $<\varepsilon^*/2$ so that if p_3^* and p_4^* are the points of A^* for which

$$(17.6)\quad \mu(U+p_3^*\, q_\eta^*) = \alpha-2\beta^*, \qquad \mu(U+p_4^*\, q_\eta^*) = \alpha+2\beta^*,$$

then $p_3^* p^* p_4^*\subset V_{\varepsilon^*/2}(p^*)$. Put $t^*_{k\zeta}=\mu(p_k^* q_\zeta^*)$ $(k=3,4)$. Note that by (17.2), the point q_β^* on A^* exists.

Take γ^* so that if $p_3'\subset V_{\gamma^*}(p_3^*)$ and $p_4'\subset V_{\gamma^*}(p_4^*)$ are points of an arc A' of $N_\lambda(q_0 q_1)$, then

$$(17.8)\qquad \varrho[p_i^*(t), p_i'(t)]<\beta^* \qquad (0\leqq t\leqq t^*_{i\beta},\ i=3,4).$$

We shall show that $p_3' q'_\eta\subset V_{\beta^*}(U+p_3^* q_\eta^*)$ and $p_4^* q_\eta^*\subset V_{\beta^*}(U+p_4' q'_\eta)$; then

$$(17.9)\qquad \mu(U+p_3' q'_\eta)<\alpha, \qquad \mu(U+p_4' q'_\eta)>\alpha,$$

and hence the point p' of A' in S (if it exists) lies on the arc $p_3' p_4'\subset V_{\beta^*}(p_3^* p^* p_4^*)\subset V_{\varepsilon^*}(p^*)$. Therefore if we choose δ^* corresponding to γ^* as we choose δ corresponding to γ, then any point p' of S on an arc A' cutting $W_{\vartheta^*}(p^*)$ lies in $V_{\varepsilon^*}(p^*)$, completing the proof.

We show first that $q^*_{\eta+\beta}\, q^*_\beta \subset U$. Say p_1^* of A^* is in $V_{\gamma_1}(p_1)$. As $p_1^*\, p_1^*(t_{1,2\eta}) \subset V_\beta(p_1\, q_{2\eta})$, and $p_1\, q_{2\eta} \cdot V_{2\eta}(q_0) = 0$, and $\beta < \eta/2$, we have $p_1^*\, p_1^*(t_{1,2\eta}) \cdot W_{\eta+\beta}(q_0) = 0$; hence $q^*_{\eta+\beta}$ lies beyond $p_1^*(t_{1,2\eta})$. As $p_1^*(t_{10}) \subset V_\beta(q_0)$, $p_1^*(t_{10})$ lies beyond q^*_β. Thus $q^*_{\eta+\beta}\, q^*_\beta \subset p_1^*(t_{1,2\eta})\, p_1^*(t_{10}) \subset V_\beta(q_{2\eta}\, q_0) \subset U$.

By (17.8), $p_3'(t^*_{3\beta}) \subset V_{\beta^*}(q^*_\beta) \subset V_\eta(q_0)$; thus $p_3'\, q_\eta' \subset p_3'\, p_3'(t^*_{3\beta})$. As $p_3'\, p_3'(t^*_{3\eta}) \subset V_{\beta^*}(p_3^*\, q^*_\eta)$ and $p_3'(t^*_{3\eta})\, p_3'(t^*_{3\beta}) \subset V_{\beta^*}(q^*_\eta\, q^*_\beta) \subset V_{\beta^*}(U)$, $p_3'\, q_\eta' \subset V_{\beta^*}(U + p_3^*\, q^*_\eta)$. As $p_4'\, p_4'(t^*_{4,\eta+\beta}) \subset V_{\beta^*}(p_4^* q^*_{\eta+\beta})$ and $p_4^* q^*_{\eta+\beta} \cdot V_{\eta+\beta}(q_0) = 0$, q_η' lies beyond $p_4'(t^*_{4,\eta+\beta})$; hence $p_4^*\, q^*_{\eta+\beta} \subset V_{\beta^*}[p_4'\, p_4'(t^*_{4,\eta+\beta})] \subset V_{\beta^*}(p_4'\, q_\eta')$. As also $q^*_{\eta+\beta}\, q^*_\eta \subset U$, $p_4^*\, q^*_\eta \subset V_{\beta^*}(U + p_4'\, q_\eta')$. This completes the proof of the theorem.

18. **Equivalence of two cross-sections.** Let p_1 and p_2 be two points, or the same point, of a curve C, and let S_1 and S_2 be cross-sections through p_1 and p_2 respectively. We suppose that the arc $p_1 p_2$ (if $p_1 \neq p_2$) contains no points of S_1 or of S_2 within it. Take an arc $q_1 q_2$ interior to C and containing p_1 and p_2 within it, short enough so that it contains no points of S_1 or S_2 other than p_1 and p_2. Take $\delta > 0$ so that every arc in $N_\delta(q_1 q_2)$ contains just one point of S_1 and one point of S_2. If S_1' and S_2' are those points of S_1 and S_2 respectively which lie on arcs of $N_\delta(q_1 q_2)$, plus limit points, then S_1' and S_2' are cross-sections through p_1 and p_2, and *they are homeomorphic.* For, to each point of one set, let correspond that point of the other set lying on the same arc of $N_\delta(q_1 q_2)$. If $p_{11}, p_{12}, \cdots$ are in S_1 and approach $p_1 \subset S_1$, and p_{1i} and p_1 lie on arcs A_i and A respectively of $N_\delta(q_1 q_2)$, then the corresponding points $p_{21}, p_{22}, \cdots$ of S_2 approach the arc A; as S_2 is closed, they approach the point p_2 of S_2 corresponding to p_1, and this correspondence is continuous.

19. **The local connectivity of cross-sections.** We show here that *if R is locally connected, then any cross-section is locally connected at all inner points* (see § 16).

If p is an inner point of S, and $q_0 q_1$ and λ are taken so as to satisfy (2), § 16, take $\lambda' \leqq \lambda$ so that the points of S on all arcs of $N_{\lambda'}(q_0 q_1)$ lie in $V_\varepsilon(p)$ ($\varepsilon > 0$ arbitrary). Take γ so that $V_\gamma(p) \subset N_{\lambda'}(q_0 q_1)$, and take δ so that any point q in $S \cdot V_\delta(p)$ is joined to p by a connected set $A \subset V_\gamma(p)$. Let A' consist of those points of S which are on arcs of $N_{\lambda'}(q_0 q_1)$ cutting A. Then A' is a continuous image of A, hence it is connected; also $A' \subset V_\varepsilon(p)$, and $p \subset A$, $q \subset A$. Therefore S is locally connected at p.

20. **Regular families in n-space.** Let F be a regular family of curves filling a region in Euclidean n-space. A fundamental question is:

Is there a cross-section through any inner point of a curve which is a closed $(n-1)$-cell (the topological image of a closed $(n-1)$-simplex)? If $n=2$, the answer is yes. For any connected cross-section S contains two points p and q such that dropping out any other point of S disconnects these points; hence S is an arc.[14] The answer is also in the affirmative if $n=3$. For $n>3$, nothing is known.

Suppose, in a given regular family, there is a cross-section S through the point p which is a closed $(n-1)$-cell. Map S onto a portion S' of the $(n-1)$-plane $x_n=0$ in n-space. If we consider only short arcs containing points of S, we can orient them similarly. Let pq be such an arc, containing the point p_0 of S; say p_0' is the corresponding point of S'. If $r \subset pq$, map r into that point of the line through p_0' perpendicular to S' for which $x_n = \pm \mu(p_0 r)$, using the $+$ or $-$ sign according as r lies in the positive or negative direction from p_0. By Theorem 13A it is seen that this mapping is continuous; hence *in the neighborhood of any such point p, the curves are equivalent to a family of straight lines filling a region of n-space.*

Example. As we noted above, a cross-section in example (e) of § 8 is an arc; we can fit a denumerable number of these arcs together to form an arc joining C to a. If we map this arc onto the radius of a circle C', we can map the curves into circles concentric with C', as in Theorem 14A. We can thus prove a theorem of Kerékjártó.[15]

III. Families of paths.

21. **Definition.** In this part we shall denote by the word *curve* the topological image of an open line segment or of a circle only. If the points of a regular oriented family of curves form an open subset R' of a locally compact[16] separable metric space R, then the curves of F together with the points of $R^* = R - R'$ form a *family of paths.* The points of R^* we call *invariant points.*

22. **Tubes.** We call a set of arcs *self-compact* if any sequence chosen from these arcs contains a subsequence $p_1q_1, p_2q_2, \cdots$ approaching an arc pq of the set: $\lim_{n\to\infty} \sigma(pq, p_nq_n) = 0$. A set of arcs of F form a *tube* T if the following conditions are satisfied:

(1) The arcs are self-compact.

(2) No two distinct arcs have common points.

(3) The set of points in T contains inner points.

[14] See Hausdorff, loc. cit. [6], p. 222, III.

[15] B. v. Kerékjártó, Topologie I, Berlin, 1923, p. 246. See also pp. 238–239.

[16] R is locally compact if each point of R is contained in a compact neighborhood of R. Euclidean space is locally compact; Hilbert space is not. R' is evidently locally compact also.

One might require also that all points of T be either inner points or limit points of inner points, and that T be connected if R' is locally connected, as in the definition of cross-sections.

The sets S_1 and S_2 of points on the negative and positive ends of the arcs of T we call the *negative* and *positive ends* of T respectively. The point sets T, S_1 and S_2 are evidently compact and closed. S_1 and S_2 are connected if T is.

LEMMA. *If T is a tube with ends S_1 and S_2 and pq is an arc of F containing at least two points of S_1+S_2, then $\delta(pq) \geqq \varrho(S_1, S_2) > 0$.*

This is certainly true if pq contains points of both S_1 and S_2. Suppose pq contains two points p' and p'' of S_1 (or of S_2), and say p'' lies in the positive direction from p'. If $p'q'$ is the arc of T containing p', then $p'q' \subset p'p''$ and $p'p''$ contains also a point of S_2, hence $\delta(pq) \geqq \varrho(S_1, S_2)$.

THEOREM 22A. *The ends S_1 and S_2 of a tube T are cross-sections.*

Take S_1 for example. If $p' \subset S_1$, take an arc $q_0'q_1'$ of diameter $< \lambda'$ containing p' within it, where $\lambda' < \varrho(S_1, S_2)/3$. Then any arc of $N_{\lambda'}(q_0'q_1')$ contains at most one point of S_1, by the lemma; hence (1), § 16 holds. Let p be a point of S_1 on an arc pq of T containing inner points of T. Then all points of pq but p and q are inner points of T. Let q_0q_1 be an arc of diameter $< \varrho(S_1, S_2)/2$ and containing p in its interior, q_1 being interior to pq. Take λ so small that $V_\lambda(q_0) \cdot T = 0$ and $V_\lambda(q_1) \subset T$. Then any arc of $N_\lambda(q_0q_1)$ contains exactly one point of S_1, and (2), § 16 is proved. We noted that S_1 was closed; hence it is a cross-section.

THEOREM 22B. *If S_0 is a compact cross-section through p_0, then for some $\alpha > 0$ the set of arcs qr for which $q = g'(p, -\alpha)$ and $r = g'(p, \alpha)$ with $p \subset S_0$ form a tube T, containing p_0 as an inner point.*

Containing each point $p' \subset S_0$ is an arc $q'r'$ such that for some λ', each arc in $N_{\lambda'}(q'r')$ contains at most one point of S_0. Take $\mu < \mu(p'+q')$ and $< \mu(p'+r')$, and take $\lambda < \lambda'$ and $< \mu/3$. Take $\delta < \lambda$ so that every point p^* of $S_0 \cdot V_\delta(p')$ is on an arc q^*r^* of $N_\lambda(q'r')$. Suppose now A is any arc of F containing such a point p^*, and $\mu(A) \leqq \lambda$. As $p'+q' \subset V_\lambda(p^*q^*)$, $\mu(p^*q^*) > \mu(p'+q') - 2\lambda > \lambda$, hence A does not contain q^*. Similarly A does not contain r^*; hence $A \subset q^*r^*$, and therefore it contains at most one point of S_0. A finite number of such sets $V_\delta(p')$ cover S_0; if 4α is the least of the corresponding numbers λ, then any arc A with $\mu(A) \leqq 4\alpha$ contains at most one point of S_0. Then α is our required number, and the arcs pq constructed as above form the required tube T.

To prove this, we see first that the arcs are self-compact (use Theorem 15A). Note now that for any arc qpr of T, $\mu(qpr) \leqq \mu(pq) + \mu(pr) = 2\alpha$, by Theorem 3F. No two arcs A', A'' of T have a common point.

For if they did, they would form an arc A with two points in S_0; hence $\mu(A) > 4\alpha$. But $\mu(A) \leqq \mu(A') + \mu(A'') \leqq 4\alpha$, a contradiction. Finally, p_0 is an inner point of T. For there is an arc $q^* r^*$ containing p_0 such that for some λ^*, each arc of $N_{\lambda^*}(q^* r^*)$ contains exactly one point of S_0, and so that $\mu[N_{\lambda^*}(q^* r^*)] < \alpha$. All points of some $V_\delta(p_0)$ are on arcs of $N_{\lambda^*}(q^* r^*)$ and are thus in T, completing the proof.

23. **First sequence of tubes.** By the last theorem, if S is a compact cross-section through p, there is an $\alpha > 0$ such that if $S_3 = g'(S, -2\alpha)$, $S_1 = g'(S, -\alpha)$, $S_2 = g'(S, \alpha)$, $S_4 = g'(S, 2\alpha)$, then there are tubes T' and T containing p, with ends S_3 and S_4, S_1 and S_2 respectively.

If R' is self-compact, put $W_1 = R'$. Otherwise, let $W_1, W_2, \cdots$ be a sequence of compact and closed sets such that each W_i is interior to W_{i+1} and $W_1 + W_2 + \cdots$ covers R'.[17] Each point of W_1 is interior to a tube T as above; a finite number of these, say $T_1, \cdots, T_{l_1}$, cover W_1. If $W_1 \neq R'$, a finite number of tubes cover W_2; let the new tubes required be $T_{l_1+1}, \cdots, T_{l_2}$, etc. We may choose the tubes so that $\delta(T_k) < 1/i$ if $k > l_i$. We define thus a (possibly finite) sequence of tubes $T_1, T_2, \cdots$ with the following properties:

(A) They cover R'.

(B) Any compact and closed set in R' has points in common with but a finite number of the tubes.

(C) $\lim_{n \to \infty} \delta(T_n) = 0$.

Corresponding to the tube T_i is the tube T_i' and cross-sections S_{i3}, S_{i1}, S_{i0} (= the above S), S_{i2}, and S_{i4}.

Note that it is possible for an arc of one tube to contain a number of points on both ends of another tube.

24. **First sequence of pseudo-tubes.** Any set of arcs satisfying (1) and (2) of § 22 we shall call a *pseudo-tube*. We shall replace the tubes $T_1, T_2, \cdots$ by pseudo-tubes $T_1^2, T_2^2, \cdots$ which have the above properties, and also the following:

(D) No two ends of pseudo-tubes have common points.

(E) Any pseudo-tube T_i^2 contains points of at most one of the ends of any preceding pseudo-tube T_k^2 $(k < i)$.

From this it follows also that any arc of any pseudo-tube T_i^2 contains at most a single point of any end of any previous pseudo-tube.

Put $T_1^2 = T_1$. Suppose we have constructed the pseudo-tubes $T_1^2, \cdots, T_{m_{i-1}}^2$, covering the tubes $T_1, \cdots, T_{i-1}$, and having properties (D) and (E) above;

[17] About each point of R' is a compact neighborhood; a denumerable number of these, $U_1, U_2, \cdots$ cover R'. Put $W_i' = \overline{U}_1 + \cdots + \overline{U}_i$. We can take a subsequence $W_1, W_2, \cdots$ of this sequence so that W_i is interior to W_{i+1}; for W_i' is contained in a finite number of the sets U_k, and is thus interior to some W_j'.

we shall construct pseudo-tubes $T^2_{m_{i-1}+1}, \cdots, T^2_{m_i}$ covering T_i and having these properties. We shall call the ends of any pseudo-tube T^2_k S^2_{k1} and S^2_{k2}.

Take a number λ less than any $\varrho(S^2_{k1}, S^2_{k2})$ $(1 \leqq k \leqq m_{i-1})$. Given any arc pq of T'_i, divide it into arcs $pq_1, q_1q_2, \cdots, q_nq$, each of diameter $< \lambda/2$, so that none of the points $q_1, \cdots, q_n$ lie on ends of previous pseudo-tubes, and q_1 lies between S_{i3} and S_{i1}, and q_n lies between S_{i2} and S_{i4}. If p_0 is the point of pq on S_{i0}, say $q_k = g'(p_0, \beta_k)$ $(k = 1, \cdots, n)$. We can take a number η, and a cross-section $S^* \subset S_{i0}$ containing all points of S_{i0} in some $V_\delta(p_0)$, (or rather, a *pseudo-cross-section*, i. e. a set of points satisfying (1) and (3) of § 16), small enough so that no cross-section $g'(S^*, \beta_k \pm \eta)$ contains points of any end $S^2_{11}, \cdots, S^2_{m_{i-1}2}$, and so that each pseudo-tube T^*_k, with ends $g'(S^*, \beta_k - \eta)$ and $g'(S^*, \beta_{k+1} + \eta)$ $(k = 1, \cdots, n-1)$, is of diameter $< \lambda$. (T^*_k is obviously a pseudo-tube.) Then these pseudo-tubes cover all points of T_i on arcs cutting S^*, and each pseudo-tube T^*_k contains points on at most one end of any pseudo-tube $T^2_1, \cdots, T^2_{m_{i-1}}$.

For each point p_0 of S_{i0} there is such an S^*; a finite number of these cross-sections cover S_{i0}. We define pseudo-tubes as above for each such S^*; the resulting pseudo-tubes cover T_i. We can choose the numbers η so that no two ends of these pseudo-tubes have common points. If one of these pseudo-tubes has ends $g'(S^*, \beta_k - \eta)$ and $g'(S^*, \beta_{k+1} + \eta)$, call its length $(\beta_{k+1} + \eta) - (\beta_k - \eta)$. Then if we arrange the new pseudo-tubes $T^2_{m_{i-1}+1}, \cdots, T^2_{m_i}$ in order of decreasing length, the pseudo-tubes $T^2_1, \cdots, T^2_{m_i}$ have properties (D) and (E). Continuing in this manner, we obtain the sequence of pseudo-tubes $T^2_1, T^2_2, \cdots$ with all the desired properties.

25. **Second sequence of pseudo-tubes.** We replace here the pseudo-tubes of the last sequence by a sequence of pseudo-tubes $T^3_1, T^3_2, \cdots$, which have the properties (A), (B) and (C) above, and also the property

(F) If any inner point p' of an arc pq of a pseudo-tube T^3_k lies in a preceding pseudo-tube T^3_l $(l < k)$, then all of pq lies in preceding pseudo-tubes.

Put $T^3_1 = T^2_1$. Having constructed the pseudo-tubes $T^3_1, \cdots, T^3_{n_{i-1}}$ with the above property, and so that the point sets $T^2_1 + \cdots + T^2_{i-1}$ and $T^3_1 + \cdots + T^3_{n_{i-1}}$ are identical, we shall construct the pseudo-tubes $T^3_{n_{i-1}+1}, \cdots, T^3_{n_i}$ with the property (F), lying in and filling out T^2_i.

The pseudo-tube T^2_i may contain points of one or the other end of each of the pseudo-tubes $T^2_1, \cdots, T^2_{i-1}$. If $T^2_{\alpha_1}, \cdots, T^2_{\alpha_M}$ are those pseudo-tubes whose negative ends contain points of T^2_i, put for simplicity $S_k = T^2_i \cdot S^2_{\alpha_k 1}$, $k = 1, \cdots, M$; if there are no such pseudo-tubes, put

$M = 0$. Similarly let $S'_1, \cdots, S'_N$ be those points of positive ends of pseudo-tubes $T^2_1, \cdots, T^2_{i-1}$ in T^2_i; if there are no such pseudo-tubes, put $N = 0$. There are 2^M subsets of the numbers $1, \cdots, M$, including the whole set and the null set. Arrange these in order, choosing first the whole set, next all sets of $M-1$ of the numbers, etc. Call these sets $C_1, \cdots, C_{2^M}$. Arrange the 2^N subsets of the numbers $1, \cdots, N$ in order similarly; call them $D_1, \cdots, D_{2^N}$. Now arrange all the 2^{M+N} pairs of these sets $(C_1, D_1), (C_1, D_2), \cdots, (C_1, D_{2^N}), (C_2, D_1), \cdots, (C_{2^M}, D_{2^N})$ in the order shown.

We shall consider each of these pairs in turn, and construct a corresponding pseudo-tube if a certain condition is satisfied. For the pair (C_s, D_t), this condition is:

There is an arc pq of T^2_i such that

(α) pq contains a point of every S_k with subscript k in the set of numbers C_s, and of every S'_l with subscript l in D_t, and

(β) if p_1 is the last point of pq on an S'_l, l in D_t, or $p_1 = p$ if D_t is the null set, and if q_1 is the first point of pq on an S_k, k in C_s, or $q_1 = q$ if C_s is the null set, then q_1 lies in the positive direction from p_1.

The condition is satisfied automatically for (C_{2^M}, D_{2^N}), there being no corresponding sets S_k, S'_l.

If the condition is satisfied for the pair (C_s, D_t), then the corresponding pseudo-tube consists of all arcs $p_1 q_1$ as named above. (The pseudo-tube corresponding to (C_{2^M}, D_{2^N}) is just T^2_i itself.) We order these pseudo-tubes $T^3_{n_{i-1}+1}, \cdots, T^3_{n_i}$ in the same manner that we ordered the pairs.

We show first that these really are pseudo-tubes. As the S_k and the S'_l are compact and closed, we can pick out from any sequence of arcs of T^3_ν $(n_{i-1}+1 \leqq \nu \leqq n_i)$ a subsequence $p_1 q_1, p_2 q_2, \cdots$, approaching an arc $p'q'$, where p_i and p' are in some fixed S'_l, l in D_t, and q_i and q' are in some fixed S_k, k in C_s (see Theorem 15 A); we must show that $p'q'$ is an arc of T^3_ν. Say $p^k q^k$ is the arc of T^2_i containing $p_k q_k$, and $p^* q^*$ is the arc of T^2_i containing $p' q'$; then also $p^k p_k \to p^* p'$, $q_k q^k \to q' q^*$. If T^3_ν corresponded to (C_s, D_t), then each $p^k p_k$ contains a point of each S'_l, l in D_t; hence the same is true of $p^* p'$. Similarly $q' q^*$ contains a point of each S_k, k in C_s. Therefore, by property (E), $p' q'$ contains within it no points of S'_l, l in D_t, or of S_k, k in C_s. Thus $p' q'$ is an arc of T^3_ν, and the arcs of T^3_ν are self-compact. As any two distinct arcs of T^3_ν lie on distinct arcs of T^2_i and no two arcs of T^2_i have common points, no two arcs of T^3_ν have common points; hence T^3_ν is a pseudo-tube.

We show now that the sequence $T^3_1, T^3_2, \cdots$ has the required properties. As each $T^3_{n_i}$ is identical with T^2_i, (A) is satisfied. (B) and (C) hold obviously. To prove (F), we must show that, if p' is a point interior to an arc $p_1 q_1$

of a pseudo-tube T^3_ν, and p' lies in a pseudo-tube T^3_λ, $\lambda<\nu$, then all of $p_1 q_1$ lies in pseudo-tubes preceding T^3_ν. Choose i so that $n_{i-1}<\nu \leqq n_i$, and say pq is the arc of T^2_i containing $p_1 q_1$.

Suppose first $\lambda \leqq n_{i-1}$; then p' lies in a pseudo-tube T^2_η, $\eta \leqq i-1$. If pq contains no points of the ends of T^2_η, then pq lies in T^2_η, and therefore $p_1 q_1$ lies in $T^3_{n_\eta} = T^2_\eta$. Suppose pq contains a point of $S^2_{\eta 1}$ or $S^2_{\eta 2}$. Define the sets S_k and S_l' for T^2_i as before. If some point of pq on an S_l' lies after some point on an S_k, then all of $p_1 q_1$ lies in pseudo-tubes $T^3_1, \cdots, T^3_{n_{i-1}}$. Suppose then p_2 is the last point of pq on any S_l', q_2 is the first point of pq on any S_k, and p_2 lies before q_2; in this case we shall show that $p_2 q_2$ lies in a pseudo-tube T^3_μ, $n_{i-1}<\mu<\nu$, from which the statement will follow again.

If (C_s, D_t) is the pair of sets of numbers to which the pseudo-tube T^3_ν corresponds, then pq contains points on all the S_l', l in D_t, and on all the S_k, k in C_s. As p' lies in T^2_η and is interior to $p_1 q_1$, either $S^2_{\eta 2}$ has a point q^* in $p' q_1$, or $S^2_{\eta 1}$ has a point in $p_1 p'$; thus pq contains a point on some $S_{l'}'$, l' not in D_t, or a point on some $S_{k'}$, k' not in C_s. Therefore if the set of all $S_k (S_l')$ with points on pq gives the set of numbers $C_{s'} (D_{t'})$, $C_{s'}$ and $D_{t'}$ contain C_s and D_t respectively, and one of them contains more numbers; hence the pair $(C_{s'}, D_{t'})$ precedes the pair (C_s, D_t). As the condition is satisfied for this pair, a corresponding pseudo-tube T^3_μ, $\mu<\nu$, was constructed, and it contains the arc $p_2 q_2$.

Suppose now $n_{i-1}<\lambda<\nu$; then the pseudo-tube T^3_λ contains a certain portion $p_2 q_2$ of the arc pq. Then pp_2 (if p_2 is not p) and $q_2 q$ (if q_2 is not q) are both contained in pseudo-tubes $T^2_1, \cdots, T^2_{i-1}$, and thus in pseudo-tubes $T^3_1, \cdots, T^3_{n_{i-1}}$. The property (F) is now shown to hold.

26. **The function τ'.** With each motion (pq) of F (see § 11) we shall associate a number $\tau'(pq)$ with the following properties:

(1) $\tau'(pq)$ is greater than, less than, or equal to zero according as (pq) is positively oriented, negatively oriented, or is the null motion.

(2) τ' is continuous, in that if $(p_i q_i) \to (pq)$ (see § 12), then $\tau'(p_i q_i) \to \tau'(pq)$.

(3)

$$\tau'[(pq)+(qr)] = \tau'(pq)+\tau'(qr). \tag{26.1}$$

Then also if (qp) is the negative of pq,

$$\tau'(qp) = -\tau'(pq). \tag{26.2}$$

We shall define τ' successively through the pseudo-tubes T^3_1, T^3_2, $\cdots$. Consider first T^3_1. If p is any point of an arc $p_0 q_0$ of T^3_1 ($p_0 \subset S^3_{11}$, $q_0 \subset S^3_{12}$), determining the motion $(p_0 p)$, put

$$\tau'(p_0 p) = \mu(p_0 p). \tag{26.3}$$

If q is another point of $p_0 q_0$, put

$$\tau'(pq) = \tau'(p_0 q) - \tau'(p_0 p). \tag{26.4}$$

τ' has so far all the required properties.

Suppose now we have defined $\tau'(pq)$ for all motions lying wholly in $T_1^3 + \cdots + T_{i-1}^3$; we shall extend its definition through T_i^3. Consider first motions (pq) lying wholly in T_i^3. If no arcs of T_i^3 are contained in former pseudo-tubes, we define $\tau'(pq)$ just as before. If all of T_i^3 is contained in former pseudo-tubes, $\tau'(pq)$ is already defined. Otherwise, we define $\tau'(pq)$ by a process similar to a method of extending the domain of continuous functions.[18]

On account of the fundamental property of the sequence of pseudo-tubes, if τ' is already defined over any part of an arc of T_i^3, it is defined over the whole arc. Let A be the set of points of S_{i1}^3 which are end points of arcs of T_i^3 over which τ' is already defined; A is closed. If p is any point of T_i^3, on the arc $p_0 q_0$ of T_i^3, put

$$p = h(p_0, s), \tag{26.5}$$

where

$$s = \frac{\mu(p_0 p)}{\mu(p_0 q_0)}.$$

s is a continuous function of p_0 and p; hence h is continuous in p_0 and s (see § 13). We shall define $\tau'(p_0 p)$ in terms of the values $\tau'(p_0' p')$, $p_0' \subset A$, $p' = h(p_0', s)$.

Given the point $p_0 \subset S_{i1}^3 - A$, put

$$a = a(p_0) = \varrho(p_0, A), \tag{26.6}$$

and

$$X_r(p_0) = A \cdot W_r(p_0) \qquad (a \leqq r \leqq 2a). \tag{26.7}$$

Given the point $p = h(p_0, s)$, let $\psi(p, r)$ be the maximum of the values $\tau'(p_0' p')$, $p' \subset h[X_r(p_0), s]$ (and hence $p_0' \subset X_r(p_0)$). We define $\tau'(p_0 p)$ by the equation

$$\tau'(p_0 p) = \frac{1}{a(p_0)} \int_a^{2a} \psi(p, r)\, dr. \tag{26.8}$$

Note that $\tau'(p_0 p)$ increases as p moves away from p_0 (as s increases).

We show now that the new τ' is continuous. Let G be the maximum of $\tau'(p_0' p')$, $p_0' \subset A$, $p' \subset T_i^3$. Take any two points $p = h(p_0, s)$ and $q = h(q_0, t)$, $p_0, q_0 \subset S_{i1}^3 - A$. If $p^* = h(p_0, t)$ and $\varrho(p_0, q_0) \leqq \delta < a(p_0)$, then, by Carathéodory, loc. cit. (end of § 541), we see that

[18] See C. Carathéodory, Reelle Funktionen, first edition, Leipzig, 1918, p. 617.

$$|a(q_0)\tau'(q_0 q) - a(p_0)\tau'(p_0 p^*)| \leqq 3G\delta,$$

which is independent of t. Obviously $\tau'(p_0 p)$ is continuous in p (p_0 fixed); hence we can take q so near p and t so near s that $3G\delta < \varepsilon/2$ and so that $a(p_0)|\tau'(p_0 p^*) - \tau'(p_0 p)| < \varepsilon/2$; it follows that

$$|a(q_0)\tau'(q_0 q) - a(p_0)\tau'(p_0 p)| < \varepsilon.$$

Therefore $a(p_0)\tau'(p_0 p)$ and thus $\tau'(p_0 p)$ is continuous for $p_0 \subset S_{i1}^3 - A$.

We must consider next the case that p_0 is in A and is a limit point of $S_{i1}^3 - A$. Take points $p_1 = h(p_0, s_1)$ and $p_2 = h(p_0, s_2)$ on opposite sides of p so that $\tau'(p_0 p_1)$ and $\tau'(p_0 p_2)$ differ from $\tau'(p_0 p)$ by less than $\varepsilon/2$. By Carathéodory, loc. cit., we see that if η is small enough and $p_i' = h(p_0', s_i) \subset V_\eta(p_i)$ ($p_0' \subset S_{i1}^3 - A$), then

$$|\tau'(p_0' p_i') - \tau'(p_0 p_i)| < \frac{\varepsilon}{2} \qquad (i = 1, 2).$$

As $\tau'(p_0' p')$ is an increasing function of p', it follows immediately that if p' is near enough p, it lies on such an arc $p_1' p_2'$, and hence $\tau'(p_0' p')$ differs from $\tau'(p_0 p)$ by less than ε. Thus τ' is continuous over its present domain of definition.

If the motion (pq) is interior to T_i^3, define $\tau'(pq)$ as in (26.4). Consider now any motion (pq) in $T_1^3 + \cdots + T_i^3$. If it does not lie in $T_1^3 + \cdots + T_{i-1}^3$, let $pp_1, p_1 p_2, \cdots, p_m q$ be a set of arcs defining (pq) (see § 11), chosen so that each $p_k p_{k+1}$ lies in some tube T_l^3. Corresponding to each $p_k p_{k+1}$ is a motion $(p_k p_{k+1})$ for which $\tau'(p_k p_{k+1})$ is defined. Noting that $(pq) = (pp_1) + (p_1 p_2) + \cdots + (p_m q)$, we put

$$\tau'(pq) = \tau'(pp_1) + \tau'(p_1 p_2) + \cdots + \tau'(p_m q). \tag{26.9}$$

This definition determines $\tau'(pq)$ uniquely. For if $pp_1', p_1' p_2', \cdots, p_n' q$ is another representation of (pq), let $q_1, q_2, \cdots, q_s$ be the set of all points p_k and p_i'; the arcs $pq_1, q_1 q_2, \cdots, q_s q$ gives another representation of (pq). Using these representations, we get corresponding numbers $\tau_1'(pq)$, $\tau_2'(pq)$, $\tau_3'(pq)$. Using (26.1), we see that $\tau_1' = \tau_2'$, also $\tau_2' = \tau_3'$; hence $\tau_1' = \tau_2'$.

It is easily seen that the resulting τ' is continuous and that (1) and (3) hold. Hence defining τ' successively through all the pseudo-tubes $T_1^3, T_2^3, \cdots$, we have defined it throughout F as required.

27. **The functions τ and f.** We shall replace the function τ' by a function τ which has all the properties of τ', and has the additional property: If pq is an arc of a pseudo-tube T_i^3 ($p \subset S_{i1}^3$, $q \subset S_{i2}^3$) and (pq) is the corresponding motion, then

$$\tau(pq) \geqq 1. \tag{27.1}$$

To do this, let Δ_i be the minimum of $\tau'(pq)$ for arcs pq of T_i^3 $(i = 1, 2, \cdots)$. We can define a continuous function $\varphi(p) > 0$ for $p \subset R'$ with the following property: If $p \subset T_i^3$, then $\varphi(p) \leqq \Delta_i$. For if Γ_k is the smallest Δ_i for any T_i^3 having points in W_k (see § 23), let $\varphi(p) = \Gamma_{k+1}$ on the boundary of each W_k, and in $W_k - W_{k-1}$, let $\varphi(p)$ be a continuous extension of its values on the boundaries of W_k and W_{k-1}, so that $\Gamma_k \geqq \varphi(p) \geqq \Gamma_{k+1}$ there. Now if $p \subset T_i^3$, and $p \subset W_k - W_{k-1}$ say, then $\varphi(p) \leqq \Gamma_k \leqq \Delta_i$, as required. (If the number of pseudo-tubes T_i^3 is finite, we need merely take $\varphi(p)$ equal to the smallest Δ_i.)

Take now any motion (pq), and express it by means of arcs $pp_1, \cdots, p_m q$, as in § 11. If we let p' move along these arcs from p to q, we obtain sub-motions (pp') of (pq); the corresponding numbers $\tau'(pp')$ increase from 0 to $\tau'(pq)$. Thus for any value of τ', $0 \leqq \tau' \leqq \tau'(pq)$, there is a corresponding point $p_{\tau'}$ for which $\tau'(pp_{\tau'}) = \tau'$. We put

$$\tau(pq) = \int_0^{\tau'(pq)} \frac{d\tau'}{\varphi(p_{\tau'})}. \tag{27.2}$$

Now if pq is any arc of a tube T_i^3, then

$$\tau(pq) \geqq \int_0^{\Delta_i} \frac{d\tau'}{\Delta_i} = 1.$$

Moreover, τ has the properties given in § 26.

We now define the function f. Take any point p of R', and any number t. Find the motion (pq) of F for which

$$\tau(pq) = t;$$

put then

$$q = f(p, t).$$

For any point p of $R^* = R - R'$, put $f(p, t) = p$.

Given a point p and a number t, there is at most a single point $q = f(p, t)$; we shall show that there always is such a point. Starting at p, follow along the curve C through p in the positive or negative direction according as t is positive or negative, until we have passed through $n \geqq |t|$ pseudo-tubes $T_{i_1}^3, \cdots, T_{i_n}^3$, such that the last point in $T_{i_k}^3$ precedes the first point in $T_{i_{k+1}}^3$. Then for the resulting motion (pq'), $|\tau(pq')| > n > |t|$; hence there is a sub-motion (pq) of (pq') for which $\tau(pq) = t$, and thus $q = f(p, t)$.

f is continuous in R', as is seen by using the proof in § 13. Of course f is continuous at all inner points of R^*. Take now any boundary point p of R^*, any $\varepsilon > 0$, and any positive integer n. Then, on account of property (C) of § 23, we can take δ so small that if q is any point of

R' in $V_\delta(p)$, on the curve C of F, we must pass through at least n distinct tubes T_i^3 before we pass out of $V_\varepsilon(p)$. It follows that for any t, $|t| \leqq n$, $f(q, t) \subset V_\varepsilon(p)$; hence $\varrho[f(p, t'), f(q, t)] = \varrho[p, f(q, t)] < \varepsilon$ for $q \subset V_\delta(p)$ and $|t| < n$, and f is shown to be continuous throughout R.

Take now any point p of R' and any two numbers t_1, t_2. Take motions (pq) and (qr) so that

$$\tau(pq) = t_1, \ \tau(qr) = t_2.$$

Then if $(pr) = (pq)+(qr)$,

$$\tau(pr) = t_1 + t_2,$$

by (26.1). As then $q = f(p, t_1)$, $r = f(q, t_2)$, and $r = f(p, t_1+t_2)$, we have $f[f(p, t_1), t_2] = f(p, t_1+t_2)$. As $f(p, 0) = p$, we have now proved

THEOREM 27A. *A function $f(p, t)$ can be defined over any family of paths, with the following properties:*

(1) *For each point p of R and any number t, $-\infty < t < +\infty$, there is a unique point*

$$q = f(p, t)$$

which lies on the curve C through p, or coincides with p if p is an invariant point. Further, for each point q of C there is a t such that $q = f(p, t)$.

(2) *$f(p, t)$ is continuous in both variables; that is, for each point p, any number t, and any $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\varrho[f(p, t), f(p', t')] < \varepsilon \text{ if } \varrho(p, p') < \delta \text{ and } |t-t'| < \delta.$$

(3) *For $p \subset R'$, $f(p, t)$ moves in the positive (negative) direction along the curve C through p as t increases (decreases).*

(4) *For each point p of R and any two numbers t_1 and t_2,*

$$f[f(p, t_1), t_2] = f(p, t_1+t_2); \tag{27.3}$$

also

$$f(p, 0) = p. \tag{27.4}$$

Any such function f we shall say defines a *flow* in R.

28. **The equivalence of two flows in R.** Let f_0 and f_1 be two flows in R. We show that *one can be deformed into the other* in the following sense: For each number x, $0 \leqq x \leqq 1$, there is a flow $f_x(p, t)$, and this set of functions is continuous in x, p and t together. That is, for any $\varepsilon > 0$, p, t and x, there is a $\delta > 0$ such that $\varrho[f_{x'}(p', t'), f_x(p, t)] < \varepsilon$ if $|x'-x| < \delta$, $\varrho(p', p) < \delta$ and $|t'-t| < \delta$.

Consider any motion (pq) of F; having defined τ_0 and τ_1 corresponding to f_0 and f_1, we define the function $u(pq)$ by the equation

$$\tau_1(pq) = \tau_0(pq) + u(pq). \tag{28.1}$$

Now for any x, $0 \leqq x \leqq 1$, put

$$\tau_x(pq) = \tau_0(pq) + xu(pq), \tag{28.2}$$

and define f_x in terms of τ_x. By writing (28.1) successively for motions (pq), (qr) and (pr), we see that

$$u(pq) + u(qr) = u(pr). \tag{28.3}$$

Noting also that $u(pq)$ is continuous, and that $\tau_0(pq) + xu(pq)$ increases as q moves in the positive direction, we see that τ_x, and hence f_x, has all the properties it should have; hence f_x defines a flow. f_x is obviously continuous in x, p and t, and the theorem is proved.

29. **Remark on cross-sections.** If we know of the existence of a function f in a family of paths, then we can find a cross-section through a point p of R' quite simply as follows. For any point q, put $q_t = f(q, t)$. Define the continuous function

$$\theta(q) = \int_0^1 \varrho(q_t, p)\, dt. \tag{29.1}$$

Using (27.3) we see that $(q_\varepsilon)_t = q_{\varepsilon+t}$. Hence, given a point q_ε on the curve through q, we find by a change of variable

$$\theta(q_\varepsilon) - \theta(q) = \int_1^{1+\varepsilon} \varrho(q_t, p)\, dt - \int_0^{\varepsilon} \varrho(q_t, p)\, dt,$$

and thus θ has a partial derivative along each curve:

$$\theta'(q) = \partial\theta(q)/\partial t = \varrho(q_1, p) - \varrho(q, p).$$

θ' is continuous and positive near p: $\theta'(p) = \varrho(p_1, p)$. Hence on each curve passing near p we can find a point q for which $\theta(q) = \theta(p)$; these points are easily seen to form a cross-section.

PRINCETON UNIVERSITY, HARVARD UNIVERSITY.

ON PROJECTIVE CONNEXIONS AND THEIR APPLICATION TO THE GENERAL FIELD-THEORY.* (GF. VI.)

By J. A. Schouten and D. van Dantzig.

Introduction.

In three previous papers (GF. I, II, III)[1] we have given a short account of a new physical theory, the "general fieldtheory" which intends to give a unification of general relativity not only with Maxwell's electromagnetical theory but also with Schrödinger's and Dirac's theory of material waves. The principal point of this theory is, that it is not founded on a metrical or an affine connexion but on a *projective* one. Hence it is closely related to the theory of Veblen and Hoffmann[2] which depends on a projective connexion as well, and to the theory of Einstein and Mayer[3] which can be brought into a projective form as we have proved in a previous paper.[4] But in contrast with these theories the geometrical conditions in our theory are reduced to a smaller number, in order to obtain the possibility of adapting the projective connexion most closely to the conditions arising from physical considerations. The method used in this paper to handle the projective connexion is the one developped by D. van Dantzig,[5] (TPZ., APD. I, II) based on the introduction of homogeneous coördinates. The method used to handle spinquantities is the one developped by J. A. Schouten.[6]

* Received July 13, 1932.

[1] GF. I, II.: J. A. Schouten and D. van Dantzig, Zum Unifizierungsproblem der Physik; Skizze einer generellen Feldtheorie, Proc. Kon. Ak. **35** (1932), pp. 642–655. Zur generellen Feldtheorie; Diracsche Gleichung und Hamiltonsche Funktion, Proc. Kon. Ak. **35** (1932), pp. 844–852. GF. III: Generelle Feldtheorie, Z. f. Ph., 78 (1932), pp. 639–667.

[2] O. Veblen and B. Hoffmann, Projective relativity, Phys. Review **36** (1931), pp. 810–822. B. Hoffmann, Projective relativity and the quantumfield, ibid. **37** (1931), pp. 88–89.

[3] A. Einstein and W. Mayer, Einheitliche Theorie von Gravitation und Elektrizität, Berlin, Sitzungsber. Pr. Ak. **25** (1931), pp. 541–557.

[4] J. A. Schouten and D. van Dantzig, Über eine vierdimensionale Deutung der neuesten Feldtheorie, Proc. Kon. Ak. **34** (1931), pp. 1398–1407.

[5] TPZ.: D. van Dantzig, Theorie des projektiven Zusammenhangs n-dimensionaler Räume, Math. Annalen **106** (1932), pp. 400–454. APD. I, II.: D. van Dantzig, Zur allgemeinen projektiven Differentialgeometrie, I. Einordnung der Affingeometrie, II. X_{n+1} mit eingliedriger Gruppe, Amsterdam, Proceedings Kon. Ak. **35** (1932), pp. 524–534, 535–542.

[6] J. A. Schouten, Über die in der Wellengleichung verwendeten hyperkomplexen Zahlen, Amsterdam, Proc. Kon. Akad. **32** (1929), pp. 105–108. Die Darstellung der Lorentzgruppe in der komplexen E_2, abgeleitet aus den Diracschen Zahlen, ibid. **33** (1930), pp. 189–197. DE.: J. A. Schouten, Dirac equations in general relativity, Journ. of Math. and Physics, **10** (1931), pp. 239–283.

In the first (geometrical) part we develop the theory of a general projective connexion with a fundamental quadric in each local space. Different kinds of homogeneous and of anholonomic systems of reference are introduced in §§ 1, 2. Instead of the vectors and affinors of affine geometry we introduce points and projectors (§ 3). They are defined with respect to two different groups: the group $\mathfrak{H}_{n+1}$ of transformations of homogeneous *coördinates* and the group $\mathfrak{F}$ of *point*-transformations. In § 4 the relations between the homogeneous coördinates and the local (projective) E_n^* on the one hand and the ordinary inhomogeneous coördinates and the local (affine) E_n on the other hand are considered. In general the E_n cannot be identified with the E_n^*, but this is possible as soon as a field of hyperplanes in the E_n^* is introduced (§ 5). Then these hyperplanes are identified with the hyperplanes "at infinity" of the E_n. If a fundamental quadric is introduced into each E_n^* (§ 6) the polar hyperplanes of the contactpoints can be used as such hyperplanes at infinity and the local space gets a euclidean metrical geometry. In § 7 an account is given of the theory of general projective connexions and this theory is used to determine in the §§ 8, 9 a class of projective connexions whose relations with the fundamental quadric are established by means of four geometrical conditions. A fifth geometrical condition, though a superfluous one, (because it follows from the first physical condition of the second part) is introduced also in § 9 because it simplifies considerably the geometrical properties of the projective connexion and its introduction enables us to give a short account of these properties at the end of the geometrical part (§ 10).

In the second (physical) part we specialise the theory for $n = 4$ and for a quadric with index $+3$. Then five physical conditions are imposed, by which from the class of projective connexions found in the first part one and only one connexion is singled out, having all the physical properties desired. The first condition (§ 12) is that the world-lines of charged masses are the geodesics of the connexion. By the second one (§ 13) the differential equation of the world-lines gets a second meaning: the point of total energy and momentum p^ν (which comes in our theory in the place of the vector of total e. a. m.) is covariant constant along a world-line. So the equation of motion is at the same time the theorem of conservation of energy and momentum. It is remarkable that in this form of the equation the indetermined potential vector φ_i does not occur anymore but is replaced by the wholly determined covariant point q_μ of the local hyperplane at infinity. § 14 contains a review of the theory of spinquantities and their covariant differentiation. This theory is applied in § 15 to Dirac's equation and in this paragraph we impose the third condition which requires that

the most simple linear differential equation that can be built be identical with Dirac's equation. It is possible to write Dirac's equation in a form which contains q_μ instead of φ_μ (§ 16). On account of the fourth condition which requires that the operator $\frac{h}{i}\nabla_\mu$ be quantumtheoretically equivalent with the point p_μ of total energy and momentum the Hamiltonian function arising from the equations of the second order derived from Dirac's equation gives the equation of the world lines in canonical form. It is remarkable that this Hamiltonian function contains the two points of intersection of the quadric with the line through the contactpoint and the point at infinity of the current-vector. If we introduce into the equation resulting from the variation principle the most simple invariant that can be built from the projector of curvature (§ 17), we get equations which can be identified with the combined equations of the gravitational and the electromagnetical field. This identification (fifth condition) fixes the value of the last unknown constant but one, so that only one constant ω remains which determines the unit of length. In the last paragraph we deal with the relations of our theory to the older theories.

I. Geometrical part.

1. **The n-dimensional manifold H_n.** The $n+1$ variables x^ν $(\varkappa, \cdots, \omega = 0, 1, \cdots, n)$, *the primary variables,* be subject to the transformations of the group $\mathfrak{H}_{n+1}$ of all transformations

$$x^{\nu'} = x^{\nu'}(x^0, x^1, \cdots, x^n), \qquad (\varkappa', \cdots, \omega' = 0', 1', \cdots, n') \tag{1.1}$$

where the $x^{\nu'}$ are *homogeneous analytical functions of degree one,*[7] regular and independent in the domain considered. A system of $n+1$ numbers x^ν of which at least one is not zero is called a *point,* the x^ν are called the (ν)-coördinates of the point. Two points x^ν and y^ν are called *coincident* if a factor exists, such that $y^\nu = \lambda x^\nu$. Each totality of all points, coincident with a definite point is called a *spot.* The totality of all ∞^n spots is called the H_n. A point of a spot of the H_n is called *point of the H_n.* There are ∞^{n+1} points of the H_n. The equation (1.1) defines the transformations of *coördinates.* But an equation of the form

$$\bar{x}^\nu = \bar{x}^\nu(x^0, x^1, \cdots, x^n), \tag{1.2}$$

[7] A homogeneous function f of degree one is not necessarily *linear,* i. e. expressible in the form $a_\nu x^\nu$ with *constant* coefficients a_ν. It is defined by the property $f(\varrho x^\nu) = \varrho f(x^\nu)$ for any ϱ and can be expressed (but not in a unique way) in the form $a_\nu x^\nu$ with coefficients a_ν which are arbitrary homogeneous functions of degree zero, i. e. only depend on the *ratios* of the x^ν.

where the $\bar{x}^{\nu}$ are homogeneous analytical functions of degree one, regular and independent in the domain considered, defines a *point-transformation* of a special kind, viz. transforming spots into spots. Of those transformations we use here those special ones

$$\bar{x}^{\nu} = \varrho x^{\nu} \tag{1.3}$$

which leave *every* spot invariant. Obviously ϱ is an arbitrary function of the x^{ν}, homogeneous of degree zero. These transformations form a group $\mathfrak{F}$.

Every function of the x^{ν} considered in this paper is *homogeneous of some degree* $\mathfrak{r}$. Any such function $f(x^{\nu})$ may or may not be invariant under transformations of $\mathfrak{H}_{n+1}$. But the different values belonging to the points of one and the same spot are always connected by the equation

$$\bar{f} = \varrho^{\mathfrak{r}} f; \qquad \bar{f} = f(\varrho x^{\nu}). \tag{1.4}$$

If $\mathfrak{r}$ is zero, f is invariant under $\mathfrak{F}$. Equivalent with (1.4) is Euler's condition of homogeneity

$$x^{\mu} \partial_{\mu} f = \mathfrak{r} f; \qquad \partial_{\mu} \overset{h}{=} \frac{\partial}{\partial x^{\mu}}, \tag{1.5}$$

[8]

thus for the $x^{\nu'}$ which are homogeneous of degree one:

$$x^{\nu} \mathcal{A}_{\nu}^{\nu'} = x^{\nu'}; \qquad \mathcal{A}_{\nu}^{\nu'} \overset{h}{=} \partial_{\nu} x^{\nu'}. \tag{1.6}$$

[9]

The partial derivatives of a homogeneous function of degree $\mathfrak{r}$ are homogeneous of degree $\mathfrak{r}-1$. Hence, the $\mathcal{A}_{\nu}^{\nu'}$ being homogeneous of degree 0,

$$x^{\mu} \partial_{\mu} \mathcal{A}_{\lambda}^{\nu'} = 0. \tag{1.7}$$

We now define several geometric objects of the H_n. A function p of the x^{ν}, invariant under $\mathfrak{H}_{n+1}$, and transforming under $\mathfrak{F}$ in the following way

$$\bar{p} = \varrho^{\mathfrak{r}} p \tag{1.8}$$

is called a *scalarfield of degree* $\mathfrak{r}$.

A system of $n+1$ functions v^{ν} or w_{λ}, which transform under $\mathfrak{H}_{n+1}$ and $\mathfrak{F}$ in the following way

$$\alpha)\ v^{\nu'} = \mathcal{A}_{\nu}^{\nu'} v^{\nu}; \qquad \beta)\ \bar{v}^{\nu} = \varrho^{\mathfrak{r}} v^{\nu} \tag{1.9}$$

$$\alpha)\ w_{\lambda'} = \mathcal{A}_{\lambda'}^{\lambda} w_{\lambda}; \qquad \beta)\ \bar{w}_{\lambda} = \varrho^{\mathfrak{r}} w_{\lambda} \tag{1.10}$$

[8] For the meaning of the sign h over $=$ see the end of § 3.

[9] The indices ν and ν' take the values $0, \cdots, n$ and $0', \cdots, n'$ resp. totally independent from each other.

resp. is called a *contra-* or *covariant pointfield of degree* $\mathfrak{r}$. The functions v^ν, w_λ are called the *(ν)-components* of the pointfield. As usual the term "field" is often dropped. A field need not be defined over the whole H_n, but we only consider fields which are defined for at least all points of one definite spot in H_n. Two contra- or covariant points of any degree, belonging to the same spot of the H_n, are called *coincident* if we get one from the other by multiplication with a scalar factor of some degree.

Now we define an E_n^*[10] with the coördinates z^ν by the transformation

$$z^{\nu'} = \mathfrak{A}_\nu^{\nu'} z^\nu, \tag{1.11}$$

in which formula the $\mathfrak{A}_\nu^{\nu'}$ are the values of $\partial x^{\nu'}/\partial x^\nu$ in a definite spot P of the H_n, and call this E_n^* the *local E_n^* belonging to this spot P*. Then to every system of coördinates x^ν in H_n belongs one and only one system of coördinates in E_n^*. The points of degree zero as defined above, which belong to the spot P are in one to one correspondance with and may therefore be identified with the points of this E_n^*. But a contravariant pointfield v^ν of degree $\mathfrak{r}$ of H_n corresponds for every point of the spot P with one point of the E_n^* and these points undergo the point-transformation (1.9β) if the point of P is transformed according to (1.3). We call a variable point of this kind a *point of degree* $\mathfrak{r}$ of E_n^*. Hence in E_n^* there is no independent group playing the role of $\mathfrak{F}$ in H_n but for every degree there exists such a group dependent on $\mathfrak{F}$, which reduces to identity if the degree is zero. These groups are representations ("Darstellungen") of $\mathfrak{F}$, just as the group of projective transformations of local coördinates belonging to $\mathfrak{A}_\nu^{\nu'}$ is a representation of $\mathfrak{H}_{n+1}$. A *spot* of E_n^* is the totality of all points of E_n^* coincident with a definite point.

According to Euler's condition of homogeneity (1.5) the x^ν themselves are not only *coördinates* of a point of H_n but also *components* of a point of degree one of E_n^*. The corresponding point in the E_n^* is identified with the point x^ν of H_n and is called the *contactpoint* of the E_n^*. Also the spots of those two points of H_n and E_n^* are identified. Because

$$dx^{\nu'} = \mathfrak{A}_\nu^{\nu'} dx^\nu, \tag{1.12}$$

$$d\bar{x}^\nu = \varrho\,(dx^\nu + x^\nu\, d\log\varrho) \tag{1.13}$$

the *dx^ν are not components of a point.* They transform in the right way under $\mathfrak{H}_{n+1}$ but in the wrong way under $\mathfrak{F}$. If $y^\nu = x^\nu + dx^\nu$ is a point of a spot in an infinitesimal neighborhood of the spot of the point x^ν in H_n, the *coördinates* $y^{\nu'}$ of y with respect to another coördinate-system (ν') are

[10] An E_n^* is an n-dimensional manifold with an ordinary projective geometry.

(1.14) $$y^{\nu'} = x^{\nu'}(y) = x^{\nu'} + dx^\mu\,\partial_\mu x^{\nu'} + \frac{1}{2}\,dx^\lambda\,dx^\mu\,\partial_{\lambda\mu}x^{\nu'} + \ldots,$$ [11]

therefore, apart from quantities of the second order

(1.15) $$y^{\nu'} = \mathcal{A}^{\nu'}_{\nu}\,y^{\nu},$$

i. e. the y^ν transform under $\mathfrak{H}_{n+1}$ like a point of the E_n^* of x^ν. Under $\mathfrak{F}$ the y^ν transform as follows

(1.16) $$\bar{y}^\nu = \bar{x}^\nu(y) = \varrho(y)\,y^\nu = (\varrho + d\varrho)\,y^\nu.$$

Because $\varrho + d\varrho$ is not a power of ϱ, the y^ν transform under $\mathfrak{F}$ unlike a point of any degree. Thus the *point* y^ν of H_n can not be identified with any *point* of the E_n^* of x^ν. But because at all events $\bar{y}^\nu$ is proportional to y^ν we can identify the *spot* y^ν of H_n with the *spot* of E_n^* which has the same coördinates. This identification is exact, apart from quantities of the second order in dx^ν.

The measuring-points belonging to the coördinates x^ν are the points of degree zero

(1.17) $$\overset{\nu}{e}_\lambda \overset{*}{=} \delta^\nu_\lambda; \qquad \overset{\nu}{e}{}^\lambda \overset{*}{=} \delta^\nu_\lambda.$$ [12]

The indices right over and under the central letter which are used only as *numerating* indices to distinguish the $n+1$ different points do not transform and are called *dead* indices, contrary to the other *transforming* ones, which may be called *living* indices.

$\overset{\nu}{e}_\lambda$ and $\overset{\nu}{e}{}^\lambda$ form the local system of reference, denoted by (ν). The components of a point with respect to (ν) are the coefficients of the measuring-points if the point is written out as a sum of multiples of measuring-points. The components of the system (ν) with respect to (ν') are

(1.18) $$\overset{\nu'}{e}_\lambda \overset{*}{=} \mathcal{A}^{\nu'}_{\lambda}; \qquad \overset{\nu}{e}{}^{\lambda'} \overset{*}{=} \mathcal{A}^{\nu}_{\lambda'}.$$

Obviously, as in the case of affinors, addition (only for projectors of the same degree), multiplication, transvection, alternating and mixing of projectors are invariant operations.

2. **Anholonomic local systems of reference.**[13] Instead of the $\mathcal{A}^{\nu'}_{\lambda}$ from (1.6) we can take $(n+1)^2$ arbitrary homogeneous functions $\mathcal{A}^{\nu'}_{\lambda}$ of the

[11] The unity-projector and -affinor and differentialoperators are not recapitulated:

$$\mathcal{A}^{\nu'\nu}_{\mu\lambda} = \mathcal{A}^{\nu'}_{\mu}\,\mathcal{A}^{\nu}_{\lambda}, \qquad \nabla_{\omega\mu} = \nabla_\omega \nabla_\mu, \qquad \partial_{\lambda\mu} = \partial_\lambda\,\partial_\mu, \text{ etc.}$$

[12] For the meaning of the $*$ over $=$ see the end of § 3.

[13] Comp. J. A. Schouten, Über nicht-holonome Übertragungen in einer L_n, Math. Zeitschr. **30** (1929), pp. 149–172; J. A. Schouten and E. R. van Kampen, Zur Einbettungs- und Krümmungstheorie nicht-holonomer Gebilde, Math. Ann. **103** (1930), pp. 752–783.

x^ν of some degree $\mathfrak{q}'$, continuous and regular in the domain considered and with the determinant $\neq 0$. We then get a local system (ν') entirely independent of the choice of the primary variables:

$$\overset{\nu'}{e}_\lambda \overset{*}{=} \mathfrak{A}^{\nu'}_\lambda; \qquad \overset{\nu}{e}_{\lambda'} \overset{*}{=} \mathfrak{A}^\nu_{\lambda'}; \qquad \mathfrak{A}^{\nu\mu'}_{\mu'\lambda} = \mathfrak{A}^\nu_\lambda, \tag{2.1}$$

and the components of a point with respect to (ν') are

$$\begin{aligned} v^{\nu'} &= \mathfrak{A}^{\nu'}_\nu v^\nu; \\ w_{\lambda'} &= \mathfrak{A}^\lambda_{\lambda'} w_\lambda. \end{aligned} \tag{2.2}$$

The

$$x^{\nu'} = \mathfrak{A}^{\nu'}_\nu x^\nu \tag{2.3}$$

are now *components* of the contactpoint with respect to (ν'). They can be used as primary *variables* if and only if

$$\Omega^{\nu'}_{\lambda'\mu'} = \mathfrak{A}^{\mu\lambda}_{\mu'\lambda'} \partial_{[\mu} \mathfrak{A}^{\nu'}_{\lambda]} \overset{*}{=} 0, \tag{2.4}$$

$$\mathfrak{q}' \overset{*}{=} 0, \tag{2.5}$$

because according to the first condition $\mathfrak{A}^{\nu'}_\nu \, dx^\nu$ is an exact differential:

$$\mathfrak{A}^{\nu'}_\nu \, dx^\nu \overset{*}{=} \frac{1}{\mathfrak{q}'+1} dx^{\nu'} \tag{2.6}$$

and according to the second condition it is the differential of $x^{\nu'}$. Hence the expression $\Omega^{\nu'}_{\lambda'\mu'}$ and $\mathfrak{q}'$ are characteristic for the anholonomy of the system (ν'), the system is holonomic if and only if they both vanish. In the anholonomic case we write

$$(dx)^{\nu'} = \mathfrak{A}^{\nu'}_\nu \, dx^\nu \neq dx^{\nu'}. \tag{2.7}$$

3. **Projectors and general geometric objects.** A system of homogeneous functions of the x^ν of some degree, given with respect to k arbitrary holonomic or anholonomic local systems of reference and defined for at least all points of one definite spot of the H_n, which transforms in a definite way if each of the local systems is transformed in some definite way, is called a field of a (projective) geometric object.[14] The functions are the *components* of the geometric object, they are called *intermediate* if $k>1$. If the number of functions is $(n+1)^p$ and if the system transforms like the product of t arbitrary contravariant points and

[14] The term geometric object introduced by J. A. Schouten and E. R. van Kampen, Math. Ann., 103 (1930), p. 758, is an alternative of the term "invariant" formerly used by the Princeton school. Comp. The Foundations of differential geometry by O. Veblen and J. H. C. Whitehead, Cambr. Tracts No. 29, 1932, p. 46.

$s = p - t$ arbitrary covariant ones the geometric object is called a *projector* of *valence*[15] p, *contravariant valence* t and *covariant valence* s. F. i. in the case $k = 2$:

$$X^{\cdot\cdots\cdot\nu_1''\cdots\nu_t''}_{\lambda_1'\cdots\lambda_s'} = \mathcal{A}^{\lambda_1\cdots\lambda_s\nu_1''\cdots\nu_t''}_{\lambda_1'\cdots\lambda_s'\nu_1\cdots\nu_t} X^{\cdot\cdots\cdot\nu_1\cdots\nu_t}_{\lambda_1\cdots\lambda_s} . \tag{3.1}$$

A point is a projector of valence one and a scalar may be considered as a projector of valence zero.

The $\mathcal{A}^{\nu'}_{\nu}$ in (1.6) and (2.2) are intermediate components of a projector called the *unity-projector;* its *non-intermediate* components $\mathcal{A}^{\nu}_{\lambda} \overset{h}{=} \partial_\lambda x^\nu$ are 1 or 0 if $\nu = \lambda$ or $\nu \neq \lambda$ resp. If we perform a point-transformation of $\mathfrak{F}$ and after this a transformation of coördinates of $\mathfrak{H}_{n+1}$ in such a way that the new coördinates of the new points are equal to the old coördinates of the old points:

$$\begin{aligned} \bar{x}^\nu &= \varrho\, x^\nu, \\ \bar{x}^{\nu'} &\overset{*}{=} \varrho^{-1}\, \bar{x}^\nu = x^\nu, \end{aligned} \tag{3.2}$$

[16]

and if the components of a geometric object $\Omega^{\cdot}_{\cdot}$[17] with respect to a *holonomic* local system after this double process for a *constant* ϱ transform into $\varrho^\varepsilon\, \Omega^{\cdot}_{\cdot}$, ε is called the excess[18] of $\Omega^{\cdot}_{\cdot}$.

Obviously the excess of a sum of projectors is equal to the excess of each term[19] and the excess of a product is equal to the sum of the excesses of the factors. The excess of a projector is invariant under partial (and, as will be proved later on, covariant) differentiation, contraction and transvection with x^ν, x^ν having the excess zero.

An equation is called a *projector-equation* if its right or left side is a projector non zero. If it admits *every* transformation of the local systems to holonomic or anholonomic other ones, and this *for each lower or upper living index separately,* then the other side is a projector too with the same valence and the same excess.

Following an idea of J. A. Schouten we will use different signs $=$ to denote (if necessary) the different groups which may leave an equation invariant:

$=$ only for equations which are invariant under transformation to *arbitrary* holonomic or anholonomic coördinates (and for *quantities,*

[15] We use the word "valence" instead of the word "degree", used in former publications, in order to avoid collisions with the notion of degree of a function.

[16] This process is a special case of the trailing along (mitschleppen) of a coördinate system, well known in problems of deformation in ordinary and Riemannian geometry.

[17] The big points stand for any number of indices.

[18] The excess corresponds with Veblens "weight" or "index". If $\mathfrak{r}$ is the degree of the components of a projector with respect to a system of reference with degree $\mathfrak{q}$, the excess ε is $\varepsilon = \mathfrak{r} - \mathfrak{q}(s - t)$. The degree $\mathfrak{r}$ changes if the degree of the system of reference changes but the excess is invariant.

[19] A sum of projectors has only a sense if the valences and the excesses of all terms are equal.

i. e. geometric objects with *linear* transformation of the components, f. i. affinors, projectors, spinors, densities, etc. even for *each index separately*), and under *arbitrary* transformations of the group $\mathfrak{F}$.

$\overset{h}{=}$ for equations which are invariant under arbitrary *holonomic* transformations (i. e. under the group $\mathfrak{H}_{n+1}$) and under $\mathfrak{F}$.

$\overset{n}{=}$ for equations which are invariant under transformation to n *arbitrary* holonomic or anholonomic nonhomogeneous coördinates (k); $(h, \cdots, m = 1, \cdots, \mathrm{n})$ (for quantities for each index separately), together with $\mathcal{A}_0^{\nu'} = \omega^{-1} x^{\nu'}$, $\mathcal{A}_\nu^0 = -q_\nu$ (comp. § 5) and under $\mathfrak{F}$.[20]

$\overset{4}{=}$ means $\overset{n}{=}$ for $n = 4$.

$\overset{a}{=}$ for the transformations belonging to $\overset{n}{=}$, if (k) is holonomic. (It is to be noted that (c) (defined p. 290) nevertheless is anholonomic.)

$\overset{o}{=}$ for the transformations belonging to $\overset{n}{=}$ if (k) (and therefore also (c)) is orthogonal.[21]

$\overset{*}{=}$ for equations which are true with respect to the system or systems of coördinates used in the equation itself, without anything being said about the question whether it admits any transformation or not.

By using these signs we will *always* mean a *minimal* statement; they never exclude the possibility that the equations admit any other transformation than those mentioned. Equations which are only true with $\overset{h}{=}$, $\overset{o}{=}$ etc. but not with $=$ are *only* used for brevity; by adding some easily calculated additional terms they can always be written with $=$. Only equations with $=$ can be differentiated covariantly without any danger of arriving at incorrect results.

4. **Inhomogeneous coördinates and local systems.** We introduce n arbitrary mutually independent functions ξ^k $(h, \cdots, m = 1, \cdots, \mathrm{n})$ of the x^ν, homogeneous of degree 0:

$$x^\mu E_\mu^k = 0; \qquad E_\mu^k \overset{h}{=} \partial_\mu \xi^k, \tag{4.1}$$

analytic, regular and independent in the domain considered. They are subject to the group $\mathfrak{G}_n$ of all transformations

$$\xi^{k'} = \xi^{k'}(\xi^1, \cdots, \xi^n) \tag{4.2}$$

where the $\xi^{k'}$ are n analytic functions, regular and independent in the domain considered. The transformation of the ξ^k is totally independent of the transformation of the x^ν, hence the ξ^k are scalars of degree zero with respect to $\mathfrak{H}_{n+1}$ and the x^ν are scalars with respect to $\mathfrak{G}_n$.

[20] If no index 0 occurs, of course $=$ may be used instead of $\overset{n}{=}$.

[21] Equations which admit orthogonal transformations of (k) without any index 0 occurring may of course be written with $\overset{o}{=}$.

A system of n numbers ξ^k is called an (affine) point, the totality of all points an X_n. There is a one to one correspondence between *points* of the X_n and *spots* of the H_n.

Anholonomic systems of reference (k') can be introduced in the X_n in the usual way, starting with a holonomic system (k) and introducing n^2 arbitrary functions $E_i^{k'}$ of the ξ^k, continuous, regular and with determinant $\neq 0$ in the domain considered. But they can be obtained also in a direct way by introducing a system of $n(n+1)$ arbitrary homogeneous functions $E_\lambda^{k'}$ of the x^ν of degree -1, continuous and regular in the domain considered, with a matrix-rank equal to n and which are subject to the condition

$$E_\mu^{k'}\, x^\mu = 0. \tag{4.3}$$

These $E_\mu^{k'}$ belong to a system of inhomogeneous coördinates $\xi^{k'}$ if and only if

$$\partial_{[\mu} E_{\lambda]}^{k'} \overset{*}{=} 0. \tag{4.4}$$

Starting with the affine transformations to holonomic or anholonomic systems of reference we define now the local E_n,[22] affine scalars, vectors, affinors and other (affine) geometric objects, just in the same way as we defined above the local E_n^*, (projective) scalars, points, projectors and other (projective) geometric objects. The affine scalars may obviously be identified with the projective scalars of degree zero.

Of course more general objects exist, having with respect to some indices the properties of an affine geometric object and with respect to the other indices those of a projective one. If they behave with respect to these indices like an affinor and a projector resp. we call them *projector-affinors*. The simplest example is

$$E_\lambda^k \overset{h}{=} \partial_\lambda \xi^k, \tag{4.5}$$

transforming like a contravariant vector in k and like a covariant point in λ and supplying the link between affinors and projectors. To every contravariant point belongs one and only one contravariant vector and to every covariant vector one and only one covariant point:

$$\begin{aligned} {}'v^k &= E_\nu^k\, v^\nu, \\ {}'w_\lambda &= E_\lambda^i\, w_i. \end{aligned} \tag{4.6}$$

Here the contravariant vector $'v^k$ depends on the *point* v^ν, not on the corresponding spot alone. Because of (4.3) $'w_\lambda$ is always incident with the contactpoint.

[22] An E_n is an n-dimensional manifold with an ordinary affine geometry.

The contactpoint and its spot correspond with the vector zero. A straight line through the contactpoint in E_n^* is in one to one correspondence with a direction in E_n. If we identify as usual the point x^ν with some point, "*the origin*" of its local E_n, there is a one to one correspondence between the points in the neighborhoods of the first order of these points in E_n and X_n, which is exact, but for quantities of the second order. As there is also an exact one to one correspondence between the points of the X_n and the spots of the H_n, there is a one to one correspondence of the spots of an infinitesimal neighborhood of the spot of the contactpoint in the E_n^* and the points of such a neighborhood in the E_n, exact but for quantities of the second order (comp. p. 276). But there is *no* such correspondence between the spots of the E_n^* and the points of the E_n *outside* of such a neighborhood. In particular there is no definite hyperplane in E_n^* which corresponds to the hyperplane at infinity of the E_n.

5. **Introduction of a field of hyperplanes in the E_n^*.**[23] A field of hyperplanes λq_λ in the local E_n^*, which do not pass through the contactpoint, determines uniquely and is determined uniquely by a covariant pointfield q_λ of excess zero, subject to the condition

$$q_\lambda x^\lambda = -\omega, \tag{5.1}$$

ω being an arbitrary positive constant, chosen once for all.[24] Such a field determines a one to one correspondence between the points of the local E_n and those spots of the local E_n^*, which do not lie in the hyperplane of q_λ. The contactpoint of the E_n^* corresponds with the origin of the E_n and on corresponding straight lines through these points; the spots in E_n^* incident with q_λ correspond with the points at infinity of the E_n. Analytically we get this correspondence in the following way. By means of E_λ^k a holonomic or anholonomic system of reference (k) is defined in the E_n. Starting with this system we define a system of measuring points (c); $(a, \cdots, h = 0, 1, \cdots, n)$ in the E_n^* by means of the equations

$$\begin{aligned} \overset{k}{e}_\lambda &\overset{*}{=} \mathfrak{A}_\lambda^k \overset{*}{=} E_\lambda^k; \\ \overset{0}{e}_\lambda &\overset{*}{=} \mathfrak{A}_\lambda^0 \overset{*}{=} -q_\lambda.^{25} \end{aligned} \tag{5.2}$$

If E_i^ν are the solutions of the equations

$$E_j^\nu E_\lambda^j = E_\lambda^\nu; \qquad E_\lambda^\nu = \mathfrak{A}_\lambda^\nu + \omega^{-1} q_\lambda x^\nu, \tag{5.3}$$
$$E_i^\nu q_\nu = 0,$$

[23] Comp. APD. I where t_λ is taken instead of our $-\omega^{-1} q_\lambda$.

[24] We prefer to take ω in the right side of (5.1) instead of 1, in order to be able to give ω a physical dimension in the physical application of the theory.

[25] All components of projectors having an index 0 differ from the corresponding components used in GF. I, II by a factor ω.

then

$$E_i^\mu \, E_\mu^k = A_i^k, \tag{5.4}$$

where A_i^k is the unity-affinor in the E_n. The contravariant measuring-points are

$$\begin{aligned} &\underset{i}{e}{}^\nu \overset{*}{=} \mathcal{A}_i^\nu \overset{*}{=} E_i^\nu; \\ &\underset{0}{e}{}^\nu \overset{*}{=} \mathcal{A}_0^\nu \overset{*}{=} \omega^{-1} x^\nu. \end{aligned} \tag{5.5}$$ [25]

The system (c) is never holonomic because the degree of $\mathcal{A}_\lambda^c$ is -1. Ω_{ab}^c vanishes if and only if the system (k) is holonomic and

$$q_{\mu\lambda} = 0; \qquad q_{\mu\lambda} = \partial_{[\mu} q_{\lambda]} + \Omega_{\lambda\mu}^\nu \, q_\nu \overset{h}{=} \partial_{[\mu} q_{\lambda]}. \tag{5.6}$$

With respect to the system (c) all components of x^ν and q_ν are zero, except $x^0 \overset{n}{=} \omega$ and $q_0 \overset{n}{=} -1$.

E_i^ν and E_λ^k establish a one to one correspondence between contra- and covariant vectors and contra- and covariant points incident with q_λ and x^ν resp. The difference between two contravariant or covariant points corresponds with a vector if and only if the *weights* of the points, defined by

$$v = v^0 = -v^\nu \, q_\nu; \qquad w = w_0 = \omega^{-1} w_\lambda \, x^\lambda \text{ [26]} \tag{5.7}$$

are equal. The point in the E_n with the radiusvector r^k corresponds with the spot of all points defined by

$$v^\nu = v(\omega^{-1} x^\nu + E_i^\nu \, r^i). \tag{5.8}$$

r^k is the difference between the point of this spot with weight 1 and $\omega^{-1} x^\nu$, and is called *the vector between* $\omega^{-1} x^\nu$ and this point. The vector $'v^k$ defined in (4.5) is the difference between the point v^ν and $\omega^{-1} v x^\nu$ and is equal to $v \, r^k$. Hence we arrive at the well known point-addition of Moebius. The linear element, the vector between x^ν and $x^\nu + dx^\nu$ is

$$d' x^\nu = E_\lambda^\nu \, dx^\lambda = E_i^\nu \, d\xi^i = dx^\nu + \omega^{-1} x^\nu \, q_\varrho \, dx^\varrho \tag{5.9}$$

where the ξ^k are holonomic or anholonomic inhomogeneous coördinates in the X_n.

The field q_λ being once introduced, we *identify* vectors and their corresponding points by giving them the same principal letter. Then the difference between E and A vanishes and the affine and projective components of a vector v^ν and w_λ are connected by the equations

[26] The weights used here differ by a factor ω from those used in APD. I and GF. I, II.

$$(5.10)\qquad \begin{aligned} v^k &= A^k_\nu v^\nu \text{ for } v^\nu q_\nu = 0; \quad v^\nu = A^\nu_k v^k; \\ w_i &= A^\lambda_i w_\lambda \text{ for } w_\lambda x^\lambda = 0; \quad w_\lambda = A^i_\lambda w_i. \end{aligned}$$

Vectors are then only a special kind of points (viz. points lying in the hyperplane at infinity), affinors and projectoraffinors a special kind of projectors.

6. **Introduction of a fundamental quadric.** In every local E_n^* we introduce a non-degenerate quadric, not passing through the contactpoint. This quadric determines uniquely and is determined uniquely by a symmetrical projector $G_{\lambda\mu}$, subject to the condition of normalisation

$$(6.1)\qquad G_{\lambda\mu} x^\lambda x^\mu = -\omega^2,$$

where ω is the constant introduced in (5.1).

$G_{\lambda\mu}$ has the excess zero and is called the *covariant fundamental projector*. The *contravariant fundamental projector* $G^{\mu\nu}$ is defined by

$$(6.2)\qquad G_{\lambda\mu} G^{\mu\nu} = \mathcal{A}^\nu_\mu.$$

We use $G_{\lambda\mu}$ and $G^{\mu\nu}$ for the raising and lowering of indices as usual and take

$$(6.3)\qquad q_\lambda = \omega^{-1} x_\lambda.$$

Then for the covariant point q_λ the equations

$$(6.4)\qquad q_\lambda x^\lambda = -\omega, \quad q_\lambda q^\lambda = \omega^{-2} x_\lambda x^\lambda = -1$$

hold. Hence in each local E_n^* we can build three geometries:

a) A non euclidean geometry, by introducing the quadric as the absolute. In the neighborhood of the contactpoint it gives the same ds^2 as the geometry mentioned under c. *This geometry is not used in this paper.*

b) A euclidean affine geometry by introducing the hyperplane of q_λ as the infinite.

c) A euclidean metrical geometry by introducing the quadric as a measuring hypersphere with radius ω in the geometry mentioned under b. The hyperplane of q_λ becomes the infinite, its intersection with the quadric the isotropic sphere at infinity.

The latter geometry is used in this paper. Hence affinors are all those and only those projectors whose transvections with q^ν and q_λ all vanish. The vanishing of Ω^c_{ab} belonging to (*c*), introduced in § 5, now depends on the vanishing of the alternating projector $q_{\mu\lambda} \overset{h}{=} \partial_{[\mu} q_{\lambda]}$. Because of

$$(6.5)\qquad q_{\lambda\mu} x^\lambda \overset{h}{=} \frac{1}{2}(x^\lambda \partial_\lambda q_\mu + \partial_\mu \omega + q_\lambda \partial_\mu x^\lambda) \overset{h}{=} 0$$

we have $q_{\mu\lambda} x^\lambda = 0$, hence $q_{\mu\lambda}$ is an *affinor*, viz. a *bivector*.

The weight v of a point v^ν is now

$$v = -v^\nu q_\nu = -\omega^{-1} v^\nu x_\nu \tag{6.6}$$

and the vector between q^ν and v^ν is

$$v'^\nu = \frac{v^\nu}{v} - q^\nu. \tag{6.7}$$

The square length of the linear element $d'x^\nu$ is given by

$$ds^2 \overset{h}{=} G_{\lambda\mu}\, d'x^\lambda\, d'x^\mu \overset{h}{=} g_{\lambda\mu}\, dx^\lambda\, dx^\mu \overset{h}{=} g_{ij}\, d\xi^i\, d\xi^j, \tag{6.8}$$

where

$$g_{\lambda\mu} = A^{\varrho\sigma}_{\lambda\mu} G_{\varrho\sigma} = G_{\lambda\mu} + q_\lambda q_\mu = A^{i\,j}_{\lambda\mu} g_{ij} \tag{6.9}$$

is the *covariant fundamental tensor*. The contravariant fundamental tensor is

$$g^{\mu\nu} = A^{\mu\nu}_{\varrho\sigma} G^{\varrho\sigma} = G^{\mu\nu} + q^\mu q^\nu = A^{\mu\nu}_{jk} g^{jk}. \tag{6.10}$$

We define as the *value* $\|v^\nu\|$ of a point v^ν

$$\|v^\nu\| = |\sqrt{v_\lambda v^\lambda}|. \tag{6.11}$$

It is zero if and only if the point lies on the quadric and is proportional to the weight of the point. If the point is a vector (i. e. lies in q_ν) its value is equal to the absolute value of its length.

7. **Projective connexions.** A linear connexion for projectors with excess ε is defined by the equations[27]

$$\begin{aligned} \nabla_\mu p &= \partial_\mu p + \varepsilon Q_\mu p; \\ \nabla_\mu v^\nu &= \partial_\mu v^\nu + \Pi^\nu_{\lambda\mu} v^\lambda + \varepsilon Q_\mu v^\nu; \\ \nabla_\mu w_\lambda &= \partial_\mu w_\lambda - \Pi^\nu_{\lambda\mu} w_\nu + \varepsilon Q_\mu w_\lambda. \end{aligned} \tag{7.1}$$

The $\Pi^\nu_{\lambda\mu}$ and Q_μ are $(n+1)^3$ and $n+1$ functions of the x^ν of excess zero (degree -1). Hence *the excess of any projector is invariant under covariant differentiation.* If p has the excess zero, $x^\mu\, \partial_\mu p$ vanishes, hence $\nabla_\mu p$ is then a vector.

The $\Pi^\nu_{\lambda\mu}$ are not components of a projector but are transformed in the usual way

$$\Pi^{\nu'}_{\lambda'\mu'} = \mathcal{A}^{\lambda\,\mu\,\nu'}_{\lambda'\mu'\nu}\, \Pi^\nu_{\lambda\mu} + \mathcal{A}^{\nu'}_{\varrho}\, \partial_{\mu'}\, \mathcal{A}^{\varrho}_{\lambda'}, \tag{7.2}$$

but the Q_μ are components of a covariant point. From (7.2) it follows that

[27] TPZ., pp. 419/420 and 451.

$$S_{\lambda\mu}^{\cdot\cdot\nu} = \Pi^{\nu}_{[\lambda\mu]} + \Omega^{\nu}_{\lambda\mu} \overset{h}{=} \Pi^{\nu}_{[\lambda\mu]} \tag{7.3}$$

is a projector of excess zero. If $S_{\lambda\mu}^{\cdot\cdot\nu}$ vanishes the connexion is called *symmetrical.*

Contrary to the affine case the expression $x^{\lambda}\,\Pi^{\nu}_{\lambda\mu}$ for a holonomic system is a projector of excess zero, which is closely connected with the invariant derivative $Q^{\nu}_{\cdot\mu}$ of x^{ν}:

$$Q^{\nu}_{\cdot\mu} = \nabla_{\mu} x^{\nu} \overset{h}{=} x^{\lambda}\,\Pi^{\nu}_{\lambda\mu} + \mathcal{A}^{\nu}_{\mu}. \tag{7.4}$$

The expression $\Pi^{\nu}_{\lambda\mu}\,x^{\mu}$ for a holonomic system is a projector of excess zero as well:

$$P^{\nu}_{\cdot\lambda} = Q^{\nu}_{\cdot\lambda} + 2\,S_{\lambda\mu}^{\cdot\cdot\nu}\,x^{\mu} \overset{h}{=} \Pi^{\nu}_{\lambda\mu}\,x^{\mu} + \mathcal{A}^{\nu}_{\lambda}, \tag{7.5}$$

which occurs in the often used relations[28]

$$\begin{aligned} x^{\mu}\,\nabla_{\mu}\,v^{\nu} &= +P^{\nu}_{\cdot\mu}\,v^{\mu} + \varepsilon\,Q\,v^{\nu}, \\ x^{\mu}\,\nabla_{\mu}\,w_{\lambda} &= -w_{\nu}\,P^{\nu}_{\cdot\lambda} + \varepsilon\,Q\,w_{\lambda} \end{aligned} \tag{7.6}$$

where

$$Q = Q_{\mu}\,x^{\mu} + 1 = \omega\,Q_0 + 1. \tag{7.7}$$

The differential dx^{ν} has *not* the properties of a point (comp. (1.13)). Hence $dx^{\mu}\,\nabla_{\mu}$ is invariant if and only if $x^{\mu}\,\nabla_{\mu}\,v^{\nu}$ is zero for *every* choice of v^{ν}, that is if $P^{\nu}_{\cdot\lambda}$ and Q vanish.[28a] But if q_{λ} is introduced there is another way to build a covariant differential by means of the vector $d'x^{\nu}$ defined in (5.9):[29]

$$\begin{aligned} \delta v^{\nu} &= d'x^{\mu}\,\nabla_{\mu}\,v^{\nu} = dv^{\nu} + \Pi^{\nu}_{\lambda\mu}\,v^{\lambda}\,dx^{\mu} + \varepsilon\,v^{\nu}\,Q_{\mu}\,dx^{\mu} \\ &\qquad + \omega^{-1}\,(P^{\nu}_{\cdot\lambda}\,v^{\lambda} + \varepsilon\,Q\,v^{\nu})\,q_{\varrho}\,dx^{\varrho}, \\ \delta w_{\lambda} &= d'x^{\mu}\,\nabla_{\mu}\,w_{\lambda} = dw_{\lambda} - w_{\nu}\,\Pi^{\nu}_{\lambda\mu}\,dx^{\mu} + \varepsilon\,w_{\lambda}\,Q_{\mu}\,dx^{\mu} \\ &\qquad + \omega^{-1}\,(-w_{\nu}\,P^{\nu}_{\cdot\lambda} + \varepsilon\,Q\,w_{\lambda})\,q_{\varrho}\,dx^{\varrho}. \end{aligned} \tag{7.8}$$

A covariant differential determines uniquely and is determined uniquely by a *displacement* viz. a projective mapping of neighboring local E_n^* on each other, characterized by the vanishing of this differential. Therefore we say in the case that $P^{\nu}_{\cdot\lambda}$ and Q vanish *that the connexion is a displacement.*

If $P^{\nu}_{\cdot\lambda}$ and Q don't vanish the connexion generally does not determine a displacement; it does only, by means of $d'x^{\nu}$ if a q_{λ} is given. But it determines *always* a *correspondence* between any contravariant pointfield v^{ν} of excess zero and its corresponding field $x^{\mu}\,\nabla_{\mu}\,v^{\nu}$. Vice versa in case q_{λ}

[28] TPZ., pp. 422 and 452.
[28a] TPZ., p. 424.
[29] APD. I, p. 531.

is given, the connexion is determined by the displacement and the correspondence combined. Because of $x^\mu \nabla_\mu \lambda = 0$ for every scalar λ of degree *zero* the projector corresponding to λv^ν is $\lambda x^\mu \nabla_\mu v^\nu$. Therefore the correspondence determines uniquely a *projectivity* for the *spots* of the E_n^*, but the projectivity determines this correspondence only apart from an arbitrary factor. In particular the point corresponding to x^ν is

$$b^\nu = P^\nu_{.\lambda} x^\lambda = Q^\nu_{.\mu} x^\mu. \tag{7.9}$$

If a field q_λ is given, the projective connexion defines uniquely an affine connexion: the affine covariant derivative of an affinor is the affinor-part of the projective covariant derivative of that affinor:

$$\overset{A}{\nabla}_j v^k = A^{ok}_{j\sigma} \nabla_\varrho v^\sigma; \quad \overset{A}{\nabla}_j w_i = A^{\varrho\pi}_{ji} \nabla_\varrho w_\pi, \tag{7.10}$$

or in projective coördinates

$$\begin{aligned} &\overset{A}{\nabla}_\mu v^\nu = A^{\varrho\nu}_{\mu\sigma} \nabla_\varrho v^\sigma \quad \text{for } v^\nu q_\nu = 0, \\ &\overset{A}{\nabla}_\mu w_\lambda = A^{\varrho\pi}_{\mu\lambda} \nabla_\varrho w_\pi \quad \text{for } w_\nu q^\nu = 0. \end{aligned} \tag{7.11}$$

This affine connexion is not a projective one because it is only defined for affinors, but it can be extended to a projective one by the additional conditions

$$\overset{A}{\nabla}_\mu q^\nu = 0, \quad \overset{A}{\nabla}_\mu q_\nu = 0, \quad \overset{A}{Q}_\mu = 0. \tag{7.12}$$

Denoting the parameters of this extended affine connexion by $\overset{A}{\Pi}{}^\nu_{\lambda\mu}$ we have

$$\begin{aligned} X^{\cdots\nu}_{\lambda\mu} &= \Pi^\nu_{\lambda\mu} - \overset{A}{\Pi}{}^\nu_{\lambda\mu} \\ &= q^\nu \nabla'_\mu q_\lambda - \omega^{-1} P'^\nu_{.\lambda} q_\mu - \omega^{-1} q_\lambda Q'^\nu_{.\mu} + \omega^{-1} q_\sigma P^\sigma_{.\varrho} A^\varrho_\lambda q_\mu q^\nu \\ &\quad + \omega^{-1} q_\sigma Q^\sigma_{.\varrho} A^\varrho_\mu q_\lambda q^\nu + \omega^{-2} q_\lambda q_\mu b'^\nu + \omega^{-2} b q_\lambda q_\mu q^\nu \end{aligned} \tag{7.13}$$

[30]

where

$$\begin{aligned} \nabla'_\mu q_\lambda &= A^{\sigma\varrho}_{\mu\lambda} \nabla_\sigma q_\varrho, \quad & P'^\nu_{.\lambda} &= A^{\pi\nu}_{\lambda\sigma} P^\sigma_{.\pi}, \\ Q'^\nu_{.\mu} &= A^{\varrho\nu}_{\mu\sigma} Q^\sigma_{.\varrho}, \quad & b'^\nu &= A^\nu_\varrho b^\varrho \end{aligned} \tag{7.14}$$

are the affinor-parts of the corresponding projectors without a prime, whereas $b = -b^\nu q_\nu$. Moreover we have

$$\overset{A}{\Pi}{}^\nu_{\lambda\mu} = A^{ij\nu}_{\lambda\mu k} \overset{A}{\Pi}{}^k_{ij} + A^\nu_k \partial_\mu A^k_\lambda - q^\nu \partial_\mu q_\lambda, \tag{7.15}$$

where $\overset{A}{\Pi}{}^k_{ij} = \Gamma^k_{ij}$ are the ordinary parameters of the affine connexion.

[30] See APD. I, p. 532.

8. **The projective connexion belonging to the fundamental quadric.** The first geometrical condition which we will impose on the projective connexion is:

1. *The fundamental quadric is covariant constant,* i. e. a covariant point s_μ of excess 0 exists, such that

$$\nabla_\mu G_{\lambda\nu} = s_\mu G_{\lambda\nu}. \tag{8.1}$$

Because of (7.6) we have for every point v^ν of excess 0:

$$-\omega^{-1} v_\nu P^\nu_{.\lambda} = q^\varrho \nabla_\varrho v_\lambda = +\omega^{-1} G_{\lambda\nu} P^\nu_{.\varrho} v^\varrho + s G_{\lambda\nu} v^\nu, \tag{8.2}$$

where

$$s = s_\mu q^\mu \tag{8.3}$$

or

$$P_{(\lambda\mu)} = -\frac{1}{2} s\omega G_{\lambda\mu}. \tag{8.4}$$

Because of (8.2) and (7.9) we have

$$q^\mu \nabla_\mu q_\lambda = \omega^{-2} b_\lambda + s q_\lambda. \tag{8.5}$$

The second geometrical condition is:

2. *The hyperplane of q_ν is invariant under the correspondence* (in other words: the field corresponding with a vectorfield is a vectorfield).

Then from (8.5) follows

$$b'_\lambda = 0. \tag{8.6}$$

But from (8.4) follows by transvection with $q^\lambda q^\mu$:

$$b = -\frac{1}{2} s\omega^2, \tag{8.7}$$

hence:

$$b^\nu = -\frac{1}{2} s\omega^2 q^\nu. \tag{8.8}$$

The third geometrical condition is:

3. *The point corresponding to any point v^ν is situated in the polar hyperplane of v^ν:*[31]

$$v_\nu P^\nu_{.\mu} v^\mu = 0. \tag{8.9}$$

Then from (8.4) we see that s_μ is a vector:

$$s = s_\mu q^\mu = 0, \tag{8.10}$$

[31] It would also be sufficient to require this only for the case v^ν is a *vector.*

hence $P_{\lambda\mu}$ is a *bivector:*

$$P_{(\lambda\mu)} = 0, \qquad b^\nu = \omega P^\nu_{.\lambda} q^\lambda = -\omega P^{\nu}_{\lambda} q^\lambda = 0. \tag{8.11}$$

From (7.3) and (8.1) we get

$$\Pi^\nu_{\lambda\mu} \overset{h}{=} \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} - s_{(\mu} \mathcal{A}^\nu_{\lambda)} + \frac{1}{2} s^\nu G_{\mu\lambda} - 2 S^\nu_{.(\lambda\mu)} + S^{\ldots\nu}_{\lambda\mu}$$
$$\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} = \frac{1}{2} G^{\nu\varrho} (\partial_\lambda G_{\mu\varrho} + \partial_\mu G_{\lambda\varrho} - \partial_\varrho G_{\lambda\mu}), \tag{8.12}$$

and from (8.11)

$$\begin{aligned} &\text{a)} \quad \omega^{-1} Q_{\lambda\mu} = q_{\mu\lambda} + S_{\mu\lambda} + 2 T_{\mu\lambda} - \frac{1}{2} s_\mu q_\lambda, \\ &\text{b)} \quad \omega^{-1} P_{\lambda\mu} = q_{\mu\lambda} + S_{\mu\lambda} - 2 Z_{\mu\lambda}, \\ &\text{c)} \quad s_\lambda \qquad = -4 S_{\lambda\mu\nu} q^\mu q^\nu, \end{aligned} \tag{8.13}$$

$S_{\mu\lambda}$, $Z_{\mu\lambda}$, $T_{\mu\lambda}$[32] and $S'^{\ldots\nu}_{\lambda\mu}$ being affinors defined by

$$\begin{aligned} &\text{a)} \quad S_{\mu\lambda} = A^{\sigma\varrho}_{\mu\lambda} S_{\sigma\varrho\nu} q^\nu = S_{\mu\lambda\nu} q^\nu + \frac{1}{2} s_{[\mu} q_{\lambda]}, \\ &\text{b)} \quad Z_{\mu\lambda} = A^{\sigma\varrho}_{\mu\lambda} S_{\nu[\sigma\varrho]} q^\nu = S_{\nu[\mu\lambda]} q^\nu - \frac{1}{4} s_{[\mu} q_{\lambda]}, \\ &\text{c)} \quad T_{\mu\lambda} = A^{\sigma\varrho}_{\mu\lambda} S_{\nu(\sigma\varrho)} q^\nu = S_{\nu(\mu\lambda)} q^\nu - \frac{1}{4} s_{(\mu} q_{\lambda)}, \\ &\text{d)} \quad S'^{\ldots\nu}_{\lambda\mu} = A^{\pi\varrho\nu}_{\lambda\mu\sigma} S^{\ldots\sigma}_{\pi\varrho}. \end{aligned} \tag{8.14}$$

Hence

$$S^{\ldots\nu}_{\lambda\mu} = S'^{\ldots\nu}_{\lambda\mu} - S_{\lambda\mu} q^\nu - 2 q_{[\lambda} (Z^{\ldots\nu}_{\mu]} + T^{\ldots\nu}_{\mu]}) - \frac{1}{2} s_{[\lambda} q_{\mu]} q^\nu. \tag{8.15}$$

For the affinor-part of $Q_{\lambda\mu}$ we write

$$Q'_{\lambda\mu} = A^{\varrho\sigma}_{\lambda\mu} Q_{\varrho\sigma} = \omega (q_{\mu\lambda} + S_{\mu\lambda} + 2 T_{\mu\lambda}). \tag{8.16}$$

The covariant derivatives of A^ν_λ and $g_{\mu\lambda}$ are of course not zero:

$$\nabla_\mu A^\nu_\lambda = \nabla_\mu q_\lambda q^\nu = \omega^{-1} Q_{\lambda\mu} q^\nu + \omega^{-1} Q^\nu_{.\mu} q_\lambda + s_\mu q_\lambda q^\nu, \tag{8.17}$$

$$\nabla_\mu g_{\lambda\nu} = 2 \omega^{-1} q_{(\nu} Q_{\lambda)\mu} + s_\mu q_\lambda q_\nu + s_\mu g_{\lambda\nu}. \tag{8.18}$$

9. The Riemannian connexion belonging to the fundamental quadric. If ξ^k are the holonomic coördinates of the X_n of § 4, then

$$d\xi^k \overset{h}{=} A^k_\nu dx^\nu; \qquad g_{ij} = A^{\lambda\mu}_{ij} g_{\lambda\mu}. \tag{9.1}$$

[32] The projectors $S_{\mu\lambda}$, $Z_{\mu\lambda}$ and $T_{\mu\lambda}$ differ from those used in GF. I, II by a factor ω.

and the X_n is a V_n with the linear element (6.8). The fundamental tensor g_{ij} defines an ordinary Riemannian connexion $\overset{R}{\nabla}$:

$$\overset{R}{\nabla}_j g_{ik} = 0; \quad \overset{R}{S}_{ij}^{\cdot\cdot k} = 0. \tag{9.2}$$

These equations can of course be translated into projective ones. Now our fourth geometrical condition is:

4. *The affine connexion, determined by the projective connexion (defined in § 7), is for projectors of excess zero identical with the ordinary Riemannian connexion.*

Accordingly we write $\overset{R}{\nabla}$ instead of the $\overset{A}{\nabla}$ of § 7. From (7.11), (8.18) and (9.2) follows

$$s_\mu = 0, \tag{9.3}$$

and from (8.13a), (8.16):

$$Q'_{\lambda\mu} = Q_{\lambda\mu}. \tag{9.4}$$

Equation (7.13) becomes now by means of (8.11), (8.13), (8.16):

$$\begin{aligned} X_{\lambda\mu}^{\cdot\cdot\nu} = \Pi_{\lambda\mu}^{\nu} - \overset{R}{\Pi}_{\lambda\mu}^{\nu} &= +\omega^{-1}(q^\nu Q_{\lambda\mu} - P_{\cdot\lambda}^{\nu} q_\mu - q_\lambda Q_\mu^{\nu}) = \\ &= q^\nu (q_{\mu\lambda} + S_{\mu\lambda} + 2T_{\mu\lambda}) - q_\mu (q_\lambda^{\cdot\nu} + S_\lambda^{\cdot\nu} - 2Z_\lambda^{\cdot\nu}) \\ &\quad - q_\lambda (q_\mu^{\cdot\nu} + S_\mu^{\cdot\nu} + 2T_\mu^{\cdot\nu}). \end{aligned} \tag{9.5}$$

Obviously the Riemannian connexion, considered *projectively* is *not* symmetrical:

$$\overset{R}{S}_{\lambda\mu}^{\cdot\cdot\nu} = q_{\lambda\mu} q^\nu. \tag{9.6}$$

From (9.3), (9.4) follows:

$$S_{\lambda\mu}^{\cdot\cdot\nu} = \overset{R}{S}_{\lambda\mu}^{\cdot\cdot\nu} + X_{[\lambda\mu]}^{\cdot\cdot\nu} = -S_{\lambda\mu} q^\nu - 2q_{[\lambda} Z_{\mu]}^{\cdot\nu} - 2q_{[\lambda} T_{\mu]}^{\cdot\nu}, \tag{9.7}$$

which is because of (8.15) equivalent with

$$S'^{\cdot\cdot\nu}_{\lambda\mu} = 0. \tag{9.8}$$

Instead of the coördinates ξ^k mentioned above we can also use an anholonomic system (k) as defined in § 4; we can choose this system in such a way that

$$\pm g_{11} \overset{o}{=} \pm g_{22} \overset{o}{=} \cdots \overset{o}{=} \pm g_{nn} \overset{o}{=} +1; \quad g_{ij} \overset{o}{=} 0 \text{ for } i \neq j. \tag{9.9}$$

The system (k) is determined but for Lorentz-Fitzgerald transformations. The operator ∂_j is for all holonomic and anholonomic systems defined by the equation

$$\partial_j = A_j^\mu \partial_\mu. \tag{9.10}$$

If the system (k) happens to be holonomic, ∂_j has the same meaning as $\frac{\partial}{\partial \xi^j}$ *if applied to functions of degree zero.* But contrary to $\frac{\partial}{\partial \xi^j}$ the operator ∂_j can be applied to functions of any degree. With respect to any holonomic system (k) the parameters of the Riemannian displacement are the Christoffel-symbols:

$$\overset{R}{\Pi}{}^k_{ij} = \Gamma^k_{ij} \overset{h}{=} \left\{ {k \atop ij} \right\}, \tag{9.11}$$

but with respect to any holonomic or anholonomic system (k) the $\overset{R}{\Pi}{}^k_{ij}$ are

$$\overset{R}{\Pi}{}^k_{ij} = \Gamma^k_{ij} = A^{i\,j\,k'}_{i'\,j'\,k} \left\{ {k' \atop i'\,j'} \right\} + A^k_{l'}\, \partial_j A^{l'}_i = \left\{ {k \atop ij} \right\} - \Omega'^k_{ij} + 2 g^{kl} g_{h(i}\, \Omega'^h_{j)l}, \tag{9.12}$$

where of course the intermediate system (k') has to be holonomic and Ω'^k_{ij} determines the anholonomy of the system (k). If in particular (k) is *orthogonal,* $\left\{ {k \atop ij} \right\} \overset{o}{=} 0$. It is well known that in this latter case the $\overset{R}{\Pi}{}^k_{ij}$ are not symmetrical in i and j:

$$\Gamma^k_{ij} = \overset{R}{\Pi}{}^k_{ij} \overset{o}{=} - \Omega'^k_{ij} + 2 g^{kl} g_{h(i}\, \Omega'^h_{j)l} \tag{9.13}$$

but that the coefficients of rotation of Ricci

$$\gamma_{ikj} \overset{o}{=} \Gamma^l_{ij}\, g_{lk} \tag{9.14}$$

are alternating in ik:

$$\gamma_{(ik)j} \overset{o}{=} 0. \tag{9.15}$$

To deduce the relations between the projective connexion and the Riemannian one we may introduce the anholonomic system of reference (c), $(a, \cdots, g = 0, 1, \cdots, \mathrm{n})$ of § 5 in such a way that with respect to (c) a contra- or covariant vector v^ν or w_λ has the components $(0, v^1, \cdots, v^\mathrm{n})$, or $(0, w_1, \cdots, w_\mathrm{n})$ resp. The components of q^ν and q_λ all vanish but for $q^0 \overset{n}{=} 1$ and $q_0 \overset{n}{=} -1$. If the system (k) is orthogonal, the only non vanishing components of $G_{\lambda\mu}$ are

$$\pm G_{00} \overset{o}{=} \pm G_{11} \overset{o}{=} \cdots \overset{o}{=} \pm G_{\mathrm{nn}} \overset{o}{=} 1. \tag{9.16}$$

The system (c) is determined but for arbitrary holonomic or anholonomic (in the special case just mentioned only orthogonal) transformations of the system (k). The components of any point v^ν or w_λ are

$$\begin{aligned} v^0 &\overset{n}{=} - v^\nu q_\nu = - \omega^{-1} v^\nu x_\nu; & v^k &\overset{n}{=} A^k_\nu v^\nu;\text{[33]} \\ w_0 &\overset{n}{=} w_\nu q^\nu = \omega^{-1} w_\nu x^\nu; & w_i &\overset{n}{=} A^\lambda_i w_\lambda. \end{aligned} \tag{9.17}$$

[33] $v^k = A^k_\nu v^\nu$ instead of $\overset{n}{=}$ would mean that v^ν were a vector, i. e. $v^0 \overset{n}{=} 0$.

Hence the intermediate components of $\mathcal{A}_\lambda^\nu$ are

$$\begin{aligned} \mathcal{A}_\lambda^0 &\overset{n}{=} -q_\lambda = -\omega^{-1}x_\lambda; & \mathcal{A}_\lambda^k &\overset{n}{=} A_\lambda^k; \\ \mathcal{A}_0^\nu &\overset{n}{=} q^\nu = \omega^{-1}x^\nu; & \mathcal{A}_i^\nu &\overset{n}{=} A_i^\nu. \end{aligned} \tag{9.18}$$

According to § 2 (2.5) this system is characterized by

$$\begin{aligned} q' &\overset{n}{=} -1, \\ \Omega_{ab}^c &\overset{n}{=} q_{ab}\,q^c + A_{abk}^{ijc}\,\Omega_{ij}^{\prime k}. \end{aligned} \tag{9.19}$$

If the system (k) is holonomic, we have

$$\Omega_{ab}^c \overset{a}{=} q_{ab}\,q^c. \tag{9.20}$$

The parameters of the Riemannian connexion $\overset{R}{\Pi}{}_{ab}^c$ with respect to this system all vanish but for the $\overset{R}{\Pi}{}_{ij}^k$:

$$\begin{aligned} \overset{R}{\Pi}{}_{0l}^c \overset{n}{=} \overset{R}{\Pi}{}_{a0}^c \overset{n}{=} \overset{R}{\Pi}{}_{ab}^0 &\overset{n}{=} 0, \\ \overset{R}{\Pi}{}_{ij}^k &\overset{n}{=} \Gamma_{ij}^k. \end{aligned} \tag{9.21}$$

The parameters of the projective connexion Π_{ab}^c are

$$\Pi_{ab}^c = \mathcal{A}_{ab\nu}^{\lambda\mu c}\,\Pi_{\lambda\mu}^\nu + \mathcal{A}_\varrho^c\,\partial_b \mathcal{A}_a^\varrho, \tag{9.22}$$

hence the only non-vanishing components of Π_{ab}^c are:

$$\begin{aligned} \Pi_{ij}^k &\overset{n}{=} \Gamma_{ij}^k \overset{a}{=} \left\{ \begin{matrix} k \\ ij \end{matrix} \right\}, \\ \Pi_{0j}^k &\overset{n}{=} \omega^{-1} Q_{\cdot j}^k, \\ \Pi_{i0}^k &\overset{n}{=} \omega^{-1} P_{\cdot i}^k, \\ \Pi_{ij}^0 &\overset{n}{=} \omega^{-1} Q_{ij} \end{aligned} \tag{9.23}$$

in accordance with (9.5).

Now we impose the last geometrical condition.[34]

5. *The point x^ν is always displaced in a direction perpendicular to $d'x^\nu$.*

Hence

$$G_{\lambda\nu}\,(\delta x^\nu)\,d'x^\lambda = (G_{\lambda\nu}\,\nabla_\mu\,x^\nu)\,d'x^\lambda\,d'x^\mu = 0 \tag{9.24}$$

[34] This geometrical condition is superfluous because it follows from the first physical condition. For that reason we will *not* use it in the physical part. We have imposed it here because of its geometrical character and in order to get a more complete division between the geometrical and the physical part of this paper.

for every direction $d'x^\nu$, from which follows using (8.13), (9.3) that

$$Q_{(\lambda\mu)} = T_{\lambda\mu} = 0, \tag{9.25}$$

hence $Q_{\lambda\mu}$ is a *bivector*.

10. **Curvature.** The projector of curvature belonging to $\Pi^\nu_{\lambda\mu}$ is

$$\begin{aligned} N^{\cdots\nu}_{\omega\mu\lambda} &= -2\,\partial_{[\omega}\Pi^\nu_{|\lambda|\,\mu]} - 2\,\Pi^\nu_{\varrho[\omega}\Pi^\varrho_{|\lambda|\,\mu]} + 2\,\Omega^\varrho_{\omega\mu}\Pi^\nu_{\lambda\varrho} \\ &\overset{h}{=} -2\,\partial_{[\omega}\Pi^\nu_{|\lambda|\,\mu]} - 2\,\Pi^\nu_{\varrho[\omega}\Pi^\varrho_{|\lambda|\,\mu]}. \end{aligned} \tag{10.1}$$

From the well known relation[35]

$$N^{\cdots\nu}_{\omega\mu\lambda} - \overset{R}{N}{}^{\cdots\nu}_{\omega\mu\lambda} = -2\overset{R}{\nabla}_{[\omega}X^{\cdots\nu}_{|\lambda|\,\mu]} - 2X^{\cdots\nu}_{\varrho[\omega}X^{\cdots\varrho}_{|\lambda|\,\mu]} + 2\overset{R}{S}{}^{\cdots\varrho}_{\omega\mu}X^{\cdots\nu}_{\lambda\varrho}, \tag{10.2}$$

where $\overset{R}{N}{}^{\cdots\nu}_{\omega\mu\lambda} = K^{\cdots\nu}_{\omega\mu\lambda}$ is the ordinary Riemannian affinor of curvature with respect to homogeneous coördinates, which can be defined similarly to (10.1) by means of the $\overset{R}{\Pi}{}^\nu_{\lambda\mu}$. From (10.2) we get by means of (9.5), (9.7)

$$\begin{aligned} N^{\cdots\nu}_{\omega\mu\lambda} - K^{\cdots\nu}_{\omega\mu\lambda} &= -2\omega^{-1}q^\nu\overset{R}{\nabla}_{[\omega}Q_{|\lambda|\mu]} + 2\omega^{-1}q_{[\mu}\overset{R}{\nabla}_{\omega]}P^\nu_{\cdot\lambda} + 2\omega^{-1}q_\lambda\overset{R}{\nabla}_{[\omega}Q^\nu_{\cdot\mu]} \\ &\quad -2\omega^{-2}Q^\nu_{\cdot[\omega}Q_{|\lambda|\,\mu]} - 2\omega^{-1}q_{\omega\mu}P^\nu_{\cdot\lambda} \\ &\quad -2\omega^{-2}P^\nu_{\cdot\varrho}Q^\varrho_{\cdot[\mu}q_{\omega]}q_\lambda + 2\omega^{-2}P_{\varrho\lambda}Q^\varrho_{\cdot[\omega}q_{\mu]}q^\nu. \end{aligned} \tag{10.3}$$

By contraction this gives:

$$\begin{aligned} N_{\mu\lambda} - K_{\mu\lambda} &= N^{\cdots\varrho}_{\varrho\mu\lambda} - K^{\cdots\varrho}_{\varrho\mu\lambda} \\ &= \omega^{-1}q_\mu\overset{R}{\nabla}_\varrho P^\varrho_{\cdot\lambda} + \omega^{-1}q_\lambda\overset{R}{\nabla}_\varrho Q^\varrho_{\cdot\mu} - q_\lambda\overset{R}{\nabla}_\mu T \\ &\quad +\omega^{-2}(Q_{\varrho\mu}P^\varrho_{\cdot\lambda} + Q^\varrho_{\cdot\mu}Q_{\lambda\varrho}) - 2\omega^{-1}q_{\varrho\mu}P^\varrho_{\cdot\lambda} \\ &\quad +\omega^{-2}q_\mu q_\lambda P_{\sigma\varrho}Q^{\varrho\sigma} - \omega^{-1}TQ_{\lambda\mu}, \end{aligned} \tag{10.4}$$

$$\begin{aligned} N - K &= N_{\mu\lambda}G^{\mu\lambda} - K_{\mu\lambda}g^{\mu\lambda} \\ &= \omega^{-2}Q_{\varrho\sigma}(2P^{\varrho\sigma} + Q^{[\varrho\sigma]}) - 2\omega^{-1}q_{\varrho\sigma}P^{\varrho\sigma} - T^2 + T_{\varrho\sigma}T^{\varrho\sigma}, \end{aligned} \tag{10.5}$$

where

$$T = \omega^{-1}Q^\varrho_{\cdot\varrho} = T_{ij}g^{ij}. \tag{10.6}$$

---

[35] J. A. Schouten, *The invariants of linear connexions with different transformations*, Amsterdam Proc. Kon. Ac. **30** (1926), pp. 276–281, formula (10). We will use the opportunity to correct some misprints in this paper: in the second member of equation (2) a factor σ should be adjoined; the second term of the right member of the first equation of (6) should be $-\frac{4}{n-2}g_{[\omega[\lambda}L_{\mu]\varrho]}g^{\varrho\nu}$; the third term in the right member of (10) should be $+2S^{\cdots\alpha}_{\omega\mu}A^{\cdots\nu}_{\lambda\alpha}$.

The projector $N_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ satisfies the following identities[36]

$$
\begin{aligned}
&\text{a)}\quad x^{\omega} N_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = \nabla_{\mu} P_{\cdot\lambda}^{\nu},\\
&\text{b)}\quad N_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} x^{\lambda} = -2\nabla_{[\omega} Q_{\cdot\mu]}^{\nu} + 2S_{\omega\mu}^{\cdot\cdot\varrho} Q_{\cdot\varrho}^{\nu},\\
&\text{c)}\quad N_{[\omega\mu\lambda]}^{\cdot\cdot\cdot\nu} = 2\nabla_{[\omega} S_{\mu\lambda]}^{\cdot\cdot\nu} + 4S_{[\omega\mu}^{\cdot\cdot\varrho} S_{\lambda]\varrho}^{\cdot\cdot\nu},\\
&\text{d)}\quad \nabla_{[\varkappa} N_{\omega\mu]\lambda}^{\cdot\cdot\cdot\nu} = -2S_{[\varkappa\omega}^{\cdot\cdot\varrho} N_{\mu]\varrho\lambda}^{\cdot\cdot\cdot\nu}
\end{aligned}
\tag{10.7}
$$

and because of (8.1), (9.3)

$$
N_{\omega\mu(\lambda\nu)} = (\nabla_{[\omega} s_{\mu]} - S_{\omega\mu}^{\cdot\cdot\varrho} s_{\varrho}) G_{\lambda\nu} = 0. \tag{10.8}
$$

From (10.7 d) follow by contraction the identities

$$
\begin{aligned}
\nabla_{\varrho} N_{\omega\mu\lambda}^{\cdot\cdot\cdot\varrho} - 2\nabla_{[\omega} N_{\mu]\lambda} &= -4S_{\varkappa[\omega}^{\cdot\cdot\varrho} N_{\mu]\varrho\lambda}^{\cdot\cdot\cdot\varkappa} - 2S_{\omega\mu}^{\cdot\cdot\varrho} N_{\varrho\lambda},\\
\nabla_{\varrho}\left(N_{\omega}^{\cdot\varrho} - \frac{1}{2} N \mathcal{A}_{\omega}^{\varrho}\right) &= -2S_{\omega\sigma}^{\cdot\cdot\varrho} N_{\varrho}^{\cdot\sigma} - S_{\mu\sigma\varrho} N_{\omega}^{\cdot\varrho\mu\sigma}.
\end{aligned}
\tag{10.9}
$$

11. **Geometrical properties.** We can decompose the quantities $Q_{\lambda\mu}$, $P_{\lambda\mu}$ and s_{μ} into:

$$
Q_{\lambda\mu} = Q'_{\lambda\mu} - q_{\varrho} Q_{\cdot\sigma}^{\varrho} A_{\mu}^{\sigma} q_{\lambda} - \omega^{-1} b'_{\lambda} q_{\mu} - \omega^{-1} b q_{\lambda} q_{\mu}; \tag{11.1}
$$

$$
P_{\lambda\mu} = P'_{\lambda\mu} - q_{\varrho} P_{\cdot\sigma}^{\varrho} A_{\mu}^{\sigma} q_{\lambda} - \omega^{-1} b'_{\lambda} q_{\mu} - \omega^{-1} b q_{\lambda} q_{\mu}; \tag{11.2}
$$

$$
s_{\mu} = s'_{\mu} - s q_{\mu}. \tag{11.3}
$$

The connexion with the parameters (9.23) has the following mutually independent geometrical properties:

Correspondence:

a) The field $x^{\mu}\nabla_{\mu} v^{\nu}$ corresponding to any vectorfield is a vectorfield. This is equivalent with

$$
q_{\varrho} P_{\cdot\sigma}^{\varrho} A_{\mu}^{\sigma} = 0. \tag{11.4}
$$

b) The vector corresponding to any vector v^{ν} is perpendicular to v^{ν}:

$$
P'_{(\lambda\mu)} = 0. \tag{11.5}
$$

Because of (8.4) and (8.7) this leads to

$$
s = 0; \quad b = 0. \tag{11.6}
$$

Because of $b = 0$ the field corresponding to *any* pointfield is a vectorfield (comp. a)).

[36] Comp. TPZ., pp. 422, 452.

c) The vector corresponding to x^ν is zero:

$$b'^\nu = 0. \tag{11.7}$$

Hence $P_{\lambda\mu} = P'_{[\lambda\mu]}$ is a bivector.

Displacement:

d) The covariant differential of x^ν is always a vector:

$$q_\varrho Q^\varrho_{.\sigma} A^\sigma_\mu = 0. \tag{11.8}$$

e) The point x^ν is displaced in a direction perpendicular to $d'x^\nu$:

$$T_{\lambda\mu} = Q'_{(\lambda\mu)} = 0. \tag{11.9}$$

Hence $Q_{\lambda\mu} = Q'_{[\lambda\mu]}$ is a bivector.

f) The covariant differential of x_λ is a covariant vector:

$$s'_\mu = 0. \tag{11.10}$$

Equivalent to e) is

e*) The covariant differential of a (contravariant) vector in the direction of this vector is a vector.

II. Physical part.

The projective connexion we have found in the first part of this paper is not wholly determined. It depends, apart from the quadric and the contactpoint, on two bivectors $P_{\lambda\mu}$ and $Q_{\lambda\mu}$, which may still be arbitrarily chosen. We might fix them by means of geometrical conditions, but we prefer not to do this for the following reason: the geometrical part as it is now is common to the several prominent unification-theories, if interpreted projectively. Moreover in all these theories the bivectors $P_{\lambda\mu}$ and $Q_{\lambda\mu}$ are proportional to $q_{\lambda\mu}$:

$$-\omega^{-1} P_{\lambda\mu} = p q_{\lambda\mu}; \quad -\omega^{-1} Q_{\lambda\mu} = q q_{\lambda\mu}; \tag{II.1}$$

where p and q are real numerical constants. *But the theories differ in the values of these constants.* As

$$q_{\lambda\mu} \overset{h}{=} \partial_{[\lambda} q_{\mu]} \overset{h}{=} \omega^{-1} \partial_{[\lambda} (G_{\mu]\nu} x^\nu), \tag{II.2}$$

$q_{\lambda\mu}$ and therefore the whole projective connexion by virtue of (II.1) depend only on *the quadric and the contactpoint.*

In the physical part we will suppose

$$n = 4 \tag{II.3}$$

and we will identify the *spots* (not the points!) *of the* H_4 with the *events* of space-time. Further we will suppose the index of $G_{\lambda\mu}$ to be 3.[37] Hence the only non-vanishing components of G_{ab} and g_{ij} are, if the system (k) is orthogonal:

$$G_{11} \overset{o}{=} G_{22} \overset{o}{=} G_{33} \overset{o}{=} 1; \qquad G_{00} \overset{o}{=} G_{44} \overset{o}{=} -1;$$
$$g_{11} \overset{o}{=} g_{22} \overset{o}{=} g_{33} \overset{o}{=} 1; \qquad g_{44} \overset{o}{=} -1.$$

12. **Geodesics.**[38] In our projective geometry geodesics in the ordinary sense do not exist, because displacement of a vector does not in general lead to a vector but to a point. But instead of displacing a linear element as in the ordinary case, we can displace an arbitrary point on the line of the element. Then we get in the neighboring E_n^* a line through the contactpoint and the displaced point and we can go on on this line with the displaced point etc. The shape of the resulting curve depends obviously on the choice of the initial point on the tangent (and not only on the *direction* of the tangent) because the contactpoint is generally not covariant constant. Now a general point p^ν on the line of any linear element $d\xi^k$ is given by the equation

$$\begin{aligned} p^0 &\overset{4}{=} -p^\nu q_\nu, \\ p^k &\overset{4}{=} p^0 \frac{l}{c} \frac{d\xi^k}{d\tau}; \end{aligned} \qquad d\tau^2 = -\frac{1}{c^2} ds^2, \tag{12.1}$$

or, in general coördinates

$$p^\nu = p^0 \left(\frac{l}{c} i^\nu + q^\nu \right); \qquad i^\nu = \frac{dx^\nu}{d\tau} = A_k^\nu \frac{d\xi^k}{d\tau}. \tag{12.2}$$

p^0 is the weight and il the length of the vector between q^ν and p^ν, which will be taken timelike. The differential equation of the geodesic, got by displacing p^ν in the direction of $d\xi^k$ is obtained by requiring $\frac{\delta p^\nu}{d\tau}$ to be coincident with p^ν:

$$\frac{\delta p^\nu}{d\tau} = \sigma p^\nu; \qquad \frac{\delta}{d\tau} = i^\mu \nabla_\mu. \tag{12.3}$$

By multiplying p^ν with $e^{-\int \sigma d\tau}$ we see that the right side of (12.3) may without restriction be supposed to be zero

$$\frac{\delta p^\nu}{d\tau} = 0, \tag{12.4}$$

[37] For the reason see GF. III. If ω^2 is chosen negative, the index is $+4$. This case is dealt with in GF. IV: J. A. Schouten, Zur generellen Feldtheorie; Ableitung des Impuls-energiestromprojektors aus einem Variationsprinzip, Z. f. Ph. **81** (1933), pp. 129–138.

[38] In this paragraph we will *not* suppose the last geometrical condition to be fulfilled, because it follows automatically from the first physical one.

in which case both the weight of p^ν and the length l will satisfy automatically some differential equation:

$$\frac{\delta}{d\tau}\log p^0 = -\left(\frac{l}{c}\,i^\nu + q^\nu\right)\frac{\delta q_\nu}{d\tau} = -\frac{l}{c}\,\omega^{-1}\,Q_{\lambda\mu}\,i^\lambda\,i^\mu. \tag{12.5}$$

$$\frac{\delta}{d\tau}\log(l^2+1) = 2\left(\frac{l}{c}\,i^\nu + q^\nu\right)\frac{\delta q_\nu}{d\tau} = 2\frac{l}{c}\,\omega^{-1}\,Q_{\lambda\mu}\,i^\lambda\,i^\mu. \tag{12.6}$$

The equation for p^0 being homogeneous, the geodesic will be *independant* of the initial *weight*, but it will depend generally on the initial *length* as the contactpoint generally is not covariant constant.

The equation (12.4) can be expressed in the Riemannian derivatives of i^ν by means of (12.5, 6) and (9.5):

$$\frac{\overset{R}{\delta}\, i^\nu}{d\tau} = -\frac{c}{l}\,\omega^{-1}\,Q^\nu_{\cdot\mu}\,i^\mu - \frac{2}{cl}\,\omega^{-1}\,Q_{\varrho\sigma}\,i^\varrho\,i^\sigma\,i^\nu, \tag{12.7}$$

or written in full:

$$\frac{d}{d\tau}\,\frac{(d\xi)^k}{d\tau} + \Gamma^k_{ij}\,\frac{(d\xi)^i}{d\tau}\,\frac{(d\xi)^j}{d\tau} = \\ = \frac{c}{l}\,(q^k_{\cdot j} + S^k_{\cdot j} - 2\,T^k_{\cdot j})\,\frac{(d\xi)^j}{d\tau} - \frac{2}{cl}\,T_{ij}\,\frac{(d\xi)^i}{d\tau}\,\frac{(d\xi)^j}{d\tau}\,\frac{(d\xi)^k}{d\tau}. \tag{12.8}$$

Now it is well known that the equation of the world-lines of points with mass m and charge e moving in a field with a gravitational tensor g_{ij} and an electromagnetic bivector

$$F_{ij} = 2\,\overset{R}{\nabla}_{[i}\,\varphi_{j]} \overset{a}{=} 2\,\partial_{[i}\,\varphi_{j]}\,^{39} \tag{12.9}$$

is

$$\frac{d}{d\tau}\,\frac{(d\xi)^k}{d\tau} + \Gamma^k_{ij}\,\frac{(d\xi)^i}{d\tau}\,\frac{(d\xi)^j}{d\tau} = -\frac{e}{mc}\,F_j^{\cdot k}\,\frac{(d\xi)^j}{d\tau}. \tag{12.10}$$

Hence if we impose the first physical condition:

1. *The world-lines of charged masses with different* $\frac{m}{e}$ *are the geodesics for different* l,

then the vectors in the right members of (12.8) and (12.10) must be equal for every l and every i^ν. Hence l must be proportional to $\frac{mc}{e}$:

$$l = k\,\frac{mc}{e} \tag{12.11}$$

[39] $\varphi_1, \varphi_2, \varphi_3$ are the components of the magnetic potential vector; $-\varphi_4$ is the electric potential.

and

$$q_{kj}+S_{kj}-\frac{k}{c}F_{kj}-2\left(T_{kj}+\frac{1}{c^2}T_{hl}\,i^h\,i^l\,g_{kj}\right) \tag{12.12}$$

vanishes by transvection with i^ν. By transvecting (12.12) with two mutually orthogonal vectors $\overset{i}{1}{}^\nu = i^\nu$, $\overset{i}{2}{}^\nu$ we see that

$$T_{ij}=0; \tag{12.13}$$

hence by (8.16)

$$Q_{ij}=Q_{[ij]}=-\omega\,(q_{ij}+S_{ij}), \tag{12.13a}$$

and

$$\boxed{\frac{k}{c}F_{ij}=-\omega^{-1}Q_{ij}}. \tag{12.14}$$

From (12.13a) and (12.6) follows that l and therefore k is constant; the latter constant has the dimension $M^{-1/2}L^{1/2}$ and is independent of l and i^ν, hence of the choice of the particular geodesic.[39a] Because of (12.5), (12.13a) implies that also the *weight* of the displaced point is a constant along the geodesic, which can therefore without loss of generality be taken equal to $\frac{e}{k}$:

$$p^\nu = m i^\nu + \frac{e}{k}q^\nu. \tag{12.15}$$

13. **The point of energy and momentum.** The electromagnetical bivector F_{ij} is related physically with the potential vector φ_i by

$$F_{ij} \overset{a}{=} 2\,\partial_{[i}\varphi_{j]}, \tag{13.1}$$

or, written in homogeneous coördinates

$$F_{\lambda\mu} \overset{h}{=} 2\,\partial_{[\lambda}\varphi_{\mu]}, \tag{13.2}$$

and geometrically with the covariant point of the plane at infinity by

$$F_{\lambda\mu}=-\frac{c}{k}\omega^{-1}Q_{\lambda\mu}=\frac{c}{k}\nabla_{[\lambda}q_{\mu]}. \tag{13.3}$$

Now let us suppose that $Q_{\lambda\mu}$ were proportional to $q_{\lambda\mu}$ with a *constant* factor of proportionality:

$$-\omega^{-1}Q_{\lambda\mu}=\nabla_{[\lambda}q_{\mu]}=q\,q_{\lambda\mu}\overset{h}{=}\partial_{[\lambda}q\,q_{\mu]}. \tag{13.4}$$

Then the difference of φ_μ and $q\frac{c}{2k}q_\mu$ would be the (projective) gradient of a scalar which we will write $\frac{c\,\omega\,q}{2k}\partial_\mu \log X$:

$$\varphi_\mu=\frac{qc}{2k}(q_\mu+\omega\,\partial_\mu \log X). \tag{13.5}$$

[39a] l is a numerical constant and has no physical dimension.

As φ_μ is a vector, transvection with q_μ shows that

$$x^\mu \partial_\mu \log X = 1, \tag{13.6}$$

hence X is homogeneous of degree (and excess) *one*.

Now $m i^\nu$ is the vector of kinetic energy and momentum and it is well known that $\frac{e}{c}\varphi^\nu$ may be considered as a vector of potential energy and momentum.[40] But by means of the equation (13.5) we could replace the *vector* $\frac{e}{c}\varphi_\lambda$ (which is only determined but for an additive gradient) by the *point* $q\frac{e}{2k}q_\lambda$ which is totally determined. Hence we might consider this point as the *"true"* point of potential energy and momentum and look upon $\frac{e}{c}\varphi_\lambda$ as the *"apparent"* vector of potential energy and momentum which appears only if we choose the gradient which may be added to $q\frac{e}{2k}q_\lambda$ in such a way that the sum is a *vector*. From this point of view the point

$$m i^\nu + q\frac{e}{2k}q^\nu \tag{13.7}$$

would be the *point of total energy and momentum*. Contrary to φ_λ the points appearing in this expression are *wholly determined*. Now we drop the assumption (13.4) and instead we impose the *second* physical condition:

2. *The point* $p_\nu = m i_\nu + \frac{e}{k}q_\nu$, *which is covariant constant along every geodesic, is the point of total energy and momentum, i. e. it differs from* $m i_\nu + \frac{e}{c}\varphi_\nu$ *only by a projective gradient:*

$$\varphi_\nu = \frac{c}{k}(q_\nu + \omega\,\partial_\nu \log X). \tag{13.8}$$

We get from this condition, using the fact that k is a constant (§ 12):

$$\frac{c}{k}\partial_{[\lambda} q_{\mu]} - \partial_{[\lambda}\varphi_{\mu]} \overset{h}{=} 0, \tag{13.9}$$

or

$$\boxed{F_{\lambda\mu} = \frac{2c}{k}q_{\lambda\mu}}. \tag{13.10}$$

[40] If the vector φ^ν is timelike and taken as time-axis, its value becomes $\frac{1}{c}$ times the ordinary potential energy.

With the aid of the condition (12.14) for the equation of the geodesics we get from (13.10)

$$-\omega^{-1} Q_{\lambda\mu} = 2 q_{\lambda\mu}. \tag{13.11}$$

Hence our provisional assumption (13.4) follows from the second physical condition and

$$q = 2. \tag{13.12}$$

The equation (12.4) which is at once *equation of motion* and *theorem of conservation of energy and momentum* allows another geometrical interpretation. According to E. Cartan[41] we can consider the displacement along a (non-closed) curve as a continuous "developing" of the local spaces on each other. According to this idea the local spaces belonging to the different points of the curve, become identified with each other; to a pointfield along the curve belongs then and only then a uniquely determined immovable point v^ν of the image-E_4^*, if it is covariant constant $\left(\text{i. e. } \frac{\delta v^\nu}{d\tau} = 0\right)$.

If we apply this to the local spaces of a geodesic, we see that the fundamental quadric and the point of energy and momentum are covariant constant. If we introduce into the E_4^* instead of the euclidean metric belonging to q^ν either the euclidean metric belonging to p^ν (i. e. with p_ν as hyperplane at infinity and p^ν as centre of the quadric) or the Lobatschevskian metric belonging to the quadric, then the motion of the material point in space-time maps itself in E_4^* as a *rotation*

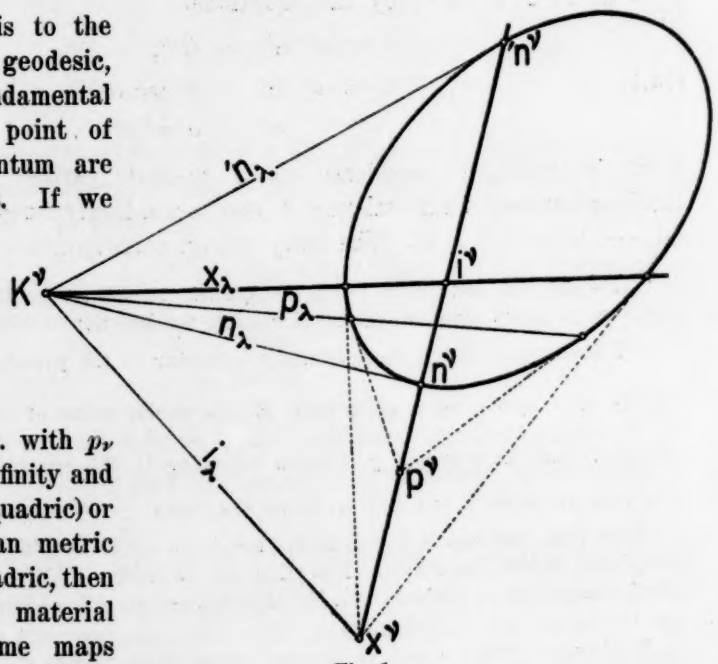

Fig. 1.

[41] Comp. for instance E. Cartan, *Sur les variétés à connexion projective*, Bull. Soc. Math. de France **52** (1924), pp. 205–241.

(of both q^ν and i^ν) around the fixed point p^ν (see Fig. 1).[42] Writing (12.4) in the form

$$\frac{\delta}{d\tau} m i^\nu = -\frac{\delta}{d\tau}\frac{e}{k} q^\nu = -\frac{e}{k} Q^\nu{}_{.\mu}\, i^\mu = k^\nu \tag{13.13}$$

we see that the four-dimensional vector (point at infinity) of force k^ν lies in the polar plane of the line through i^ν and q^ν and signifies apart from constant factors the *velocity* of i^ν and q^ν under the rotation mentioned above.

According to (6.11) the "value" of the point $\frac{m}{c} i^\nu$ or mci^ν is the proper mass m or the proper energy mc^2 resp. of the material point. Therefore perhaps it would be plausible to consider the value of the remaining part of cp^ν, viz. $\left\| \frac{ce}{k} q^\nu \right\| = \frac{ce}{k}$ as a kind of *electrogravitational* proper energy of the material point.[43]

For the electron this energy is about 10^{20} times as large as the mass-energy mc^2 but it does not give a contribution to the inertia.

14. **Spinquantities.**[44,44a] We introduce the Dirac-numbers α^c, $(a, \cdots, g = 0, 1, 2, 3, 4)$[45] (by the equations)

$$\begin{aligned} \alpha^{(a}\,\alpha^{b)} &= G^{ab}, \\ (\alpha^a\,\alpha^b)\,\alpha^c &= \alpha^a\,(\alpha^b\,\alpha^c), \\ \alpha^0 &\overset{o}{=} \alpha^1\,\alpha^2\,\alpha^3\,\alpha^4. \end{aligned} \tag{14.1}$$

Each α^c may be considered as a co-contravariant spinor (affinor of local spinspace) with valence 2 and accordingly may be written $\alpha^{cC}_{\cdot\cdot A}$, $(A, \cdots, G = 5, 6, 7, 8)$. The unity spinor, corresponding to 1 be written α^C_A.

[42] The points n^ν and $'n^\nu$ in which the quadric intersects the straight line through q^ν and p^ν are in reality complex conjugated, because the direction of this straight line is timelike.

[43] If the integral of the energy-density belonging to the potential $\frac{e}{4\pi r}$ over the space outside of a sphere has a given value E, the double radius of the sphere is $\frac{e^2}{4\pi E}$. On the other hand the radius of gravitation belonging to this energy E is $\frac{k}{8\pi c^2} E$. These two radii are equal if and only if E has the value $\frac{ec}{k}$ mentioned above.

[44] See D E., and also J. A. Schouten, *Über die in der Wellengleichung verwendeten hyperkomplexen Zahlen,* Amsterdam, Proc. Kon. Ak. **32** (1929), pp. 105–108. *Die Darstellung der Lorentzgruppe in der komplexen E_2, abgeleitet aus den Diracschen Zahlen,* ibid. **33** (1930), pp. 189–197.

[44a] (February 1933.) A more recent development of the anylisis of spinquantities, including the cases where the index of $G_{\lambda\mu}$ is $+4$ $(\omega^2 < 0)$ and $+1$ is to be found in GF. IV and in GF.V: J. A. Schouten, Zur generellen Feldtheorie. Raumzeit und Spinraum, Z. f. Ph. **81** (1933), pp. 405–417.

[45] In §§ 14, 15, 16 the system (k) is always taken as an *orthogonal* one.

α^0 determines two invariant planes E_2 and $\overline{E}_2$ in spinspace.[46] Every spinor can be decomposed with respect to these planes f. i.

$$\begin{aligned} \alpha &= \iota + \overline{\iota}, \\ \alpha^0 &\overset{o}{=} i\iota + i\overline{\iota}, \\ \alpha^k &\overset{o}{=} \beta^k + \overline{\beta}^k, \quad \beta^k = \iota\,\alpha^k\,\overline{\iota}, \quad \overline{\beta}^k = \overline{\iota}\,\alpha^k\,\iota. \end{aligned} \tag{14.2}$$

ι and $\overline{\iota}$ are the unity spinors in E_2 and $\overline{E}_2$. The coördinates in these planes can be chosen in such a way that the components of β^k and $\overline{\beta}^k$ and of ι and $\overline{\iota}$ are complex conjugate. We write a contravariant spinvector in the first plane $v^{\mathfrak{a}}$, $(\mathfrak{a}, \cdots, \mathfrak{g} = 5, 6)$ and the spinvector with complex conjugate components in the other plane $v^{\mathfrak{G}}$, $(\mathfrak{A}, \cdots, \mathfrak{G} = \overline{5}, \overline{6})$. The components of ι, $\overline{\iota}$, β^k and $\overline{\beta}^k$ are

$$\begin{aligned} &\iota^{\mathfrak{a}}, \quad \beta^{k\mathfrak{c}}_{\cdots\mathfrak{A}}, \\ &\overline{\iota}^{\mathfrak{G}}, \quad \overline{\beta}^{k\mathfrak{G}}_{\cdots\mathfrak{a}}. \end{aligned} \tag{14.3}$$

β^k and $\overline{\beta}^k$ lie with one index in one plane and with the other one in the other plane. Obviously[47]

$$\begin{aligned} \beta^{(ij)} &= g^{ij}\,\iota, \quad & \beta^{ij} &= \beta^i\,\overline{\beta}^j, \\ \overline{\beta}^{(ij)} &= g^{ij}\,\overline{\iota}, \quad & \overline{\beta}^{ij} &= \overline{\beta}^i\,\beta^j. \end{aligned} \tag{14.4}$$

A covariant differentiation for contravariant spinvector-densities of weight $+\frac{1}{4}$ and excess (and degree) ε and covariant ones of weight $-\frac{1}{4}$ and of excess (and degree) ε is given by[48]

$$\begin{aligned} \nabla_\mu X^C &= \partial_\mu X^C + \Lambda^C_{A\mu}\,X^A + \varepsilon\,Q_\mu\,X^C, \\ \nabla_\mu X_C &= \partial_\mu X_A - \Lambda^C_{A\mu}\,X_C + \varepsilon\,Q_\mu\,X_A, \end{aligned} \tag{14.5}$$

where

$$\Lambda^C_{C\mu} = 0 \tag{14.6}$$

and $\Lambda^C_{A\mu}$ is transformed as follows under arbitrary transformations of coördinates in space-time and spinspace, which are totally independant of each other:

$$\Lambda^{C'}_{A'\mu'} = \mathfrak{A}^{\mu}_{\mu'}\left\{\alpha^{A\,C'}_{A'\,C}\,\Lambda^C_{A\mu} + \alpha^{C'}_B\,\partial_\mu\,\alpha^B_{A'} - \left(\frac{1}{4}\,\alpha^{D'}_B\,\partial_\mu\,\alpha^B_{D'}\right)\alpha^{C'}_{A'}\right\}. \tag{14.7}$$

Such a covariant differentiation of spinquantities may be fixed by the condition that the covariant differential of $\alpha^{cC}_{\cdots A}$ vanish. It is well known,

[46] Comp. DE., p. 251.
[47] DE., p. 252.
[48] DE., p. 280.

that because of $\partial_b G^{ac} \overset{o}{=} 0$ the coördinates in the local spinspaces can be chosen in such a way that the ordinary differential of $\alpha^{cC}_{\cdot\cdot A}$ vanishes. If Λ^C_{Ab} are the parameters of the covariant differentiation of contravariant spin-vectordensities of weight $\frac{1}{4}$ with respect to the system (c) in space-time and with respect to *these special* coördinates in spinspace, then we have

$$(14.8)\qquad \nabla_b \alpha^{cC}_{\cdot\cdot A} = \partial_b \alpha^{cC}_{\cdot\cdot A} + \Pi^c_{ab} \alpha^{aC}_{\cdot\cdot A} + \Lambda^C_{Bb} \alpha^{cB}_{\cdot\cdot A} - \alpha^{cC}_{\cdot\cdot B} \Lambda^B_{Ab} = 0.$$

Now because of (14.6) for every choice of b the sedenion Λ^C_{Ab} may be written as a sum of multiples of the five sedenions $\alpha^{cC}_{\cdot\cdot A}$ and the ten sedenions $\alpha^{[a\,c]\,C}_{\cdot\cdot\cdot A}$:

$$(14.9)\qquad \Lambda^C_{Ab} = \left(\frac{1}{4} \Lambda^D_{Eb} \alpha^{\cdot E}_{c\cdot D}\right) \alpha^{cC}_{\cdot\cdot A} + \left(\frac{1}{8} \Lambda^D_{Eb} \alpha^{\cdot\cdot\cdot E}_{[c\,a]\cdot D}\right) \alpha^{[a\,c]\,C}_{\cdot\cdot\cdot A}.$$

Substituting (14.9) in (14.8) we get

$$(14.10)\qquad \Lambda^C_{Ab} \overset{*}{=} -\frac{1}{4} \Pi^c_{ab} \alpha^{a\cdot C}_{c\cdot A}.$$

From this equation follow the values of the parameters $\Lambda^{C'}_{A'\mu}$ with respect to any system of coördinates (C') in spinspace by means of (14.7).[48a] For the $\overset{R}{\Lambda}{}^C_{Ab}$ which belong to the Riemannian connexion we find in the same way

$$(14.11)\qquad \overset{R}{\Lambda}{}^C_{Ab} \overset{*}{=} -\frac{1}{4} \overset{R}{\Pi}{}^c_{ab} \alpha^{a\cdot C}_{c\cdot A}$$

hence by means of (9.5):

$$(14.12)\qquad \Lambda_\mu - \overset{R}{\Lambda}_\mu = -\frac{1}{4} X^{\cdot\cdot\nu}_{\lambda\mu} \alpha^{\lambda}_{\cdot\nu} \overset{4}{=} -\frac{1}{4} \omega^{-1}(-2 Q_{\lambda\mu} \alpha^{\lambda 0} - q_\mu P_{\nu\lambda} \alpha^{\lambda\nu}).$$

From (14.11) we see that $\overset{R}{\Lambda}{}^C_{Ab} \overset{*}{=} 0$ in cartesian coördinates in case there is no gravitation. By decomposition of (14.10) we get

$$(14.13)\qquad \begin{aligned} \Lambda^{\mathfrak{c}}_{ab} &\overset{*}{=} -\frac{1}{4} \Pi^k_{ib} \beta^{i\cdot\mathfrak{c}}_{\cdot k\cdot a}, \\ \Lambda^{\mathfrak{C}}_{\mathfrak{A}b} &\overset{*}{=} -\frac{1}{4} \Pi^k_{ib} \bar{\beta}^{i\cdot\mathfrak{C}}_{\cdot k\cdot\mathfrak{A}}, \\ \Lambda^{\mathfrak{c}}_{\mathfrak{A}b} &\overset{*}{=} +\frac{1}{2} i \Pi^i_{0b} \beta^{\cdot\mathfrak{c}}_{i\cdot\mathfrak{A}}, \\ \Lambda^{\mathfrak{C}}_{ab} &\overset{*}{=} -\frac{1}{2} i \Pi^i_{0b} \bar{\beta}^{\cdot\mathfrak{C}}_{i\cdot a}. \end{aligned}$$

[48a] The general formula is

$$\Lambda^C_{A\mu} = -\frac{1}{4} \Pi^\nu_{\lambda\mu} \alpha^{\lambda\cdot C}_{\nu\cdot A} + \frac{1}{4} \alpha^{\nu C}_{\cdot\cdot B} \partial_\mu \alpha^{\cdot B}_{\nu\cdot A}$$

In GF. III, p. 661 (9.7, 9, 13, 14) and note [2] the sign is erroneous. (February 1933.)

A *pseudovectordensity* is a vectordensity which is determined but for an arbitrary scalar factor. The parameters of the covariant differentiation of contravariant pseudovectordensities of weight $+\frac{1}{4}$ and degree 0 have the form

$$\Lambda^{C}_{A\mu} + \frac{ei}{ch}\alpha^{C}_{A}\varphi^{*}_{\mu}, \tag{14.14}$$

φ^{*}_{μ} being a covariant point transforming in the following way

$$'\varphi^{*}_{\mu} = \varphi^{*}_{\mu} + \frac{ch}{e}\partial_{\mu}\varphi \tag{14.15}$$

if the vectordensity gets a factor $e^{i\varphi}$.

15. **Dirac's equations.** Dirac's equations for Riemannian space-time are[49]

$$\beta^{jc}_{\cdot\cdot\mathfrak{B}}\left\{\frac{h}{i}\left(\partial_j\psi^{\mathfrak{B}} + \overset{R}{\Lambda}{}^{\mathfrak{B}}_{\mathfrak{A}j}\psi^{\mathfrak{A}}\right) - \frac{e}{c}\varphi_j\psi^{\mathfrak{B}}\right\} - mc\psi^{c} = 0, \tag{15.1}$$

$$\bar{\beta}^{j\mathfrak{C}}_{\cdot\cdot b}\left\{\frac{h}{i}\left(\partial_j\psi^{b} + \overset{R}{\Lambda}{}^{b}_{aj}\psi^{a}\right) - \frac{e}{c}\varphi_j\psi^{b}\right\} + mc\psi^{\mathfrak{C}} = 0. \tag{15.2}$$

They can be combined into the one more simple equation

$$\left\{\alpha^{i}\left(\frac{h}{i}\partial_i + \frac{h}{i}\overset{R}{\Lambda}_{i} - \frac{e}{c}\varphi_i\right) - mci\alpha^{0}\right\}\psi \overset{4}{=} 0. \tag{15.3}$$

This equation can be brought into the simple form[50]

$$\frac{h}{i}\alpha^{\mu}\overset{R}{\nabla}_{\mu}\psi = 0 \tag{15.4}$$

by taking

$$\varphi^{*}_{i} \overset{o}{=} \varphi_i; \qquad \varphi^{*}_{0} \overset{o}{=} i\frac{mc^{2}}{e}. \tag{15.5}$$

We now impose the third physical condition

3. *The equation*

$$\frac{h}{i}\alpha^{\mu}\nabla_{\mu}\psi = 0, \tag{15.6}$$

ψ being a pseudovectordensity of weight $+\frac{1}{4}$, is identical with Dirac's equation for Riemannian space-time.

[49] DE., p. 265; contrary to that paper e denotes here the positive charge, hence $e<0$ for electrons.

[50] DE., p. 283.

Now (15.6) can be written, using (14.12)

$$0 = \frac{h}{i}\alpha^\mu \overset{R}{\nabla}_\mu \psi + \frac{h}{i}\alpha^\mu (\Lambda_\mu - \overset{R}{\Lambda}_\mu)\psi \overset{4}{=}$$

(15.7)

$$\overset{4}{=} \frac{h}{i}\alpha^\mu \overset{R}{\nabla}_\mu \psi + \frac{1}{4}\frac{h}{i}\omega^{-1}(P_{\lambda\mu} - 2Q_{\lambda\mu})\alpha^{\lambda\mu 0}\psi$$

hence the equation (15.6) is equivalent with Dirac's equation if and only if

(15.8) $$\boxed{P_{\lambda\mu} = 2Q_{\lambda\mu}}$$ [51]

which is equivalent because of (8.13) and (12.13a) to

(15.9) $$Z_{ij} = \frac{1}{2}\omega^{-1}Q_{ij} = -\frac{1}{2}(q_{ij} + S_{ij}).$$

Using (8.13), (12.14), (13.10) and (15.8) we see that all bivectors which occur in our theory are now proportional to $q_{\lambda\mu}$:

(15.10) $$\omega^{-1}P_{\lambda\mu} = -pq_{\lambda\mu}; \qquad \omega^{-1}Q_{\lambda\mu} = -qq_{\lambda\mu}.$$

(15.11) $$S_{\lambda\mu} = (q-1)q_{\lambda\mu}; \qquad Z_{\lambda\mu} = \frac{1}{2}(q-p)q_{\lambda\mu}.$$

(15.12) $$F_{\lambda\mu} = q\frac{c}{k}q_{\lambda\mu},$$

with

(15.13) $$p = 2q = 4.$$

From (15.12) and $\partial_{[\lambda} q_{\mu\nu]} \overset{h}{=} 0$ follows, q being a constant, of course Maxwell's second equation:

(15.14) $$\overset{R}{\nabla}_{[\lambda} F_{\mu\nu]} = 0.$$

The connexion for quantities of excess zero is now wholly determined. It is remarkable that because of (9.5) and (15.9)

(15.15) $$S_{[\lambda\mu\nu]} = \overset{R}{S}_{[\lambda\mu\nu]}.$$

Now from (14.12) follows

(15.16) $$\alpha^\mu \Lambda_\mu - \alpha^\mu \overset{R}{\Lambda}_\mu \overset{4}{=} \frac{1}{4}\omega^{-1}(P_{\lambda\mu} - 2Q_{\lambda\mu})\alpha^{\lambda\mu 0}$$

or, using (9.7), (12.13) and (15.11) and writing in full:

(15.17) $$\alpha^{\mu C}_{\cdot\cdot B}\Lambda^B_{A\mu} - \alpha^{\mu C}_{\cdot\cdot B}\overset{R}{\Lambda}{}^B_{A\mu} = \frac{1}{4}(S_{[\lambda\mu\nu]} - \overset{R}{S}_{[\lambda\mu\nu]})\alpha^{\lambda\mu\nu C}_{\cdot\cdot\cdot A}.$$

---

[51] If this condition were not fulfilled, Dirac's equation would contain an additional term of the form $-\frac{1}{4}\frac{h}{i}(p-2q)q_{\lambda\mu}\alpha^{\lambda\mu 0}$, which would give rise to a kind of spin-term.[51a]

[51a] (February 1933.) The additional term is dealt with in GF. V. If we impose the condition that the divergence of the current vector vanishes, it can only appear if the index of $G_{\lambda\mu}$ is $+4$ ($\omega^2<1$) or $+1$.

Hence (15.8) is equivalent to the identity of the operators $\alpha^\mu \Lambda_\mu$ and $\alpha^\mu \overset{R}{\Lambda}_\mu$. If we had introduced instead of the third physical condition the condition of the identity of these operators we should have got immediately the equation (15.8).

16. **Different forms of Dirac's equations.** The form (15.6) of the Dirac equations, written in full and in general coördinates:

$$(16.1)\qquad \alpha^{\mu C}_{\cdot\cdot B}\left\{\frac{h}{i}(\partial_\mu \psi^B + \Lambda^B_{A\mu}\psi^A) - \frac{e}{c}\varphi_\mu \psi^B + imcq_\mu \psi^B\right\} = 0$$

is inconvenient for two reasons. Firstly the displacement of projectors and affinors occurring in the macrocosmic equations is independent of e and m (they occur only in the differentiated object), but the covariant differentiation itself used in (15.6) depends on e and m. Secondly the indetermined vector φ_μ could be eliminated from the macrocosmic equation of the geodesics and replaced by the determined point q_μ. But in (16.1) both φ_μ and q_μ do occur. The second inconvenience can be removed by introducing instead of the *pseudo*spinvector-density ψ^C a *spinvectordensity* Ψ^C of degree $-i\omega\frac{e}{hk}$ [52]

$$(16.2)\qquad \Psi^C = X^{-i\omega\frac{e}{hk}}\psi^C.$$

Then (16.1) passes because of (13.5) into

$$(16.3)\qquad \alpha^{\mu C}_{\cdot\cdot B}\left\{\frac{h}{i}(\partial_\mu \Psi^B + \Lambda^B_{A\mu}\Psi^A) - \left(\frac{e}{k} - imc\right)q_\mu \Psi^B\right\} = 0,$$

or, using (14.5):

$$(16.4)\qquad \alpha^\mu\left\{\frac{h}{i}\nabla_\mu - \left(\frac{e}{k} - imc\right)q_\mu + \frac{e}{k}\omega Q_\mu\right\}\Psi = 0.$$

We now impose the *fourth physical condition*, which allows us to determine Q_μ:

4. *The operator* $\frac{h}{i}\nabla_\mu$ [53] *is quantumtheoretically equivalent to the point of total energy and momentum* p_μ: [54] *the algebraical equation equivalent to* (16.4) *has always a solution* $\Psi \neq 0$. [55]

[52] This is the only place where the constant ω appears in an essential way; in all other formulae it is merely an irrelevant dimensional constant.

[53] Because of the identity of $\alpha^\mu \Lambda_\mu$ and $\alpha^\mu \overset{R}{\Lambda}_\mu$ proved in § 15 we can take $\frac{h}{i}\overset{R}{\nabla}_\mu$ just as well.

[54] This condition is plausible because the influence of the charge e consists (because of (12.15)) in changing the kinetic energy and momentum mi^ν from p^ν into $p^\nu - \frac{e}{k}q^\nu$.

[55] In this case it has at once two linearly independent solutions.

This algebraical equation has the form

$$\alpha^\mu n_\mu \psi = 0 \tag{16.5}$$

with

$$n_\mu = p_\mu - \left(\frac{e}{k} - i m c\right) q_\mu + \frac{e}{k} \omega Q_\mu. \tag{16.6}$$

Now an equation of the form (16.5) has a solution $\psi \neq 0$ if and only if $n_\mu n^\mu = 0$ as is seen at once by operating on (16.5) with $\alpha^\lambda n_\lambda$.

Hence

$$\| n_\mu \|^2 = \left\| m(i_\mu + i c q_\mu) + \frac{e}{k} \omega Q_\mu \right\|^2 = 0 \tag{16.7}$$

or

$$\frac{e^2}{k^2} \omega^2 Q_\mu Q^\mu + \frac{2 e m}{k} \omega Q_\mu i^\mu + 2 i \frac{e m c}{k} \omega Q_\mu q^\mu = 0. \tag{16.8}$$

Because this must hold for every direction of the vector i^ν, the second term must itself be zero for every i^ν; i. e. the vectorpart of Q_μ must be zero. Because Q_μ must be independent of $\frac{e}{m}$ the two remaining terms must vanish separately, hence

$$Q_\mu = 0. \tag{16.9}$$

Indeed

$$\left.\begin{matrix} n^\nu \\ {}'n^\nu \end{matrix}\right\} = p^\nu - \left(\frac{e}{k} \mp i m c\right) q^\nu \tag{16.10}$$

are the two points of intersection of the quadric with the straight line through p^ν and q^ν (see fig. 1).

The Dirac equation becomes now

$$\alpha^\mu \left\{\frac{h}{i} \nabla_\mu - \left(\frac{e}{k} - i m c\right) q_\mu\right\} \psi = 0. \tag{16.11}$$

It is to be noticed that e occurs explicity only apparently because (16.11) can be written

$$\left(\frac{h}{i} \alpha'^\mu \partial_\mu + \frac{h}{i} \alpha^\mu A_\mu + i m c \alpha^\mu q_\mu\right) \psi = 0, \quad \alpha'^\mu = A^\mu_\nu \alpha^\nu, \tag{16.12}$$

where e occurs only implicitly (in the degree of ψ). Although the equation (16.12) is actually real, it is *not* possible to consider only real solutions ψ, because the degree of ψ is imaginary, i. e. ψ must be a common solution of both (16.12) and

$$x^\mu \partial_\mu \psi = -i \omega \frac{e}{hk} \psi. \tag{16.13}$$

The Hamiltonian function is

$$H = \frac{1}{4m\omega} G_{\lambda\mu}(n^\lambda n^\mu + {}'n^\lambda {}'n^\mu)$$

$$= \frac{1}{2m\omega}\left\{p_\lambda p_\mu G^{\lambda\mu} - \frac{2e}{k} p_\lambda q^\lambda + \left(\frac{e^2}{k^2} - m^2c^2\right) q^\lambda q^\mu G_{\lambda\mu}\right\}. \tag{16.14}$$

By partial differentiation with respect to p_λ holding q^ν, $G_{\lambda\mu}$ and $G^{\mu\nu}$ constant and by partial differentiation with respect to q_ν holding p_λ constant and considering the $G_{\lambda\mu}$, $G^{\lambda\mu}$ as functions of $x^\nu = \omega q^\nu$, we get the equation of the worldlines of charged masses

$$\frac{\partial H}{\partial p_\nu} = \frac{1}{\omega m}\left(p^\nu - \frac{e}{k} q^\nu\right) = \frac{d' q^\nu}{d\tau}, \qquad \frac{d'}{d\tau} = i^\mu \partial_\mu,$$

$$\frac{\partial H}{\partial q^\lambda} = -\Pi^\nu_{\lambda\mu} p_\nu i^\mu \qquad = -\frac{d' p_\lambda}{d\tau}. \tag{16.15}$$

From (16.10) and (16.14) follows

$$H = 0. \tag{16.16}$$

By iteration of the operator in (16.4), we get the wave equations of the second order in the form

$$\alpha^{\lambda\mu}\left(\frac{h}{i}\nabla_\lambda - \frac{e}{k} q_\lambda + i m c q_\lambda\right)\left\{\frac{h}{i}\nabla_\mu - \frac{e}{k} q_\mu + i m c q_\mu\right\}\psi = 0, \tag{16.17}$$

ψ being a normalized spinvectordensity of weight $+\frac{1}{4}$ and excess $-i\omega\frac{e}{hk}$. By going back to a ψ with excess zero, $\frac{e}{k} q_\mu$ in (16.17) changes into $\frac{e}{c}\varphi_\mu$. Decomposing then α into β and $\bar{\beta}$ we get the two equations of the second order in the ordinary (Riemannian) form.[56] Of course these equations can be obtained as well by the elimination of ψ^c and $\psi^{\mathfrak{C}}$ from the two equations resulting from the decomposition of (16.11).

The Dirac-equation (16.11) can be brought into another quite remarkable form. If we operate on the left side with $\alpha^0 \overset{4}{=} -\alpha^\mu q_\mu$ we get

$$-\frac{h}{i} q_\lambda \alpha^{\lambda\mu} \nabla_\mu \psi = \left(\frac{e}{k} - i m c\right)\psi. \tag{16.18}$$

This equation shows that the solutions of Dirac's equation are the "eigenfunctions" of the operator $\frac{h}{i}\alpha^{0\mu}\nabla_\mu$ which belong to the "eigenvalue" $\frac{e}{k} - imc$.

[56] DE., p. 281.

If we use in (16.18) the Riemannian differential operator $\overset{R}{\nabla}_\mu$ instead of ∇_μ (which is permissible because of (15.8)), we get because of

$$q^b \overset{R}{A}_b \overset{o}{=} -\frac{1}{4} q^b \overset{R}{H}{}^c_{ab} \alpha^a_{.c} \overset{o}{=} 0:$$

$$\frac{h}{i} q^\mu \overset{R}{\nabla}_\mu \psi = -\frac{e}{k} \psi, \tag{16.19}$$

hence (16.18) takes the form

$$+\frac{h}{i} q_\lambda \alpha^\lambda \alpha'^\mu \overset{R}{\nabla}_\mu \psi = imc\psi; \qquad \alpha'^\mu = A^\mu_\nu \alpha^\nu, \tag{16.20}$$

which has the remarkable property that the mass itself is eigenvalue of an operator.

17. **Variation.** The most simple principle of variation is

$$\delta I = \delta \int N \sqrt{\mathfrak{G}}\, dx^0\, dx^1\, dx^2\, dx^3\, dx^4 = 0; \qquad \mathfrak{G} = \mathrm{Det}\,(G_{\lambda\mu}), \tag{17.1}$$

where $G_{\lambda\mu}$ has to be varied and x^ν and ω to be kept constant.

$$\delta G^{\lambda\mu} x_\lambda x_\mu = x^\lambda x^\mu \delta G_{\lambda\mu} = -x_\lambda x_\mu \delta G^{\lambda\mu} = 0. \tag{17.2}$$

We suppose that $P_{\lambda\mu}$ and $Q_{\lambda\mu}$ differ only by constant factors from $q_{\lambda\mu}$:

$$-\omega^{-1} P_{\lambda\mu} = p q_{\lambda\mu}; \qquad -\omega^{-1} Q_{\lambda\mu} = q q_{\lambda\mu}, \tag{17.3}$$

but we will *not* yet use the values $p = 4$, $q = 2$ found in § 15 and § 13.

Denoting $dx^0\, dx^1\, dx^2\, dx^3\, dx^4$ by $d\tau$ we can split up δI into three parts:

$$\delta I = \delta I_1 + \delta I_2 + \delta I_3, \tag{17.4}$$

where

$$\begin{aligned}
\delta I_1 &= \int (\delta \mathfrak{G}^{1/2}) N d\tau + \int \mathfrak{G}^{1/2} (\delta G^{\lambda\mu}) N_{\lambda\mu} d\tau \\
&= \int \mathfrak{G}^{1/2} \left(N_{(\lambda\mu)} - \frac{1}{2} N G_{\lambda\mu} \right) \delta G^{\lambda\mu} d\tau \\
&= \int \mathfrak{G}^{1/2} \left[\left(K_{\lambda\mu} - \frac{1}{2} K g_{\lambda\mu} \right) - (q^2 - pq + 2p) q^\varrho_\lambda q_{\mu\varrho} \right. \\
&\quad \left. + \frac{1}{2} (q^2 - 2pq + 2p) q_{\varrho\sigma} q^{\varrho\sigma} q_{\lambda\mu} + (p+q) q_\lambda \overset{R}{\nabla}_\varrho q^\varrho_\mu \right] (\delta G^{\lambda\mu})\, d\tau,
\end{aligned} \tag{17.5}$$

$$\begin{aligned}
\delta I_2 &= \int 2 \mathfrak{G}^{1/2} S_\nu^{\lambda\mu} \delta \left\{ {\nu \atop \lambda\mu} \right\} d\tau \\
&= \int \mathfrak{G}^{1/2} (2 - p - q) \left(q_\lambda \overset{R}{\nabla}_\varrho q^\varrho_\mu - q^\varrho_\lambda q_{\mu\varrho} \right) (\delta G^{\lambda\mu})\, d\tau,
\end{aligned} \tag{17.6}$$

$$\delta I_3 = \int 2\mathfrak{G}^{1/2} S_\nu^{\lambda\mu}\, \delta\left(\Pi_{\lambda\mu}^{\nu} - \left\{\begin{matrix}\nu\\ \lambda\,\mu\end{matrix}\right\}\right) d\tau$$

$$= \int \mathfrak{G}^{1/2}(q-1)\Big\{-2(2p-q-1)\, q_\lambda \overset{R}{\nabla}_\varrho q_\mu^\varrho + (3p-q-2)\, q_{\lambda\varrho}\, q_\mu^\varrho\Big\}(\delta G^{\lambda\mu})\, d\tau, \tag{17.7}$$

in which formulae integrals of divergences have been omitted. Hence we have

$$\delta I = \int \mathfrak{G}^{1/2}(\delta G^{\lambda\mu})\, d\tau \Big[\Big(K_{\lambda\mu} - \frac{1}{2} K g_{\lambda\mu}\Big) - 2(q^2 - 2pq + 2p)\Big(q_\lambda^\varrho\, q_{\mu\varrho} - \frac{1}{4} q_{\varrho\sigma} q^{\varrho\sigma} g_{\lambda\mu}\Big) + 2(q^2 - 2pq + 2p)\, q_\lambda \overset{R}{\nabla}_\varrho q_\mu^\varrho\Big] = 0. \tag{17.8}$$

Hence the variational equations are

$$\Big(K_{\lambda\mu} - \frac{1}{2} K g_{\lambda\mu}\Big) - 2\,\frac{q^2 - 2pq + 2p}{q^2}\,\frac{k^2}{c^2}\Big(F_\lambda^\varrho\, F_{\mu\varrho} - \frac{1}{4} F_{\varrho\sigma} F^{\varrho\sigma} q_{\lambda\mu}\Big) = 0; \tag{17.9}$$

$$2(q^2 - 2pq + 2p)\,\frac{k}{qc}\overset{R}{\nabla}_\varrho F_\mu^\varrho = 0. \tag{17.10}$$

On the other hand the field equations for the gravitational and electromagnetic fields in empty space are (written in homogeneous coördinates)

$$K_{\lambda\mu} - \frac{1}{2} K g_{\lambda\mu} + \frac{\varkappa}{c^2}\Big(F_\lambda^\varrho\, F_{\mu\varrho} - \frac{1}{4} F_{\varrho\sigma} F^{\varrho\sigma} q_{\lambda\mu}\Big) = 0. \tag{17.11}$$

Now we impose the fifth physical condition:

5. *The affinor part of the equation resulting from the variational principle* (17.1) *is equivalent to the combined equations of the gravitational and the electromagnetic fields in empty space.*

This condition is then and only then fulfilled if

$$-2(q^2 - 2pq + 2p)\, k^2 = \varkappa q^2. \tag{17.12}$$

From this equation follows that for real values of k

$$q^2 - 2pq + 2p < 0. \tag{17.13}$$

Substituting the values of p and q deduced in § 15 and § 13 we get

$$\boxed{k = \sqrt{\frac{\varkappa}{2}}}. \tag{17.14}$$

The only remaining constant is ω, determining the unit of length.

The remaining variational equation (17.10) gives Maxwell's first set of equations for the electromagnetic field in empty space, whereas Maxwell's second set has already been found in § 15.

18. **Relations to the older theories.** The theories of Kaluza-Klein, of Veblen and Hoffmann and of Einstein and Mayer (the latter in the projective form in which we have brought it) are closely related to our theory. The whole geometrical part of this paper is equally applicable to all these theories. They can all be characterized by the relation

$$\nabla_\mu G_{\lambda\nu} = 0, \tag{18.1}$$

together with relations of the form

$$-\omega^{-1} Q_{\lambda\mu} = q q_{\lambda\mu}; \qquad -\omega^{-1} P_{\lambda\mu} = p q_{\lambda\mu}, \tag{18.2}$$

$$F_{\lambda\mu} = f \cdot \frac{c}{k} q_{\lambda\mu}, \tag{18.3}$$

with *constant* f, p and q. Apart from the first-mentioned theory they differ (apart from details which are for our present purpose not relevant) *only* in the values of p and q.

Now we have seen that the right form of the geodesics requires the relation

$$f = q, \tag{18.4}$$

that the theorem of the point of energy and momentum requires

$$q = 2, \tag{18.5}$$

that the form of Dirac's equation without additional terms requires

$$p = 2q \tag{18.6}$$

and that the variational principle requires

$$k = \sqrt{\frac{\varkappa}{2}}, \quad q \neq 0, \tag{18.7}$$

where as the $+$-sign of ω^2 demands

$$q^2 - 2pq + 2p < 0. \tag{18.8}$$

Equations (18.3) and (18.4) can be considered as to define the physical quantity $F_{\lambda\mu}$ by means of the geometrical quantity $q_{\lambda\mu}$.

The theory of Veblen and Hoffmann is characterized by the property of symmetry $S_{\lambda\mu}^{\cdot\cdot\nu} = 0$, hence $p = q = 1$. It therefore gives neither the

theorem of he point of energy and momentum, nor Dirac's equation without additional term, nor the $+$-sign of ω^2.[57]

The theory of Einstein and Mayer in its projective form is characterized by the condition that an ordinary covariant differential (by means of dx^ν, not of $d'x^\nu$) exists, hence it requires $p = 0$. It therefore gives neither Dirac's equation, because this would require $q = 0$, contradictory to (18.8) nor the $+$-sign of ω^2. The value of q can't be fixed by the data Einstein and Mayer have given. It can be fixed f. i. as

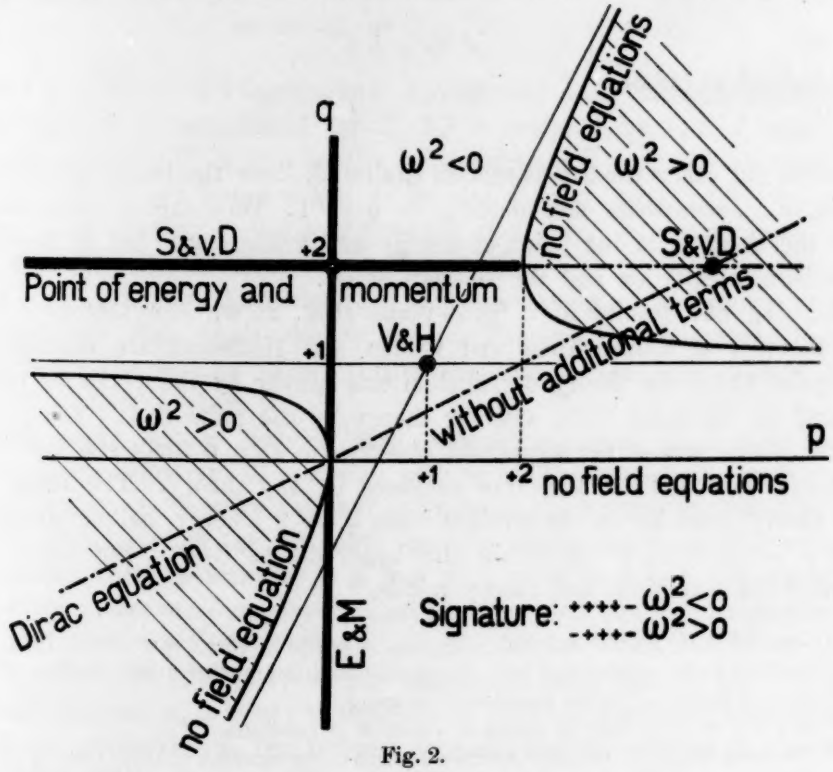

Fig. 2.

$q = 2$, in which case it gives the theorem of the point of energy and momentum.

The theory of Kaluza-Klein is not a fourdimensional-projective one, but a fivedimensional-stationary one. Now the difference between those two kinds of theories is quite unessential. For a fivedimensional Riemannian

[57] This refers only to the projective connection actually applied on physics in the papers cited in note [2] on page 271. The more general theory of projective connections developped but not yet physically exploited in Veblens Generalised Projective Geometry, Journ. Lond. Math. Soc., 4 (1929), pp. 140—160, covers the whole (p, q)-plane, as does our theory if p and q are not specialized.

geometry is stationary if it allows a one-parameter group of motions. The paths of this group form a fourdimensional manifold which can be considered as a general *projective* manifold.[58] By "projectivising" it in this way the theory of Kaluza-Klein is brought into accord with our general scheme, and is found to satisfy all our geometrical conditions. But the geodesics are given by

(18.9) $$p^\mu \nabla_\mu p^\nu = 0,$$

instead of $p'^\mu \nabla_\mu p^\nu = 0$. Hence instead of the condition (18.4) arises

(18.10) $$f = p+q$$

and instead of (18.5)

(18.11) $$p+q = 2,$$

whereas the other conditions remain unaltered. Now the theory of Kaluza-Klein is a *symmetrical* one, hence $p = q = 1$. Their theory gives therefore the theorem of the point of energy and momentum[59] but it does *not* give Dirac's equation and the $+$-sign of ω^2.

Mapping our results in a (p, q)-plane (Fig. 2) we can therefore say: the theories of Kaluza-Klein and Veblen and Hoffmann are mapped by the point (1,1); the theory of Einstein and Mayer by the q-axis or particularly by the point (0,2), and our theory by the point (2,4). The points of the p-axis and of the hyperbola $q^2-2pq+2p=0$ don't give the right field-equations and are therefore excluded by any theory. The image of the theory must lie on the straight lines $q = 2$ (energy and momentum)[60] and $p = 2q$ (Dirac) and within the hyperbola ($\omega^2 > 0$). But these are exactly the conditions our theory fulfills.[61]

[58] Comp. APD. II, pp. 538 and 542.

[59] But it is to be noticed that in a *fourdimensional* interpretation the equation of the geodesics (18.9) has no simple geometrical meaning.

[60] Or $p+q = 2$ if (18.9) is chosen as equation of geodesics.

[61] (February 1933.) If we allow additional terms in the equation of Dirac then for $\omega^2<0$ all points $q = 2$, $p<2$ are possible. But for $\omega^2>0$ only the point $q = 2$, $p = 4$ is possible if we hold to the condition that the divergence of the current vector vanishes. (GF. V).

A CORRESPONDENCE BETWEEN CLASSES OF IDEALS AND CLASSES OF MATRICES.[1]

By Claiborne G. Latimer and C. C. MacDuffee.

1. By the well-known Hamilton-Cayley theorem, a square matrix D, is a root of

$$f(x) \equiv x^n + k_1 x^{n-1} + \cdots\cdots + k_n = 0, \tag{1}$$

where $\pm f(x)$ is the characteristic determinant of D, it being understood that when D is substituted for x, k_n is replaced by $k_n I$, I being the identity matrix.[2] If A is a matrix of determinant ± 1, the elements of $D' = ADA^{-1}$ are integers and D' is also a root of (1). The question arises as to whether or not every matric root may be thus obtained from one by proper choice of A.

If D, D' are related as above, A as before being a unimodular matrix, we shall say that D' is similar to D. This relation is symmetric and transitive. The totality of all matrices similar to the same matrix will be said to form a class.

Since every square matrix, with rational elements, is a root of its characteristic equation, it is a root of a uniquely determined equation of lowest degree, $\leqq n$, with rational coefficients, the leading coefficient being unity. This is called the minimum equation of the matrix. If D and D' are similar matrices and one is a root of a given equation, the other is also a root. Hence a necessary condition that two matrices be similar is that they have the same minimum equation.

Hereafter, let $f(x)$ in (1) be any polynomial such that (a) the k's are rational integers, $k_n \neq 0$, (b) the factors of $f(x)$, with rational coefficients, which are irreducible in the rational field are distinct. We shall consider only those matric roots of (1) for which (1) is the minimum equation. It will be shown that there is a one-to-one correspondence between the classes of such matric roots and the classes of ideals in a set $\mathfrak{G}$ of elements in a certain linear algebra. For the case where $f(x)$ is irreducible, $\mathfrak{G}$ is equivalent to the ring defined by an algebraic root of (1).

[1] Received, June 24, 1932.

[2] It will be understood throughout this paper, unless the contrary is explicitly stated, that all matrices referred to are square matrices of the same order n, the elements of which are rational integers; also that the coefficients of all polynomials referred to are rational.

2. Consider the matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots\cdots & & 1 \\ -k_n & -k_{n-1} & \cdots\cdots & & -k_1 \end{pmatrix}.$$

The characteristic determinant of this matrix is $|C-xI| = (-1)^n f(x)$. In the matrix $(C-xI)$, the determinant of order $n-1$ in the upper right-hand corner is unity. Hence (1) is the minimum equation of C.[3]

Let $\mathfrak{A}$ be the algebra, the elements of which are all polynomials in C. An element $\varphi(C)$, of $\mathfrak{A}$ is zero if and only if $\varphi(x)$ is divisible by $f(x)$. Since the irreducible factors of $f(x)$ are distinct, it follows that if $\varphi^a(C) = 0$, where a is a positive integer, then $\varphi(C) = 0$. Therefore $\mathfrak{A}$ contains no nilpotent elements and is semi-simple.

Let $\mathfrak{G}$ be the set of all polynomials in C with rational integral coefficients. In a paper, hereafter referred to as Tr., MacDuffee obtained certain results on ideals in a set of integral elements in a rational semi-simple algebra.[4] He employed Dickson's definition of such a set.[5] Under this definition, such a set is characterized by four properties, R, C, U, M. However the property M, the maximal property, was not employed in Tr. and the results are valid for sets with the properties R, C, U; in particular, they are valid for the set $\mathfrak{G}$ considered here.

We employ the same definitions of ideal, class of ideals and non-singular ideal as given in Tr., noting that $\mathfrak{A}$ is a commutative algebra. For the case where $f(x)$ is irreducible, $\mathfrak{G}$ is evidently equivalent to the ring $\mathfrak{R} = [1, \theta, \theta^2, \cdots, \theta^{n-1}]$, i. e. the set of all polynomials with integral coefficients in an algebraic root θ, of (1). The present definitions of ideal and class of ideals are then equivalent to the usual definitions.

We may take as a basis of $\mathfrak{G}$, $C^{i-1}\,(i = 1, 2, \cdots, n)$. Then the matrices C_p of p. 76, Tr., are $C_p = C^{p-1}\,(p = 1, 2, \cdots, n)$. If $\mathfrak{K}$ is a non-singular ideal in $\mathfrak{G}$ with the basis

$$\omega_i = g_{i1} I + g_{i2} C + \cdots + g_{in} C^{n-1} \qquad (i = 1, 2, \cdots, n),$$

and if $G \equiv (g_{ij})$, by Theorem 4, Tr., $G\,C^{p-1}\,G^{-1} = D^{p-1}\,(p = 1, 2, \cdots, n)$, where the elements of $D = (d_{ij})$ are rational integers. These n powers of D are the class matrices of $\mathfrak{K}$, relative to this basis. (1) is the minimum

[3] Dickson; Algebras and Their Arithmetics, p. 109, Theorem 1.

[4] An Introduction to the Theory of Ideals in Linear Associative Algebras; Transactions of the American Mathematical Society, Vol. 31 (1929), pp. 71–90.

[5] Dickson, ibid, pp. 141–142.

equation of D since the same is true of C. We find $C\omega_i = \sum_j d_{ij}\omega_j$ $(i = 1, 2, \cdots, n)$. If D' is in the same class as D, $D' = ADA^{-1}$ where $A \equiv (a_{ij})$ is unimodular. Then $\omega_i' = \sum_j a_{ij}\omega_j$ $(i = 1, 2, \cdots, n)$ form a basis of $\mathfrak{R}$ and the class matrices of $\mathfrak{R}$, relative to this basis are powers of D'. Thus for every class of non-singular ideals in $\mathfrak{G}$, there is a corresponding class of matric roots of (1) of the required type. Furthermore, by Theorem 6, Tr., this class of matrices is unique.

It remains to show that if (1) is the minimum equation of a matrix D, then D is in one of the classes referred to above. Since $f(x)$ divides the characteristic determinant $|D - xI|$ of D, it follows that $|D - xI| = (-1)^n f(x)$ and the g. c. d. of the $(n-1)$-rowed minors of this determinant is unity.[6] Then the invariant factors of D are all unity except $I_n = f(x)$. The same is true of the invariant factors of C. It follows that there is a non-singular matrix G, the elements of which are rational, such that $GCG^{-1} = D$.[7] After multiplying G by a properly chosen scalar, we may assume that all its elements are rational integers. By Theorem 4, Tr., G is the matrix of an ideal, $\mathfrak{R}$. It follows that D is in the class of matric roots of (1) corresponding to the class of ideals containing $\mathfrak{R}$. We have then the

THEOREM. *Let* $f(x) = x^n + k_1 x^{n-1} + \cdots + k_n$, *where the* k*'s are rational integers,* $k_n \neq 0$, *and assume that the algebraic roots of*

$$f(x) = 0 \tag{1}$$

are distinct. (1) *is the minimum equation of a matrix* C. *There is a one-to-one correspondence between the classes of square matrices, of order* n *and with rational integral elements only, which have* (1) *as their minimum equation and the classes of non-singular ideals in the set* $\mathfrak{G}$ *of all polynomials in* C *with rational integral coefficients.*

From the above conditions on $f(x)$, we have $f(x) = f_1(x) \cdot f_2(x) \cdot \cdots \cdot f_s(x)$, where the f_i are distinct polynomials, with rational integral coefficients, which are irreducible in the rational field. If θ_i is an algebraic root of $f_i(x) = 0$, let $\mathfrak{R}_i$ be the set of all polynomials in θ_i with rational integral coefficients $(i = 1, 2, \cdots, s)$. There is a uniquely determined polynomial φ_i, of degree less than that of f_i, such that $\varphi_i(x) \cdot f(x)/f_i(x) \equiv 1$, modulo $f_i(x)$ $(i = 1, 2, \cdots, s)$. If h_i is the number of classes of ideals in $\mathfrak{R}_i$, it may be shown that there are exactly $h = h_1 \cdot h_2 \cdot \cdots \cdot h_s$ classes of ideals in $\mathfrak{G}$ or there are more than h classes according as all or not all of the coefficients of the s φ's are integers.

[6] Dickson, ibid., p. 109, Lemma and Theorem 1.

[7] Dickson, Modern Algebraic Theories, p. 104.

If F is an algebraic field of class number unity, the results of the above theorem may be generalized to the case where the k's and the elements of the matrices are integers of F. We define similar matrices as before except that the determinant of A is a unit of F.

3. In this paragraph we shall give two numerical illustrations of the theorem. It will be noted that if $\omega_1, \omega_2, \cdots, \omega_n$ form a basis of an ideal and if G is the corresponding ideal matrix, the matrix $GCG^{-1} = D \equiv (d_{ij})$ is given directly by $C\omega_i = \sum_j d_{ij}\omega_j \ (i = 1, 2, \cdots, n)$.

It is well-known that the field defined by a primitive fifth root of unity θ, is of class number unity and $1, \theta, \theta^2, \theta^3$ form a basis of the set $\mathfrak{G}'$ of all integral numbers in the field. Hence for the case $f(x) = x^4+x^3+x^2+x+1$, the set $\mathfrak{G}$ of the theorem is equivalent to $\mathfrak{G}'$ and there is a single class of fourth order matrices which are roots of $f(x) = 0$. Then every such matric root is similar to

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

Consider the case $f(x) = x^2 - x + 6$. It is known that (a) $1, \theta$ form a basis of the set $\mathfrak{G}'$ of all integral numbers in the field defined by a root θ, of $f(x) = 0$, (b) $\mathfrak{G}'$ is of class number three, (c) representative ideals from the three classes are $\{1\} = [1, \theta]$, $\mathfrak{R} = [2, 1+\theta]$, $\mathfrak{R}^2 = [4, 1+\theta]$.[8] The set $\mathfrak{G}$ of the theorem is equivalent to $\mathfrak{G}'$. Forming the class matrices for each of these ideals, we obtain

$$D_1 = \begin{pmatrix} 0 & 1 \\ -6 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -1 & 2 \\ -4 & 2 \end{pmatrix}, \quad D_3 = \begin{pmatrix} -1 & 4 \\ -2 & 2 \end{pmatrix}.$$

Then every second order matric root of $x^2 - x + 6 = 0$ is similar to one and only one of these D's.

[8] Hecke, Vorlesungen über die Theorie der Algebraischen Zahlen, pp. 177–178.

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ON LINEAR FUNCTIONAL OPERATIONS AND THE MOMENT PROBLEM FOR A FINITE INTERVAL IN ONE OR SEVERAL DIMENSIONS.[1]

By T. H. Hildebrandt and I. J. Schoenberg.

The note by T. H. Hildebrandt[2] on the moment problem for a finite interval, which deduces the Hausdorff theorem on moments from the theorem of Riesz on linear functional operations, suggests the question whether conversely the Riesz result is derivable from the Hausdorff theorem in a simple way, thus giving a new proof of the Riesz theorem. In the first part of this paper we show that this is actually the case. Incidentally, there is suggested a very simple and elegant method of proving the Riesz theorem directly.

The object of the second part of this paper is to extend to the case of several variables, the methods of the note by I. J. Schoenberg[3] on the moment problem, as well as the results indicated above, for the one-dimensional case. Detailed consideration of this extension seems also to be justified by the fact that it requires the deduction of properties of functions of bounded variation as related to Stieltjes integrals in several variables.

For convenience, the results obtained by this extension are stated and proved for the case of two variables only. The extension to more than two variables presents no essential difficulty.

1. **A derivation of Riesz' theorem from the Hausdorff theorem.** Let $L[f]$ be a linear continuous functional operation on continuous functions $f(t)$ on $(0, 1)$. Then there exists an M such that for every f:

$$|L[f]| \leqq M\|f\|,$$

where $\|f\|$ is the maximum of $|f|$ on $(0, 1)$. Let $L[t^n] = \mu_n$. Then

$$L[t^n(1-t)^m] = \mu_n - \binom{m}{1}\mu_{n+1} + \cdots + (-1)^m \mu_{n+m} = \Delta^m \mu_n.$$

Now observe that since

$$\sum_{m=0}^{n} \binom{n}{m} t^m (1-t)^{n-m} = (t+1-t)^n = 1$$

[1] Received, July 9, 1932.

[2] T. H. Hildebrandt, *On the moment problem for a finite interval,* Bulletin of the Amer. Math. Soc., vol. 38 (1932), pp. 269-270.

[3] I. J. Schoenberg: *On finite and infinite completely monotonic sequences,* Bulletin of the Amer. Math. Soc., vol. 38 (1932), pp. 72-76.

and $t^m(1-t)^{n-m} \geqq 0$ for $0 \leqq t \leqq 1$, we have

$$\left| \sum_{m=0}^{n} \varepsilon_m \binom{n}{m} t^m (1-t)^{n-m} \right| \leqq 1$$

if $\varepsilon_m = \pm 1$. Consequently by choosing the ε_m properly, we obtain

$$(1) \qquad \sum_{m=0}^{n} \binom{n}{m} |\Delta^{n-m} \mu_m| = \left| \sum_{m=0}^{n} \binom{n}{m} \varepsilon_m L[t^m(1-t)^{n-m}] \right| \leqq M$$

for every n. Hence the Hausdorff hypothesis is satisfied, and there exists a function of bounded variation $\chi(t)$, such that

$$L[t^n] = \mu_n = \int_0^1 t^n \, d\chi(t).$$

By using the Weierstrass approximation theorem and the continuity properties of $\int f \, d\chi$ and of $L[f]$, we then obtain the Riesz theorem, that for every continuous function $f(t)$

$$L[f] = \int_0^1 f(t) \, d\chi(t).$$

2. **A direct proof of Riesz' theorem.** It is however, more elegant to proceed directly using the Bernstein polynomials

$$B_n(f) = \sum_{m=0}^{n} f\left(\frac{m}{n}\right) \binom{n}{m} t^m (1-t)^{n-m}.$$

Let us define a sequence of functions φ_n as follows:

(a) $\varphi_n(0) = 0$;

(b) $\varphi_n\left(\frac{m}{n}+0\right) - \varphi_n\left(\frac{m}{n}-0\right) = \binom{n}{m} L[t^m(1-t)^{n-m}]$;

(c) $\varphi_n(t)$ is constant in the intervals $\frac{m}{n} < t < \frac{m+1}{n}$ and $\varphi_n(t) = \varphi_n(t+0)$ holds for $0 < t < 1$.

Then

$$L[B_n(f)] = \sum_{m=0}^{n} f\left(\frac{m}{n}\right) \binom{n}{m} L[t^m(1-t)^{n-m}] = \int_0^1 f(t) \, d\varphi_n(t).$$

Now on account of the inequality (1) the functions φ_n are uniformly of bounded variation. Hence applying the theorem of Helly on compactness of a set of functions of bounded variation, there exists a subsequence φ_{n_k} of the φ_n, converging to a function of bounded variation φ. Then by the properties of Stieltjes integrals[4], $\int f \, d\varphi_{n_k}$ converges to $\int f \, d\varphi$, while

[4] E. Helly, Wiener Sitzungsberichte, vol. 121 IIa (1912), pp. 225–226. See also the paper of H. E. Bray, *Elementary properties of the Stieltjes integral*, Annals of Math. (2), vol. 20, pp. 180-181.

from the continuity of $L[f]$ and the uniform convergence of $B_n(f)$ to $f(t)$, it follows that $L[B_{n_k}(f)]$ converges to $L[f]$, so that

$$L[f] = \int_0^1 f(t)\, d\varphi(t).$$

Since $\int f\, d\varphi$ is not affected by changing φ at a denumerable set of points, this φ will agree with the above function χ excepting possibly at a denumerable set of points interior to (0, 1).

3. **On Stieltjes integrals and functions of bounded variation of two variables.** In order to extend the previous considerations to functions of two variables, some properties of Stieltjes integrals and of functions of two variables of bounded variation are needed. Let $f(x, y)$ and $\chi(x, y)$ be defined and finite in the unit-square $U(0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1)$. Let

$$\begin{aligned} &0 = x_0 < x_1 < \cdots < x_{p-1} < x_p = 1, \\ &0 = y_0 < y_1 < \cdots < y_{q-1} < y_q = 1 \end{aligned} \tag{2}$$

and let us consider the so called double increment of χ

$$\Delta(\chi; x_{i+1}, y_{j+1}, x_i, y_j) = \chi(x_{i+1}, y_{j+1}) - \chi(x_{i+1}, y_j) - \chi(x_i, y_{j+1}) + \chi(x_i, y_j).$$

The Stieltjes integral is by definition the limit

$$\lim_{\delta \to 0} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} f(\xi_{ij}, \eta_{ij})\, \Delta(\chi; x_{i+1}, y_{j+1}, x_i, y_j) = \int_0^1 \int_0^1 f(x, y)\, d_x d_y\, \chi(x, y) \tag{3}$$

in case this limit exists, where δ is the largest subinterval of both divisions (2), while ξ_{ij} and η_{ij} are any numbers subject to the restrictions $x_i \leqq \xi_{ij} \leqq x_{i+1}$, $y_j \leqq \eta_{ij} \leqq y_{j+1}$. Fréchet[5] has shown that this integral exists if

(α) $f(x, y)$ is continuous in U;

(β) The double variation of $\chi(x, y)$ in U

$$V(\chi) = \text{Least upper bound of} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} |\Delta(\chi; x_{i+1}, y_{j+1}, x_i, y_j)|$$

is finite.

Let us suppose that these conditions are satisfied. It follows from the definition (3) that neither $V(\chi)$ nor the Stieltjes integral change, if $\chi(x, y)$ is replaced by $\chi(x, y) + \alpha(x) + \beta(y)$, where $\alpha(x)$ and $\beta(y)$ are arbitrary finite functions for $0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1$, respectively. In particular, replacing $\chi(x, y)$ by $\chi(x, y) - \chi(x, 0) - \chi(0, y) + \chi(0, 0)$, we see that without loss of generality we may suppose also

(γ) $\chi(x, 0) = \chi(0, y) = 0$ for $0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1$.

[5] M. Fréchet, Nouvelles Annales de Mathématiques (4), vol. 10 (1910), pp. 247–248.

In the present paper we shall say that $\chi(x, y)$ is *of bounded variation*, if $\chi(x, y)$ satisfies both conditions (β) and (γ).[6]

A function $\varphi(x, y)$ defined in U will be called *monotonic* if

(β') $\quad \Delta(\varphi; \xi, \eta, x, y) \geqq 0$ for $0 \leqq x < \xi \leqq 1,\ 0 \leqq y < \eta \leqq 1,$

(γ') $\varphi(x, 0) = \varphi(0, y) = 0$ for $0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1.$

It follows in particular that a monotonic function $\varphi(x, y)$ is non-decreasing in x, for y fixed, and also non-decreasing in y, for x fixed. Hence $x \leqq x'$, $y \leqq y'$ imply $\varphi(x, y) \leqq \varphi(x', y')$. Hardy[7] has shown that a function $\chi(x, y)$ of bounded variation may be expressed in the form

$$\chi(x, y) = \varphi(x, y) - \psi(x, y), \tag{4}$$

where $\varphi(x, y)$ and $\psi(x, y)$ are both monotonic.

We shall need the following

LEMMA 1. *Let $\{\chi_n(x, y)\}$ be a sequence of functions which are uniformly of bounded variation in U. Then there exists a subsequence $\{\chi_m(x, y)\}$ of the given sequence and a function $\chi(x, y)$ of bounded variation in U such that throughout U*

$$\lim_{m \to \infty} \chi_m(x, y) = \chi(x, y). \tag{5}$$

If $f(x, y)$ is continuous in U, then

$$\lim_{m \to \infty} \int_0^1 \int_0^1 f(x, y)\, dx\, dy\, \chi_m(x, y) = \int_0^1 \int_0^1 f(x, y)\, dx\, dy\, \chi(x, y). \tag{6}$$

Let us first consider the case when the functions $\chi_n(x, y)$ are monotonic (in the sense described above) and uniformly bounded. The set R of points (r, s) of U with both coördinates rational is denumerable. By the usual diagonal method of Cantor we construct a subsequence $\{\chi_{n'}(x, y)\}$ converging at all rational points (r, s) to a limit which we call $\bar{\chi}(r, s)$. The function $\bar{\chi}(r, s)$ thus defined over the set R obviously satisfies the monotoneity condition (β') and (γ') with respect to this set R. A theorem of W. H. Young[8] shows that $\bar{\chi}(r, s)$ will have a definite limit at all points

[6] According to the definition of Hardy and Krause a function $\chi(x, y)$ is called of bounded variation if (β) is satisfied and both $\chi(x, 0)$ and $\chi(0, y)$ are of bounded variation in the sense of Jordan. See E. W. Hobson, *Functions of a real variable*, vol. 1 (3rd edition, 1927), § 254. However, the more restricted condition (γ) will be very convenient for our present purpose. See also C. A. Fisher, *Linear functionals of n-spreads*, Annals of Math. (2), vol. 19 (1917–18), pp. 38–39.

[7] G. H. Hardy, *On double Fourier series*, Quarterly Journal of Math., vol. 37 (1906), pp. 57–59.

[8] W. H. Young, *On multiple integrals*, Proc. of the Royal Soc., vol. 93 (1917), p. 31. See also E. W. Hobson, loc. cit., § 308. Young's proof of his theorem, which is stated for a function defined in every point of U, obviously holds also for a function $\bar{\chi}(r, s)$ defined on the set R.

of U with the possible exception of a set S of points contained in a denumerable set of line segments of the form

$$V_i: \quad x = \xi_i, \quad 0 \leqq y \leqq 1 \quad (0 \leqq \xi_i \leqq 1;\ i = 1, 2, 3, \cdots),$$
$$H_j: \quad y = \eta_j, \quad 0 \leqq x \leqq 1 \quad (0 \leqq \eta_j \leqq 1;\ j = 1, 2, 3, \cdots).$$

Extend by continuity the definition of $\bar{\chi}(r, s)$ at all points not in S. Then $\chi_{n'}(x, y)$ converges to $\bar{\chi}(x, y)$ at every such point. Indeed, $\varepsilon > 0$ being given, one may choose two points (r_1, s_1), and (r_2, s_2) with $r_1 \leqq x \leqq r_2$, $s_1 \leqq y \leqq s_2$ and such that

$$\bar{\chi}(x, y) - \varepsilon < \bar{\chi}(r_1, s_1) \leqq \bar{\chi}(x, y) \leqq \bar{\chi}(r_2, s_2) < \bar{\chi}(x, y) + \varepsilon.$$

On the other hand

$$\chi_{n'}(r_1, s_1) \leqq \chi_{n'}(x, y) \leqq \chi_{n'}(r_2, s_2),$$

and hence, on account of the convergence at rational points,

$$\bar{\chi}(x, y) - \varepsilon < \chi_{n'}(r_1, s_1), \qquad \chi_{n'}(r_2, s_2) < \bar{\chi}(x, y) + \varepsilon$$

for n' sufficiently large, that is to say,

$$|\bar{\chi}(x, y) - \chi_{n'}(x, y)| < \varepsilon \text{ for } n' \text{ sufficiently large.}$$

Hence $\chi_{n'}(x, y) \to \bar{\chi}(x, y)$ at all the points where this function is defined. On the line segment V_1, the $\chi_{n'}$ are functions of y alone and we may get a subsequence $\chi_{n_i'}$ converging on V_1, by applying the theorem of Helly. In the same way we get a subsequence $\chi_{n_2'}$ (of $\chi_{n_1'}$) converging at every point of H_1; then a subsequence $\chi_{n_3'}$ (of $\chi_{n_2'}$) converging at every point of V_2, and so on. The diagonal sequence of this sequence of sequences, say χ_m, converges at every point of U to a function $\chi(x, y)$ which is necessarily monotonic. We have $\chi = \bar{\chi}$ on the set R and also outside of S.

If $\chi_n(x, y)$ are uniformly of bounded variation, then

$$\chi_n(x, y) = \varphi_n(x, y) - \psi_n(x, y),$$

where φ_n and ψ_n are monotonic and uniformly bounded. There is a subsequence $\varphi_{n'}$ (of φ_n) converging to a monotonic function φ, and similarly a subsequence ψ_m (of $\psi_{n'}$) converging to a monotonic function ψ. Then $\chi_m = \varphi_m - \psi_m \to \chi = \varphi - \psi$ throughout U and the first part of our lemma is proved.

To prove (6) it suffices to use the relation (5) only for the points of R. Let M denote an upper bound for the total variations of χ_m and χ. Let

$O(\delta)$ be the least upper bound of the oscillations of $f(x, y)$ in every rectangle $x_1 \leqq x \leqq x_2$, $y_1 \leqq y \leqq y_2$ of both dimensions $\leqq \delta$. For $\varepsilon > 0$, the uniform continuity of $f(x, y)$ implies the existence of a δ with $O(\delta) < \varepsilon/3M$. Let the numbers x_i and y_j defining the divisions (2) be rational and all subintervals $\leqq \delta$. H. E. Bray has shown (loc. cit. His proof for simple integrals applies for double integrals as well) that

$$\left| \int_0^1 \int_0^1 f\, dx\, dy\, \chi_m - \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} f(\varrho_i, \sigma_j)\, \Delta(\chi_m; r_{i+1}, s_{j+1}, r_i, s_j) \right| \leqq O(\delta)\, M < \varepsilon/3$$

and also

$$\left| \int_0^1 \int_0^1 f\, dx\, dy\, \chi - \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} f(\varrho_i, \sigma_j)\, \Delta(\chi; r_{i+1}, s_{j+1}, r_i, s_j) \right| \leqq O(\delta)\, M < \varepsilon/3.$$

By (5), the two double sums will differ by less than $\varepsilon/3$ for m sufficiently large, hence

$$\left| \int_0^1 \int_0^1 f\, dx\, dy\, \chi_m - \int_0^1 \int_0^1 f\, dx\, dy\, \chi \right| < \varepsilon$$

for m sufficiently large. Hence Lemma 1 is completely proved.

We prove next the following

LEMMA 2.[9] *Let $\chi(x, y)$ be a function of bounded variation in U. A necessary and sufficient condition so that*

$$\int_0^1 \int_0^1 f(x, y)\, dx\, dy\, \chi(x, y) = 0 \tag{7}$$

for every function $f(x, y)$ continuous in U, is that $\chi(x, y)$ shall vanish throughout U with the possible exception of a set of points lying on two denumerable sets of line segments of the form

$$\begin{aligned} &V_i\colon\ x = \xi_i, \quad 0 < y \leqq 1 \qquad (0 < \xi_i < 1;\ i = 1, 2, 3, \cdots), \\ &H_j\colon\ y = \eta_j, \quad 0 < x \leqq 1 \qquad (0 < \eta_j < 1;\ j = 1, 2, 3, \cdots). \end{aligned} \tag{8}$$

The sufficiency of this condition is immediately proved, since one may take the divisions (2), which lead to the definition of our integral (7), so as to avoid the line segments V_i and H_j. Then the double sum (3) is always zero and hence (3) leads to (7).

To prove the necessity, we set $f(x, y)$ equal to $f(x)$ for every y, or $g(y)$ for every x, where $f(x)$ and $g(y)$ are arbitrary continuous functions on (0, 1), and obtain

$$\int_0^1 f(x)\, dx\, \chi(x, 1) = 0 \text{ and } \int_0^1 g(y)\, dy\, \chi(1, y) = 0,$$

[9] This lemma is due to C. A. Fisher, loc. cit.

respectively. From this we conclude[10] that $\chi(x, 1)$ and $\chi(1, y)$ vanish identically except for a denumerable set of values ξ_i' of x and η_j' of y interior to the interval $(0, 1)$.

If now $f(x, y)$ be assumed to be any continuous function of bounded variation, then the $\int_0^1\int_0^1 \chi(x, y)\,d_x d_y f(x, y)$ exists also and the integration by parts process is applicable[11], which because of the behavior of $\chi(x, y)$ on the boundary and at the vertices of U gives:

$$\int_0^1\int_0^1 f(x, y)\,d_x d_y \chi(x, y) = \int_0^1\int_0^1 \chi(x, y)\,d_x d_y f(x, y). \tag{9}$$

If now, following Riesz, we allow $f(x, y)$ to range over the set of monotonic continuous functions:

$$f(x, y) = \begin{cases} xy & \text{for } 0 \leqq x \leqq \xi, \quad 0 \leqq y \leqq \eta, \\ \xi y & \text{for } \xi \leqq x \leqq 1, \quad 0 \leqq y \leqq \eta, \\ x\eta & \text{for } 0 \leqq x \leqq \xi, \quad \eta \leqq y \leqq 1, \\ \xi\eta & \text{for } \xi \leqq x \leqq 1, \quad \eta \leqq y \leqq 1, \end{cases}$$

then we obtain for every (ξ, η):

$$\int_0^\xi\int_0^\eta \chi(x, y)\,dx\,dy = 0.$$

As a consequence, we have

$$\frac{1}{\delta\xi\,\delta\eta}\int_\xi^{\xi+\delta\xi}\int_\eta^{\eta+\delta\eta} \chi(x, y)\,dx\,dy = 0$$

from which we conclude that $\chi(x, y) = 0$ at all points of continuity of $\chi(x, y)$. Utilizing the continuity properties of $\chi(x, y)$ and the fact that $\chi(x, 1)$ and $\chi(1, y)$ vanish except at a denumerable set of points interior to the interval, we obtain the desired result that $\chi(x, y)$ must vanish at every point of U excepting perhaps points lying on one of the segment V_i and H_j of (8).

From the fact that a continuous function can be uniformly approximated by a sequence of polynomials, it follows that equation (7) is equivalent to

$$\int_0^1\int_0^1 x^m y^n\,d_x d_y\,\chi(x, y) = 0 \text{ for } m, n = 0, 1, 2, \cdots. \tag{10}$$

By applying Lemma 2, we obtain information concerning the determination of a function of bounded variation by its moments in the following corollary which is equivalent to Lemma 2:

[10] See F. Riesz, Annales de l'École Normale Supérieure, ser. 3, vol. 28 (1911), pp. 38-39.

[11] See Hobson, *Functions of a Real Variable*, § 448. If in the definition of the Stieltjes integral, we assume that ξ_{ij} is independent of j, and η_{ij} independent of i, then the condition that $f(x, y)$ be of bounded variation may be omitted.

COROLLARY. *If $\chi(x, y)$ and $\bar{\chi}(x, y)$ are any two functions of bounded variation, then a necessary and sufficient condition that they have the same moments, i. e. that the set of equations*

$$\int_0^1\int_0^1 x^m y^n \, d_x d_y \, \chi(x, y) = \int_0^1\int_0^1 x^m y^n \, d_x d_y \, \bar{\chi}(x, y) \qquad (m, n = 0, 1, 2, \cdots) \tag{11}$$

shall hold, is that $\chi(x, y) = \bar{\chi}(x, y)$ everywhere in U with the possible exception of a set of points contained in a set of line segments of the form (8).

4. **The theorem of Hausdorff for double sequences.** Let μ_{mn} $(m, n = 0, 1, 2, \cdots)$ be a real double sequence. We attribute to the expression

$$\Delta_1^k \Delta_2^h \mu_{mn}$$

the usual meaning where the first operator Δ_1^k applies to the first subscript m and the second operator Δ_2^h applies independently to the second subscript n. We shall say that the given double sequence is *completely monotonic* if all the inequalities

$$\Delta_1^k \Delta_2^h \mu_{mn} \geqq 0 \qquad (k, h, m, n = 0, 1, 2, \cdots),$$

are satisfied. It is readily seen that μ_{mn} is completely monotonic in each subscript for every fixed value of the other subscript. The most general completely monotonic sequence is given by the following

THEOREM 1. *Let $\mu_{mn}(m, n = 0, 1, 2, \cdots)$ be a given real double sequence. A necessary and sufficient condition that there exists a monotonic function $\chi(x, y)$ satisfying the system*

$$\mu_{mn} = \int_0^1\int_0^1 x^m y^n \, d_x \, d_y \, \chi(x, y) \qquad (m, n = 0, 1, 2, \cdots), \tag{12}$$

is that the system of linear inequalities

$$\Delta_1^k \Delta_2^h \mu_{mn} \geqq 0 \qquad (k, h, m, n = 0, 1, 2, \cdots), \tag{13}$$

shall be satisfied.

A necessary and sufficient condition that there exists a function $\chi(x, y)$ of bounded variation satisfying the system (12) *is that the sequence*

$$\sum_{m,n=0}^{p} \binom{p}{m}\binom{p}{n} |\Delta_1^{p-m} \Delta_2^{p-n} \mu_{mn}| \qquad (p = 0, 1, 2, \cdots) \tag{14}$$

shall be bounded.

The proof given by Schoenberg (loc. cit.) for the similar theorem regarding simple sequences is readily extended. We show that μ_{mn} defined by (12) with χ monotonic is the most general solution of the system of linear inequalities (13). The identity

$$\Delta_1^k \Delta_2^h \mu_{mn} = \int_0^1 \int_0^1 x^m y^n (1-x)^k (1-y)^h \, dx \, dy \, \chi(x, y)$$

shows that (12) actually defines a solution of (13). To prove the converse that every solution μ_{mn} of (13) may be expressed in the form (12), we first solve the finite system

(15) $$\Delta_1^k \Delta_2^h \mu_{mn} \geqq 0 \qquad (k+m \leqq p;\ h+n \leqq p),$$

involving the $(p+1)^2$ variables $\mu_{mn}\,(m, n = 0, 1, \cdots, p)$. The identities

$$\Delta_1^k \Delta_2^h \mu_{mn} = \Delta_1^k \Delta_2^h \mu_{m+1,n} + \Delta_1^{k+1} \Delta_2^h \mu_{mn},$$
$$\Delta_1^k \Delta_2^h \mu_{mn} = \Delta_1^k \Delta_2^h \mu_{m,n+1} + \Delta_1^k \Delta_2^{h+1} \mu_{mn},$$

show that (15) is a consequence of its partial system

(16) $$\Delta_1^{p-m} \Delta_2^{p-n} \mu_{mn} \geqq 0 \qquad (m, n = 0, 1, \cdots, p).$$

The solution of this last system amounts to the inversion of the linear transformation

(17) $$\varrho_{pmn} = \Delta_1^{p-m} \Delta_2^{p-n} \mu_{mn} \qquad (m, n = 0, 1, \cdots, p).$$

Let us consider the new operator

$$\nabla^k u_m = u_m + \binom{k}{1} u_{m+1} + \cdots + u_{m+k}.$$

We have seen in the case of simple sequences that the two linear transformations

$$v_m = \Delta^{p-m} u_m \text{ and } u_m = \nabla^{p-m} v_m \qquad (m = 0, 1, \cdots, p)$$

are inverse to each other (Schoenberg, loc. cit., systems (6) and (7)). Applying this result to (17) we get successively

$$\Delta_2^{p-n} \mu_{mn} = \nabla_1^{p-m} \varrho_{pmn}, \qquad \mu_{mn} = \nabla_2^{p-n} \nabla_1^{p-m} \varrho_{pmn}.$$

Hence

(18) $$\mu_{mn} = \nabla_1^{p-m} \nabla_2^{p-n} \varrho_{pmn} \qquad (m, n = 0, 1, \cdots, p)$$

is the linear transformation which is inverse to (17). Hence (18), with $\varrho_{pmn} \geqq 0$, represents the most general solution of the system (15). The explicit form of (18) is

$$\mu_{mn} = \sum_{u=0}^{p-m} \sum_{v=0}^{p-n} \binom{p-m}{u} \binom{p-n}{v} \varrho_{p,m+u,n+v}$$

or

$$\mu_{mn} = \sum_{r=m}^{p} \sum_{s=n}^{p} \binom{p-m}{r-m} \binom{p-n}{s-n} \varrho_{prs}.$$

Introducing the quantities

$$\lambda_{pmn} = \binom{p}{m}\binom{p}{n}\varrho_{pmn} = \binom{p}{m}\binom{p}{n}\Delta_1^{p-m}\Delta_2^{p-n}\mu_{mn} \tag{19}$$

the last system becomes

$$\mu_{mn} = \sum_{r,s=0}^{p} \frac{r(r-1)\cdots(r-m+1)}{p(p-1)\cdots(p-m+1)} \cdot \frac{s(s-1)\cdots(s-n+1)}{p(p-1)\cdots(p-n+1)}\lambda_{prs}, \tag{20}$$

$$(m, n = 0, 1, \cdots, p).$$

Let us define in U a step-function $\chi_p(x, y)$ as follows:

(a) $\chi_p(x, 0) = \chi_p(0, y) = 0$.

(b) $\Delta\left(\chi_p; \frac{m}{p}+0, \frac{n}{p}+0, \frac{m}{p}-0, \frac{n}{p}-0\right) = \lambda_{pmn} \quad (m, n = 0, 1, \cdots, p)$.

(c) $\chi_p(x, y)$ is constant in each of the rectangles $m/p < x < (m+1)/p$, $n/p < y < (n+1)/p$, as well as on each of the line segments $m/p < x < (m+1)/p$, $y = 1$; $x = 1$, $n/p < y < (n+1)/p$; and $\chi_p(x, y) = \chi_p(x+0, y+0)$ for $0 < x \leqq 1$, $0 < y \leqq 1$.

From (16) and (19) we see that $\chi_p(x, y)$ is monotonic in U.

Suppose that μ_{mn} is a solution of (13). Then (20) holds for any value of p with $\lambda_{prs} \geqq 0$ and

$$\chi_p(1, 1) = \sum_{r,s=0}^{p} \lambda_{prs} = \mu_{00}.$$

On the other hand, for m and n fixed and for $p \geqq m$, $p \geqq n$, (20) may be written in the form

$$\mu_{mn} = \int_0^1\int_0^1 \frac{x\left(x-\frac{1}{p}\right)\left(x-\frac{2}{p}\right)\cdots\left(x-\frac{m-1}{p}\right)}{\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right)\cdots\left(1-\frac{m-1}{p}\right)}$$

$$\times \frac{y\left(y-\frac{1}{p}\right)\cdots\left(y-\frac{n-1}{p}\right)}{\left(1-\frac{1}{p}\right)\cdots\left(1-\frac{n-1}{p}\right)} d_x\, d_y\, \chi_p(x, y) \tag{21}$$

$$= \int_0^1\int_0^1 x^m y^n\, d_x\, d_y\, \chi_p(x, y) + O\left(\frac{1}{p}\right).$$

The remark that the sequence of monotonic functions $\chi_p(x, y)$ is uniformly bounded and Lemma 1 readily complete the proof of the first part of our theorem. The proof for the second part of the theorem which is concerned with the solution of bounded variation is quite similar to the proof in the case of a simple sequence and needs no further explanation.

For statistical purposes it might be of interest to know in what case the monotonic function $\chi(x, y)$ in (12) is also continuous. W. H. Young has shown (loc. cit.) that the continuity of both functions $\chi(x, 1)$ $(0 \leqq x \leqq 1)$ and $\chi(1, y)$ $(0 \leqq y \leqq 1)$ is necessary and sufficient for the continuity of $\chi(x, y)$ in U. Criteria for the continuity of these functions may be derived from the formulas

$$\mu_{m0} = \int_0^1 x^m \, d_x \, \chi(x, 1), \qquad \mu_{0n} = \int_0^1 y^n \, d_y \, \chi(1, y) \qquad (m, n = 0, 1, 2, \cdots),$$

(which follow from (12), for $f(x, y) = x^m$ and y^n respectively, by integration by parts), applying one of the criteria for the similar problem for simple sequences.[12]

5. The theorem of F. Riesz on linear functional operations for continuous functions in two variables. The following extension of the Riesz theorem is due to C. A. Fisher (loc. cit.):

THEOREM 2. *Every continuous linear functional operation $L[f]$ defined for functions $f(x, y)$ which are continuous in the unit-square $U(0 \leqq x \leqq 1, 0 \leqq y \leqq 1)$ may be represented as a Stieltjes integral*

$$L[f] = \int_0^1 \int_0^1 f(x, y) \, d_x \, d_y \, \chi(x, y), \tag{22}$$

with $\chi(x, y)$ of bounded variation in U. The function $\chi(x, y)$ is uniquely defined in the sense of Lemma 2.

A first proof of this theorem follows immediately from our Theorem 1 by a direct extension of the considerations of § 1 and needs no further explanations. We prefer to extend the direct proof for the Riesz theorem given in § 2.

Suppose that

$$L[f] \leqq M \cdot \|f\| \tag{23}$$

holds for every f continuous in U. The Bernstein polynomials

$$B_{pq}(f) = \sum_{m=0}^{p} \sum_{n=0}^{q} f\left(\frac{m}{p}, \frac{n}{q}\right) \binom{p}{m} \binom{q}{n} x^m y^n (1-x)^{p-m} (1-y)^{q-n} \tag{24}$$

converge uniformly[13] to $f(x, y)$ as $p \to \infty$, $q \to \infty$. Let us define in U a step-function $\chi_p(x, y)$ by the conditions (a) and (c) of Section 4, while (b) is replaced by the following condition

[12] See Th. Kaluza, Math. Zeitschrift, vol. 28 (1928), pp. 200–202, and I. J. Schoenberg, Math. Zeitschrift, vol. 30 (1929), pp. 761–767.

[13] Bernstein's original proof for the case of one variable can be readily extended.

$$\text{(b')} \qquad \begin{aligned} &\Delta\left(\chi_p;\ \frac{m}{p}+0,\ \frac{n}{p}+0,\ \frac{m}{p}-0,\ \frac{n}{p}-0\right) \\ &= \binom{p}{m}\binom{p}{n} L[x^m y^n (1-x)^{p-m}(1-y)^{p-n}] \qquad (m, n = 0, 1, \cdots, p). \end{aligned}$$

For suitable values $\varepsilon_{mn} = \pm 1$ we get as a consequence of (23)

$$\begin{aligned} V(\chi_p) &= \sum_{m,n=0}^{p} \varepsilon_{mn} \binom{p}{m}\binom{p}{n} L[x^m y^n (1-x)^{p-m}(1-y)^{p-n}] \\ &= L\left[\sum_{m,n} \varepsilon_{mn} \binom{p}{m}\binom{p}{n} x^m y^n (1-x)^{p-m}(1-y)^{p-n}\right] \leqq M, \end{aligned}$$

which shows that the functions χ_p are uniformly of bounded variation in U. On the other hand from (24) we get

$$\begin{aligned} L[B_{pp}(f)] &= \sum_{m,n=0}^{p} f\left(\frac{m}{p}, \frac{n}{p}\right)\binom{p}{m}\binom{p}{n} L[x^m y^n (1-x)^{p-m}(1-y)^{p-n}] \\ &= \int_0^1\int_0^1 f(x, y)\, d_x\, d_y\, \chi_p(x, y). \end{aligned}$$

The uniform convergence of the $B_{pp}(f)$ to $f(x, y)$ and Lemma 1 complete the proof as in Section 2.

We finally remark that Hildebrandt's derivation of the Hausdorff theorem from the theorem of Riesz (loc. cit.) may also be immediately extended to prove our Theorem 1. It is based on a property of the Bernstein polynomials expressed by the relation

$$B_{pp}(P_{rs}) = P_{rs}(x, y) + \sum_{u=1}^{r-1}\sum_{v=1}^{s-1} A_{uv}(x, y)/p^{u+v},$$

where P_{rs} is a polynomial in (x, y) of degree (r, s) and the A_{uv} are polynomials in (x, y), of degrees not exceeding (r, s), which do not depend on p.

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(ADDED IN PROOF, MARCH 1933). J. D. Tamarkin has kindly called our attention to the fact that Lemma I is not new but was given by J. Radon, Wiener Sitzungsberichte, vol. 122 IIa (1913), pp. 1337-1342 and vol. 128 IIa (1919), pp. 1092-1094. Radon's paper should also be mentioned in connection with Fisher's Theorem 2.

ON THE SUMMABILITY OF FOURIER SERIES. II.*

By Einar Hille and J. D. Tamarkin.

1. **Introduction.** The present paper is the second of a series of memoirs devoted to the general theory of summability of Fourier series. The first and third memoirs of the series[1] are dealing with the problem of summation of Fourier series with the aid of Nörlund's and Hausdorff's means respectively. Here we are concerned with general definitions of summation which satisfy the condition of being "effective" in a certain sense when used for summation of Fourier series. We establish relationships between definitions of summation possessing various effectiveness properties. Most of our results are valid not only for trigonometric Fourier series, but for general Fourier series associated with arbitrary orthonormal sets.

The following terminology and notation will be employed. Let

$$u_0 + u_1 + \cdots + u_n + \cdots \tag{1.01}$$

be an arbitrary series, convergent or not, and $\mathfrak{A} = (a_{mn})$ be a matrix of a definition of summation, which consists in constructing the "m-th transform"

$$t_m = \sum_{n=0}^{\infty} a_{mn}\, u_n \tag{1.02}$$

of (1.01) and evaluating the "generalized sum" of (1.01) as defined by

$$s = \lim_{m\to\infty} t_m. \tag{1.03}$$

For brevity we shall speak simply of a "transformation $\mathfrak{A}$".

Let $f(x)$ be an arbitrary function integrable over a given interval (a, b). If in addition

$$\int_a^b |f(x)|^p\, dx < \infty, \quad p \geqq 1,$$

we shall say that $f(x) \subset L_p$. It is convenient to allow $p = \infty$ and to introduce the corresponding class L_∞. It is customary to identify this class with the class B of functions measurable and essentially bounded on

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[1] Hille-Tamarkin [1, 2]. The numbers in brackets refer to the Bibliography at the end of the present paper.

(a, b) (when sets of values of x of measure zero can be disregarded). For reasons to be explained below, we shall depart from this usual notation and shall designate by L_∞ the class of all functions continuous on (a, b).

The class L_p becomes a linear vector metric complete space[2] if we introduce the norm of the element $f \subset L_p$,

$$\|f\|_p \equiv \left[\frac{1}{b-a}\int_a^b |f(x)|^p\,dx\right]^{1/p} \text{ if } p<\infty; \qquad \|f\|_\infty = \max_{a\leq x\leq b} |f(x)|, \tag{1.04}$$

and define the distance between two "points" f, g of the space L_p as

$$(f, g) \equiv \|f-g\|_p. \tag{1.05}$$

Let

$$\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x), \cdots; \quad \int_a^b \varphi_i(x)\,\overline{\varphi_j(x)}\,dx = \delta_{ij} \tag{1.06}$$

be an arbitrary orthonormal set for the interval (a, b). It will be assumed throughout this paper that the set $\{\varphi_n(x)\}$ satisfies the following conditions.

(α) The functions $\varphi_n(x)$ are continuous.

(β) The set $\{\varphi_n(x)\}$ is closed in L_∞, which means that an arbitrary $f(x) \subset L_\infty$ can be approximated in L_∞ as closely as we like by means of finite linear combinations of the functions $\varphi_n(x)$.

As a consequence of condition (β) we state

(γ) The set $\{\varphi_n(x)\}$ is complete in L_1, which means that the conditions

$$\int_a^b f(x)\,\overline{\varphi_n(x)}\,dx = 0, \qquad n = 1, 2, \cdots; \quad f \subset L_1 \tag{1.07}$$

imply $f(x) = 0$ almost everywhere.[3]

Let $f(x)$ be an arbitrary function $\subset L_p$. On setting

$$f_n = \int_a^b f(x)\,\overline{\varphi_n(x)}\,dx, \qquad n = 1, 2, \cdots \tag{1.08}$$

we designate by $S(x; f)$ the formal Fourier series of f associated with the set $\{\varphi_n(x)\}$,

$$f(x) \sim S(x; f) \equiv \sum_{n=1}^{\infty} f_n\,\varphi_n(x). \tag{1.09}$$

We shall say that this series $S(x; f) \subset L_p$ if its generating function $f(x) \subset L_p$

Let us apply the transformation $\mathfrak{A}$ to the series (1.09). We shall be concerned with the problem of convergence in L_p of the m-th transform

[2] We refer to Hildebrandt [1] and Banach [1] concerning the terminology of the general theory of linear operations and transformations used in the present paper.

[3] Orlicz 1, pp. 4, 6.

of this series to the function $f(x)$. There is no difficulty in computing the m-th transform of (1.09) when $\mathfrak{A}$ is of finite reference. When $\mathfrak{A}$ is of infinite reference, the series

$$\sum_{n=1}^{\infty} a_{mn} f_n \varphi_n(x) \tag{1.10}$$

defining this m-th transform may not converge. Still it can be used for our purposes provided (1.10) is also a Fourier series $\subset L_p$. In this case we define as the m-th transform of (1.09) the function which generates (1.10), and put

$$\tau_m(x; f) \sim \sum_{n=1}^{\infty} a_{mn} f_n \varphi_n(x).$$

With this notation we can formulate the following

DEFINITION 1.1. *A transformation $\mathfrak{A}$ is said to be effective in the mean of order p, or simpler, (L_p)-effective, if it satisfies the conditions*

(E. 1) *The series*

$$\tau_m(x; f) \sim \sum_{n=1}^{\infty} a_{mn} f_n \varphi_n(x) \subset L_p, \qquad m = 1, 2, \cdots \tag{1.11}$$

whenever $f(x) \subset L_p$;

(E. 2) *The m-th transform $\tau_m(x; f)$ converges to $f(x)$ in L_p,*

$$\|\tau_m(x; f) - f(x)\|_p \to 0 \quad \text{as} \quad m \to \infty. \tag{1.12}$$

If conditions (E. 1), (E. 2) *are satisfied not for all $f(x) \subset L_p$, but only for those of a certain sub-class L_p^0 of L_p, we shall say that $\mathfrak{A}$ is (L_p)-effective on L_p^0.*

It is customary to designate by p' the conjugate exponent,

$$p' = \frac{p}{p-1};\ \infty;\ \text{or } 1 \text{ according as } 1 < p < \infty;\ p = 1;\ \text{or } p = \infty \tag{1.13}$$

respectively, so that always

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{1.14}$$

It will be assumed that $1 \leqq p \leqq \infty$.

Our main result can now be stated as follows:

(A) *The classes of (L_p)- and of $(L_{p'})$-effective transformations are identical.*

(B) *If $\mathfrak{A}$ is (L_p)-effective, then $\mathfrak{A}$ is also (L_q)-effective for an arbitrary q between p and p'.*

We see therefore that the property of being (L_p)-effective is symmetric with respect to $p = 2$, and that the class of (L_p)-effective transformations, considered as a function of p, increases or decreases, according as p increases from 1 to 2, or from 2 to ∞, reaching its possible maximum for $p = 2$.

There is a considerable number of articles in the literature dealing with (L_p)-effectiveness, and particularly, with (L_1)- and (L_∞)-effectiveness of

various specified classes of transformations.[4] To the best of our knowledge, however, the relationship between the various classes of "effective" transformations has not been investigated.

The reader will notice that our Theorems (A) and (B) strongly resemble some well-known results of the theory of factor-sequence transformations of Fourier series.[5] It will be seen in the sequel that the resemblance is not merely formal, and that these results are fundamental for the proofs of our theorems.

Our discussion is considerably simplified by using some simple facts of the general theory of linear transformations in abstract spaces. In order not to interrupt the exposition we collect all necessary preliminaries, in the form of lemmas, in § 2. The general results concerning (L_p)-effective transformations are derived in § 3. The remainder of the paper deals with trigonometric Fourier series. In § 4 we give the general form of certain, fairly extended, subclasses of (L_1)-, and (L_∞)-effective transformations for trigonometric Fourier series. In the last § 5 we discuss the (L_1)-effectiveness of the "convergence" transformation for some special classes of sine and cosine Fourier series. From our present point of view there is an essential difference between the "identity" and the "convergence" transformations, whose matrices $\mathfrak{A}^i$ and $\mathfrak{A}^c$ are represented respectively by

$$(\mathfrak{A}^i) \qquad a_{mn} = 1, \qquad m, n = 1, 2, \cdots;$$

$$(\mathfrak{A}^c) \qquad a_{mn} = \begin{cases} 1, & n = 1, 2, \cdots, m, \\ 0, & n = m+1, \cdots. \end{cases}$$

[4] Without any pretense of being exhaustive we mention the following memoirs which have points of contact with the ideas or methods of the present paper: Gross [1]; Hahn [1]; Orlicz [1]; W. H. and G. C. Young [1]; Zygmund [1]. The definition of (L_∞)-effectiveness is closely related, but not identical, to the definition of (F)-effectiveness introduced in our memoirs [1, 2]. It should be observed that the property of being (L_∞)-effective implies the uniform convergence of the m-th transform $\tau_m(x; f)$ to the generating (continuous) function $f(x)$. This property is apparently more restrictive than (L_1)-effectiveness, so that the identity of these two classes of transformations is far from being obvious a priori. In fact, the present paper has originated in an attempt to construct a transformation $\mathfrak{A}$ which is (L_1)-effective without being (L_∞)-effective. As observed before, our definition of the class L_∞ deviates from the usual one where L_∞ designates the class B of all essentially bounded measurable functions $g(x)$ with the metric $\| g \|_B =$ upper measurable bound of $| g(x) |$. In such a space, however, even the simplest and best known transformations obviously are not effective since, in general, the m-th transform $\tau_m(x; f)$ does not converge uniformly to $f(x)$. That is why we are compelled to restrict the space L_∞ to contain only continuous functions. [Nevertheless, the problem of effectiveness of a transformation in the space B can be attacked successfully as will be shown in an addition to the present paper, to appear in the next number of these Annals. Added in the proof.]

[5] M. Riesz [1] pp. 487–490, where other references are found. See also Orlicz [1]. Our notation will differ slightly from that of Riesz.

The matrix $\mathfrak{A}^l$ is of course always (L_p)-effective, $1 \leqq p \leqq \infty$, while $\mathfrak{A}^c$ is (L_p)-effective only for $1 < p < \infty$.

2. **Auxiliary propositions.** I. We start with some indispensable notions and facts of the general theory of transformations in abstract spaces. Let $U(x)$ be a transformation which coördinates with each point x of a linear vector metric complete space $\mathfrak{S}$[5a] a unique point y of another linear vector metric complete space $\mathfrak{S}_1$. The transformation U will be designated as an operation, if $\mathfrak{S}_1$ reduces to the space of real or, more generally, of complex numbers; in this case $\|U(x)\| = |U(x)|$. Without any danger of confusion we shall use the same notation for the norms in the spaces $\mathfrak{S}$ and $\mathfrak{S}_1$. We say that

1. $U(x)$ is continuous at x if $\|U(x') - U(x)\| \to 0$ whenever $\|x' - x\| \to 0$, and continuous (in $\mathfrak{S}$) if it is continuous at each $x \subset \mathfrak{S}$.
2. $U(x)$ is limited, with the bound M_U if

$$M_U = \underset{0 \neq x \subset \mathfrak{S}}{\text{l. u. b.}} \frac{\|U(x)\|}{\|x\|} = \underset{\|x\|=1}{\text{l. u. b.}} \|U(x)\| < \infty. \tag{2.01}$$

3. $U(x)$ is linear if

$$U(c_1 x_1 + c_2 x_2) = c_1 U(x_1) + c_2 U(x_2). \tag{2.02}$$

4. A sequence of transformations $\{U_n(x)\}$ is bounded on a set $E \subset \mathfrak{S}$ if there exists a constant C_E depending on E such that

$$\|U_n(x)\| \leqq C_E, \qquad x \subset E. \tag{2.03}$$

The sequence $\{U_n(x)\}$ converges to $U_0(x)$ on E if

$$\|U_n(x) - U_0(x)\| \to 0 \quad \text{as} \quad n \to \infty; \qquad x \subset E. \tag{2.04}$$

Of course E may reduce to a single point x.

5. A sequence of linear limited transformations $\{U_n(x)\}$ is uniformly limited if

$$M_{U_n} \leqq M < \infty \tag{2.05}$$

which is equivalent to the condition

$$\|U_n(x)\| \leqq M\|x\|, \qquad n = 1, 2, \cdots \tag{2.06}$$

where M does not depend on x or n.

The following lemma is fundamental in the theory of sequences of linear limited transformations.

LEMMA 2.1. *If a sequence of linear limited transformations $\{U_n(x)\}$ converges at each point of $\mathfrak{S}$ then the sequence $\{U_n(x)\}$ is uniformly limited.*[6] *The sequence $\{U_n(x)\}$ then converges to a linear limited transformation $U_0(x)$.*

[5a] A space of type (B) in the terminology of Banach.

[6] Banach [1], p. 80, Theorem 5.

It is well-known that in order that a linear transformation $U(x)$ be limited it is necessary and sufficient that either (i) $U(x)$ be bounded on the boundary of the unit sphere $\|x\| = 1$, or (ii) $U(x)$ be bounded in an arbitrary fixed sphere $\|x - x_0\| \leqq r_0$, or else (iii) $U(x)$ be continuous at a single point x_0.

We note that the most general form of a linear limited operation $l(f)$ on L_p $(1 \leqq p < \infty)$ is given by

$$l(f) = \int_a^b f(t)\, g(t)\, dt \tag{2.07}$$

with

$$g(t) \subset L_{p'}, \qquad M_l = \|g\|_{p'} \tag{2.08}$$

where, in the case $p = 1$, $g(t)$ is an arbitrary *bounded* function.

In the space L_∞ we have

$$l(f) = \int_a^b f(s)\, d\alpha(s), \qquad M_l = \int_a^b |d\alpha(s)| \equiv V(\alpha), \tag{2.09}$$

where $\alpha(s)$ is a function of bounded variation, normalized according to the condition

$$\alpha(s_0) = \frac{1}{2}[\alpha(s_0 + 0) + \alpha(s_0 - 0)], \qquad a < s_0 < b. \tag{2.10}$$

II. We shall need some properties of linear limited transformations on L_p to a sub-set of L_p.

LEMMA 2.2. *Let $U(f)$ be a linear limited transformation on L_p to a sub-set of L_p and $W(g)$ a linear limited transformation on $L_{p'}$ to a sub-set of $L_{p'}$, $1 \leqq p \leqq \infty$. If the relation*

$$\int_a^b g\, U(f)\, dx = \int_a^b f\, W(g)\, dx \tag{2.11}$$

is satisfied for an arbitrary pair of functions $f \subset L_p$, $g \subset L_{p'}$, then the transformations U and W have the same bounds

$$M_U = M_W. \tag{2.12}$$

Proof. Let M be the bound of the bilinear operation

$$Q(f, g) \equiv \frac{1}{b-a}\int_a^b g\, U(f)\, dx = \frac{1}{b-a}\int_a^b f\, W(g)\, dx,$$

viz.,

$$M = \text{l. u. b.}\, |Q(f, g)|, \qquad \|f\|_p = \|g\|_{p'} = 1.$$

Since

$$\left|\frac{1}{b-a}\int_a^b \varphi(x)\, \psi(x)\, dx\right| \leqq \|\varphi\|_p \|\psi\|_{p'}; \quad \varphi \subset L_p, \quad \psi \subset L_{p'},$$

we have

$$|Q(f,g)| \leqq \begin{cases} \|g\|_{p'} \|U(f)\|_p \leqq M_U, \\ \|f\|_p \|W(g)\|_{p'} \leqq M_W, \end{cases}$$

whence

$$M \leqq M_U, \qquad M \leqq M_W.$$

On the other hand, an arbitrarily small $\varepsilon > 0$ being given, we can find two functions $f_\varepsilon \subset L_p$, $g_\varepsilon \subset L_{p'}$ such that

$$\|U(f_\varepsilon)\|_p = M_U - \varepsilon, \qquad \|W(g_\varepsilon)\|_{p'} = M_W - \varepsilon, \qquad \|f_\varepsilon\|_p = \|g_\varepsilon\|_{p'} = 1.$$

At this juncture we have to consider separately the cases $1 < p < \infty$, $p = 1$, $p = \infty$. Without loss of generality we may assume that all the functions concerned are real-valued. Let $1 < p < \infty$. Upon putting

$$f'(x) = [\|W(g_\varepsilon)\|_{p'}]^{1-p'} |W(g_\varepsilon)|^{p'-1} \operatorname{sgn} W(g_\varepsilon), \qquad \|f'\|_p = 1,$$
$$g'(x) = [\|U(f_\varepsilon)\|_p]^{1-p} |U(f_\varepsilon)|^{p-1} \operatorname{sgn} U(f_\varepsilon), \qquad \|g'\|_{p'} = 1,$$

we find

$$Q(f', g_\varepsilon) = \|W(g_\varepsilon)\|_{p'} = M_W - \varepsilon < M,$$
$$Q(f_\varepsilon, g') = \|U(f_\varepsilon)\|_p = M_U - \varepsilon < M,$$

whence it follows $M = M_U = M_W$.

Now let $p = 1$, $p' = \infty$. There exists a function $g'' \subset L_\infty$ such that

$$Q(f_\varepsilon, g'') \geqq \|U(f_\varepsilon)\|_1 - \varepsilon \geqq M_U - 2\varepsilon, \qquad \|g''\|_\infty = 1.$$

Indeed, on putting $g_0 = \operatorname{sgn} U(f_\varepsilon)$ we have

$$Q(f_\varepsilon, g_0) = \|U(f_\varepsilon)\|_1.$$

The function g_0 is measurable and bounded and $\|g_0\|_B = 1$. It is plain that there exists a sequence of continuous functions $\{g_\nu(x)\}$ which converges to $g_0(x)$ almost everywhere, while $\|g_\nu\|_\infty = 1$. We have then

$$\int_a^b g_\nu(x)\, U(f_\varepsilon)\, dx \to \int_a^b g_0(x) U(f_\varepsilon)\, dx \quad \text{as} \quad \nu \to \infty.$$

Hence, for ν sufficiently large,

$$|Q(f_\varepsilon, g_\nu) - Q(f_\varepsilon, g_0)| < \varepsilon,$$

and we can take $g''(x) = g_\nu(x)$.

There also exists a function $f'' \subset L$, such that

$$Q(f'', g_\varepsilon) \geqq \|W(g_\varepsilon)\|_\infty - \varepsilon \geqq M_W - 2\varepsilon, \qquad \|f''\|_1 = 1.$$

Indeed, $W(g_\varepsilon) \subset L_\infty$ is continuous; there will exist an interval $I \subset (a, b)$ in which

$$\| W(g_\varepsilon) \|_\infty - \varepsilon \leqq | W(g_\varepsilon) | \leqq \| W(g_\varepsilon) \|_\infty .$$

Now define

$$f''(x) = \begin{cases} \dfrac{(b-a) \operatorname{sgn} W(g_\varepsilon)}{mI}, & x \subset I, \quad mI = \text{length of } I, \\ 0 \text{ elsewhere.} \end{cases}$$

Obviously

$$Q(f'', g_\varepsilon) = \frac{1}{mI} \int_I | W(g_\varepsilon) | \, dx \geqq \| W(g_\varepsilon) \|_\infty - \varepsilon, \qquad \| f'' \|_1 = 1.$$

Thus we conclude again that $M = M_U = M_W$. The case $p = \infty$, $p' = 1$ can be treated in an analogous fashion and Lemma 2.2 is now completely established.[7]

III. We now pass to the theory of factor-sequence transformations. A sequence of constants $\{a_n\}$ is designated as a sequence of class (p, p), $1 \leqq p \leqq \infty$, with respect to the orthonormal set $\{\varphi_n(x)\}$ if, when being applied to an arbitrary Fourier series

$$\sum_{n=1}^{\infty} f_n \varphi_n(x)$$

of class L_p, it leads to a Fourier series

$$\sum_{n=1}^{\infty} a_n f_n \varphi_n(x)$$

of class L_p. It is well-known[8] that the classes of factor-sequences (p, p) and (p', p') are identical.

LEMMA 2.3. *If a factor-sequence* $\{a_n\} \subset (p, p)$, $1 \leqq p \leqq \infty$, *then the transformation* $T_a(x; f) \equiv T_a(f)$ *defined by*

$$f(x) \sim \sum_{n=1}^{\infty} f_n \varphi_n(x) \subset L_p, \tag{2.13}$$

$$T_a(x; f) \sim \sum_{n=1}^{\infty} a_n f_n \varphi_n(x) \subset L_p \tag{2.14}$$

[7] In the case $1 < p < \infty$ Lemma 2.2 is also immediately derived from the fact that the transformations $U(f)$, $W(g)$ satisfying (2.11) are adjoint of each other, and so have the same bounds. Cf. Hildebrandt [1], p. 200; Banach [1], p. 100. In the case $p = 1$ (or $p = \infty$) a difficulty arises owing to the fact that the space L_∞ (or L_1) is only a non-dense part of the space where the adjoint of U (or of W) is defined. We prefer to give a direct proof instead of investigating this point, which can easily be done, however.

[8] M. Riesz [1], p. 489, in the case of trigonometric Fourier series. In the general case, Orlicz [1], pp. 22–24. It should be observed that our notation (∞, ∞) corresponds to (C, C) of Orlicz.

is a linear limited transformation on L_p to a sub-set of L_p. The corresponding transformation defined by

$$g(x) \sim \sum_{n=1}^{\infty} g_n \varphi_n(x) \subset L_{p'}, \tag{2.15}$$

$$T_a(x; g) \sim \sum_{n=1}^{\infty} a_n g_n \varphi_n(x) \subset L_{p'}, \tag{2.16}$$

is also a linear limited transformation on $L_{p'}$ to a sub-set of $L_{p'}$, and we have

$$M_a^{(p)} \equiv \underset{\|f\|_p=1}{\text{l. u. b.}} \| T_a(f) \|_p = M_a^{(p')} \equiv \underset{\|g\|_{p'}=1}{\text{l. u. b.}} \| T_a(g) \|_{p'}. \tag{2.17}$$

Furthermore, the relation

$$\int_a^b g(x)\, T_a(x; f)\, dx = \int_a^b f(x)\, T_a(x; f)\, dx \tag{2.18}$$

holds for an arbitrary pair of functions $f(x) \subset L_p$, $g(x) \subset L_{p'}$.

Proof. The facts stated in this lemma are partly known and partly are readily derived from various passages of the above-mentioned papers by M. Riesz and Orlicz. The fact that the transformations $T_a(f)$ and $T_a(g)$ are limited follows immediately from a general result of Banach.[9] When $1 \leqq p < \infty$ the relation (2.18) has been proved by Orlicz.[10] Being symmetric in f, g it holds also when $p = \infty$. Now it only remains to apply Lemma 2.2.

LEMMA 2.4. *If $\{a_n\} \subset (p, p) = (p', p')$, $1 \leqq p \leqq \infty$, then also $\{a_n\} \subset (q, q)$ for any q between p and p'. The transformation $T_a(x; h)$, $h \subset L_q$, is limited on every L_q and has the bound*

$$M_a^{(q)} \leqq M_a^{(p)} = M_a^{(p')}. \tag{2.19}$$

The expression $\log M_a^{(q)}$ is a convex function of q.[11]

3. **(L_p)-effective transformations.** We return to the notation of § 1 and first investigate conditions under which a transformation $\mathfrak{A}$ is (L_p)-effective. We observe that, in view of the above discussion, the m-th transforms $\tau_m(x; f)$, $\tau_m(x; g)$ of the Fourier series

$$f(x) \sim \sum_{n=1}^{\infty} f_n \varphi_n(x) \subset L_p, \tag{3.01}$$

$$g(x) \sim \sum_{n=1}^{\infty} g_n \varphi_n(x) \subset L_{p'} \tag{3.02}$$

[9] [1], p. 113, Theorem 8, proof.

[10] [1], p. 23.

[11] This lemma is merely a special case of a fundamental result of M. Riesz, [1], p. 481, Theorem V.

determine linear limited transformations on L_p, $L_{p'}$, respectively, provided condition (E.1) is satisfied for a fixed value of p or p'. Hence the problem of (L_p)-effectiveness reduces to the investigation of the convergence of the sequence of linear limited transformations $\tau_m(x;f)$ in the space L_p.

LEMMA 3.1. *If a transformation* $\mathfrak{A}$ *is* (L_p)*-effective for a fixed* p, $1 \leqq p \leqq \infty$, *then*

$$a_{mn} \to 1 \quad as \quad m \to \infty, \qquad n = 1, 2, \cdots. \tag{3.03}$$

Proof. If $\mathfrak{A}$ is (L_p)-effective, we must have

$$\|\tau_m(x;f) - f(x)\|_p \to 0 \quad \text{as} \quad m \to \infty$$

for an arbitrary function $f(x) \subset L_p$. On setting here $f(x) = \varphi_n(x)$ we obtain

$$\|\tau_m(x;\varphi_n) - \varphi_n(x)\|_p = |a_{mn} - 1| \, \|\varphi_n\|_p \to 0$$

which yields the desired result.

THEOREM 3.1. *In order that a transformation* $\mathfrak{A}$ *be* (L_p)*-effective,* $1 \leqq p \leqq \infty$, *it is necessary and sufficient that* (i) $\mathfrak{A}$ *be* (L_p)*-effective on every sub-set* L_p^0 *which is dense in* L_p, *and* (ii) *that the sequence of the associated m-th transforms of the series* (3.01) *be a uniformly limited sequence of transformations on* L_p. *Insofar as the sufficiency is concerned, condition* (i) *can be replaced by a less restrictive one, viz.,* (i′) $\mathfrak{A}$ *is* (L_p)*-effective on a (single) sub-set* L_p^0 *dense in* L_p.

Proof. The necessity of the condition (i) is obvious. The necessity of (ii) follows immediately from Lemma 2.1, since the (L_p)-effectiveness of the transformation $\mathfrak{A}$ implies

$$\|\tau_m(x;f) - f(x)\|_p \to 0 \quad \text{as} \quad m \to \infty,$$

i. e. the convergence in L_p of the sequence of linear limited transformations $\tau_m(x;f)$ at each point f of the space L_p.

Assume now that conditions (i′) and (ii) are satisfied. Then an arbitrary function $f(x) \subset L_p$ can be approximated in L_p as closely as we like by a function of L_p^0. Hence, if $f(x) \subset L_p$ and an arbitrarily small $\varepsilon > 0$ are given, we can find a function $\psi_\varepsilon(x) \subset L_p^0$ such that

$$\|f - \psi_\varepsilon\|_p \leqq \frac{\varepsilon}{2(M+1)}, \qquad M_{\tau_m} \leqq M.$$

Since $\mathfrak{A}$ is assumed to be (L_p)-effective on L_p^0 there exists an $m_0 = m_0(\varepsilon)$ such that

$$\|\tau_m(x;\psi_\varepsilon) - \psi_\varepsilon(x)\|_p \leqq \frac{\varepsilon}{2} \quad \text{for} \quad m \geqq m_0.$$

Then

$$\|\tau_m(x;f)-f(x)\|_p \leqq \|\tau_m(x;\psi_\varepsilon)-\psi_\varepsilon(x)\|_p+\|\tau_m(f-\psi_\varepsilon)\|_p+\|f-\psi_\varepsilon\|_p$$
$$\leqq \frac{\varepsilon}{2}+M\|f-\psi_\varepsilon\|_p+\|f-\psi_\varepsilon\|_p \leqq \varepsilon$$

which proves the sufficiency of conditions (i'), (ii).[12]

Remark 3.1. If condition (E.1) is satisfied, $\tau_m(x;f)$ is a linear limited transformation on L_p. Let $M_m^{(p)}$ be the bound of $\tau_m(x;f)$. Condition (ii) of Theorem 3.1 states that the sequence $\{M_m^{(p)}\}$ must be bounded. We set

$$\overline{\lim_{m\to\infty}}\, M_m^{(p)} \equiv M_{\mathfrak{A}}^{(p)} \tag{3.04}$$

and designate this constant as the (L_p)-bound of the transformation $\mathfrak{A}$.

THEOREM 3.2. *The classes of (L_p)-effective and of $(L_{p'})$-effective transformations are identical, and have the same (L_p)- and $(L_{p'})$-bounds.*

Proof. Let $\mathfrak{A}$ be (L_p)-effective. First we see, by Lemma 3.1, that

$$a_{mn}\to 1 \quad \text{as} \quad m\to\infty; \qquad n=1,2,\cdots. \tag{3.03}$$

Now consider the set L^0 consisting of all finite linear combinations of the functions $\varphi_n(x)$. By property (γ) (p. 330) of our orthonormal set $\{\varphi_n(x)\}$ the set L^0 is dense in L_∞. Since $\|f\|_{r_1} \leqq \|f\|_{r_2}$ whenever $r_1 \leqq r_2$, the set L^0 is also dense in any L_q, $1 \leqq q \leqq \infty$. It is readily seen that $\mathfrak{A}$ is $(L_{p'})$-effective on L^0. Indeed, let

$$\psi(x)=\sum_{\nu=1}^{s} c_\nu\,\varphi_\nu(x)$$

be an arbitrary function $\subset L^0$. Then, by (3.03),

$$\|\tau_m(x;\psi)-\psi(x)\|_{p'} = \left\|\sum_{\nu=1}^{s}(a_{m\nu}-1)\,c_\nu\,\varphi_\nu(x)\right\|_{p'}$$
$$\leqq \sum_{\nu=1}^{s}|a_{m\nu}-1|\,|c_\nu|\,\|\varphi_\nu\|_{p'} \to 0 \quad \text{as} \quad m\to\infty.$$

Hence condition (i') of Theorem 3.1 for the $(L_{p'})$-effectiveness of $\mathfrak{A}$ is satisfied. In view of the (L_p)-effectiveness of $\mathfrak{A}$ condition (E.1) is satisfied, and so the transformations $\tau_m(x;f)$, $\tau_m(x;g)$, $f\subset L_p$, $g\subset L_{p'}$, are limited, and, by Lemma 2.3, have the same bounds $M_m^{(p)}=M_m^{(p')}$. By Theorem 3.1 we have

$$M_{\mathfrak{A}}^{(p')}=\overline{\lim_{m\to\infty}}\, M_m^{(p')}=\overline{\lim_{m\to\infty}}\, M_m^{(p)}=M_{\mathfrak{A}}^{(p)}<\infty.$$

[12] The argument which we use here is analogous to an argument of Orlicz [1], pp. 13–14. Hahn [1] has considered a general class of representations of a function $f(x)$ by means of singular integrals which possess the property of converging in the space L_1 to $f(x)$. It is readily seen that in the case of Fourier series the singular integrals of Hahn reduce to transformations which satisfy the sufficient conditions of our Theorem 3.1.

It follows that condition (ii) of Theorem 3.1 for the $(L_{p'})$-effectiveness is also satisfied, and so $\mathfrak{A}$ is (L_p)-effective.

Remark 3.2. If the condition (α) (p. 330) of continuity of the functions $\varphi_n(x)$ is replaced by the condition: (α') The function $\varphi_n(x)$ are essentially bounded (not necessarily uniformly in n), then an analogous argument will show that *every matrix* $\mathfrak{A}$ *which is* (L_∞)*-effective with respect to the set* $\{\varphi_n(x)\}$ *is also* (L_1)*-effective.* We do not know whether the converse holds under these, more general, assumptions.

THEOREM 3.3. *If a transformation* $\mathfrak{A}$ *is* (L_p)*-effective,* $1 \leqq p \leqq \infty$, *then* $\mathfrak{A}$ *is also* (L_q)*-effective for any* q *between* p *and* p', *and*

$$M_{\mathfrak{A}}^{(q)} \leqq M_{\mathfrak{A}}^{(p)} = M_{\mathfrak{A}}^{(p')}. \tag{3.05}$$

Proof. Using the same set L^0 as in the proof of Theorem 3.2 we see that condition (i') for the (L_q)-effectiveness is satisfied. Condition (ii) is also satisfied; indeed the (L_p)-effectiveness of $\mathfrak{A}$ implies, in view of Lemma 2.4, that $\tau_m(x; h)$, $h \subset L_q$, is a limited transformation on L_q, while

$$M_m^{(q)} \leqq M_m^{(p)}, \quad M_{\mathfrak{A}}^{(q)} \leqq \overline{\lim_{m\to\infty}} M_m^{(p)} = M_{\mathfrak{A}}^{(p)}.$$

We close this section by proving

THEOREM 3.4. *The condition* (3.03) *and*

$$|a_{mn}| \leqq A, \qquad m, n = 1, 2, \cdots \tag{3.06}$$

are necessary and sufficient in order that the matrix $\mathfrak{A}$ *be* (L_2)*-effective. In particular every regular transformation is* (L_2)*-effective.*[13]

Proof. Let (3.03), (3.06) be satisfied. If $f(x)$ is an arbitrary function $\subset L_2$, we have by Parseval's identity

$$\|\tau_m(x; f)\|_2 = \left[\sum_{n=1}^{\infty} |a_{mn}|^2 |f_n|^2\right]^{1/2} \leqq A\left[\sum_{n=1}^{\infty} |f_n|^2\right]^{1/2} = A\|f\|_2.$$

Now let

$$s_n(x; f) \equiv s_n \equiv \sum_{\nu=1}^{n} f_\nu \varphi_\nu(x)$$

be the n-th partial sum of the series (1.09). Then

$$\begin{aligned}\|\tau_m(f)-f\|_2 &\leqq \|\tau_m(f)-\tau_m(s_n)\|_2 + \|s_n-f\|_2 + \|\tau_m(s_n)-s_n\|_2 \\ &\leqq (A+1)\|s_n-f\|_2 + \|\tau_m(s_n)-s_n\|_2.\end{aligned}$$

Since $\|s_n-f\|_2 \to 0$ as $n\to\infty$ and, in view of (3.03), $\|\tau_m(s_n)-s_n\|_2 \to 0$ as $m\to\infty$, n being fixed, it follows that $\|\tau_m(f)-f\|_2 \to 0$, and so, conditions (3.03), (3.06) are sufficient.

[13] A regular transformation is one which transforms every convergent series into a convergent series with the same sum. See the discussion in § 5.

To prove the necessity of these conditions we observe that the transformation which carries $f(x) \subset L_2$ into $\tau_m(x;f) \subset L_2$ may be considered as a transformation which carries an arbitrary element $\xi = (x_1, x_2, \cdots)$ of the space l_2 with the metric $\|\xi\| = [\sum |x_\nu|^2]^{1/2}$ into the element

$$\tau_m(\xi) = \eta^{(m)} = (a_{m1}x_1, a_{m2}x_2, \cdots) \subset l_2.$$

It is well-known that a necessary and sufficient condition that this transformation be limited is that the sequence $\{a_{mn}\}$ (for fixed m) be bounded. Furthermore

$$M_{\tau_m} = \underset{n=1,2,\cdots}{\text{l. u. b.}} |a_{mn}|.$$

The condition of the (L_2)-effectiveness of $\mathfrak{A}$ can be interpreted by stating that $\|\tau_m(\xi) - \xi\| \to 0$ at each point $\xi \subset l_2$. By Lemma 2.1 the sequence of transformations $\{\tau_m(\xi)\}$ must be uniformly limited, and (3.06) follows at once. Necessity of (3.03) follows from Lemma 3.1.

4. **(L_1)- and (L_∞)-effectiveness for trigonometric Fourier series.** In the present section we are concerned with trigonometric Fourier series

$$f(x) \sim S(x;f) \equiv f_0 + \sum_{n=1}^{\infty} (f_n e^{inx} + f_{-n} e^{-inx}), \quad f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt.$$

The interval (a, b) reduces here to $(-\pi, \pi)$ and $f(x)$ is assumed to be periodic. The transform of $S(x;f)$ by means of a factor sequence $\{a_0, a_1, \cdots, a_n, \cdots\}$ is defined at present by

$$T_a(x;f) \equiv T_a(f) \sim a_0 f_0 + \sum_{n=1}^{\infty} a_n (f_n e^{inx} + f_{-n} e^{-inx})$$

while the m-th transform of $S(x;f)$ by means of a matrix $\mathfrak{A}$ is

$$\tau_m(x;f) \sim a_{m0} f_0 + \sum_{n=1}^{\infty} a_{mn} (f_n e^{inx} + f_{-n} e^{-inx}).$$

It is obvious how the results of § 3 should be modified with the present notation.

As concerns the factor-sequences of class $(1, 1) = (\infty, \infty)$ we state

LEMMA 4.1. *A necessary and sufficient condition that a factor-sequence* $\{a_n\} \subset (1, 1) = (\infty, \infty)$ *is that*

$$a_n = \int_{-\pi}^{\pi} e^{int}\, d\alpha(t), \qquad n = 0, 1, \cdots \tag{4.01}$$

where $\alpha(t)$ *is an arbitrary function of bounded variation such that*

$$\alpha(-t) = -\alpha(t). \tag{4.02}$$

It can be assumed without loss of generality that $\alpha(t)$ is normalized according to

$$\alpha(t) = \frac{1}{2}[\alpha(t+0)+\alpha(t-0)], \quad -\pi<t<\pi. \tag{4.03}$$

The transformation $T_\alpha(f)$ is then represented by

$$T_\alpha(x;f) = \int_{-\pi}^{\pi} f(x+t)\,d\alpha(t).^{14} \tag{4.04}$$

We observe that the integral of the right-hand member is taken over the closed interval $(-\pi,\pi)$ in the sense of Young-Lebesgue-Stieltjes, and reduces to the classical Riemann-Stieltjes integral when $f\subset L_\infty$. In the case $f\subset L_1$ this integral exists for almost all x and represents a function $\subset L_1$.

In the following discussion we shall consider the class $\mathfrak{S}$ of transformations $\mathfrak{A}$ which sum the Fourier series of an arbitrary step-function $\varphi(x)$ to the value $\varphi(x)$ at each point of continuity of $\varphi(x)$.

THEOREM 4.1. *The most general expression of the elements of a matrix $\mathfrak{A}\subset\mathfrak{S}$ which is (L_1)- or (L_∞)-effective is given by*

$$a_{mn} = \int_{-\pi}^{\pi} e^{int}\,d\alpha_m(t); \qquad m, n = 0, 1, 2, \cdots. \tag{4.05}$$

Here $\alpha_m(t)$ are functions of bounded variation such that

$$\alpha_m(t) = \frac{1}{2}[\alpha_m(t+0)+\alpha_m(t-0)], \qquad -\pi<t<\pi,$$
$$\alpha_m(-t) = -\alpha_m(t), \qquad \alpha_m(0) = 0, \tag{4.06}$$

$$\alpha_m(x)\to\frac{1}{2}, \quad \alpha_m(x+0)-\alpha_m(x-0)\to 0 \quad as \quad m\to\infty, \quad 0<x\leqq\pi, \tag{4.07}$$

$$V_m \equiv 2\int_0^{\pi} |d\alpha_m(t)| \leqq A<\infty. \tag{4.08}$$

The (L_1)- and (L_∞)-bounds of $\mathfrak{A}$ are given by

$$M_{\mathfrak{A}}^{(1)} = M_{\mathfrak{A}}^{(\infty)} = \overline{\lim_{m\to\infty}}\, V_m. \tag{4.09}$$

Proof. If $\mathfrak{A}$ is (L_1)- or (L_∞)-effective, conditions (E.1) and (E.2) are satisfied with $p = 1$ and $p = \infty$. As in the proof of Theorems 3.1, 3.2 it follows that the sequence of transformations $\{\tau_m(x;f)\}$ is uniformly limited (in L_1 and in L_∞). Since

[14] M. Riesz [1], p. 488. The notation of Riesz is slightly changed here. The function $\alpha(t)$ is defined only for $-\pi\leqq t\leqq\pi$. We shall assume, without loss of generality, that outside of this interval $\alpha(t)$ equals $\alpha(\pm\pi)$ according as $t>\pi$ or $t<-\pi$. The same agreement will be made tacitly concerning the functions $\alpha_m(t)$ below.

$$M_m^{(1)} = M_m^{(\infty)} = \int_{-\pi}^{\pi} |d\alpha_m(t)| \tag{4.10}$$

we conclude that (4.09) holds; (4.08) then follows.

By hypothesis $\mathfrak{A} \subset \mathfrak{S}$. If we choose for $\varphi(t)$ the periodic step-functions defined by

$$\varphi_1(t) = \begin{cases} 1 & \text{if } 0 \leqq |t| < x \leqq \pi, \\ 0 & \text{if } |t| \geqq x, \end{cases}$$

$$\varphi_2(t) = \begin{cases} 0 & \text{if } -\pi \leqq t < x, \quad 0 < x \leqq \pi, \\ 1 & \text{if } t = x, \\ 0 & \text{if } t > x, \end{cases}$$

we find respectively (Lemma 4.1) that

$$\tau_m(0; \varphi_1) = \int_{-x+0}^{x-0} d\alpha_m(t) = 2\alpha_m(x-0) \to \varphi_1(0) = 1,$$

$$\tau_m(0; \varphi_2) = \alpha_m(x+0) - \alpha_m(x-0) \to \varphi_2(0) = 0,$$

which in view of (4.06) establishes the necessity of (4.07). The necessity of (4.05) and (4.06) follows from Lemma 4.1.

Assume that conditions (4.05)–(4.08) are satisfied. It follows immediately from (4.07) that $\mathfrak{A} \subset \mathfrak{S}$. We now pass on to the proof that the transformation $\mathfrak{A}$ is (L_∞)- and (L_1)-effective. First let $f(x)$ be an arbitrary function $\subset L_\infty$, viz., periodic and continuous. In view of (4.05)–(4.07) we have, as $m \to \infty$,

$$\tau_m(x; f) - f(x) = \int_0^{\pi} \omega(x, t)\, d\alpha_m(t) + o(1),$$

where

$$\omega(x, t) \equiv f(x+t) + f(x-t) - 2f(x).$$

Let us investigate the integral

$$I \equiv \int_0^{\pi} \omega(x, t)\, d\alpha_m(t) = \int_0^{\delta} + \int_{\delta}^{\pi} \equiv I_1 + I_2.$$

Since $f(x)$ is continuous, being given an arbitrarily small $\varepsilon > 0$, we can fix a δ so small that

$$|I_1| \leqq A \max_{0 \leq t \leq \delta} |\omega(x, t)| \leqq \frac{\varepsilon}{3}.$$

Let $\varphi(x)$ be an arbitrary step-function and

$$\omega_1(x, t) \equiv \varphi(x+t) + \varphi(x-t) - 2\varphi(x).$$

We can write

$$I_2 = \int_{\delta}^{\pi} [\omega(x, t) - \omega_1(x, t)]\, d\alpha_m(t) + \int_{\delta}^{\pi} \omega_1(x, t)\, d\alpha_m(t) \equiv I_2' + I_2''.$$

The number δ being fixed, select the function $\varphi(x)$ in such a way that

$$|\omega(x, t) - \omega_1(x, t)| \leqq \eta, \quad |I_2'| \leqq \eta A \leqq \frac{\varepsilon}{3}.$$

In view of (4.07), after the function $\varphi(x)$ has been fixed,

$$\int_\delta^\pi \omega_1(x, t)\, d\alpha_m(t) \to 0 \quad \text{as} \quad m \to \infty,$$

uniformly in x. Hence, for sufficiently large values of m and uniformly in x,

$$|I_2''| \leqq \frac{\varepsilon}{3}.$$

On combining these results we obtain

$$|\tau_m(x; f) - f(x)| \leqq \varepsilon,$$

uniformly in x, or

$$\|\tau_m(x; f) - f(x)\|_\infty \to 0 \quad \text{as} \quad m \to \infty$$

which establishes the (L_∞)-effectiveness of $\mathfrak{A}$.

The (L_1)-effectiveness of $\mathfrak{A}$ follows from Theorem 3.2.

The transformations $\mathfrak{A}$ which transform convergent series into convergent series with the same sum are called regular. We designate by $\mathfrak{R}$ the class of all regular transformations. It is plain then that $\mathfrak{R} \subset \mathfrak{S}$. Necessary and sufficient conditions for the regularity of a transformation $\mathfrak{A}$ are given by

(R.1) $$a_{mn} \to 1 \quad \text{as} \quad m \to \infty; \qquad n = 0, 1, 2, \cdots;$$

(R.2) $$\sum_{n=0}^{\infty} |a_{mn} - a_{mn+1}| \leqq A$$

where A is a fixed positive constant.[15]

THEOREM 4.2. *The most general expression of the elements of a regular matrix* $\mathfrak{A}$ *which is* (L_1)- *or* (L_∞)-*effective is given by* (4.05), *where the function* $\alpha_m(t)$, *in addition to conditions* (4.06)–(4.08), *satisfies also the condition*

$$\alpha_m(t) = a_m \eta(t) + \int_0^t \beta_m(u)\, du, \tag{4.11}$$

where a_m *is a constant and* $\eta(t)$ *is the function defined by*

$$\eta(t) = \begin{cases} \frac{1}{2} \text{ if } t > 0, \\ 0 \text{ if } t = 0, \\ -\frac{1}{2} \text{ if } t < 0, \end{cases} \quad |t| < \pi; \quad \eta(t + 2\pi) = \eta(t), \tag{4.12}$$

[15] Cf. Hahn [2], pp. 33–34.

and $\beta_m(t)$ *is a function* $\subset L_1$ *such that the series*

$$\sum_{n=0}^{\infty} |\gamma_{mn}|, \qquad \gamma_{mn} = \int_{-\pi}^{\pi} e^{nit}(1-e^{it})\,\beta_m(t)\,dt \tag{4.13}$$

converges for each $m = 0, 1, 2, \cdots$ *and has a sum which is uniformly bounded in* m.

Proof. We have only to investigate the additional conditions (4.11)–(4.13). Since $\mathfrak{A}$ is (L_∞)-effective, we have for $f(x) = e^{inx}$,

$$a_{mn} = \int_{-\pi}^{\pi} e^{int}\,d\alpha_m(t) = \tau_m(0; f) \to f(0) = 1$$

so that condition (R. 1) of the regularity of $\mathfrak{A}$ is automatically satisfied. For simplification we define a_{mn} for negative values of n by putting $a_{m,-n} = a_{mn}$. Then

$$a_{mn} - a_{mn+1} = \int_{-\pi}^{\pi} e^{int}(1-e^{it})\,d\alpha_m(t) = \int_{-\pi}^{\pi} e^{int}\,dA_m(t)$$

where

$$A_m(t) = \int_0^t (1-e^{iu})\,d\alpha_m(u).$$

We set

$$B(t) = ct + A_m(t)$$

and choose the constant c under the condition

$$B(\pi) = B(-\pi).$$

Then

$$a_{mn} - a_{mn+1} = \int_{-\pi}^{\pi} e^{int}\,dB(t) = -in\int_{-\pi}^{\pi} e^{int}B(t)\,dt \equiv \pi b_n, \quad n \neq 0,$$
$$b_0 \equiv 0.$$

Condition (R. 2) of the regularity of $\mathfrak{A}$ implies, and is implied by, the convergence of $\sum_{-\infty}^{\infty} |b_n|$. Hence $\sum_{-\infty}^{\infty} b_n e^{inx}$ is as an absolutely convergent Fourier series and represents a continuous function which equals

$$\frac{d}{dx}\sum_{n\neq 0}\frac{b_n}{-in}e^{-inx} = \frac{dB(x)}{dx}.$$

Thus

$$B(t) = ct + \alpha_m(t)(1-e^{it}) + \int_0^t e^{iu}\,\alpha_m(u)\,du$$

is absolutely continuous and has a continuous derivative, which shows that

$$\alpha_m(t)(1-e^{it})$$

is absolutely continuous on $(-\pi, \pi)$. Hence $\alpha_m(t)$ is itself absolutely continuous on $(-\pi, \pi)$, except perhaps at $t = 0$, where $\alpha_m(t)$ may have

a jump, a_m say. Thus we have (4.11) and the proof of the last statement of Theorem 4.2 follows at once.

5. **(L_1)-effectiveness of convergence.** We have already mentioned in § 1 that the "convergence" transformation which is defined by

$$a_{mn} = 1, \text{ or } 0 \text{ according as } n \leqq m \text{ or } n > m, \quad m = 0, 1, 2, \cdots, \tag{5.01}$$

$$\tau_m(x; f) = s_m(x; f) \equiv s_m(x) \equiv s_m \equiv f_0 + \sum_{n=1}^{m} (f_n e^{inx} + f_{-n} e^{-inx}), \tag{5.02}$$

is not (L_1)-effective. We shall indicate two sub-sets of L_1 on which the convergence is (L_1)-effective. These sub-sets will be constituted by sine- or cosine-series

$$S(x) \equiv \sum_{n=1}^{\infty} q_n \sin nx, \tag{5.03}$$

$$C(x) \equiv \sum_{n=1}^{\infty} q_n \cos nx \tag{5.04}$$

where the sequence $\{q_n\}$, in addition to being positive and $\downarrow 0$, is subjected to certain other restrictions. Instead of saying that the convergence transformation is (L_1)-effective we shall say that the corresponding Fourier series converge in the mean to their generating functions.

THEOREM 5.1. *In order that the sine-series*

$$S(x) = \sum_{n=1}^{\infty} q_n \sin nx, \quad q_n \downarrow 0, \tag{5.05}$$

converge in the mean to a function $f(x) \subset L_1$, it is necessary and sufficient that the series

$$\sum_{n=1}^{n} \frac{q_n}{n} \tag{5.06}$$

converge.

Proof. The condition (5.06) is necessary and sufficient in order that (5.05) be a Fourier series, and hence that $S(x) \subset L_1$.[16] Now, if $\|f - s_n\|_1 \to 0$ then $S(x) \subset L_1$ which shows the necessity of (5.06) for the convergence of (5.05) in the mean.

To prove the sufficiency of (5.06) we first observe that (5.06) implies that

$$\sum_{n=1}^{\infty} (q_n - q_{n+1}) \log n < \infty \tag{5.07}$$

and that

$$q_n \log n \to 0 \quad \text{as} \quad n \to \infty. \tag{5.08}$$

[16] Young [1], p. 46.

This follows from

$$\sum_{\nu=1}^{n} \frac{q_\nu}{\nu} = \sum_{\nu=1}^{n-1} \sigma_\nu (q_\nu - q_{\nu+1}) + \sigma_n q_n,$$

$$\sum_{\nu=m}^{n} \frac{q_\nu}{\nu} > q_n (\sigma_n - \sigma_{m-1}) \sim q_n \log\left(\frac{n}{m}\right),$$

where

$$\sigma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \sim \log n.$$

Now we have

$$2 \sin \frac{x}{2} [s_n(x) - s_{m-1}(x)] = \sum_{\nu=m}^{n} q_\nu \, 2 \sin \frac{x}{2} \sin \nu x$$

$$= \sum_{\nu=m}^{n} q_\nu [t_{\nu+1}(x) - t_\nu(x)] = -q_m t_m(x) + q_n t_{n+1}(x) + \sum_{\nu=m}^{n-1} (q_\nu - q_{\nu+1}) t_{\nu+1}(x),$$

$$t_\nu(x) = 1 - \cos\left(\nu - \frac{1}{2}\right) x,$$

whence

$$\| s_n(x) - s_m(x) \|_1 \leqq q_m I_m + q_n I_{n+1} + \sum_{\nu=m}^{n-1} (q_\nu - q_{\nu+1}) I_{\nu+1},$$

where

$$I_\nu = \int_0^{\pi} \left[1 - \cos\left(\nu - \frac{1}{2}\right) x\right] \left[\sin \frac{x}{2}\right]^{-1} dx = O(\log \nu).$$

From (5.07), (5.08) it follows that the sequence $\{s_n(x)\}$ converges in the mean. There will exist then a function $f(x) \subset L_1$, to which $s_n(x)$ converges in the mean. Since $f(x)$ has the same Fourier coefficients as $S(x) \subset L_1$, these two functions must be equal almost everywhere. Hence $\| S - s_n \|_1 \to 0$, and Theorem 5.1 is established.

In the case of the cosine-series (5.04) the situation is less satisfactory owing to the fact that no simple necessary and sufficient conditions are known which have to be imposed on the sequence in order to make $C(x) \subset L_1$. It is known[17] that the condition $q_n \downarrow 0$ is not sufficient that (5.04) be a Fourier series. On the other hand simple sufficient conditions have been found by Young and Kolmogoroff.[18] They consist in the assumption that $q_n \to 0$ while the series

$$\sum_{n=1}^{\infty} n |\Delta^2 q_n|, \qquad \Delta q_n = q_n - q_{n+1}, \qquad \Delta^2 q_n = \Delta q_n - \Delta q_{n+1},$$

converges. These conditions are satisfied in the particular case where $q_n \downarrow 0$, $\Delta q_n \downarrow 0$ (the sequence $\{q_n\}$ is convex). For this class of cosine

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whence

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where

$$I_\nu = \int_0^\pi \left[1 - \cos\left(\nu - \frac{1}{2}\right) x\right] \left[\sin \frac{x}{2}\right]^{-1} dx = O(\log \nu).$$

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converges. These conditions are satisfied in the particular case where $q_n \downarrow 0$, $\Delta q_n \downarrow 0$ (the sequence $\{q_n\}$ is convex). For this class of cosine

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series it has been proved by Kolmogoroff that a necessary and sufficient condition that (5.04) converge in the mean to $f(x) \subset L$ is given by $q_n \log n \to 0$.

Bibliography.

S. Banach.
1. Théorie des opérations linéaires. Warsaw, 1932.

W. Gross.
1. Zur Poissonschen Summierung. Sitzungsber. Wiener Akademie, Abt. IIa, 124 (1915), pp. 1017–1037.

H. Hahn.
1. Über die Darstellung gegebener Funktionen durch singuläre Integrale. II. Mitt., Denkschr. Wiener Akademie, Math.-Naturw. Kl., 93 (1916), pp. 657–692.
2. Über Folgen linearer Operationen. Monatshefte Math. u. Ph., 32 (1922), pp. 1–88.

H. T. Hildebrandt.
1. Linear functional transformations in general spaces. Bull. Amer. Math. Soc., 37 (1931), pp. 185–212.

E. Hille and J. D. Tamarkin.
1. On the summability of Fourier series. I. Trans. Amer. Math. Soc., 34 (1932), pp. 757–783.
2. On the summability of Fourier series. III. To appear in the Mathemat. Annalen.

A. Kolmogoroff.
1. Sur l'ordre de grandeur des coefficients de la série de Fourier. Bull. Intern. de l'Acad. Polonaise, Classe des sc. math. 1923, pp. 83–86.

W. Orlicz.
1. Beiträge zur Theorie der Orthogonalentwickelungen. Studia Math., 1 (1929), pp. 1–39.

M. Riesz.
1. Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires. Acta Math. 49 (1926), pp. 465–497.

M. Steinhaus.
1. Sur les développements orthogonaux. Bull. Intern. Ac. Polon. Crac., 1926, pp. 11–39.

S. Szidon.
1. Reihentheoretische Sätze und ihre Anwendungen in der Theorie der Fourierschen Reihen. Math. Zeitschr. 10 (1921), pp. 121–127.

W. H. Young.
1. On the Fourier series of bounded functions. Proc. London Math. Soc., (2) 12 (1913), pp. 41–70.

A. Zygmund.
1. Sur les fonctions conjuguées. Fund. Math., 13 (1929), pp. 284–303.

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SUMMATION OF FUNCTIONS OF A COMPLEX VARIABLE.[1]

By R. D. Carmichael.

Introduction. The object of this paper is to develop the theory of the principal solutions of the equations,

$$F(x+\omega) - F(x) = \omega\varphi(x), \qquad G(x+\omega) + G(x) = 2\varphi(x),$$

under two classes of hypotheses concerning the given function $\varphi(x)$ and to give as an application a theory of expansions in series of Bernoulli-Hurwitz functions.

In part I a definition of principal sums is employed which is closely related to that of Nörlund (l. c.) but differs from his in putting the emphasis upon solutions of approximating equations rather than upon processes of summing divergent series and integrals. The theory of such principal sums is developed, by a very direct method, for the case of functions $\varphi(x)$ having the asymptotic character (1.15).

In part II are found several theorems concerning integral functions of exponential type. A definition of principal sums of such functions is proposed. A general theory of the summation of these functions is developed by means of modified or generalized sum formulas of the Euler-Maclaurin type, here introduced for the first time. A very direct method is given for developing the Fourier expansion theory of principal sums of functions of exponential type.

Finally, in part III, necessary and sufficient conditions are constructed for the expansion of functions in series of Bernoulli-Hurwitz functions, including the special case of Bernoulli polynomials.

I. Summation of functions having a defined asymptotic character.

1.1. **Principal sums.** If ω is a given constant different from zero and if $\varphi(x)$ is a given function then solutions of the equations

$$\omega\Delta_\omega F(x) \equiv F(x+\omega) - F(x) = \omega\varphi(x), \tag{1.1}$$

$$2\nabla_\omega G(x) \equiv G(x+\omega) + G(x) = 2\varphi(x), \tag{1.2}$$

[1] Received, May 14, 1932.—Presented to the American Mathematical Society at the Summer Meeting in Los Angeles in 1932.

will be called *sums* and *alternating sums* of $\varphi(x)$ respectively. The constant ω is called the span of the symbols Δ_ω and ∇_ω. The general solution of (1.1) [(1.2)] is obviously the sum of a particular solution and an arbitrary function $P(x)$ [$\pi(x)$] such that $P(x+\omega)-P(x)=0$ [$\pi(x+\omega)+\pi(x)=0$]. Solutions of these equations obviously exist. Owing to the nature of the arbitrary elements in the general solutions, the essential problem[2] of *summation* is that of determining solutions having characteristic properties of particular interest. It will take different forms for different classes of functions $\varphi(x)$. In each case such special solutions will be called *principal sums.*

If the series in the equations,

$$(1.3)\quad F(x) = -\omega \sum_{k=0}^{\infty} \varphi(x+k\omega), \qquad G(x) = 2\sum_{k=0}^{\infty}(-1)^k \varphi(x+k\omega),$$

converge the sums $F(x)$ and $G(x)$ defined by them will be called respectively the *principal sum* and the *principal alternating sum* of $\varphi(x)$. More generally if the series in the equations,

$$(1.4)\quad F_\eta(x\,|\,\omega) = f(\eta) - \omega \sum_{k=0}^{\infty} \varphi(x+k\omega)\, e^{-\eta(x+k\omega)},$$

$$(1.5)\quad G_\eta(x\,|\,\omega) = 2\sum_{k=0}^{\infty}(-1)^k \varphi(x+k\omega)\, e^{-\eta(x+k\omega)},$$

where η is such that $\eta\omega>0$ and where $f(\eta)$ is independent of x, converge they define functions $F_\eta(x\,|\,\omega)$ and $G_\eta(x\,|\,\omega)$ satisfying respectively the approximating equations

$$(1.6)\quad \Delta_\omega F(x) = \varphi(x)\, e^{-\eta x}, \qquad \nabla_\omega G(x) = \varphi(x)\, e^{-\eta x};$$

and these solutions are such that if they exist for every η such that $\eta\omega>0$ and if the limits,

$$(1.7)\quad \lim_{\eta=0} F_\eta(x\,|\,\omega) = F(x\,|\,\omega), \qquad \lim_{\eta=0} G_\eta(x\,|\,\omega) = G(x\,|\,\omega), \qquad \eta\omega>0,$$

exist the functions $F(x\,|\,\omega)$ and $G(x\,|\,\omega)$ defined by them are solutions of equations (1.1) and (1.2) respectively. These solutions will be called respectively the *principal sum* and the *principal alternating sum* of $\varphi(x)$. The second is uniquely defined by the named process while the first is defined only up to an arbitrary additive constant, this being admitted owing to the possibility of varied choice in selecting the function $f(\eta)$. *Except for this additive constant the solutions defined in the more general*

[2] Nörlund, *Acta Mathematica* 44 (1922), pp. 71–211. See other papers referred to in this memoir.

case exist and are identical with those in (1.3) *whenever the latter exist,* as one may readily show.

The principal solutions thus defined are identical with those of Nörlund (l. c.) when the latter exist, though in the case of $F(x|\omega)$ our definition is different in form from that of Nörlund. In Nörlund's definition the principal emphasis is upon certain processes of summing divergent series and integrals; in ours it is upon the passage from the approximating equations (1.6) to equations (1.1) and (1.2) by letting η approach 0 in such way that $\arg\eta = -\arg\omega$.

By changing x to $x-\omega$ we may put equations (1.1) and (1.2) into the forms

$$\Delta_{-\omega} F(x) = \varphi(x-\omega), \quad \nabla_{-\omega} G(x) = \varphi(x-\omega). \tag{1.8}$$

Whereas the original equations involve the span ω these have the span $-\omega$. If the function $\varphi(x)$ has suitable properties we may treat the new form of the equation as we have already treated the original form and thus arrive at the second principal solutions (in contrast with which the foregoing may be called the first principal solutions) in the form

$$F(x|-\omega) = \lim_{\eta=0} F_\eta(x|-\omega), \quad G(x|-\omega) = \lim_{\eta=0} G_\eta(x|-\omega), \quad \eta\omega<0, \tag{1.9}$$

where

$$F_\eta(x|-\omega) = f_1(\eta) + \omega \sum_{k=1}^{\infty} \varphi(x-k\omega)\, e^{-\eta(x-k\omega)}, \tag{1.10}$$

$$G_\eta(x|-\omega) = 2\sum_{k=1}^{\infty} (-1)^{k-1} \varphi(x-k\omega)\, e^{-\eta(x-k\omega)}. \tag{1.11}$$

Results of interest will arise from an examination of the relations of the second principal solutions to the first.

In the case when $f(\eta)$ and $f_1(\eta)$ may be chosen independent of $|\omega|$ it is easy to verify the following relations (Nörlund, l. c.) by first establishing them for the case of the approximating equations and then taking the limit as η approaches zero:

$$\sum_{s=0}^{n-1} F\left(x+\frac{s\omega}{n}\,\middle|\,\omega\right) = nF\left(x\,\middle|\,\frac{\omega}{n}\right); \tag{1.12}$$

$$\sum_{s=0}^{n-1} (-1)^s F\left(x+\frac{s\omega}{n}\,\middle|\,\omega\right) = -\frac{1}{2}\omega G\left(x\,\middle|\,\frac{\omega}{n}\right), \quad n \text{ even}; \tag{1.13}$$

$$\sum_{s=0}^{n-1} (-1)^s G\left(x+\frac{s\omega}{n}\,\middle|\,\omega\right) = G\left(x\,\middle|\,\frac{\omega}{n}\right), \quad n \text{ odd}; \tag{1.14}$$

and the equations obtained from them by replacing ω by $-\omega$.

The following three theorems are readily established (the second and third being proved by aid of the first):

THEOREM 1.1. *If $S\{\varphi(x)\}$ denotes a principal sum (or alternating sum) of $\varphi(x)$ in accordance with any one of the foregoing four definitions, then*

$$S\{a\varphi(x)+b\psi(x)\} = aS\{\varphi(x)\}+bS\{\psi(x)\},$$

where a and b are constants, provided that the indicated sums in the second member of the equation exist and provided that the arbitrary constant (when it occurs) is suitably determined for the sum in the first member.

THEOREM 1.2. *If $Q(x)$ is a principal sum of $\Delta_\omega Q(x)$ and is such that the series in the relation*

$$F(x\,|\,\omega) = Q(x)-\omega\sum_{k=0}^{\infty}\{\varphi(x+k\omega)-\Delta_\omega Q(x+k\omega)\}$$

converges, then the function $F(x\,|\,\omega)$ is a principal sum of $\varphi(x)$.

THEOREM 1.3. *If $P(x)$ is the principal alternating sum of $\nabla_\omega P(x)$ and is such that the series in the relation*

$$G(x\,|\,\omega) = P(x)+2\sum_{k=0}^{\infty}(-1)^k\{\varphi(x+k\omega)-\nabla_\omega P(x+k\omega)\}$$

converges, then the function $G(x|\omega)$ is the principal alternating sum of $\varphi(x)$.

It is obvious that theorems corresponding to the last two may be stated for the second principal sums.

1.2. **Formal solutions in domains $D_i(\omega)$.** By a domain $D_0(\omega)$ we shall mean a region of the complex x-plane such that if x_0 is in the region then $x_0+s\omega$ is also in the region where s is a variable real number whose range depends on x_0 and consists either of all real numbers or all real numbers s such that $s\geqq b$ where b is a non-positive real number depending on x_0. A domain $D_0(\omega)$ will also be called a domain $D_1(\omega)$ if there exists a sector formed by two rays from 0 to ∞ and containing in its interior the ray from 0 (exclusive) through $-\omega$ to ∞ such that the sector and the domain have no point in common. A domain $D_0(\omega)$ will also be called a domain $D_2(\omega)$ if for every x_0 there exists a real number s such that $x_0+s\omega$ is in the domain. By $D_3(\omega)$ we shall denote a region which is at the same time a domain $D_1(\omega)$ and a domain $D_2(\omega)$. We shall sometimes write $D_i(\omega)$, without assigning the value of i, when it is a matter of indifference to the truth of the proposition which type of domain is considered. A domain $D_i(-\omega)$ may be obtained from a corresponding domain $D_i(\omega)$ by reflection on the line through 0 perpendicular to the line joining 0 and ω. It is obvious that suitable sectors afford particular domains $D_i(\omega)$.

Let us now suppose that $\varphi(x)$ has the asymptotic form[3]

$$\varphi(x) \sim p(x) + q(x)\log x + c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots \tag{1.15}$$

in some domain $D_i(\omega)$, where $p(x)$ and $q(x)$ are polynomials in x with the possibility that one or both of them may be identically zero. Without loss of generality we may (and we do) assume that $p(0) = 0$. Then there exist unique polynomials $r(x)$ (with $r(0) = 0$) and $s(x)$ and a unique set of constants $a_0 (= 0)$, $a_1, a_2, \cdots$ such that the functions

$$Q_\mu(x) = r(x) + s(x)\log x + a_0 + \frac{a_1}{x} + \cdots + \frac{a_{\mu-1}}{x^{\mu-1}}, \quad \mu = 1, 2, 3, \cdots, \tag{1.16}$$

have the property that

$$\Delta_\omega Q_\mu(x) = p(x) + q(x)\log x + c_0 + \frac{c_1}{x} + \cdots + \frac{c_\mu}{x^\mu} + \frac{\gamma_{\mu+1}}{x^{\mu+1}} + \cdots, \quad |x| > |\omega|.$$

Likewise there exist unique polynomials $\varrho(x)$ (with $\varrho(0) = 0$) and $\sigma(x)$ and a unique set of constants $\alpha_0, \alpha_1, \alpha_2, \cdots$ such that the functions

$$P_\mu(x) = \varrho(x) + \sigma(x)\log x + \alpha_0 + \frac{\alpha_1}{x} + \cdots + \frac{\alpha_\mu}{x^\mu}, \quad \mu = 0, 1, 2, \cdots, \tag{1.17}$$

have the property that

$$\nabla_\omega P_\mu(x) = p(x) + q(x)\log x + c_0 + \frac{c_1}{x} + \cdots + \frac{c_\mu}{x^\mu} + \frac{\delta_{\mu+1}}{x^{\mu+1}} + \cdots, \quad |x| > |\omega|.$$

If we let μ become infinite in (1.16) and (1.17) the second members become infinite series (in general divergent) which we denote by $S(x)$ and $R(x)$ respectively. We shall say that these series afford formal solutions of (1.1) and (1.2) respectively.

1.3. **Actual solutions in domains $D_i(\omega)$.** We shall now prove the first of the following two theorems, omitting the similar proof of the second:

THEOREM 1.4. *If $\varphi(x)$ has the asymptotic form* (1.15) *in some domain $D_i(\omega)$ and if for a given positive integer μ the function $Q_\mu(x)$ is defined as in* (1.16), *then equation* (1.1) *has the solution*

[3] One might assume a more general asymptotic form

$$\varphi(x) \sim \sum_{s=0}^{n} (\log x)^s \left\{p_s(x) + c_{0s} + \frac{c_{1s}}{x} + \frac{c_{2s}}{x^2} + \cdots\right\},$$

where n is a positive integer and the $p_s(x)$ are polynomials in x, and proceed by a suitable modification of the methods of the text to obtain results comparable with those here presented; but the interest attaching to the more general problem is not increased in proportion to the added complication.

$$(1.18) \qquad F(x) = Q_\mu(x) - \omega \sum_{k=0}^{\infty} \{\varphi(x+k\omega) - \Delta_\omega Q_\mu(x+k\omega)\}.$$

If $S(x)$ is defined as in the last paragraph of § 1.2 *then $F(x) \sim S(x)$ in any domain $D_1(\omega)$ contained in $D_i(\omega)$. This solution $F(x)$ is the only solution of* (1.1) *in $D_i(\omega)$ for which the difference $F(x) - Q_1(x)$ approaches* 0 *as x becomes infinite in $D_1(\omega)$. It is therefore independent of μ.*

THEOREM 1.5. *If $\varphi(x)$ has the asymptotic form* (1.15) *in some domain $D_i(\omega)$ and if for a given non-negative integer μ the function $P_\mu(x)$ is defined as in* (1.17), *then equation* (1.2) *has the solution*

$$(1.19) \quad G(x) = P_\mu(x) + 2 \sum_{k=0}^{\infty} (-1)^k \{\varphi(x+k\omega) - \nabla_\omega P_\mu(x+k\omega)\}.$$

If $R(x)$ is defined as in the last paragraph of § 1.2 *then $G(x) \sim R(x)$ in any domain $D_1(\omega)$ contained in $D_i(\omega)$. This solution $G(x)$ is the only solution of* (1.2) *in $D_i(\omega)$ for which the difference $G(x) - P_0(x)$ approaches* 0 *as x becomes infinite in $D_1(\omega)$. It is therefore independent of μ.*

We employ the following lemma,[4] now classic in the difference calculus:

LEMMA. If $t = u + iv$ where u and v are real and if $\nu \geqq 2$, we have

$$\sum_{k=0}^{\infty} \frac{1}{|t+k|^\nu} < \frac{1}{|t|^{\nu-1}} \left(\frac{1}{|t|} + \frac{\pi}{2} \right) \quad \text{if} \quad u \geqq 0 \quad \text{and} \quad t \neq 0,$$

$$\sum_{k=0}^{\infty} \frac{1}{|t+k|^\nu} < \frac{1}{|v|^{\nu-1}} \left(\frac{4}{|v|} + \pi \right) \quad \text{if} \quad u \leqq 0 \quad \text{and} \quad v \neq 0.$$

There exists a constant M such that

$$|\varphi(x+k\omega) - \Delta_\omega Q_\mu(x+k\omega)| < M|x+k\omega|^{-\mu-1}$$

when x is in $D_i(\omega)$ and $|x+k\omega|$ is large. Hence, since $\mu \geqq 1$, the series in (1.17) converges and defines a function $F(x)$. That $F(x)$ is a solution of (1.1) may be verified by direct substitution. Replacing x by $t\omega$ in the foregoing inequality and employing the lemma, one shows that $F(x) \sim S(x)$ in $D_1(\omega)$. Since the difference of any two solutions of (1.1) is a periodic function of period ω, the remainder of Theorem 1.4 is readily established.

It will be observed that if we start out with a domain $D_1(\omega)$ then the sums $F(x)$ and $G(x)$ of the function $\varphi(x)$, obtained in the preceding two theorems, belong to the same class of functions as that to which $\varphi(x)$ itself belongs. This remains true if we replace $F(x)$ by the sum $F(x) + c$ where c is an arbitrary constant. Hence the processes of summation employed in forming $F(x) + c$ and $G(x)$ from $\varphi(x)$ may be applied to these sum functions themselves, and then to the repeated sums thus obtained,

[4] See, for instance, Batchelder's *Linear Difference Equations*, p. 24.

and so on indefinitely. If the polynomial $q(x)$ in (1.15) is identically zero, the process of alternating summation in Theorem 1.5 may be repeated for the class of functions $\varphi(x)$ for which $q(x) \equiv 0$; but this is not true in general for the process of summation in Theorem 1.4.

The formal solutions obtained in § 1.2 are unchanged if we take equations (1.1) and (1.2) in the form (1.8). But the solutions corresponding to (1.18) and (1.19) are different, at least if we use the span $-\omega$ (rather than ω) as is natural owing to the form of (1.8). Corresponding to the last two theorems we therefore have the following two which we state without proof:

THEOREM 1.6. *If $\varphi(x)$ has the asymptotic form* (1.15) *in some domain $D_i(-\omega)$ and if for a given positive integer μ the function $Q_\mu(x)$ is defined as in* (1.16), *then equation* (1.1) *has the solution*

$$\bar{F}(x) = Q_\mu(x) + \omega \sum_{k=0}^{\infty} \{\varphi(x - \omega - k\omega) - \Delta_{-\omega} Q_\mu(x - k\omega)\}. \tag{1.20}$$

If $S(x)$ is defined as in the last paragraph of § 1.2 *then $\bar{F}(x) \sim S(x)$ in any domain $D_1(-\omega)$ contained in $D_i(-\omega)$. This solution $\bar{F}(x)$ is the only solution of* (1.1) *in $D_i(-\omega)$ for which the difference $\bar{F}(x) - Q_1(x)$ approaches* 0 *as x becomes infinite in $D_1(-\omega)$. It is therefore independent of μ.*

THEOREM 1.7. *If $\varphi(x)$ has the asymptotic form* (1.15) *in some domain $D_i(-\omega)$ and if for a given non-negative integer μ the function $P_\mu(x)$ is defined as in* (1.17), *then equation* (1.2) *has the solution*

$$\bar{G}(x) = P_\mu(x) + 2 \sum_{k=0}^{\infty} (-1)^k \{\varphi(x - \omega - k\omega) - \nabla_{-\omega} P_\mu(x - k\omega)\}. \tag{1.21}$$

If $R(x)$ is defined as in the last paragraph of § 1.2 *then $\bar{G}(x) \sim R(x)$ in any domain $D_1(-\omega)$ contained in $D_i(-\omega)$. This solution $\bar{G}(x)$ is the only solution of* (1.2) *in $D_i(-\omega)$ for which the difference $\bar{G}(x) - P_0(x)$ approaches* 0 *as x becomes infinite in $D_1(-\omega)$. It is therefore independent of μ.*

With respect to the processes of summation in the last two theorems one may make remarks concerning repeated summation similar to those made in the case of the first two theorems of this section.

When the asymptotic relation (1.15) holds both for $D_i(\omega)$ and for $D_i(-\omega)$ there is an important problem of the relation of $F(x)$ and $\bar{F}(x)$ and of $G(x)$ and $\bar{G}(x)$, to be treated in § 1.5.

1.4. **Principal sums of $\varphi(x)$.** We shall now prove the following theorems:

THEOREM 1.8. *The first and second principal sums of $\Delta_\omega Q_\mu(x)$ are both equal to $Q_\mu(x) + c$ where c is a constant.*

THEOREM 1.9. *The first and second principal alternating sums of $\nabla_\omega P_\mu(x)$ are both equal to $P_\mu(x)$.*

The function $Q_\mu(x)$ $[P_\mu(x)]$ is the sum of $Q_1(x)$ $[P_0(x)]$ and a polynomial in $1/x$. From Theorem 1.1 and the proposition stated at the end of the second paragraph of § 1.1 it follows therefore that it is sufficient to prove these two theorems for the cases $\mu = 1$ and $\mu = 0$ respectively.

We have

$$
\begin{aligned}
&-\omega \sum_{k=0}^{\infty} \Delta_\omega\, Q_1(x+k\omega)\, e^{-\eta(x+k\omega)} \\
&\quad = \sum_{k=0}^{\infty} \{Q_1(x+k\omega) - Q_1(x+k\omega+\omega)\}\, e^{-\eta(x+k\omega)} \\
&\quad = Q_1(x)\, e^{-\eta x} + \sum_{k=1}^{\infty} Q_1(x+k\omega)\, \{e^{-\eta(x+k\omega)} - e^{-\eta(x+k\omega-\omega)}\} \\
&\quad = Q_1(x)\, e^{-\eta x} + (1-e^{\eta\omega}) \sum_{k=1}^{\infty} Q_1(x+k\omega)\, e^{-\eta(x+k\omega)} \\
&\quad = Q_1(x)\, e^{-\eta x} + (1-e^{\eta\omega})\, g(x,\eta), \text{ say.}
\end{aligned}
$$

Therefore in order to establish Theorem 1.8 for the first principal sum it is sufficient to show that

$$(1-e^{\eta\omega})\, \{g_n(x,\eta) - g_n(0,\eta)\} \quad \text{and} \quad (1-e^{\eta\omega})\, \{h_n(x,\eta) - h_n(0,\eta)\}$$

both approach 0 with η, where

$$
\begin{aligned}
g_n(x,\eta) &= \sum_{k=1}^{\infty} (x+k\omega)^n\, e^{-\eta(x+k\omega)}, \\
h_n(x,\eta) &= \sum_{k=1}^{\infty} (x+k\omega)^n \log(x+k\omega)\, e^{-\eta(x+k\omega)}
\end{aligned}
$$

and n is a non-negative integer.

Now if D_η^n denotes the nth derivative with respect to η we have

$$g_n(x,\eta) = (-1)^n \sum_{k=1}^{\infty} D_\eta^n\, e^{-\eta(x+k\omega)} = (-1)^n\, D_\eta^n\, \{e^{-\eta(x+\omega)}/(1-e^{-\eta\omega})\}.$$

Hence $g_n(x,\eta) - g_n(0,\eta)$ is $(-1)^n$ times the nth derivative with respect to η of a function which is analytic at $\eta = 0$, whence it follows that $g_n(x,\eta)$ has the property needed for the proof.

We have

$$
\begin{aligned}
h_n(x,\eta) &= \sum_{k=1}^{\infty} (x+k\omega)^n\, \{\log\omega + \log(k+x/\omega)\}\, e^{-\eta(x+k\omega)} \\
&= \log\omega \sum_{k=1}^{\infty} (x+k\omega)^n\, e^{-\eta(x+k\omega)} + \omega^n\, l_n(x,s),
\end{aligned}
$$

where

$$l_n(x,s) = \sum_{k=1}^{\infty} (k+x/\omega)^n \log(k+x/\omega)\, e^{-s(k+x/\omega)}, \qquad s = \eta\omega.$$

From the property of $g_n(x, \eta)$ just established it follows that it is now sufficient to show that as s approaches 0 through positive values we have

$$\lim_{s=0} (1-e^s)\{l_n(x, s) - l_n(0, s)\} = 0.$$

Using the well known formula

$$\log(k+x/\omega) = \int_0^\infty t^{-1}\{e^{-t} - e^{-t(k+x/\omega)}\}\, dt, \qquad R(k+x/\omega) > 0,$$

we have for s greater than 0 the relation

$$\begin{aligned} l_n(x, s) &= (-1)^n D_s^n \sum_{k=1}^{\infty} e^{-s(k+x/\omega)} \int_0^\infty t^{-1}\{e^{-t} - e^{-t(k+x/\omega)}\}\, dt \\ &= (-1)^n D_s^n \int_0^\infty \left\{\frac{e^{-t}}{t}\,\frac{e^{-s(1+x/\omega)}}{1-e^{-s}} - \frac{e^{-(s+t)(1+x/\omega)}}{t(1-e^{-(s+t)})}\right\} dt. \end{aligned}$$

Hence $l_n(x, s) - l_n(0, s)$ is $(-1)^n$ times the nth derivative with respect to s of a function of s which is analytic at $s = 0$. Therefore $l_n(x, s)$ has the required property.

Thus we have established Theorem 1.8 for the case of first principal sums. The other part of the theorem is similarly proved.

Only obvious modifications of the foregoing arguments are needed for the proof of Theorem 1.9.

By aid of Theorems 1.1, 1.8 and 1.9 and the proposition at the end of the second paragraph of § 1.1 a ready proof is obtained of the following theorem:

THEOREM 1.10. *The solutions* $F(x)$, $G(x)$, $\bar{F}(x)$, $\bar{G}(x)$ *in Theorems* 1.4, 1.5, 1.6, 1.7 *are respectively the principal sums* $F(x|\omega)$, $G(x|\omega)$, $F(x|-\omega)$, $G(x|-\omega)$ *of the function* $\varphi(x)$ *satisfying the hypotheses of the respective theorems.*

1.5. **Relations between first and second principal sums.** These relations take their most interesting form when $\varphi(x)$ is a rational function; this is the case which we shall treat. Then the asymptotic forms of the functions $F(x|\omega)$ and $G(x|\omega)$ [$F(x|-\omega)$ and $G(x|-\omega)$] of Theorem 1.10 are valid in a region $D_1(\omega)$ [$D_1(-\omega)$] which coincides with a sector of angle $2\pi - \delta$ where $0 < \delta < \frac{1}{2}\pi$. We shall suppose that $\log x$ has the same determination in the two sectors on the ray whose points have the argument $\frac{1}{2}\pi + \arg\omega$. Then the periodic functions $F(x|\omega) - F(x|-\omega)$ and $G(x|\omega) - G(x|-\omega)$ have no essential singularity in either end of a periodic strip, as one sees by aid of the asymptotic forms of the functions involved.

But the singularities of $F(x|\omega)$ and $G(x|\omega)$ [$F(x|-\omega)$ and $G(x|-\omega)$] in the finite plane consist of poles at the points $\alpha_i - k\omega$ [$\alpha_i + k\omega + \omega$]

where $k = 0, 1, 2, \cdots$ and where the α_i are the poles of $\varphi(x)$ in the finite plane, the orders of the poles in the set congruent to α_i being not greater than the order of the pole of $\varphi(x)$ at $x = \alpha_i$ or a congruent point. This is proved by observing that in the case of any one of these functions there is a half-plane in which it is analytic and of such sort that it may be continued analytically from this half-plane into the complementary half-plane by aid of the simple difference equation satisfied by the function.

Therefore the periodic function $F(x|\omega) - F(x|-\omega)$ $[G(x|\omega) - G(x|-\omega)]$ is a rational function $\sigma(t)$ [odd rational function $\varrho(t)$] of t where $t = e^{2\pi i x/\omega}$ $[t = e^{\pi i x/\omega}]$.

Moreover, when $\varphi(x)$ is a particular given rational function of x then the positions of the poles of $F(x|\omega)$, $F(x|-\omega)$, $G(x|\omega)$, $G(x|-\omega)$ are in each case completely determined together with the principal parts at these poles. It may be shown that these descriptive properties of these functions regarding their singularities together with their asymptotic representation by $Q_1(x)$ or $P_1(x)$, as the case may be, and the fact that $F(x|\omega) - F(x|-\omega)$ is of period ω while $G(x|\omega) - G(x|-\omega)$ changes sign when x is replaced by $x + \omega$ are sufficient to completely characterize these four functions independently of the equations by means of which the functions were defined in the first case. An illustration of this proposition is given in the next section.

1.6. **Applications to the sums of $1/x$.** For the case when $\varphi(x) = 1/x$ and $\omega = 1$ the functions $F(x|\omega)$ and $F(x|-\omega)$ of Theorem 1.10 will be denoted by $\psi(x)$ and $\bar{\psi}(x)$ respectively. The forms of these functions, as given in Theorems 1.4 and 1.6, lead readily to the formulas

$$\psi(x) = -\gamma - \sum_{k=0}^{\infty} \left\{ \frac{1}{x+k} - \frac{1}{1+k} \right\},$$

$$\bar{\psi}(x) = \pi i - \gamma + \sum_{k=1}^{\infty} \left\{ \frac{1}{x-k} + \frac{1}{k} \right\},$$

where γ denotes Euler's constant. The leading term in the asymptotic form of both these functions is $\log x$. We have

$$\bar{\psi}(x) = \psi(x) + \frac{2\pi i\, e^{2\pi i x}}{e^{2\pi i x} - 1}.$$

These functions have the following properties:

(1) $\psi(x)$ is single-valued and analytic throughout the finite plane except for poles of the first order with residue -1 at the points $0, -1, -2, \cdots$;

(2) $\bar{\psi}(x)$ is single-valued and analytic throughout the finite plane except for poles of the first order with residue $+1$ at the points $1, 2, 3, \cdots$;

(3) $\lim_{x=\infty}\{\psi(x)-\log x\}=0, \quad -\frac{1}{2}\pi-\varepsilon\leqq\arg x\leqq\frac{1}{2}\pi+\varepsilon,$
ε being a small positive quantity;

(4) $\lim_{x=\infty}\{\bar{\psi}(x)-\log x\}=0, \quad \frac{1}{2}\pi-\varepsilon\leqq\arg x\leqq\frac{3}{2}\pi+\varepsilon;$

(5) $\bar{\psi}(x)-\psi(x)$ is a periodic function of period unity.

Here it is understood that $\log x$ has its principal determination when $\arg x=\frac{1}{2}\pi$ and that in (3) [(4)] it is then rendered unique by a cut from 0 to ∞ along the negative [positive] axis of reals.

It is readily proved that $\psi(x)$ and $\bar{\psi}(x)$ are completely characterized by these properties in the sense that if $g(x)$ and $\bar{g}(x)$ are functions having the properties of $\psi(x)$ and $\bar{\psi}(x)$ here stated then $g(x)\equiv\psi(x)$ and $\bar{g}(x)\equiv\bar{\psi}(x)$.

By considering similarly the first and second principal alternating sums of $1/x$ one is led to the functions $\beta(x)$ and $\bar{\beta}(x)$,

$$\beta(x)=2\sum_{k=0}^{\infty}\frac{(-1)^k}{x+k}, \quad \bar{\beta}(x)=2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{x-k},$$

$$\bar{\beta}(x)=\beta(x)-\frac{2\pi}{\sin\pi x},$$

with the following characterizing properties:

(1) $\beta(x)$ is single-valued and analytic throughout the finite plane except for poles of the first order at the points $0, -1, -2, \cdots$, the residues being alternately $+2$ and -2;

(2) $\bar{\beta}(x)$ is single-valued and analytic throughout the finite plane except for poles of the first order at the points $1, 2, 3, \cdots$, the residues being alternately $+2$ and -2;

(3) $\lim_{x=\infty}\beta(x)=0, \quad -\frac{1}{2}\pi-\varepsilon\leqq\arg x\leqq\frac{1}{2}\pi+\varepsilon;$

(4) $\lim_{x=\infty}\bar{\beta}(x)=0, \quad \frac{1}{2}\pi-\varepsilon\leqq\arg x\leqq\frac{3}{2}\pi+\varepsilon;$

(5) the function $P(x)$, $P(x)=\bar{\beta}(x)-\beta(x)$, has the property that $P(x+1)+P(x)=0$.

By means of the properties of the four functions treated in this section several classic expansions of trigonometric functions are readily obtained, as, for instance, the partial fractions expansion of $\cot\pi x$ and $\sec\pi x$.

1.7. **Applications to the sums of** $\log x$. If one uses the span 1 and forms the first and second principal sums of $\log x$ and takes the exponential of the resulting functions, then,[5] when the arbitrary constants are determined

[5] This is carried out for $\Gamma(x)$ by Nörlund, *Vorlesungen über Differenzenrechnung*, pp. 109 ff. One prefers to make a (not difficult) modification of the method of Nörlund so as to avoid employing the product expansion of the sine function in deriving the results.

in a convenient way, one has the functions $\Gamma(x)$ and $\bar{\Gamma}(x)$ with the same asymptotic form (but in different sectors) where

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{k=1}^{\infty}\left(1+\frac{x}{k}\right)e^{-x/k}, \quad \bar{\Gamma}(x) = -2\pi i\, e^{(\pi i-\gamma)x}\prod_{k=1}^{\infty}\left(1-\frac{x}{k}\right)e^{x/k},$$

$$\bar{\Gamma}(x) = (1-e^{2\pi i x})\,\Gamma(x).$$

These functions satisfy the equation $g(x+1) = xg(x)$. Or these functions may also be obtained by integrating $\psi(x)$ and $\bar{\psi}(x)$ and taking the exponential of the resulting functions. The fundamental properties of these functions may be very rapidly developed from this point of view. It may be shown that they are completely characterized by the possession of the following properties:

(1) $\Gamma(x)$ is single-valued and analytic throughout the finite plane except for poles of the first order at 0, 1, 2, $\cdots$, with residue 1 at 0, and it vanishes nowhere in the finite plane;

(2) $\bar{\Gamma}(x)$ is an integral function;

(3) $\lim_{x=\infty} \Gamma(x)x^{-x+1/2}e^{x} = (2\pi)^{1/2}, \quad -\frac{1}{2}\pi-\varepsilon \leqq \arg x \leqq \frac{1}{2}\pi+\varepsilon;$

(4) $\lim_{x=\infty} \bar{\Gamma}(x)x^{-x+1/2}e^{x} = (2\pi)^{1/2}, \quad \frac{1}{2}\pi-\varepsilon \leq \arg x \leqq \frac{3}{2}\pi+\varepsilon;$

(5) $\bar{\Gamma}(x)/\Gamma(x)$ is periodic of period 1.

If one uses the span 1 and forms the first and second principal alternating sums of $\frac{1}{2}\log x$ and takes the exponential of the resulting functions, then[6] one has the functions $\gamma(x)$ and $\bar{\gamma}(x)$, namely:

$$\gamma(x) = (2e^{-\gamma})^{1/2}\prod_{k=1}^{\infty}\frac{x+2k-2}{x+2k-1}e^{\frac{1}{2k}}, \quad \bar{\gamma}(x) = i(2e^{-\gamma})^{1/2}\prod_{k=1}^{\infty}\frac{x-2k+1}{x-2k}e^{\frac{1}{2k}}.$$

Moreover we have

$$\gamma(x+1)\gamma(x) = x, \qquad \bar{\gamma}(x+1)\bar{\gamma}(x) = x, \qquad \bar{\gamma}(x) = i\cot\frac{1}{2}\pi x\cdot\gamma(x).$$

The leading term in the asymptotic representation of the functions $\log\gamma(x)$ and $\log\bar{\gamma}(x)$ is $\frac{1}{2}\log x$,

It may be shown that $\gamma(x)$ and $\bar{\gamma}(x)$ are completely defined by the following properties:

(1) $\gamma(x)$ is single-valued and analytic throughout the finite plane except for poles of the first order at -1, -3, -5, $\cdots$; it has zeros of the first order at 0, -2, -4, $\cdots$;

[6] See Nörlund's *Vorlesungen*, pp. 115 ff., for $\gamma(x)$.

(2) $\bar{\gamma}(x)$ is single-valued and analytic throughout the finite plane except for poles of the first order at 2, 4, 6, $\cdots$; it has zeros of the first order at 1, 3, 5, $\cdots$;

(3) $\lim_{x=\infty} \gamma(x)x^{-1/2} = 1, \quad -\frac{1}{2}\pi-\varepsilon \leqq \arg x \leqq \frac{1}{2}\pi+\varepsilon;$

(4) $\lim_{x=\infty} \bar{\gamma}(x)x^{-1/2} = 1, \quad \frac{1}{2}\pi-\varepsilon \leqq \arg x \leqq \frac{3}{2}\pi+\varepsilon;$

(5) $\bar{\gamma}(x)/\gamma(x)$ is periodic of period 2.

By means of the properties of the four functions treated in this section several classic expansions of trigonometric functions are readily obtained, as, for instance, the product expansion of the sine function.

II. Summation of functions of exponential type.

2.1. **Functions of exponential type.**[7] If $f(x)$ is an analytic function which is regular at x_0 and x_1, then by expanding $f(x)$ in powers of $x-x_0$ and suitably dominating the expansion it is easily shown that

$$\limsup_{n=\infty} |f^{(n)}(x_0)|^{1/n} = \limsup_{n=\infty} |f^{(n)}(x_1)|^{1/n},$$

where the superscripts denote derivatives with respect to x. If these superior limits have the finite value q ($q \geqq 0$) then $f(x)$ is an integral function; in such a case we shall say that $f(x)$ is of exponential type q. This terminology is justified by the following theorem:

THEOREM 2.1. *A necessary and sufficient condition that the function $f(x)$ shall be of exponential type q is* (1) *that numbers σ shall exist for which it is true that for every positive number ε there exists a quantity M, depending on ε and σ in general but independent of x, such that for all (finite) values of x we have*

$$|f(x)| < Me^{(\sigma+\varepsilon)|x|} \tag{2.1}$$

and (2) *that q shall be the least possible value for such numbers σ. Moreover, when $f(x)$ is of exponential type q, we have*

$$|f^{(n)}(x)| < M(q+\varepsilon)^n e^{(q+\varepsilon)|x|}, \qquad n = 0, 1, 2, \cdots, \tag{2.2}$$

where M is independent of x and n.

Inequality (2.2) is readily proved by aid of the expansion of $f^{(n)}(x)$ in powers of x. Taking $n = 0$ we see that (2.1) is satisfied for every σ such that $\sigma \geqq q$. We complete the proof of the theorem by showing

[7] See Hilb, *Mathematische Annalen* 82 (1920), pp. 1–39, 85 (1922), pp. 89–98; Perron, *ibid.* 84 (1921), pp. 31–42; Polya, *ibid.* 89 (1923), pp. 179–191; Pincherle, *Acta Mathematica* 48 (1926), pp. 279–304 (first published in 1888); Hurwitz, *ibid.* 20 (1897), pp. 285–312.

that (2.1) can not be satisfied for any σ less than q. From (2.1) and Cauchy's inequalities for the coefficients of the power series expansion of $f(x)$ it follows that for every positive number ϱ we have

$$|f^{(n)}(0)| < M\varrho^{-n}\, e^{(\sigma+\varepsilon)\varrho} \cdot n!.$$

Putting $\varrho = n/(\sigma+\varepsilon)$ and using the fact (easily proved by elementary means) that

$$\lim_{n=\infty} (n!\, e^n\, n^{-n})^{1/n} = 1,$$

we find that

$$\limsup_{n=\infty} |f^{(n)}(0)|^{1/n} \leqq \sigma+\varepsilon.$$

Hence (2.1) can not be satisfied when $\sigma < q$. The theorem is therefore established.

We shall have occasion to employ the following (readily demonstrated) theorems stated here without proof:

THEOREM 2.2. *If $f(x)$ and $g(x)$ are two analytic functions of exponential type not exceeding q and if a is a constant, then the function $h(x)$,*

$$h(x) = \int_a^x g(t) f(x-t)\, dt,$$

is of exponential type not exceeding q.

THEOREM 2.3. *If $f(x)$ is of exponential type q and if the radius ϱ of convergence of the power series $\sum a_k t^k$ is greater than q then the series*

$$\sum_{k=0}^{\infty} a_k f^{(k)}(x)$$

is uniformly convergent in any whatever finite region of the x-plane and defines a sum function $g(x)$ of exponential type not exceeding q.

THEOREM 2.4. *Let $f(x)$ be of exponential type not exceeding q and let it have the period α. Let n denote the greatest integer not exceeding $|\alpha| q/(2\pi)$. Then $f(x)$ has the form*

$$f(x) = \sum_{k=-n}^{n} c_k\, e^{2k\pi i x/\alpha},$$

where the c's are constants. Moreover every function $f(x)$ of the latter form is of exponential type not exceeding q and is periodic with period α.

2.2. **Sum formulas of the Euler-Maclaurin type.** By expanding $F(x+\omega)$ and $G(x+\omega)$ in powers of ω by means of Taylor's theorem and employing the operator D to denote differentiation with respect to x, the equations,

(2.3) $$\Delta_\omega F(x) = \varphi(x), \qquad \nabla_\omega G(x) = \varphi(x),$$

may be written in the symbolic forms

$$(e^{\omega D}-1)\,F(x) = \omega\,\varphi(x), \qquad (e^{\omega D}+1)\,G(x) = 2\,\varphi(x),$$

whence one has the symbolic formal solutions

$$F(x) = \omega(e^{\omega D}-1)^{-1}\,\varphi(x), \qquad G(x) = 2(e^{\omega D}+1)^{-1}\,\varphi(x). \tag{2.4}$$

Now we may write

$$\frac{\omega}{e^{\omega t}-1} = -\frac{1}{2}\,\omega + \frac{1}{t} + \sum_{\nu=1}^{m}\left\{\frac{1}{t+2\nu\pi i/\omega} + \frac{1}{t-2\nu\pi i/\omega}\right\} + \sum_{\nu=2}^{\infty} B_\nu^{(m)}\,\frac{\omega^\nu}{\nu!}\,t^{\nu-1}, \tag{2.5}$$

$$\frac{2}{e^{\omega t}+1} = 1 - \frac{2}{\omega}\sum_{\nu=1}^{m}\left\{\frac{1}{t+(2\nu-1)\pi i/\omega} + \frac{1}{t-(2\nu-1)\pi i/\omega}\right\} + \sum_{\nu=1}^{\infty} A_\nu^{(m)}\,\frac{\omega^\nu}{\nu!}\,t^\nu, \tag{2.6}$$

where m is a non-negative integer. When $m = 0$ we shall understand in each case that the finite sum as to ν is omitted. We shall conventionally allow m to be infinite, understanding that the second line in each formula is then to be omitted. When m is finite the radius of convergence of the infinite power series in (2.5) [(2.6)] is $(2m+2)\pi/|\omega|$ [$(2m+1)\pi/|\omega|$], as one sees from the analytic character of the function represented by the series.

Replacing t in the last two equations by D and substituting the resulting symbolic expansions for the corresponding symbolic operators in (2.4) and then applying these expansions term by term to $\varphi(x)$ in the respective cases and simplifying the parts of the solutions contributed by the finite sums in accordance with the method which is usual in the symbolic solution of non-homogeneous linear differential equations, we obtain the following formal solutions of equations (2.3):

$$F(x) = -\frac{1}{2}\,\omega\varphi(x) + \int_a^x \varphi(u)\,du + 2\sum_{\nu=1}^{m}\int_a^x \cos\frac{2\nu\pi(u-x)}{\omega}\,\varphi(u)\,du + \sum_{\nu=2}^{\infty} B_\nu^{(m)}\,\frac{\omega^\nu}{\nu!}\,\varphi^{(\nu-1)}(x), \tag{2.7}$$

$$G(x) = \varphi(x) - \frac{4}{\omega}\sum_{\nu=1}^{m}\int_a^x \cos\frac{(2\nu-1)\,\pi(u-x)}{\omega}\,\varphi(u)\,du + \sum_{\nu=1}^{\infty} A_\nu^{(m)}\,\frac{\omega^\nu}{\nu!}\,\varphi^{(\nu)}(x), \tag{2.8}$$

where a is any conveniently chosen constant. It may be shown directly, by aid of relations obtained from the identities (2.5) and (2.6), that these

expressions afford formal solutions of equations (2.3). Conventional interpretations are to be given to these formulas when $m = 0$ and when $m = \infty$ in accordance with the statements following (2.5) and (2.6).

When $m = 0$ we have in (2.5) and (2.7) formulas of the Euler-Maclaurin and Boole-Nörlund types respectively. Formula (2.7) for $m = \infty$ has been treated by Wedderburn.[8] In the general case of finite m we shall find the formulas useful in the summation of functions of exponential type q. We shall call (2.7) [(2.8)] the modified Euler-Maclaurin [Boole-Nörlund] sum formula.

Putting $a = 0$ in (2.7) and (2.8), replacing u by $x - t$ in the integrals and then in each equation combining the terms involving integrals and simplifying by aid of the trigonometric identities,

$$1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos m\theta = \sin\left(m + \frac{1}{2}\right)\theta / \sin\frac{1}{2}\theta,$$

$$2\cos\theta + 2\cos 3\theta + \cdots + 2\cos(2m-1)\theta = \sin 2m\theta / \sin\theta,$$

we have the formal solutions

$$F(x) = -\frac{1}{2}\omega\varphi(x) + \int_0^x \frac{\sin(2m+1)\pi t/\omega}{\sin \pi t/\omega}\varphi(x-t)\,dt + \sum_{\nu=2}^{\infty} B_\nu^{(m)} \frac{\omega^\nu}{\nu!}\varphi^{(\nu-1)}(x), \tag{2.9}$$

$$G(x) = \varphi(x) - \frac{1}{\omega}\int_0^x \frac{\sin 2m\pi t/\omega}{\sin \pi t/\omega}\varphi(x-t)\,dt + \sum_{\nu=1}^{\infty} A_\nu^{(m)} \frac{\omega^\nu}{\nu!}\varphi^{(\nu)}(x). \tag{2.10}$$

Taking $m = \infty$ we have the following modification of the Wedderburn formulas:

$$F(x) = -\frac{1}{2}\omega\varphi(x) + \lim_{m=\infty}\int_0^x \frac{\sin(2m+1)\pi t/\omega}{\sin \pi t/\omega}\varphi(x-t)\,dt, \tag{2.11}$$

$$G(x) = \varphi(x) - \frac{1}{\omega}\lim_{m=\infty}\int_0^x \frac{\sin 2m\pi t/\omega}{\sin \pi t/\omega}\varphi(x-t)\,dt. \tag{2.12}$$

On replacing these limits by the corresponding limits of averages, as in the case of Cesàro summability of order 1, and simplifying by aid of the trigonometric identities,

$$\sin\theta + \sin 3\theta + \cdots + \sin(2m-1)\theta = \frac{1 - \cos 2m\theta}{2\sin\theta},$$

$$\sin\theta + \sin 2\theta + \cdots + \sin m\theta = \frac{\cos\frac{1}{2}\theta - \cos(m+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta},$$

[8] Wedderburn, *Annals of Mathematics* (2), 16 (1914), pp. 82-85.

we have the following formal solutions:

$$(2.13)\quad F(x) = -\frac{1}{2}\omega\varphi(x) + \lim_{m=\infty}\int_0^x \frac{1-\cos 2m\pi t/\omega}{2m\sin^2\pi t/\omega}\varphi(x-t)\,dt,$$

$$(2.14)\quad G(x) = \varphi(x) - \frac{1}{\omega}\lim_{m=\infty}\int_0^x \frac{\cos\pi t/\omega - \cos(2m+1)\pi t/\omega}{2m\sin^2\pi t/\omega}\varphi(x-t)\,dt.$$

We shall now prove the following two theorems:

THEOREM 2.5. *If $\varphi(x)$ is of exponential type not exceeding q and if m is the integer defined by the relation*

$$2\pi m \leqq |\omega|q < 2\pi(m+1),$$

then the first equation (2.3) *has a solution $F(x)$ defined by the modified Euler-Maclaurin sum formula* (2.7); *and $F(x)$ is of exponential type not exceeding q. If $\varphi(x)$ is of exponential type q then $F(x)$ is of exponential type q.*

THEOREM 2.6. *If $\varphi(x)$ is of exponential type not exceeding q and if m is the integer defined by the relation*

$$(2m-1)\pi \leqq |\omega|q < (2m+1)\pi,$$

then the second equation (2.3) *has a solution $G(x)$ defined by the modified Boole-Nörlund sum formula* (2.8); *and $G(x)$ is of exponential type not exceeding q. If $\varphi(x)$ is of exponential type q then $G(x)$ is of exponential type q.*

In order to prove Theorem 2.5 consider the second member of equation (2.7) when $\varphi(x)$ and m satisfy the conditions named in the theorem. The first two terms represent functions of exponential type not exceeding q. That each of the terms in the finite sum as to ν (when this sum is present) represents a function of exponential type not exceeding q follows at once from Theorem 2.2 and the obvious fact that $\cos ax$ is of exponential type $|a|$. That the infinite series in (2.7) converges and represents a function of exponential type not exceeding q follows from Theorem 2.3 and the known region of convergence of the power series in the second member of equation (2.5). Therefore the second member of equation (2.7) is a sum of a finite number of functions each of exponential type not exceeding q, whence it follows that the function $F(x)$ defined by this equation is of exponential type not exceeding q. That this function satisfies the first equation (2.3) may be shown by direct substitution. If $F(x)$ is of exponential type q_1 then $F(x+\omega)-F(x)$ is of exponential type not exceeding q_1. Hence $F(x)$ is of exponential type q when $\varphi(x)$ has this property. This establishes Theorem 2.5.

A similar argument leads to the proof of Theorem 2.6.

2.3. **Principal sums of functions of exponential type.** When $\varphi(x)$ is of exponential type q we shall say that a solution of the first [second] equation (2.3) is a *principal sum* [*principal alternating sum*] of $\varphi(x)$ if the solution is itself of exponential type q. No solution of either equation can be of lower type than that of $\varphi(x)$.

The following theorems are easy consequences of Theorems 2.5, 2.6, and 2.4:

THEOREM 2.7. *If $\varphi(x)$ is of exponential type q and if m is the greatest integer not exceeding $|\omega|q/(2\pi)$ then the most general principal sum of $\varphi(x)$ is the algebraic sum of any particular principal sum (always existent) and the function*

$$\sum_{k=-m}^{m} c_k e^{2k\pi i x/\omega}$$

where the c's are arbitrary constants.

THEOREM 2.8. *If $\varphi(x)$ is of exponential type q and if m is the greatest integer not exceeding $\frac{1}{2}+|\omega|q/(2\pi)$ then the principal alternating sum of $\varphi(x)$ is unique when $m=0$ and is otherwise the algebraic sum of a particular principal alternating sum (always existent) and the function*

$$\sum_{k=-m+1}^{m} c_k e^{(2k-1)\pi i x/\omega}$$

where the c's are arbitrary constants.

These two theorems, for the case $m=0$, are given by Hilb (l. c., pp. 90–91). In the general case they are, in view of our Theorem 2.1, equivalent to theorems obtained by J. R. Purdy[9] by quite different means and under hypotheses of a different form.

The sums of $\varphi(x)$ referred to in Theorems 2.7 and 2.8 are of the same type as the function $\varphi(x)$ itself. Hence the indicated process of summation may be applied in turn to these sums and in this way repeated sums of any order may be obtained.

There are other ways of forming principal sums [alternating sums] of a given function $\varphi(x)$ of exponential type q. If we write

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 + \cdots,$$

$$f(t) = \frac{a_0}{t} + \frac{a_1}{t^2} + \cdots + \frac{n!\, a_n}{t^{n+1}} + \cdots, \qquad |t| > q,$$

and if C denotes a circle about 0 as center and of radius ϱ greater than q then we have

$$\varphi(x) = \frac{1}{2\pi i}\int_C e^{xt} f(t)\, dt.$$

[9] J. R. Purdy, "The treatment of finite integration by means of the Cauchy integral theorem," (unpublished) Illinois dissertation, 1930.

Pincherle (l. c., pp. 285–86) and Hurwitz (l. c.) have observed that equations (2.3) have the respective solutions

$$(2.15)\quad F(x) = \frac{1}{2\pi i}\int_C \frac{\omega e^{xt}}{e^{\omega t}-1} f(t)\,dt, \qquad G(x) = \frac{1}{2\pi i}\int_C \frac{2e^{xt}}{e^{\omega t}+1} f(t)\,dt,$$

provided that C does not pass through a singularity of an integrand.

We shall now prove the following theorem:

THEOREM 2.9. *If C is the circle $|t| = q+\varepsilon$, where ε is positive, and if neither of the functions $e^{\omega t}-1$ and $e^{\omega t}+1$ has a zero in the circular ring $q<|t|\leqq q+\varepsilon$, then the function $F(x)$ [$G(x)$] in the first [second] equation* (2.15) *is a principal sum [alternating sum] of the given function $\varphi(x)$ of exponential type q.*

From (2.15) it follows that a constant M_ε exists such that

$$|F(x)| < M_\varepsilon\, e^{(q+\varepsilon)|x|}, \qquad |G(x)| < M_\varepsilon\, e^{(q+\varepsilon)|x|}.$$

Then from Theorem 2.1 we see that $F(x)$ and $G(x)$ are of exponential type not exceeding $q+\varepsilon$. But these functions are unaltered as ε is decreased provided that it remains positive. Hence the functions are of type not exceeding q; and therefore they are of type q since $\varphi(x)$ is itself of this type.

If in the integrals in (2.15) we replace $\omega(e^{\omega t}-1)^{-1}$ and $2(e^{\omega t}+1)^{-1}$ by their expansions in (2.5) and (2.6) respectively, the integer m having first been chosen so that in the respective cases we have $(2m+2)\pi/|\omega|$ and $(2m+1)\pi/|\omega|$ greater than q, and if we perform the integrations term by term, after having multiplied the factor $e^{xt}f(t)$ into the series, we have formulas which readily yield in the following results:

$$(2.16)\quad F(x) = -\frac{1}{2}\omega\varphi(x) + \frac{1}{2\pi i}\sum_{\nu=-m}^{m}\int_C \frac{e^{xt}f(t)}{t+2\nu\pi i/\omega}\,dt + \sum_{\nu=2}^{\infty} B_\nu^{(m)}\frac{\omega^\nu}{\nu!}\varphi^{(\nu-1)}(x),$$

$$(2.17)\quad G(x) = \varphi(x) - \frac{2}{\omega\pi i}\sum_{\nu=1}^{m}\int_C \left\{\frac{1}{t+(2\nu-1)\pi i/\omega} + \frac{1}{t-(2\nu-1)\pi i/\omega}\right\} e^{xt}f(t)\,dt + \sum_{\nu=1}^{\infty} A_\nu^{(m)}\frac{\omega^\nu}{\nu!}\varphi^{(\nu)}(x).$$

These formulas are essentially equivalent to (2.7) and (2.8) respectively, as we shall now show. For the case of (2.16) we observe that the function

$$u_\nu(x) = \frac{1}{2\pi i}\int_C \frac{e^{xt}f(t)}{t+2\nu\pi i/\omega}\,dt$$

verifies the relation

$$u_\nu'(x) + 2\nu\pi i u_\nu(x)/\omega = \varphi(x).$$

Thence it follows readily that the first line in the second member of (2.16) defines the same function as the corresponding part of (2.7) except for an exponential sum of the same form as that in Theorem 2.7 with coefficients c. For the case of (2.17) a similar remark is available. Hence formulas (2.16) and (2.17) are essentially equivalent to (2.7) and (2.8) respectively.

2.4. **Other formal solutions.** It is convenient to record here other types of formal solutions. For the sake of brevity and simplicity we present them mainly for the case of the equation

$$F(x+1)-F(x) = \varphi(x). \tag{2.18}$$

If $\varphi(x)$ has the form

$$\varphi(x) = \int_C \frac{\Gamma(x+t)}{\Gamma(x-t)} g(t)\,dt$$

then equation (2.18) has the solution

$$F(x) = \int_C \frac{\Gamma(x+t)}{\Gamma(x-t-1)} \frac{g(t)}{2t+1}\,dt$$

provided that the path C is independent of x and passes through no singularity of either integrand.

Similarly, if

$$\varphi(x) = \frac{1}{\Gamma(x)} \int_C \Gamma(x+t) g(t)\,dt$$

then (2.18) has the solution

$$F(x) = \frac{1}{\Gamma(x-1)} \int_C t^{-1}\Gamma(x+t) g(t)\,dt.$$

Again, if

$$\varphi(x) = \Gamma(x) \int_C \frac{g(t)\,dt}{\Gamma(x+t)}$$

then (2.18) has the solution

$$F(x) = \Gamma(x) \int_C \frac{g(t)}{1-t} \frac{dt}{\Gamma(x+t-1)}.$$

Finally, if we employ the transformation of Nörlund[10] and write

$$\varphi(x) = \frac{1}{\Gamma(x)} \int_C \frac{g(t)\,dt}{\Gamma(t-x)},$$

then the equation $G(x+1)+G(x) = \varphi(x)$ has the solution

$$G(x) = \frac{1}{\Gamma(x-1)} \int_C \frac{g(t)\,dt}{(t-2)\,\Gamma(t-x)}.$$

[10] Nörlund, *Enzyklop. d. Math. Wiss.* II C 7 (1923), p. 720.

2.5. **Fourier series expansions of principal sums.** It is convenient in this section (and involves no real loss of generality) to take $\omega = 1$; and this we do. For the circle C in the first equation (2.15) we take the circle C_r about 0 as a center and of radius $(2r+1)\pi$, greater than q, where r is an integer. We write

$$F_r(x) = \frac{1}{2\pi i}\int_{C_r} \frac{e^{xt}}{e^t-1} f(t)\,dt. \tag{2.19}$$

We form similarly the functions $F_{r+p}(x)$, $p = 1, 2, 3, \cdots$. First we prove that

$$\lim_{p=\infty} F_{r+p}(x) = 0 \tag{2.20}$$

when $0<x<1$; when $a_0 = 0$ we show that x may have the range $0 \leqq x \leqq 1$.

Write $t = u+iv$ where u and v are real. Then the two parabolas $v = u^2$ and $v = -u^2$ cut the circle C_{r+p} in four points, thus separating it into four arcs. The two of these arcs which cut the real axis we call major arcs; the other two we call minor arcs. We write $\theta = \arg t$. As p becomes infinite the range of θ on a minor arc approaches zero.

Take first the case when $a_0 = 0$. Then a constant M_1 exists such that $|f(t)| < M_1 |t|^{-2}$ for all t on all the circles C_{r+p}, $p = 0, 1, 2, \cdots$. There also exists a constant M_2 such that $|e^{tx}(e^t-1)^{-1}| < M_2$ for all x in the range $0 \leqq x \leqq 1$ and for all t on all the circles C_{r+p}. Then

$$|F_{r+p}(x)| < (2\pi)^{-1} M_1 M_2 \int_{C_{r+p}} |t|^{-2}|\,dt|.$$

Thence we have (2.20) for the case when $a_0 = 0$.

In order to complete the proof of (2.20) it is evidently sufficient to show further that it is true when $f(t) = 1/t$ and x is in the range $0<x<1$. For the minor arcs it is sufficient to employ the inequality $|e^{tx}(e^t-1)^{-1}| < M_2$ and to make use of the fact that the range of θ on a minor arc is infinitesimal as p becomes infinite. Thus the part of the value of $F_{r+p}(x)$ contributed by the integral along a minor arc is seen to be infinitesimal as p becomes infinite. That the part contributed by the integral along a major arc is also infinitesimal as p becomes infinite follows readily from the value of $f(t)$ and the fact that for a fixed x in the range $0<x<1$ there exists a constant M_3 such that $|te^{xt}(e^t-1)^{-1}| < M_3$ for all t on major arcs of all circles C_{r+p}, $p = 0, 1, 2, \cdots$.

Having established (2.20) we note that the difference $F_r(x) - F_{r+p}(x)$ is equal to the negative of the sum of the residues of the integrand in (2.19) in the circular ring bounded by the circles C_r and C_{r+p}. As p becomes infinite this sum of residues becomes an infinite series, while from (2.20)

it follows that this series is convergent. We are thus led to the following result:

$$(2.21)\quad F_r(x) = -\sum_{k=r+1}^{\infty}\{f(2k\pi i)\,e^{2k\pi ix}+f(-2k\pi i)\,e^{-2k\pi ix}\},\quad 0<x<1.$$

Moreover, when $a_0 = 0$ this holds for the range $0 \leqq x \leqq 1$. It may be shown that for $x = 0$ or $x = 1$ the sum of the series is $\frac{1}{2}\{F_r(0)+F_r(1)\}$; this is done by observing that the proposition is already proved when $a_0 = 0$ and that it is easily established when $f(t) = 1/t$ and hence is true in general.

Thus we have established the Fourier expansion of $F_r(x)$ for the interval (0, 1). The result is obtained in a different way by Hurwitz (l. c.), whose analysis assumes that $F_r(x)$ is known to have a Fourier series development, whereas our argument establishes that fact.

In the case of the second equation in (2.15) we take C to be the circle Γ_r about 0 as center and of radius $2r\pi$, greater than q, where r is a positive integer, and we write

$$(2.22)\qquad G_r(x) = \frac{1}{2\pi i}\int_{\Gamma_r}\frac{2e^{xt}}{e^t+1}f(t)\,dt.$$

Proceeding as in the previous case one may prove that

$$(2.23)\quad G_r(x) = 2\sum_{k=r}^{\infty}\{f(2k\pi i+\pi i)\,e^{(2k+1)\pi ix}+f(-2k\pi i-\pi i)\,e^{-(2k+1)\pi ix}\},\quad 0<x<1,$$

while for $x = 0$ and $x = 1$ the sum of the series is $\frac{1}{2}\{G_r(0)+G_r(1)\}$. Moreover, when $a_0 = 0$, the equation (2.23) is valid for the range $0 \leqq x \leqq 1$.

In equation (2.21) [(2.23)] we have the Fourier expansion of a particular principal sum [alternating sum] of $\varphi(x)$. By applying Theorems 2.7 and 2.8 one may easily obtain the Fourier expansion of any other particular principal sum of $\varphi(x)$ and indeed of any sum of exponential type; what is requisite in each case is to determine the coefficients in a finite number of terms at the beginning of the series, the expansion being already known to exist.

In a recent paper (Trans. Amer. Math. Soc.) I have generalized the theory of Fourier series in a direction suggested by the results just given.

As an application of the theory here given we have the Fourier expansion of x^n, where n is a non-negative integer, in the form:

$$x^n = \frac{1}{n+1}-\sum_{k=1}^{\infty}\{f_n(2k\pi i)\,e^{2k\pi ix}+f_n(-2k\pi i)\,e^{-2k\pi ix}\},\quad 0<x<1,$$

where

$$f_n(t) = \frac{1}{t}+\frac{n}{t^2}+\frac{n(n-1)}{t^3}+\cdots+\frac{n(n-1)(n-2)\cdots 3\cdot 2}{t^n}.$$

This is obtained by taking $\varphi(x) = (x+1)^n - x^n$.

III. Expansions in Bernoulli-Hurwitz Functions.

3.1. **The Bernoulli-Hurwitz functions.** The Bernoulli polynomials $B_\nu(x)$ may be defined by means of the identity

$$\frac{te^{tx}}{e^t-1} = \sum_{\nu=0}^{\infty} B_\nu(x)\frac{t^\nu}{\nu!}, \tag{3.1}$$

whence it follows that

$$B_\nu(x) = \frac{\nu!}{2\pi i}\int_C \frac{e^{tx}}{e^t-1}\frac{dt}{t^\nu}, \qquad \nu = 0, 1, 2, \cdots, \tag{3.2}$$

where C is a circle about 0 as center and of radius less than 2π. The Bernoulli-Hurwitz functions $B_{\nu,r}(x)$ we shall define by the relations

$$B_{\nu,r}(x) = \frac{\nu!}{2\pi i}\int_{C_r} \frac{e^{tx}}{e^t-1}\frac{dt}{t^\nu}, \qquad \nu, r = 0, 1, 2, \cdots, \tag{3.3}$$

where C_r denotes the circle of radius $\pi(2r+1)$ about 0 as a center. Then we have

$$\begin{cases} B_{\nu,0}(x) = B_\nu(x), & \nu = 0, 1, 2, \cdots, \\ B_{\nu,r}(x) = B_\nu(x) + \dfrac{\nu!}{(2\pi i)^\nu}\displaystyle\sum_{k=1}^{r} k^{-\nu}\{e^{2k\pi ix} + (-1)^\nu e^{-2k\pi ix}\}, & \\ \qquad\qquad \nu = 0, 1, 2, \cdots, \ r = 1, 2, 3, \cdots. \end{cases} \tag{3.4}$$

From § 2.5 it follows readily that

$$B_{2\nu,r}(x) = \frac{(-1)^{\nu+1}\,2\cdot(2\nu)!}{(2\pi)^{2\nu}}\sum_{k=r+1}^{\infty}\frac{\cos 2k\pi x}{k^{2\nu}}, \quad 0 \leqq x \leqq 1, \quad \nu > 0; \tag{3.5}$$

$$B_{2\nu+1,r}(x) = \frac{(-1)^{\nu+1}2\cdot(2\nu+1)!}{(2\pi)^{2\nu+1}}\sum_{k=r+1}^{\infty}\frac{\sin 2k\pi x}{k^{2\nu+1}}, \quad \begin{cases} 0 \leqq x \leqq 1, & \nu > 0, \\ 0 < x < 1, & \nu = 0. \end{cases} \tag{3.6}$$

Moreover, from (3.1), (3.2) and (3.4) we have the identity

$$\frac{te^{tx}}{e^t-1} + \sum_{k=1}^{r}\left(\frac{2k\pi ie^{2k\pi ix}}{2k\pi i - t} + \frac{2k\pi ie^{-2k\pi ix}}{2k\pi i + t}\right) = \sum_{\nu=0}^{\infty} B_{\nu,r}(x)\frac{t^\nu}{\nu!}, \qquad r > 0. \tag{3.7}$$

The function of t in the left member of this equation is analytic in the interior of the circle $|t| = 2\pi(r+1)$ and has a singularity on its boundary. Therefore $2\pi(r+1)$ is the radius of convergence of the power series in the second member. Hence, for every x, we have

$$\limsup_{\nu=\infty}\left|\frac{B_{\nu,r}(x)}{\nu!}\right|^{1/\nu} = \frac{1}{2\pi(r+1)}, \qquad r = 0, 1, 2, \cdots. \tag{3.8}$$

More precise information is available for certain values of x. From (3.8) and the value of $B_{\nu,r+1}(x)$ in terms of $B_{\nu,r}(x)$ afforded by (3.4) we have

$$\limsup_{\nu=\infty}\left|\frac{B_{\nu,r}(x)}{\nu!} + \frac{e^{2(r+1)\pi ix}}{(2(r+1)\pi i)^\nu} + \frac{e^{-2(r+1)\pi ix}}{(-2(r+1)\pi i)^\nu}\right|^{1/\nu} = \frac{1}{2\pi(r+2)}.$$

Hence if σ is such that $(r+1)/(r+2)<\sigma<1$ then for sufficiently large values of ν we have

$$|B_{\nu,r}(x)[2\pi i(r+1)]^{\nu}/\nu! + \{e^{2(r+1)\pi ix} + (-1)^{\nu} e^{-2(r+1)\pi ix}\}| < \sigma^{\nu}.$$

Therefore

$$\lim_{\nu=\infty} B_{2\nu,r}(x)[2\pi i(r+1)]^{2\nu}/(2\nu)! = -2\cos 2(r+1)\pi x, \tag{3.9}$$

$$\lim_{\nu=\infty} B_{2\nu+1,r}(x)[2\pi i(r+1)]^{2\nu+1}/(2\nu+1)! = -2i\sin 2(r+1)\pi x. \tag{3.10}$$

Hence if x_1 and x_2 are such that $\cos 2(r+1)\pi x_1 \neq 0$ and $\sin 2(r+1)\pi x_2 \neq 0$, then we have

$$\lim_{\nu=\infty} \left| \frac{B_{2\nu,r}(x_1)}{(2\nu)!} \right|^{1/(2\nu)} = \frac{1}{2\pi(r+1)}, \tag{3.11}$$

$$\lim_{\nu=\infty} \left| \frac{B_{2\nu+1,r}(x_2)}{(2\nu+1)!} \right|^{1/(2\nu+1)} = \frac{1}{2\pi(r+1)}. \tag{3.12}$$

From (3.3) it may readily be shown (compare Hurwitz, l. c., p. 287), by a method essentially equivalent to that employed in establishing Cauchy's inequalities for the coefficients of a power series, that if x is confined to a given region S of the complex plane and if $\varrho \geqq e^{\pi|x|}$ in S and is independent of x, then an absolute constant M exists such that

$$|B_{\nu,r}(x)| < M\varrho^{2r+1}\pi^{-\nu}\nu!/(2r+1)^{\nu-1} \tag{3.13}$$

for all x in S.

3.2. **Connection with the problem of summation.** If $\varphi(x)$ denotes the integral function

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

then since $B_{\nu,r}(x+1) - B_{\nu,r}(x) = \nu x^{\nu-1}$ the equation

$$f(x+1) - f(x) = \varphi(x)$$

obviously has the formal solution

$$f(x) = \sum_{\nu=1}^{\infty} a_{\nu-1}\nu^{-1} B_{\nu,r}(x), \tag{3.14}$$

where r is now understood to be a function of ν. Hurwitz (l. c.) shows that this is an actual solution and that $f(x)$ is an integral function when r is properly chosen as a function of ν and in particular when $r = \nu$. This proposition is very easily proved by aid of (3.13). Hurwitz also determines a necessary and sufficient condition that (3.14) shall afford a valid sum of $\varphi(x)$ when r in (3.14) is held fixed and the series in (3.14) is required to converge uniformly in every finite region of the plane. From (3.11) and (3.12) it follows at once that it is necessary to this result that

$\varphi(x)$ be of exponential type not greater than $2\pi(r+1)$ while from (3.8) it follows that it is sufficient if $\varphi(x)$ be of exponential type less than $2\pi(r+1)$. What remains to complete the analysis will appear incidentally as we proceed.

The series which we shall investigate are of the form of the series in (3.14) with r held fixed. The case $r = 0$ has been treated by Nielsen[11] by a method different from that which follows here.

3.3. **Series of Bernoulli-Hurwitz functions.** Let us now consider series of the form

$$\sum_{\nu=0}^{\infty} c_\nu B_{\nu, r}(x)/\nu! \tag{3.15}$$

where r is a fixed non-negative integer.

If $x = x_1$ is a point at which series (3.15) converges and if $4(r+1)x_1$ is not an odd integer then from (3.11) it follows that

$$\limsup_{\nu=\infty} |c_{2\nu}|^{1/(2\nu)} \leqq 2\pi(r+1). \tag{3.16}$$

If $x = x_2$ is a point at which series (3.15) converges and if $2(r+1)x_2$ is not an integer then from (3.12) it follows that

$$\limsup_{\nu=\infty} |c_{2\nu+1}|^{1/(2\nu+1)} \leqq 2\pi(r+1). \tag{3.17}$$

When conditions (3.16) and (3.17) are satisfied and p is a positive integer it follows from (3.8) that the series

$$\sum_{\nu=0}^{\infty} c_\nu B_{\nu, r+p}(x)/\nu! \tag{3.18}$$

converges absolutely for all values of x, while from (3.13) it follows that this series converges absolutely and uniformly in every preassigned finite region.

Taking $p = 1$ and employing (3.4) and (3.15) to (3.18) we see that the series

$$\sum_{\nu=0}^{\infty} c_\nu \left\{ \frac{e^{2(r+1)\pi i x}}{(2(r+1)\pi i)^\nu} + \frac{e^{-2(r+1)\pi i x}}{(-2(r+1)\pi i)^\nu} \right\} \tag{3.19}$$

converges whenever series (3.15) converges. Hence if x_1 and x_2 are points of convergence of series (3.15) and if $4(r+1)x_1$ is not an odd integer and neither of the quantities $2(r+1)x_2$ and $2(r+1)(x_1 - x_2)$ is an integer, then both the series

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\alpha\nu} c_\nu}{(2(r+1)\pi i)^\nu}, \qquad \alpha = 0, 1, \tag{3.20}$$

[11] Nielsen, *Mathematische Annalen* 59 (1904), pp. 103–109.

converge. Moreover, these series converge absolutely provided that (3.15) converges absolutely for $x = x_1$ and $x = x_2$. Thence it follows that series (3.19) converges (or converges absolutely, as the case may be) for all values of x and that it converges uniformly in every finite region.

From the named properties of series (3.18) and (3.19) and their relation to series (3.15), we now conclude to the following theorem:

THEOREM 3.1. *If the series* (3.15) *converges [converges absolutely] for* $x = x_1$ *and* $x = x_2$, *where* $4(r+1)x_1$ *is not an odd integer and* $2(r+1)x_2$ *and* $2(r+1)(x_1 - x_2)$ *are not integers, then in every finite region of the* x*-plane this series converges [converges absolutely] and uniformly. Moreover the series*

$$\sum_{\nu=0}^{\infty} c_\nu t^{-\nu} \tag{3.21}$$

converges [converges absolutely] for $t = \pm 2\pi i(r+1)$ *and hence also when* $|t| > 2\pi(r+1)$.

The hypotheses of the theorem are satisfied whenever there is any region (however small) in which series (3.15) converges [converges absolutely]. But it is possible for this series to converge at *one* point ($x = 0$, for instance) without converging in any region.

3.4. **Remark on certain series of polynomials.** The following example will serve to illuminate the hypotheses in Theorem 3.1. Let $g(t)$ be a function of t having the following properties: (a) it is analytic in the circle $|t| < \varrho$; (b) it vanishes to the first order at $t = \alpha$ where $0 < |\alpha| < \varrho$; (c) it vanishes nowhere else in the circle $|t| < \varrho$; (d) at $t = 0$ it has the value 1. Form the expansions

$$\frac{e^{xt}}{g(t)} = \sum_{\nu=0}^{\infty} P_\nu(x) t^\nu, \qquad \frac{e^{xt}}{g(t)} - \frac{e^{\alpha x}}{(t-\alpha)g'(\alpha)} = \sum_{\nu=0}^{\infty} Q_\nu(x) t^\nu,$$

where $g'(t)$ is the derivative of $g(t)$ with respect to t. Then $P_\nu(x)$ is a polynomial in x of degree ν and

$$Q_\nu(x) = P_\nu(x) + \alpha^{-\nu-1} e^{\alpha x}/g'(\alpha).$$

From function-theoretic considerations it follows that

$$\limsup_{\nu=\infty} |P_\nu(x)|^{1/\nu} = |\alpha|^{-1}, \qquad \limsup_{\nu=\infty} |Q_\nu(x)|^{1/\nu} \leqq \varrho^{-1} < |\alpha|^{-1}.$$

From these superior limits and the given relation between $P_\nu(x)$ and $Q_\nu(x)$ it follows that

$$\lim_{\nu=\infty} |P_\nu(x)|^{1/\nu} = |\alpha|^{-1}.$$

Now if the series

$$\sum_{\nu=0}^{\infty} c_\nu P_\nu(x) \tag{3.22}$$

converges [converges absolutely] at $x = x_1$ it follows that $\limsup |c_\nu|^{1/\nu} \leqq |\alpha|$ and hence that the series

$$\sum_{\nu=0}^{\infty} c_\nu Q_\nu(x)$$

converges absolutely and uniformly in every preassigned finite region, the uniformity being established by aid of the Cauchy inequalities for the coefficients $Q_\nu(x)$ in the power series $\sum Q_\nu(x) t^\nu$. Therefore the series $\sum c_\nu e^{\alpha x}$ converges [converges absolutely] for $x = x_1$; whence it follows that it converges uniformly in every finite region. We are thus led to the theorem:

THEOREM 3.2. *If the series* (3.22) *converges [converges absolutely] at any single point $x = x_1$ then in any whatever finite region it converges [converges absolutely] and uniformly.*

By differentiating with respect to x both members of the defining relation for the polynomials $P_\nu(x)$ we see that

$$P_\nu(x) = P'_{\nu+1}(x), \qquad \nu = 0, 1, 2, \cdots.$$

The case $g(t) = 1 - t$ is of interest. In this case $P_\nu(x)$ has the value

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^\nu}{\nu!},$$

and this is a "section" of the power series expansion of e^x.

3.5. **Expansions in Bernoulli-Hurwitz functions.** If a function $F(x)$ should have two distinct expansions in series (3.15) for the same value of r, then $F(x+1) - F(x)$ would have two distinct expansions in non-negative integral powers of x. Since this is impossible, we have the following theorem:

THEOREM 3.3 *No function $F(x)$ can have two distinct expansions in series of the form* (3.15) *for the same value of r.*

Suppose that $F(x)$ has an expansion of the form

$$F(x) = \sum_{\nu=0}^{\infty} c_\nu B_{\nu,r}(x)/\nu! \tag{3.23}$$

where the series meets the convergence conditions described in Theorem 3.1. Then we have

$$F(x+1) - F(x) = \sum_{\nu=0}^{\infty} c_{\nu+1} x^\nu/\nu! \equiv \varphi(x), \tag{3.24}$$

where $\varphi(x)$ is defined by this relation.

converge. Moreover, these series converge absolutely provided that (3.15) converges absolutely for $x = x_1$ and $x = x_2$. Thence it follows that series (3.19) converges (or converges absolutely, as the case may be) for all values of x and that it converges uniformly in every finite region.

From the named properties of series (3.18) and (3.19) and their relation to series (3.15), we now conclude to the following theorem:

THEOREM 3.1. *If the series* (3.15) *converges [converges absolutely] for* $x = x_1$ *and* $x = x_2$, *where* $4(r+1)x_1$ *is not an odd integer and* $2(r+1)x_2$ *and* $2(r+1)(x_1 - x_2)$ *are not integers, then in every finite region of the x-plane this series converges [converges absolutely] and uniformly. Moreover the series*

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converges [converges absolutely] for $t = \pm 2\pi i(r+1)$ *and hence also when* $|t| > 2\pi(r+1)$.

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where $g'(t)$ is the derivative of $g(t)$ with respect to t. Then $P_\nu(x)$ is a polynomial in x of degree ν and

$$Q_\nu(x) = P_\nu(x) + \alpha^{-\nu-1} e^{\alpha x}/g'(\alpha).$$

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$$\limsup_{\nu=\infty} |P_\nu(x)|^{1/\nu} = |\alpha|^{-1}, \qquad \limsup_{\nu=\infty} |Q_\nu(x)|^{1/\nu} \leqq \varrho^{-1} < |\alpha|^{-1}.$$

From these superior limits and the given relation between $P_\nu(x)$ and $Q_\nu(x)$ it follows that

$$\lim_{\nu=\infty} |P_\nu(x)|^{1/\nu} = |\alpha|^{-1}.$$

Now if the series

$$\sum_{\nu=0}^{\infty} c_\nu P_\nu(x) \tag{3.22}$$

converges [converges absolutely] at $x = x_1$ it follows that $\limsup |c_\nu|^{1/\nu} \leqq |\alpha|$ and hence that the series

$$\sum_{\nu=0}^{\infty} c_\nu Q_\nu(x)$$

converges absolutely and uniformly in every preassigned finite region, the uniformity being established by aid of the Cauchy inequalities for the coefficients $Q_\nu(x)$ in the power series $\sum Q_\nu(x)t^\nu$. Therefore the series $\sum c_\nu e^{\alpha x}$ converges [converges absolutely] for $x = x_1$; whence it follows that it converges uniformly in every finite region. We are thus led to the theorem:

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By differentiating with respect to x both members of the defining relation for the polynomials $P_\nu(x)$ we see that

$$P_\nu(x) = P'_{\nu+1}(x), \qquad \nu = 0, 1, 2, \cdots.$$

The case $g(t) = 1 - t$ is of interest. In this case $P_\nu(x)$ has the value

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^\nu}{\nu!},$$

and this is a "section" of the power series expansion of e^x.

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THEOREM 3.3 *No function $F(x)$ can have two distinct expansions in series of the form* (3.15) *for the same value of r.*

Suppose that $F(x)$ has an expansion of the form

$$F(x) = \sum_{\nu=0}^{\infty} c_\nu B_{\nu,r}(x)/\nu! \tag{3.23}$$

where the series meets the convergence conditions described in Theorem 3.1. Then we have

$$F(x+1) - F(x) = \sum_{\nu=0}^{\infty} c_{\nu+1} x^\nu/\nu! \equiv \varphi(x), \tag{3.24}$$

where $\varphi(x)$ is defined by this relation.

From (3.24) we have the values

$$c_\nu = F^{(\nu)}(1) - F^{(\nu)}(0), \qquad \nu = 1, 2, \cdots,$$

where $F^{(\nu)}(x)$ is the νth derivative of $F(x)$, while from (3.23) it follows that c_0 is then defined by the relation

$$(2r+1)\,c_0 = F(0) - \sum_{\nu=1}^{\infty} c_\nu\, B_{\nu,r}(0)/\nu!.$$

If we write

$$q = \limsup_{\nu=\infty} |\, c_\nu\,|^{1/\nu}$$

it follows from (3.16) and (3.17) that $q \leqq 2\pi(r+1)$. The function $\varphi(x)$ in (3.24) is of exponential type q.

Write

$$f(t) = \frac{c_1}{t} + \frac{c_2}{t^2} + \cdots, \qquad |t| > q, \tag{3.25}$$

and form the function

$$H(x) = \frac{1}{2\pi i} \int_C \frac{e^{xt}}{e^t - 1} f(t)\, dt$$

where C is a circle about 0 as a center and of radius $q+\varepsilon$ where ε is a small positive quantity and $q+\varepsilon < 2\pi\varrho$, ϱ being the least integer such that $2\pi\varrho > q$. Then, since $H(x+1) - H(x) = \varphi(x)$, it follows from Theorem 2.9 that $H(x)$ is of exponential type q. Moreover, we have

$$H(x) = \sum_{\nu=1}^{\infty} c_\nu\, B_{\nu,\varrho-1}(x)/\nu!,$$

as one sees by using the series in (3.25) and performing a term-by-term integration in evaluating $H(x)$.

If $\varrho = r+1$ then $F(x) - H(x) = c_0\, B_{0,r}(x)$ and $F(x)$ is of exponential type not exceeding q. If $\varrho \leqq r$ we have

$$F(x) - H(x) = c_0\, B_{0,\varrho-1}(x) + \sum_{\nu=0}^{\infty} \frac{c_\nu}{(2\pi i)^\nu} \sum_{k=\varrho}^{r} k^{-\nu} \{e^{2k\pi i x} + (-1)^\nu e^{-2k\pi i x}\}.$$

Then $F(x)$ is of exponential type not exceeding the greater of q and $2\pi r$. A more precise statement concerning the type of $F(x)$ can be obtained from the last equation in terms of properties of the coefficients c_ν.

The following theorem contains a part of the preceding results:

THEOREM 3.4. *If a function $F(x)$ has the convergent expansion* (3.23) *then $F(x)$ is of exponential type not exceeding $2\pi(r+1)$.*

Let us next consider the case of a given function $F(x)$ of exponential type q where $2\pi r \leqq q < 2\pi(r+1)$ and let us write

$$\varphi(x) \equiv F(x+1) - F(x) = \sum_{\nu=1}^{\infty} c_{\nu+1}\, x^\nu/\nu!.$$

Define $f(t)$ as in (3.25). Then the equation

$$h(x+1)-h(x) = \varphi(x)$$

has a solution of the form

$$h(x) = \frac{1}{2\pi i}\int_C \frac{e^{xt}}{e^t-1} f(t)\,dt$$

where C is a circle about 0 of radius $q+\varepsilon$ less than $2\pi(r+1)$ where ε is a small positive quantity. This function $h(x)$ is of exponential type not exceeding q and it has the expansion

$$h(x) = \sum_{\nu=1}^{\infty} c_\nu B_{\nu,r}(x)/\nu!\,.$$

Then $F(x)-h(x)$ is a periodic function of period 1 and of exponential type not exceeding q. From Theorem 2.4 it follows that it has the form

$$F(x)-h(x) = \sum_{k=-r}^{r} a_k e^{2k\pi ix}$$

where the a_k are constants.

Hence we have the following theorem:

THEOREM 3.5. *Any given function $F(x)$ of exponential type q such that $2\pi r \leqq q < 2\pi(r+1)$ has an expansion of the form*

$$F(x) = \sum_{k=-r}^{r} a_k e^{2k\pi ix} + \sum_{\nu=1}^{\infty} c_\nu B_{\nu,r}(x)/\nu!, \tag{3.26}$$

where the a's and c's are (uniquely determined) constants.

From § 2.5 it follows that the Fourier expansion of $h(x)$ in the interval (0,1) involves no term in $e^{2k\pi ix}$ for which $|k| < r+1$. Hence the finite sum in the foregoing expansion of $F(x)$ is the sum of the first $r+1$ terms of the Fourier expansion of $F(x)$ in the interval (0,1).

If p is a positive integer we have from (3.4) the relation

$$B_{\nu,r}(x) = B_{\nu,r+p}(x) - \frac{\nu!}{(2\pi i)^\nu}\sum_{k=r+1}^{r+p} k^{-\nu}\{e^{2k\pi ix}+(-1)^\nu e^{-2k\pi ix}\}.$$

On substituting this value for $B_{\nu,r}(x)$ in (3.26) we are led readily to the following corollary:

COROLLARY. *The function $F(x)$ has an expansion of the form* (3.26) *with r replaced throughout by $r+p$ where p is any non-negative integer. In each case the finite sum as to k consists of certain terms of the Fourier series for $F(x)$ in the interval* (0,1).

Finally let us suppose that $F(x)$ is a given function of exponential type $2\pi(r+1)$. Then, as we have seen, we have an expansion of the form

$$F(x) = \sum_{k=-r-1}^{r+1} a_k e^{2k\pi ix} + \sum_{\nu=1}^{\infty} c_\nu B_{\nu,r+1}(x)/\nu! \tag{3.27}$$

in which the a's and c's are uniquely determined. If $F(x)$ also has an expansion of the form (3.26) then the coefficients c_ν must be the same in the two cases, as one sees from the power series expansion of $F(x+1)-F(x)$. Hence the expansions (3.26) and (3.27) can coexist only if the series

$$\sum_{\nu=1}^{\infty} \frac{c_\nu}{(2\pi i)^\nu (r+1)^\nu} \{e^{2(r+1)\pi i x} + (-1)^\nu e^{-2(r+1)\pi i x}\}$$

converges. Moreover, when this series converges it gives the negative of the terms for $k = \pm(r+1)$ in the finite sum in (3.27); furthermore, when this series converges the two expansions (3.26) and (3.27) coexist.

These considerations lead to the following theorem:

THEOREM 3.6. *Let $F(x)$ be any given function of exponential type $2\pi(r+1)$. Write*

$$c_\nu = F^{(\nu)}(1) - F^{(\nu)}(0), \qquad \nu = 1, 2, \cdots.$$

Then a necessary and sufficient condition that $F(x)$ shall have an expansion of the form (3.26) *is that each of the following series shall converge:*

$$\sum_{\nu=1}^{\infty} \frac{(-1)^{\alpha\nu} c_\nu}{(2\pi i)^\nu (r+1)^\nu}, \qquad \alpha = 0, 1.$$

The theorems and corollary in this section 3.3 afford necessary and sufficient conditions for the expansion of functions in series of Bernoulli-Hurwitz functions. Such series together with a *finite* number of terms from the Fourier series afford expansions, valid throughout the finite plane, of any given function of any exponential type q; and no function outside of this class has such expansions.

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EIN BEISPIEL ZU NÖRLUNDS SUMMATIONSVERFAHREN.[1]

VON G. SZEGÖ.

Im Juli-Heft der „Annals" lese ich eine Note von Herrn R. E. Gilman,[2] in der eine Potenzreihe

$$P_0 + P_1 z + P_2 z^2 + \cdots + P_n z^n + \cdots$$

mit folgenden Eigenschaften konstruiert wird:

1. Für $n \to \infty$ gilt $P_n \to \infty$.
2. $\dfrac{P_{n+1}}{P_n} \to 1$.
3. $\sum_{\nu=1}^{n} |P_\nu - P_{\nu-1}| = O(|P_n|)$.
4. Bei radialer Annäherung $r \to 1 - 0$ ist

$$\liminf \frac{|\sum P_n r^n|}{\sum |P_n| r^n} = 0.$$

Dadurch gelingt es ihm, die Tragweite der Summationsverfahren von Nörlund und Abel miteinander zu vergleichen.

Man gewinnt leicht ein Beispiel der verlangten Art, indem man sich auf das folgende Resultat von Herrn O. Perron stützt:[3] Es sei

$$f(x; z) = \sum_{n=0}^{\infty} c_n(x) z^n = (1-z)^{-\frac{3}{2}} e^{\frac{x}{1-z}};$$

dann gilt die asymptotische Formel

$$c_n(x) = \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{2}} e^{\frac{x}{2}} e^{2\sqrt{xn}} \left[1 + O\left(n^{-\frac{1}{2}}\right)\right].$$

Der Fall x reell und $x \leqq 0$ ist dabei auszuschließen, unter den Potenzen und Wurzeln sind die Hauptwerte zu verstehen. Für reelle und positive x sind die Koeffizienten $c_n(x)$ offenbar positiv. Sie hängen mit den Laguerreschen Polynomen folgendermaßen zusammen: $c_n(x) = e^x L_n^{\left(\frac{1}{2}\right)}(-x)$.[4] Sie können also nur für reelle und negative x verschwinden.

[1] Received, October 26, 1932. — Auszug aus einem Briefe an Herrn J. D. Tamarkin.

[2] A remark on Nörlund's method of summation. Diese Annals, (2), 33 (1932), S. 429—432.

[3] Über das Verhalten einer ausgearteten hypergeometrischen Reihe bei unbegrenztem Wachstum eines Parameters. Crelle's Journal, 151 (1921), S. 63–78; s. insbesondere S. 77.

[4] Bezüglich der Bezeichnung vgl. Pólya-Szegö, Aufgaben und Lehrsätze aus der Analysis (Berlin 1925), Bd. II, S. 294.

Wir setzen nun

$$P_n = c_n(x), \qquad x = \varrho e^{i\varphi}, \qquad \varrho > 0, \qquad 0 < \varphi < \pi.$$

Wegen $2\Re\sqrt{x} = c > 0$ ist dann 1. richtig. Ferner ist auch 2. erfüllt, sogar in der schärferen Form

$$\frac{P_{n+1}}{P_n} = 1 + O\left(n^{-\frac{1}{2}}\right).$$

Hieraus folgt

$$\sum_{\nu=1}^{n} |P_\nu - P_{\nu-1}| = \sum_{\nu=1}^{n} |P_{\nu-1}| \left| \frac{P_\nu}{P_{\nu-1}} - 1 \right| = O\left[\sum_{\nu=1}^{n} e^{c\sqrt{\nu}}\, \nu^{-\frac{1}{2}}\right],$$

also 3., da ja

$$\sum_{\nu=1}^{n} e^{c\sqrt{\nu}}\, \nu^{-\frac{1}{2}} = O\left[\sum_{\nu=1}^{n} (e^{c\sqrt{\nu}} - e^{c\sqrt{\nu-1}})\right] = O(e^{c\sqrt{n}}).$$

Wir beweisen schließlich 4., *und zwar mit* lim *statt* lim inf. Setzt man nämlich $x' = \varrho \cos^2 \frac{\varphi}{2}$, so wird $\Re(\sqrt{x} - \sqrt{x'}) = 0$, folglich

$$\frac{|c_n(x)|}{c_n(x')} = \cos\frac{\varphi}{2} \exp\left[\frac{\varrho}{2}\left(\cos\varphi - \cos^2\frac{\varphi}{2}\right)\right]\left[1 + O\left(n^{-\frac{1}{2}}\right)\right].$$

Es gibt also eine nur von x abhängige positive Zahl A, derart daß

$$|c_n(x)| > A\, c_n(x')$$

gilt, d. h.

$$\sum_{n=0}^{\infty} |P_n|\, r^n > A f(x'; r).$$

Hieraus schließt man

$$\frac{\left|\sum P_n r^n\right|}{\sum |P_n|\, r^n} < A^{-1} \frac{|f(x; r)|}{f(x'; r)} = A^{-1} \exp\left(\frac{\varrho\cos\varphi - x'}{1 - r}\right) \to 0,$$

da $\varrho\cos\varphi < \varrho\cos^2\frac{\varphi}{2}$. Es sind somit sämtliche Forderungen 1.—4. erfüllt (sogar wesentlich mehr).

Schließlich sei bemerkt, daß die Funktion $\sum P_n r^n$ bei dem obigen Grenzübergang für $\frac{\pi}{2} < \varphi < \pi$ gegen Null konvergiert, während $\sum |P_n|\, r^n$ für alle $0 < \varphi < \pi$ unendlich wird.

KÖNIGSBERG, PR.

APPARENT VARIABLES FROM THE STANDPOINT OF COMBINATORY LOGIC.[1]

By H. B. CURRY.[2]

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1. Introduction.

The problem of analyzing the complex of notions associated with the use of variables in logic has already been solved to a certain extent in my previous papers. In fact, this has been accomplished by constructing a symbolic theory in whose formal developments variables do not appear explicitly. It is not, however, convenient to continue the study of logic in that manner. The notation for variables, although it involves assumptions capable of analysis, has nevertheless numerous advantages; and it is the object of the present paper to show how they may be introduced into the formal developments without loss of precision.

Variables[3] have been eliminated from the formal developments hitherto because their use, without an underlying analysis, is incompatible with one of the main aims of logical investigation, viz., to state the presuppositions of logic with the greatest possible definiteness and simplicity. That this incompatibility exists may be seen from the two following considerations.

1) As used in ordinary logic, variables are not the names of any entities whatever, but are "incomplete symbols", whose function is to indicate possibilities of substitution. Every logical proposition can, indeed, be thought of as a statement solely concerned with the constant entities which appear therein.[4]

[1] Received April 16, 1932.—Presented to the Am. Math. Society, Dec. 31, 1930. For abstract see Bull. Am. Math. Society, Vol. 37, p. 28 (Jan. 1931).

[2] National Research Fellow 1931–32.

[3] The word variable is here used in its mathematical sense, meaning a special kind of symbol distinguished from constant. Cf. Convention 3 below.

[4] Thus consider the proposition $\vdash (x)(x = x)$. This may be stated in the form, "the property of being equal to itself has universal validity". So stated it expresses a relation between the three constants 1) universality, 2) equality, and 3) the operation which converts a relation into the property of being an entity having that relation to itself. These three constants are denoted in combinatory notation by Π, Q, and W, respectively; and the above law may be stated in the form $\vdash \Pi(WQ)$. This example shows, incidentally, how

It follows that an abstract theory involving variables as primitive ideas[5] cannot have a direct logical interpretation.[6] This consideration suggests, and the existence of a variable-free combinatory logic proves, that such primitive ideas are superfluous. Furthermore, the elimination of variables is an improvement in that the number of variables must be infinite if all their functions are to be performed,[7] while the same results can be achieved in a variable-free logic containing only a finite number of primitive ideas.[8]

2) The ordinary use of variables is such that significant logical facts, being inherent in the notation itself, escape explicit formulation. We cite, as an example, the fact of being a relation or property (special cases of above mentioned possibilities of substitution). Moreover, certain equivalences exist between different sequences of substitution operations.[9] No apparatus is available whereby the propositions expressing such facts may be formally

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the ideas in this and the following paragraphs may be verified from an inspection of the symbolic theory above mentioned; but as a matter of fact they seem at least inherently probable on their own account, and where a part of the motivating influences which preceded the construction of the formal theory.

[5] I. e., primitive ideas essentially similar to variables in their functions. Such theories have been constructed by Hilbert and his school. See especially J. v. Neumann, Zur Hilbertschen Beweistheorie, Math. Zeitschr., vol. 26 (1927), pp. 1–46.

[6] I. e., an interpretation such that each primitive idea of the abstract theory is interpreted as denoting a definite logical concept.

[7] For if the number of variables were finite, say 100, then we should be able to construct only functions with not more than 100 variables; whereas by the constructions outlined in II A (see reference below), we can construct functions of any number of variables, so long as we have functions of at least two variables to start with. The system would, therefore, lack a certain closure property which is of theoretical importance, even if it should be possible to evolve all the logic we wanted for practical purposes by means of functions of not more than 100 variables. It is worth noting that, in the system of von Neumann above cited, only a finite number of the constituents, other than variables, are really primitive; the rest are essentially defined, and can theoretically be eliminated entirely. Thus the fact that there are an infinite number of primitive ideas in von Neumann's primitive frame is due to the presence of variables therein; and on this circumstance, presumably, rests his statement (l. c., p. 4) that the positive integers with all their properties are "inhaltlich unvermeidbar".

[8] A remark of Schönfinkel's (Math. Annalen vol. 92, p. 315 [1924]) is essentially to the effect that, in any theory containing the non-formal primitive ideas of combinatory logic, the number of formal primitive ideas, if originally denumerable, can be reduced to one. Fot let $X_1, X_2, X_3, \cdots$ be the original formal primitive ideas. If we introduce the new primitive idea Y, then we can define $X_1 \equiv YY$, $X_2 \equiv YX_1$, $X_3 \equiv YX_2$, etc. But there is certainly a sense in which such an elimination of formal primitive ideas is a mere triviality, while the elimination of variables to form a combinatory logic free from such symbols is not.

[9] Thus, to repeat an example already mentioned (Amer. Jour. Math., vol. 51 (1929), p. 369), the following two ways of combining the three entities $\varphi(x, y)$, a, b are equivalent: i) substitution of a for x to obtain $\varphi(a, y)$, then b for y; ii) substitution of b for y to obtain $\varphi(x, b)$ and then a for x. This equivalence can be seen in no other way than by observing

proved. On the contrary, entities are represented by composite symbols involving variables, these symbols being neither simple names, since they have structure,[10] nor descriptions of the entity concerned as the end product of a unique construction process, since there are usually several such processes, each equally compatible with the symbol; but they form graphical representations, so to speak, such that a variety of properties of the entity represented have to be apprehended from the intuitive nature of the graphing medium. The ideal of simplicity and definiteness requires that such dependence on intuition be reduced to a minimum. Indeed, a formal theory should conform as closely as possible to the following criteria: a) every entity should be denoted either by a simple symbol not presupposing any of its properties, or else by a symbol which is a description of the entity by some one of its characteristics, in particular as the result of a unique process of construction out of the primitive entities; b) all the properties of entities should be stated in propositions which are either a part of the primitive frame, or are formally derivable therefrom according to the rules. Whether or not it is possible to actually attain this standard,[11] it is at least possible to approximate to it more closely than the ordinary use of variables will allow.[12]

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that the symbol for the end product, viz. $\varphi(a, b)$, is the same in both cases. The distinction between i) and ii) is moreover more than a mere question of order in which processes are performed; for in the ordinary conception the substitution of a constant for a variable when there is only one variable present is a process different from that when there are two, and in the latter case it makes a difference whether the first variable be used for the substitution or the second. In the notation of combinatory logic the result of process i) is $\varphi a b$, that of ii) is $C\varphi b a$, the equivalence being explicitly provable by means of Rule C.

[10] By structure is meant structure having significance. The simpler words of ordinary discourse have structure, but this structure is—barring inflections and the like-purely phonetic or typographical, so that these words are simple symbols.

[11] It may be argued, for example, that even the notation of combinatory logic does not meet the requirement exactly. Thus suppose a, b, c, d, are entities, and consider $((a\,b)\,(c\,d))$. To construct it we have the following operations to perform: i) formation of $(a\,b)$ from a and b, ii) formation of $(c\,d)$ from c and d, iii) formation of $((a\,b)\,(c\,d))$ from $(a\,b)$ and $(c\,d)$. The notation does not specify in which order the operations i) and ii) are to be performed. This is, however, merely a question of order in performing operations; the operations themselves do not differ as was the case in the previous footnote.

[12] These conclusions have previously been expressed by saying that the rules for a logic with variables would necessarily be more complex than for one without variables. But the complexity of rules involves two things: a) the complexity of statement (number, etc.); b) amount of intuitive information presupposed for their understanding. Of these the second is naturally more important than the first; indeed the rules for a logic with variables would perhaps not be essentially more complicated, in the first respect, than those for a logic without them. The point is that the rules for a variable logic, if stated, would necessarily presuppose more information of the second kind; moreover it is probable that no matter how carefully the rules were formulated, there would still be some intuitive information presupposed by the very use of the symbolism. But this is essentially the same idea as above.

The preceding considerations do not, however, militate against the introduction of variables by means of definitions on the basis of an underlying analysis. In fact, some of them argue in favor of such a course, since the fact that entities will then be denoted by expressions which display certain of their properties intuitively is an advantage. In the present case the underlying analysis is furnished by the combinatory logic of my previous papers; the actual introduction of variables is the thesis of the present one. This introduction amounts, essentially, to the following: a) definition of expressions involving variables by specifying uniquely how the entity to be represented is to be constructed out of the primitive entities; b) proof that the entities so represented actually have the properties which one is accustomed to infer from their representation. The theory so modified lacks none of the precision of the underlying analysis, i. e., it conforms to the ideals stated above to precisely the same extent as does the underlying analysis. Moreover, the proof mentioned in b) comes to this, that a process is given whereby any inference which we can make, directly or indirectly, by the use of variables may be resolved into an aggregate — — not infrequently a complicated one — — of inferences, each of the exceedingly simple sort constituted by the rules of the underlying theory.

The variables so introduced are here called apparent; because the true significance of the term apparent in connection with logical variables is, as I see it, simply that the variables are incomplete symbols serving as convenient ways of expressing propositions which theoretically can be expressed without them. The position taken above is therefore that all variables appearing in the formulas of a logical theory are apparent, at least in interpretation. But not all types of logical variables are here considered. Attention is confined to those types whose meaning in use can be defined in terms of the entities already postulated. Existential apparent variables, not being of that character, are not considered; but a theory of the ordinary kind of such variables can be superposed on the treatment here given by defining them in terms of negation.[15]

The developments of this paper have a distinctly formal character, in that they are practically translations into the new notation of a synthesis already completed. In the latter part of the paper, to be sure, the synthesis is pushed a little further, and new axioms are introduced; however, these developments are essentially the establishment of formulas having as their interpretations certain principles previously proved as rules. On the other hand, the theorems of this paper constitute an explicit formulation of the rules governing apparent variables of unrestricted generality. As such they form the basis for further developments of combinatory logic now in course of preparation; indeed the reader who is not interested in the

underlying analysis can take them as fundamental. It will be found that these theorems, together with the primitive frame for the algebra of propositions, form a complete foundation for the ordinary calculus of functions,[13] provided one bears in mind the admission in the 2nd edition of the Principia Mathematica[14] that there is no need for introducing "free" variables into logic at all.[15]

A final remark concerns the difference between the use of variables here contemplated and that indulged in hitherto. It is not quite correct to say that variables have not yet appeared in the formal propositions explicitly; they have not appeared at all, either explicitly or implicitly. Every formal proposition of the present theory is a statement that a given particular entity is an assertion; its formal demonstration consists of a series of steps, each of which is an application of one of the rules, whereby the proposition may be derived from the axioms. The only use of variables so far has been in connection with theorems and their proofs. These theorems are statements about the theory which are capable of being judged by an intelligent mind surveying the theory from outside; their proofs are intended simply to convince such a mind that the statements made are true. The variables have made it possible to make complicated statements of this character, and in this they stand on the same footing as the words of ordinary language. But in the future variables will be used according to prescribed rules as constituents of symbols which denote actual entities, and hence will appear in the symbolic expression of formal propositions. Yet the reader should note, in the first place, that the definitions enable us to eliminate these variables and, in the second place, that it is still always possible, in any particular case arising under the

[13] I. e., the engere Funktionenkalkül of Hilbert and Ackermann (Grundzüge der theoretischen Logik, Berlin, 1928, Kap. 3).

[14] Introduction, p. XIII.

[15] If we make the agreement that, when $\mathfrak{X}$ is an expression involving the variables $x_1, x_2, \cdots, x_m$, we are to interpret Hilbert and Ackermann's assertion of $\mathfrak{X}$ to mean the assertion of $(x_1, x_2, \cdots, x_m)\ \mathfrak{X}$ (as defined below in Def. 4), then the relevant rules and propositions of the "engere Funktionenkalkül" are consequences of the theorems proved and axioms assumed below as follows:

Regel α: follows from Theorem 7.

Regel β: $=$ Theorem 12.

Regel γ, first part: follows from Theorem 15.

Axiom e: $=$ Ax. Π_0.

The propositions and rules of Hilbert and Ackermann concerning the existential quantifier can be proved by defining

$$\Sigma = N \cdot \Pi \cdot BN, \qquad (N \text{ being negation}),$$

$$(Ex)\ \mathfrak{X} \equiv \Sigma([x]\,\mathfrak{X}) \qquad (\mathfrak{X} \text{ being an expression involving } x).$$

This however, will not be discussed here.

theorems, for a mathematical Methuselah to construct the formal proof of the assertion concerned in the course of n years. The theorems, in fact, uniquely describe how the gentleman should proceed.

2. Preliminary Conventions.

Reference will be made to my previous papers as follows:[16]

I. Grundlagen der kombinatorischen Logik, Kapitel I. *Am. Journ. Math.*, vol. 52, pp. 509–525 (July, 1930).

II. Same, Kapitel II. *Am. Journ. Math.*, vol. 52, pp. 526–536, 789–834, (July and October, 1930).

IIa. Some additions to the theory of combinators. Change as indicated. *Am. Journ. Math.*, vol. 54, pp. 551–558 (July 1932).

III. The universal quantifier in combinatory logic. *Annals of Math.*, Series 2, vol. 32, pp. 154–180 (1931).

Symbols following the above numerals will refer to the divisions and subdivisions of the paper concerned according to the scheme used in that paper itself.

DEFINITION 1. The symbol $\mathfrak{P}_n$ defined in III 4, Def. 4 shall henceforth be replaced by P_n^*, i. e.,

$$P_n^* \equiv B_2 \Pi_n P_n.$$

CONVENTION 1. The word *assertion* will be used henceforth as the English equivalent of "Formel" in I C. *Formula* will be used for propositions of the form $\vdash X$, $\vdash_x \mathfrak{X}$ (as in IIa, Convention 4), etc.

CONVENTION 2. If X is any entity, the notation $\overset{C}{\vdash} X$ shall mean that the formula $\vdash X$ follows combinatorially (IIa, Convention 5). Similarily, if $\mathfrak{X}$ is an expression involving the variables $x_1, x_2, \cdots, x_m$, the notation $\overset{C}{\vdash_x} \mathfrak{X}$ shall mean that the formula $\vdash_x \mathfrak{X}$ follows combinatorially, and so on.

CONVENTION 3. The term *constant* shall hereafter denote any symbol which either stands for a definite entity, or else is to be regarded as so doing throughout a given context in contradistinction to the variables of that context. By context is here meant a definition, theorem, or the like. In applying such a context to a particular case, the unspecified constants may be interpreted either as entities, or else as the variables,—or expressions involving the variables—, of some other context.[17] In the latter case the formulas, if any, of the original context must be understood to hold formally with the respect to these extraneous variables as in IIa, Convention 4;

[16] For an excellent summary of I, II, and III, see review of III by Arnold Schmidt in Zentralblatt für Math., vol. 1, p. 261 (1931).

[17] An interpretation of this character may of course be vacuous in that the hypotheses of the context are not fulfilled.

and of course the variables so used to interpret constants must be different from those which take the place of the original variables.

Furthermore the definitions and rules formulated hitherto, which apply to unspecified entities, shall be regarded as extended to arbitrary constants; a similar extension for the theorems then follows as a consequence.[18]

CONVENTION 4. The term *primitive constant* shall denote a constant whose interpretations are either primitive entities or variables of some other context.

The following conventions and definition have to do with the systematizing of the rules for the insertion and removal of unnecessary parentheses.

CONVENTION 5. By an *operational symbol* or *operator* shall be meant a symbol of one of the two following types:

a) An *ω-operator* shall be a symbol not defined in isolation, but introduced by a definition of the form

$$\mathfrak{x}\,\omega\,\mathfrak{y} \equiv \mathfrak{Z},$$

where $\mathfrak{x}$ and $\mathfrak{y}$ are expressions, and $\mathfrak{Z}$ is an expression derived from $\mathfrak{x}$ and $\mathfrak{y}$ according to rules prescribed in the definition. The expressions $\mathfrak{x}$ and $\mathfrak{y}$ shall be called the *operands*. The term ω-operation shall also include simple juxtaposition which indicates application (a primitive idea).

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[18] The purpose of this convention is to provide a technique for extending to expressions contexts primarily concerned with symbols for arbitrary entities. Such a technique is made necessary by the introduction of apparent variables. For suppose a symbol $\mathfrak{x}$ containing certain constants $u_1, u_2, \cdots, u_p$ be defined to mean the same as a combination X of these constants. It frequently happens that we wish to define a $\mathfrak{y}$, containing $\mathfrak{x}$ as a constituent, such that certain of the u_j are apparent variables. This means that we must be able to interpret the definition of $\mathfrak{x}$ for the case where certain of the u_j are variables—or else, which is inconvenient, adopt a new definition. Thus, to define $(x)\,(y)\,uxy$ (u a constant), we must first define $(y)\,uxy$ by Def. 4 below, and then interpret the definition for the case where x is a variable.

The distinction between constant and variable implied by the above convention must not be confused with some other distinctions sometimes made. Thus in philosophic usage a constant is a symbol whose meaning is completely determinate. From this point of view all the symbols of any abstract theory are variables because the meanings of the primitive ideas are not completely specified. In another sense a symbol may be regarded as constant with respect to a given theory; thus, in a theory of real numbers $1, 2, e$, and $\sqrt{\pi}$ are constants; letters generally are variables. (These constants are, however, variables in the first sense so long as the term "real number" is undefined). On the other hand, the use of the term constant in ordinary mathematics is substantially the same as here. The terms variable and constant have no significance except in relation to a given context, and the constants of one context are often the variables of the next. (In this sense it is significant to talk about parameters, viz., symbols which are variables in one of two closely related contexts and constants in the other.) Furthermore I think it may be shown that variables have the same functions in mathematics as in logic, and that any fact which may be expressed with their aid may also, by a sufficient circumlocution, be expressed without them.

b) A *φ-operator* shall be a symbol, likewise not defined in isolation, but introduced by a definition of the form

$$\varphi \mathfrak{X} \equiv \mathfrak{Z},$$

where $\mathfrak{X}$ is an expression, and $\mathfrak{Z}$ is an expression derived from $\mathfrak{X}$ according to prescribed rules. In this case $\mathfrak{X}$ shall be called the *operand.*

CONVENTION 6. Among the operational symbols there will be established an arbitrary ranking which shall be called *seniority.* For the operators already introduced this ranking shall be determined by the following list, the one first mentioned being senior to the others, and the second senior to the third:

1) Equality (=), introduced in I C 4, Def. 2.
2) Composite Product ($\cdot$), introduced in II B 4, Def. 1.
3) Application (indicated by simple juxtaposition), introduced in I C 2 (primitive).

The position in this ranking of operators later introduced will be stated at the time of introduction.

DEFINITION 2. Compound symbols which are otherwise ambiguous because of the omission of parentheses shall be interpreted according to the following rules: except as otherwise indicated by parentheses the left operand of an ω-operator shall extend toward the left to either the beginning of the entire expression or to an operator senior to the ω-operator; the right operand of any given operator shall extend toward the right to either the end of the expression or to an operator ψ of the same rank as or senior to the given operator, provided that, when the operator ψ is a φ-operator, the operand of this φ shall first be determined, and the operand of the given operator shall include both the φ and its operand. Any expression enclosed in parentheses shall of course be treated as a single non-operational symbol.

Remark 1. It can be shown that a given compound symbol $\mathfrak{A}$ can be interpreted (i. e., defined) in only one way. In other words, if we take the operators in $\mathfrak{A}$ *seriatim,* determine the scopes of their operands and mark them off, where necessary, by parentheses, then the result is independent of the order in which the operators are taken. It is not necessary, however, that the reader should prove this; we may agree that we shall take the operators in order from left to right in so far as the definition allows this to be done.[19]

[19] In order that the process may not break down due to certain operations not having an operand or the like, $\mathfrak{A}$ must satisfy certain conditions, as, for example, that an ω-operator always has an operand immediately on its left. These conditions are not here stated; for the purpose is to state a process of definition and not a theorem, and the conditions will be automatically fulfilled whenever the definition is to be applied.

Remark 2. The foregoing definition includes as special cases I C 4, Def. 1, II B 4 Defs. 2 and 3, and the remarks relative to the insertion of parentheses in I C 4, Def. 2. It therefore does not affect any definitions previously made, but will be useful for systematizing definitions in the future.

Remark 3. It is possible, by writing all operational symbols to the left of all their operands, to dispense with the necessity of having parentheses at all. Such a method of writing logical formulas has indeed been proposed by Lukasiewicz. But it seems to me that the formulas so written are not as perspicuous as those written according to the foregoing method.

Examples.

$$X = Y \cdot BZ \equiv X = (Y \cdot (BZ)).$$

The following are all the possible cases where only two operations occur: If ω_1 is senior to ω_2,

(1) $$\mathfrak{x}\,\omega_1\,\mathfrak{y}\,\omega_2\,\mathfrak{z} \equiv \mathfrak{x}\,\omega_1\,(\mathfrak{y}\,\omega_2\,\mathfrak{z}),$$

(2) $$\mathfrak{x}\,\omega_2\,\mathfrak{y}\,\omega_1\,\mathfrak{z} \equiv (\mathfrak{x}\,\omega_2\,\mathfrak{y})\,\omega_1\,\mathfrak{z}.$$

If φ_1 is senior to ω_2, and ω_1 senior to φ_2,

(3) $$\varphi_1\,\mathfrak{x}\,\omega_2\,\mathfrak{y} \equiv \varphi_1\,(\mathfrak{x}\,\omega_2\,\mathfrak{y}),$$

(4) $$\varphi_2\,\mathfrak{x}\,\omega_1\,\mathfrak{y} \equiv (\varphi_2\,\mathfrak{x})\,\omega_1\,\mathfrak{y}.$$

Without regard to seniority,

(5) $\mathfrak{x}\,\omega\,\mathfrak{y}\,\omega\,\mathfrak{z} \equiv (\mathfrak{x}\,\omega\,\mathfrak{y})\,\omega\,\mathfrak{z}$ (i. e., where the two ω's have the same rank),

(6) $\mathfrak{x}\,\omega\,\varphi\,\mathfrak{y} \equiv \mathfrak{x}\,\omega\,(\varphi\,\mathfrak{y})$,

(7) $\varphi_1\,\varphi_2\,\mathfrak{x} \equiv \varphi_1\,(\varphi_2\,\mathfrak{x})$.

3. Fundamental Properties of Apparent Variables.

In this section the apparent variables are introduced and their elementary properties—those which do not depend on properties of implication—are investigated. The types of variables introduced are two in number and may be called functional and universal apparent variables respectively; the second type requires no preliminary discussion.

The variables of the first type are analogous to those appearing in the Principa Mathematica in expressions like $\varphi\hat{x}$ and $\varphi(\hat{x}, \hat{y})$. Such variables are not ordinarily called "apparent", but from the present point of view they form, in a sense, the fundamental kind of apparent variables.

The above notation is ambiguous in that an expression like $\varphi(\hat{x}, \hat{y})$ does not distinguish between φ and its converse. One might think that if the former were denoted by $\varphi(\hat{x}, \hat{y})$ the latter should be denoted by $\varphi(\hat{y}, \hat{x})$; but, since the letters used for variables are irrelevant, these two expressions are not distinct. One might avoid this difficulty by specifying that special

letters, in a definite order, be used for variables,[20] but it is not convenient to so restrict ourselves. This is the reason for the prefix in the following definition. As here defined, if $[x, y]\varphi xy$ denote a given entity, $[y, x]\varphi yx$ will denote the same entity (Th. 1), while the converse (i. e., $C\varphi$) will be denoted by either of the notations, $[x, y]\varphi yx$ or $[y, x]\varphi xy$.[21]

DEFINITION 3. If $\mathfrak{X}$ is a combination of the variables $x_1, x_2, \cdots, x_n$ and constants, then

$$[x_1, x_2, \cdots, x_n]\mathfrak{X}$$

shall denote the entity X constructed as follows: let all the defined symbols and symbol combinations appearing in $\mathfrak{X}$ be replaced by their definitions; let $u_1, u_2, \cdots, u_p$ be the distinct primitive constants appearing in the new expression in the order in which they first appear there; let U denote the normal combinator[21a] such that the expression $UIu_1u_2\cdots u_p x_1 x_2 \cdots x_n$ reduces formally to $\mathfrak{X}$; then

$$X \equiv UIu_1u_2\cdots u_p.^{22}$$

The prefix $[x_1, x_2, \cdots, x_n]$ shall be regarded as a symbol of operation extending to the right (i. e., a φ-symbol of Convention 5); as such it shall be junior to $=$ and senior to all other operational symbols so far introduced.

DEFINITION 4. If $\mathfrak{X}$ is a combination of variables $x_1, x_2, \cdots, x_n$ and constants, then

$$(x_1, x_2, \cdots, x_n)\mathfrak{X} \equiv \Pi_n([x_1, x_2, \cdots, x_n]\mathfrak{X}).$$

The prefix $(x_1, x_2, \cdots, x_n)$, regarded as a φ-operator, shall have the same seniority as that considered in Def. 3. The definition shall hold for $n = 0$ in the same sense as in Def. 3.

THEOREM 1. *If $\mathfrak{X}$ is a combination of the variables $x_1, x_2, \cdots, x_n$ and constants, and $\mathfrak{Y}$ is the expression obtained by substituting for $x_1, x_2, \cdots, x_n$ respectively the variables $y_1, y_2, \cdots, y_n$ (the substitution being made for every occurrence of the x_i), then*

$$\vdash^{C} [x_1, x_2, \cdots, x_n]\mathfrak{X} = [y_1, y_2, \cdots, y_n]\mathfrak{Y},$$
$$\vdash^{C} (x_1, x_2, \cdots, x_n)\mathfrak{X} = (y_1, y_2, \cdots, y_n)\mathfrak{Y}.$$

Proof. Follows immediately from the definitions.

[20] This is essentially what was done in II A. Cf. second footnote on p. 526 of that paper.

[21] A similar notation is used by the authors of the Principia for relations, but not, strangely enough, for functions.

[21a] Defined in IIa, Convention 3.

[22] The part of the above definition relating to the construction of X makes sense when $n = 0$. Accordingly when a general theorem is stated below concerning expressions of the form just defined in which n is indeterminate, it will be understood, unless specified to the contrary, that n can take on the value 0, in which case the defined expression shall be understood as denoting the X above defined (which is then equal to $\mathfrak{X}$ combinatorially).

THEOREM 2. *If $\mathfrak{X}$ is a combination of constants and the variables $x_1, x_2, \cdots, x_n$, and X is a constant, then a necessary and sufficient condition that*

(1) $$\vdash^C X x_1 x_2 \cdots x_n = \mathfrak{X}$$

is that

(2) $$\vdash^C X = [x_1, x_2, \cdots, x_n]\,\mathfrak{X}.$$

Proof. Let

$$Y \equiv [x_1, x_2, \cdots, x_n]\,\mathfrak{X}.$$

Then $Y x_1 x_2 \cdots x_n$ may be transformed into $\mathfrak{X}$ by a formal reduction process (Def. 3). Hence if (2) holds combinatorially so does (1). The converse follows by II a, Theorem 5.

COROLLARY. *If X is a constant, then*

$$\vdash^C [x_1, x_2, \cdots, x_n]\, X x_1 x_2 \cdots x_n = X.$$

THEOREM 3. *If $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are combinations of constants and the variables $x_1, x_2, \cdots, x_n$, then a necessary and sufficient condition that*

(1) $$\vdash^C_x \mathfrak{X}_1 = \mathfrak{X}_2$$

is that

(2) $$\vdash^C [x_1, x_2, \cdots, x_n]\,\mathfrak{X}_1 = [x_1, x_2, \cdots, x_n]\,\mathfrak{X}_2.$$

Proof. Let

$$X_1 \equiv [x_1, x_2, \cdots, x_n]\,\mathfrak{X}_1$$

and

$$X_2 \equiv [x_1, x_2, \cdots, x_n]\,\mathfrak{X}_2.$$

Then

$$\vdash^C_x X_1 x_1 x_2 \cdots x_n = \mathfrak{X}_1 \qquad \text{(Th. 2)}$$
$$= \mathfrak{X}_2 \qquad \text{(Hp.);}$$

whence, by Theorem 2,

$$\vdash^C X_1 = [x_1, x_2, \cdots, x_n]\,\mathfrak{X}_2$$
$$= X_2 \qquad \text{(by def.).}$$

Conversely suppose (2) holds. Then

$$\vdash^C X_1 = X_2 \qquad \text{(Hp. and def.),}$$
$$\vdash^C X_1 x_1 x_2 \cdots x_n = \mathfrak{X}_1 \qquad \text{(Th. 2),}$$
$$\vdash^C X_2 x_1 x_2 \cdots x_n = \mathfrak{X}_2 \qquad \text{(Th. 2).}$$

From these we may obtain (1), q. e. d.

THEOREM 4. *If $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are combinations of constants and the variables $x_1, x_2, \cdots, x_n$, and*

$$\vdash^C_x \mathfrak{X}_1 = \mathfrak{X}_2,$$

then

$$\vdash^{C} (x_1, x_2, \cdots, x_n)\, \mathfrak{X}_1 = (x_1, x_2, \cdots, x_n)\, \mathfrak{X}_2.$$

Proof. Follows immediately from Theorem 3 and Definition 4.

COROLLARY 1. *If $\mathfrak{X}$ is a combination of constants and the variables $x_1, x_2, \cdots, x_n$, and X is a constant such that*

$$\vdash^{C}_{x} X x_1 x_2 \cdots x_n = \mathfrak{X},$$

then

$$\vdash^{C} (x_1, x_2, \cdots, x_n)\, \mathfrak{X} = \Pi_n X.$$

Proof. Follows by Theorem 2 and Definition 4.

COROLLARY 2. *If X is any constant, then*

$$\vdash^{C} (x_1, x_2, \cdots, x_n)\, X x_1 x_2 \cdots x_n = \Pi_n X.$$

THEOREM 5. *If* 1) *$\mathfrak{X}_1$ and $\mathfrak{X}_2$ are combinations of constants and the variables $x_1, x_2, \cdots, x_m$,* 2) *$\mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_m$ are combinations of constants and the variables $y_1, y_2, \cdots, y_n$,*[23] 3) *$\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ are the expressions obtained from $\mathfrak{X}_1$ and $\mathfrak{X}_2$ respectively by substituting $\mathfrak{z}_i$ for x_i $(i = 1, 2, \cdots, m)$ in all its occurrences, and* 4)

$$\vdash^{C} [x_1, x_2, \cdots, x_m]\, \mathfrak{X}_1 = [x_1, x_2, \cdots, x_m]\, \mathfrak{X}_2,$$

then

$$\vdash^{C}_{y} \mathfrak{Y}_1 = \mathfrak{Y}_2,$$

$$\vdash^{C} [y_1, y_2, \cdots, y_n]\, \mathfrak{Y}_1 = [y_1, y_2, \cdots, y_n]\, \mathfrak{Y}_2,$$

$$\vdash^{C} (y_1, y_2, \cdots, y_n)\, \mathfrak{Y}_1 = (y_1, y_2, \cdots, y_n)\, \mathfrak{Y}_2.$$

Proof. By Theorem 3,

$$\vdash^{C}_{x} \mathfrak{X}_1 = \mathfrak{X}_2.$$

If throughout this deduction we substitute $\mathfrak{z}_i$ for x_i, we shall have a proof that

$$\vdash^{C}_{y} \mathfrak{Y}_1 = \mathfrak{Y}_2.$$

The rest follows by Theorems 3 and 4.

COROLLARY. *If $\mathfrak{X}_1, \mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_m$ and $\mathfrak{Y}_1$ are as in the theorem, and if*

$$X_1 \equiv [x_1, x_2, \cdots, x_m]\, \mathfrak{X}_1,$$

then

$$\vdash^{C}_{y_1} \mathfrak{Y}_1 \equiv X_1\, \mathfrak{z}_1\, \mathfrak{z}_2 \cdots \mathfrak{z}_m,$$

$$\vdash^{C} [y_1, y_2, \cdots, y_n]\, \mathfrak{Y}_1 = [y_1, y_2, \cdots, y_n] X_1\, \mathfrak{z}_1\, \mathfrak{z}_2 \cdots \mathfrak{z}_m,$$

$$\vdash^{C} (y_1, y_2, \cdots, y_n)\, \mathfrak{Y}_1 = (y_1, y_2, \cdots, y_n) X_1\, \mathfrak{z}_1\, \mathfrak{z}_2 \cdots \mathfrak{z}_m.$$

Proof. This is the special case where $\mathfrak{X}_2 \equiv X_1 x_1 x_2 \cdots x_m$.

[23] $n \geqq 0$. Cf. remark after Theorem 7.

THEOREM 6. *If* $\mathfrak{X}$ *is a combination of constants and the variables* $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n$, *then*

a) $\overset{C}{\vdash} [x_1, x_2, \cdots, x_m][y_1, y_2, \cdots, y_n]\mathfrak{X} = [x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n]\mathfrak{X}$,

b) $\overset{C}{\vdash} (x_1, x_2, \cdots, x_m)(y_1, y_2, \cdots, y_n)\mathfrak{X} = (x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n)\mathfrak{X}$.

Proof. a) Let $X \equiv [x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n]\mathfrak{X}$. Then

(1) $\overset{C}{\underset{xy}{\vdash}} X x_1 x_2 \cdots x_m y_1 y_2 \cdots y_n = \mathfrak{X}$ (Th. 2),

$\therefore \overset{C}{\underset{x}{\vdash}} [y_1, y_2, \cdots, y_n]\mathfrak{X} = X x_1 x_2 \cdots x_m$ (Th. 2; cf. Conv. 3),

$\overset{C}{\vdash} [x_1, x_2, \cdots, x_m][y_1, y_2, \cdots, y_n]\mathfrak{X} = X$, q. e. d. (Th. 2).

b) Let X be defined as in the preceding case. Then (1) is true and hence by Theorem 4, Cor. 1,

$$\begin{aligned} \overset{C}{\underset{x}{\vdash}} (y_1, y_2, \cdots, y_n)\mathfrak{X} &= \Pi_n (X x_1 x_2 \cdots x_m) \\ &= B_m \Pi_n X x_1 x_2 \cdots x_m \qquad \text{(II B 1, Satz 3).} \end{aligned}$$

$$\begin{aligned} \therefore \overset{C}{\vdash} (x_1, x_2, \cdots, x_m)(y_1, y_2, \cdots, y_n)\mathfrak{X} & \\ = \Pi_m (B_m \Pi_n X) & \qquad \text{(Th. 4, Cor. 1)} \\ = (\Pi_m \cdot B_m \Pi_n) X & \qquad \text{(II B 4, Satz 1)} \\ = \Pi_{m+n} X & \qquad \text{(III 2, Th. 3)} \\ = (x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n)\mathfrak{X}, & \text{q.e.d. ((1); Th. 4, Cor. 1).} \end{aligned}$$

THEOREM 7. *If* 1) $\mathfrak{X}$ *is a combination of constants and the variables* $x_1, x_2, \cdots, x_m$, 2) $\mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_m$ *are combinations of constants and the variables* $y_1, y_2, \cdots, y_n$,[24] 3) $\mathfrak{Y}$ *is the expression obtained from* $\mathfrak{X}$ *by substituting* $\mathfrak{z}_i$ *for* x_i $(i = 1, 2, \cdots, m)$ *in all its occurrences, and* 4)

$$\vdash (x_1, x_2, \cdots, x_m)\mathfrak{X},$$

then

$$\vdash (y_1, y_2, \cdots, y_n)\mathfrak{Y}.$$

Proof. Let $X \equiv [x_1, x_2, \cdots, x_m]\mathfrak{X}$. Then by Hp. 4 and Definition 4,

(1) $$\vdash \Pi_m X.$$

Also, by Theorem 5, Cor.,

(2) $$\overset{C}{\underset{y}{\vdash}} \mathfrak{Y} = X \mathfrak{z}_1 \mathfrak{z}_2 \cdots \mathfrak{z}_m.$$

Now let $a_1, a_2, \cdots, a_p$ be constants and U be a combinator such that the expression

$$U X a_1 a_2 \cdots a_p \; y_1 y_2 \cdots y_n$$

reduces formally—the symbols $X, a_1, a_2, \cdots, a_p, y_1, y_2, \cdots, y_n$, being all treated formally—to the right-hand side of (2). Since X is left isolated in

[24] $n \geqq 0$. Cf. Remark below.

the first position we may suppose U is regular. Then it is of order $n+p$ and degree m (III 3, Conv. 1). Hence by III 7, Theorem 6 and Convention 1,[25]

$$\vdash \Pi_{n+p}(UX).$$

Therefore

$$\vdash \Pi_n(UX\,a_1\,a_2\cdots a_p) \qquad \text{(III 2, Th. 4)}. \tag{3}$$

But, by definition of U and Theorem 4, Cor. 1,

$$\begin{aligned} \overset{C}{\vdash} \Pi_n(UX\,a_1\,a_2\cdots a_p) &= (y_1, y_2, \cdots, y_n)\,(X\,\mathfrak{z}_1\,\mathfrak{z}_2\cdots\mathfrak{z}_m) \\ &= (y_1, y_2, \cdots, y_n)\,\mathfrak{Y} \qquad \text{(Th. 5)}. \end{aligned} \tag{4}$$

From (3), (4), and Rule Q_1 the formula to be proved follows.

Remark. The preceding proof, and also that of Theorem 5, are valid for the case $n = 0$; i. e., for the case where the $\mathfrak{z}_i$, and therefore $\mathfrak{Y}$, are constants. (See footnote to Definition 3.) The conclusion is, in this special case, that $\mathfrak{Y}$ is itself an assertion.

The preceding theorem is a formal statement of the principle of substitution already proved informally in III 1. It is of such generality that it may be well to mention here some other special cases of it. Suppose, for instance, that $\mathfrak{z}_i \equiv x_i$ and the x's are a transformation (Umwandlung) of the y's. In particular let the x's be a permutation of the y's. Then the theorem states that the order of the variables in the prefix is immaterial. (This is of course not true for the [ ] prefix.) Again the y's may contain additional variables not contained in the x's, so that, the theorem states, additional variables may be introduced into the prefix *ad libitum*. Likewise variables may be repeated in the prefix, or, conversely, repetitions of variables appearing more than once or variables appearing there but not in $\mathfrak{X}$ may be omitted. These are, moreover, only a few of the simplest cases which can arise under the theorem.

Convention 7. The above theorem will be referred to hereafter as *Subst.* In the proofs, Subst. $\begin{pmatrix} \mathfrak{z}_1, & \mathfrak{z}_2, & \cdots, & \mathfrak{z}_m \\ x_1, & x_2, & \cdots, & x_m \end{pmatrix} A$, where A is the name of a formula, shall mean that the proposition under discussion follows from A by substitution of $\mathfrak{z}_i$ for x_i and application of this theorem. When certain of the x's are omitted from the above symbol it shall be understood that such an x is to be left unchanged.

Theorem 8. *If $\mathfrak{X}$ and $\mathfrak{Y}$ are combinations of constants and the variables $x_1, x_2, \cdots, x_n$ such that*

$$\overset{C}{\underset{x}{\vdash}} \mathfrak{X} = \mathfrak{Y},$$

then

$$\vdash (x_1, x_2, \cdots, x_n)\,(\mathfrak{X} = \mathfrak{Y}).$$

[25] The convention cited is unfortunately misstated. The letter Z should be inserted after the words "A regular combinator".

Proof.

$$\begin{aligned} &\vdash \Pi(W(CQ)) && \text{(Ax. } Q\text{)} \\ &= \vdash (x)\, W(CQ)\, x && \text{(Th. 4, Cor. 2)} \\ &= \vdash (x)\, Qxx && \text{(Rules } W \text{ and } C\text{; Th. 4)} \\ &= \vdash (x)\ x = x && \text{(I. C 4, Def. 2).} \\ \therefore\ &\vdash (x_1, x_2, \cdots, x_m)(\mathfrak{x} = \mathfrak{x}) && \left[\text{Subst.} \begin{pmatrix} \mathfrak{x} \\ x \end{pmatrix}\right] \\ = &\vdash (x_1, x_2, \cdots, x_m)(\mathfrak{x} = \mathfrak{y}), \quad \text{q. e. d.} && \text{(Hp.; Th. 4).} \end{aligned}$$

4. Some Properties of Formal Implication.

DEFINITION 5. If X and Y are entities, then

$$X \supset Y \equiv PXY.$$

As a symbol of operation, $\supset$ shall be senior to all others introduced so far.

CONVENTION 8. The notation

$$\vdash X \supset \vdash Y$$

shall be understood as an abbreviation for the argument:

$$\begin{aligned} &\vdash X, \\ &\vdash X \supset Y, \\ \therefore\ &\vdash Y && \text{(Rule } P\text{).} \end{aligned}$$

Similarly for $\vdash X_1 \supset \vdash X_2 \supset \vdash X_3 \cdots$, etc.

THEOREM 9.

a) $\overset{C}{\vdash} P = [x, y](x \supset y)$,

b) $\overset{C}{\vdash} P_n = [x, y, t_1, t_2, \cdots, t_n](x\, t_1\, t_2 \cdots t_n \supset y\, t_1\, t_2 \cdots t_n)$.

Proof. By definition, a) is identical with

$$\overset{C}{\vdash} P = [x, y]\, Pxy;$$

it therefore follows immediately by Theorem 2, Cor.

To prove b): In III 4, Theorem 6, it was shown to follow combinatorially that

$$\begin{aligned} \vdash_{x,y,t} P_n xy\, t_1\, t_2 \cdots t_n &= P_0\,(x\, t_1\, t_2 \cdots t_n)\,(y\, t_1\, t_2 \cdots t_n) \\ &= (x\, t_1\, t_2 \cdots t_n \supset y\, t_1\, t_2 \cdots t_n) && \text{(Def. 5).} \end{aligned}$$

The formula b) then follows by Theorem 2.

THEOREM 10. *For any constants u and v,*

$$\vdash^{C}_{u,v} P_n^* uv = (t_1, t_2, \cdots, t_n)(x\, t_1\, t_2 \cdots t_n \supset y\, t_1\, t_2 \cdots t_n).$$

Proof. By Definition 1, $P_n^* \equiv B_2 \Pi_n P_n$.

$$\begin{aligned}
\therefore \vdash^C_{u,v} P_n^* u v &= B_2 \Pi_n P_n u v \\
&= \Pi_n (P_n u v) && \text{(II B 1, Satz 3)} \\
&= (t_1, t_2, \cdots, t_n) P_n u v t_1 t_2 \cdots t_n && \text{(Th. 4, Cor. 2)} \\
&= (t_1, t_2, \cdots, t_n)(u t_1 t_2 \cdots t_n \supset v t_1 t_2 \cdots t_n) && \text{(Ths. 9, 2, and 4).}
\end{aligned}$$

COROLLARY. $\vdash^C P_n^* = [u, v]\,(t_1, t_2, \cdots, t_n)\,(x t_1 t_2 \cdots t_n \supset y t_1 t_2 \cdots t_n)$.

Proof. Follows from Theorem 10 by Theorem 2.

THEOREM 11. *If* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *are combinations of constants and the variables* $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n$ *such that*

$$\vdash (x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n)(\mathfrak{X} \supset \mathfrak{Y}),$$

then

$$\vdash (x_1, \cdots, x_m)((y_1, \cdots, y_n)\mathfrak{X} \supset (y_1, \cdots, y_n)\mathfrak{Y}).$$

Proof. Let

$$X \equiv [x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n]\mathfrak{X}$$

and

$$Y \equiv [x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n]\mathfrak{Y}.$$

Then, by Theorem 2,

$$\begin{aligned}
(1)\qquad & \vdash^C_{x,y} X x_1 x_2 \cdots x_m y_1 y_2 \cdots y_n = \mathfrak{X}, \\
& \vdash^C_{x,y} Y x_1 x_2 \cdots x_m y_1 y_2 \cdots y_n = \mathfrak{Y}. \\
\therefore\ & \vdash (x_1, \cdots, x_m, y_1, \cdots, y_n)(\mathfrak{X} \supset \mathfrak{Y}) && \text{(Hp.)} \\
=\ & \vdash (x_1, \cdots, x_m, y_1, \cdots, y_n)(X x_1 \cdots x_m y_1 \cdots y_n \supset Y x_1 \cdots x_m y_1 \cdots y_n) \\
& && \text{((1); Th. 4)} \\
=\ & \vdash P^*_{m+n} X Y && \text{(Th. 10).} \\
\therefore\ & \vdash P_m^* (B_m \Pi_n X)(B_m \Pi_n Y) && \text{(III 6, Th. 2)} \\
=\ & \vdash (x_1, \cdots, x_m)(B_m \Pi_n X x_1 \cdots x_m \supset B_m \Pi_n Y x_1 \cdots x_m) && \text{(Th. 10)} \\
(2)\quad =\ & \vdash (x_1, \cdots, x_m)(\Pi_n (X x_1 \cdots x_m) \supset \Pi_n (Y x_1 \cdots x_m)) \\
& && \text{(II B 4, Satz 3; Th. 4).}
\end{aligned}$$

But by (1) and Theorem 4, Cor. 1,

$$(3)\qquad \vdash^C_x \Pi_n (X x_1 \cdots x_m) = (y_1, y_2, \cdots, y_n)\mathfrak{X},$$

$$(4)\qquad \vdash^C_x \Pi_n (Y x_1 \cdots x_m) = (y_1, y_2, \cdots, y_n)\mathfrak{Y}.$$

From (2), (3), (4) the conclusion follows by Theorem 4.

COROLLARY. *If* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *are combinations of constants and variables* $x_1, x_2, \cdots, x_n$ *such that*

$$\vdash (x_1, x_2, \cdots, x_n)(\mathfrak{X} \supset \mathfrak{Y}),$$

then

$$\vdash (x_1, x_2, \cdots, x_n)\mathfrak{X} \supset (x_1, x_2, \cdots, x_n)\mathfrak{Y}.$$

Proof. This is the special case $m = 0$ of the theorem. That the variables are denoted by x's instead of y's of course makes no difference.

THEOREM 12. *If* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *are combinations of constants and the variables* $x_1, x_2, \cdots, x_n$ *such that*

$$\vdash (x_1, x_2, \cdots, x_n)\,\mathfrak{X}$$

and

$$\vdash (x_1, x_2, \cdots, x_n)\,(\mathfrak{X} \supset \mathfrak{Y}),$$

then

$$\vdash (x_1, x_2, \cdots, x_n)\,\mathfrak{Y}.$$

Proof. Follows from Theorem 11, Cor., and Rule P.

NEW AXIOMS FOR IMPLICATION.

Ax. P_0. $\quad \vdash (x)\,(x \supset x)$.

Ax. (PB). $\quad \vdash ((x, y, z)\,(x \supset y) \supset ((z \supset x) \supset (z \supset y)))$.

DISCUSSION OF THESE AXIOMS. These axioms are at variance with my previous statements in two points, as follows:

1) They state principles which belong to the calculus of propositions; whereas in III 1 I stated that such properties do not properly belong within the domain of combinatory logic. But a further consideration of the matter shows that it is not feasible to rigidly separate the algebra of propositions from combinatory logic. Accordingly these axioms are introduced here, and will be used in the sequel.

2) They allow us in certain cases to assert PXY when X and Y are not propositions.[26] This contradicts both our common sense conception of implication and a statement made in I A.[27] But this is a question of how the entity P is to be interpreted; strictly speaking there can be no logical objection to interpreting it in whatever way is most convenient. The present interpretation simplifies the formal developments and is here adopted for that reason. The narrower interpretation for implication can, perhaps, be associated with an entity P' defined as

[26] This same character is also possessed by Ax. (ΠK).

[27] Page 514, top. It may be remarked incidentally that the whole paragraph referred to is in need of revision. For it is possible—the desirability is another matter—to define a proposition as that which is either true or false,—i. e., the property of being a proposition as

$$[x]\ Vx\,(Nx) \qquad \text{(or even as } [x]\ ((x \supset Nx) \supset Nx)\text{)}$$

where V is alternation and N negation—and in terms of this propositional functions of all orders, shapes, and sizes (I expect to show this in a later paper). Even so, it does not follow without proof that the Schönfinkel primitive ideas suffice for the whole of logic, even when one goes to the extreme of indulgence with regard to definitions such as

$$p \supset q \equiv_{Df} -p \vee q,$$
$$f(\iota x)\,(\varphi x) \equiv_{Df} (\exists b) : \psi x \equiv_x x = b : fb.$$

$$P' \equiv [x, y]\,(\mathrm{Pr}\,x \,\&\, \mathrm{Pr}\,y \,\&\, P\,x\,y),$$

where Pr means being a proposition and & is an operational symbol for conjunction. It may be possible, at a later stage, to revise the theory so that P itself can have this narrower interpretation; but that question will not be touched upon in this paper.

THEOREM 13.

a) $\vdash (x)\, P_n^*\, x\, x$.

b) *If* $\mathfrak{X}$ *is an expression involving the variables* $x_1, x_2, \cdots, x_n$, *then*

$$\vdash (x_1, x_2, \cdots, x_n)\,(\mathfrak{X} \supset \mathfrak{X}).$$

Proof. b) follows by Subst. $\binom{\mathfrak{X}}{x}$ Ax. P_0. To obtain a), let us regard $\mathfrak{X}$ as $x\,t_1\,t_2 \cdots t_n$, so that we have

$$\begin{aligned} &\vdash (x,\ t_1, t_2, \cdots, t_n)\ (x\,t_1\,t_2 \cdots t_n \supset x\,t_1\,t_2 \cdots t_n) \\ &= \vdash (x)\,(t_1, t_2, \cdots, t_n)\ (x\,t_1\,t_2 \cdots t_n \supset x\,t_1\,t_2 \cdots t_n) \qquad \text{(Th. 6).} \end{aligned}$$

Again it follows from Theorem 10 that

$$\overset{C}{\vdash} P_n^*\, x\, x = (t_1, t_2, \cdots, t_n)\ (x\,t_1\,t_2 \cdots t_n \supset x\,t_1\,t_2 \cdots t_n).$$

From the last two formulas a) follows by Theorem 4.

THEOREM 14. *If* $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ *are combinations of constants and the variables* $x_1, x_2, \cdots, x_n$ *such that*

$$\text{i)}\quad \vdash (x_1, x_2, \cdots, x_n)\,(\mathfrak{X} \supset \mathfrak{Y}),$$

$$\text{ii)}\quad \vdash (x_1, x_2, \cdots, x_n)\,(\mathfrak{Y} \supset \mathfrak{Z}),$$

then

$$\vdash (x_1, x_2, \cdots, x_n)\,(\mathfrak{X} \supset \mathfrak{Z}).$$

Proof. By Subst. $\binom{\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}}{z, x, y}$ Ax. (PB),

$$\vdash (x_1, x_2, \cdots, x_n)\ ((\mathfrak{Y} \supset \mathfrak{Z}) \supset ((\mathfrak{X} \supset \mathfrak{Y}) \supset (\mathfrak{X} \supset \mathfrak{Z}))).$$

$$\therefore \vdash (x_1, x_2, \cdots, x_n)\ ((\mathfrak{X} \supset \mathfrak{Y}) \supset (\mathfrak{X} \supset \mathfrak{Z})) \qquad \text{(Hp. ii), Th. 12}_\prime.$$

$$\therefore \vdash (x_1, x_2, \cdots, x_n)\ (\mathfrak{X} \supset \mathfrak{Z}),\ \text{q. e. d.} \qquad \text{(Hp. i), Th. 12).}$$

THEOREM 15. *If* 1) $\mathfrak{X}$ *is a combination of constants and the variables* $x_1, x_2, \cdots, x_m$, *and* 2) $\mathfrak{Y}$ *is a similar combination which may, however, involve the variables* $y_1, y_2, \cdots, y_n$ *not appearing in* $\mathfrak{X}$, *and* 3)

$$\vdash (x_1, x_2, \cdots, x_m,\ y_1, y_2, \cdots, y_n)\,(\mathfrak{X} \supset \mathfrak{Y}),$$

then

$$\vdash (x_1, x_2, \cdots, x_m)\,(\mathfrak{X} \supset (y_1, y_2, \cdots, y_n)\,\mathfrak{Y}).$$

Proof. By Hp. 3) and Theorem 11,

(1) $\quad \vdash (x_1, x_2, \cdots, x_m)\,((y_1, y_2, \cdots, y_n)\,\mathfrak{X} \supset (y_1, y_2, \cdots, y_n)\,\mathfrak{Y})).$

Let

$$V_n \equiv K_n \cdot K_{n-1} \cdot K_{n-2} \cdot \cdots \cdot K_1.$$

Then

$$\vdash^{C}_{x,y} V_n\,\mathfrak{X}\,y_1\,y_2 \cdots y_n = \mathfrak{X},$$

and hence ($\mathfrak{X}$ does not involve the variables y_i)

(2) $\quad \vdash^{C}_{x} (y_1, y_2, \cdots, y_n)\,\mathfrak{X} = \Pi_n (V_n\,\mathfrak{X})$ $\qquad$ (Th. 4, Cor. 1).

Now V_n is of order n and degree 0. Therefore, by III 7, Theorem 8[28] (the presupposed transitivity of P_n^* has been proved in Theorem 14),

$$\begin{aligned} &\vdash P_1^*\,\Pi_0\,(\Pi_n \cdot V_n) \\ &= \vdash (t)\,(t \supset \Pi_n\,(Vt)) \qquad \text{(Th. 10; Def. of } \Pi_0\text{; Th. 4).} \\ &\therefore \vdash (x_1, x_2, \cdots, x_n)\,(\mathfrak{X} \supset \Pi_n\,(V\mathfrak{X})) \qquad \left(\text{Subst.}\begin{pmatrix}\mathfrak{X}\\ t\end{pmatrix}\right) \\ &= \vdash (x_1, x_2, \cdots, x_n)\,(\mathfrak{X} \supset (y_1, y_2, \cdots, y_n)\,\mathfrak{X}) \qquad (3) \end{aligned}$$

by (2) and Theorem 4. The proposition to be proved follows from (3), (1), and Theorem 14.

5. Properties of Substitution and Implication as Formulas.

In this section I shall derive several theorems whose effect is to show that many of the preceding theorems may be formally stated as propositions. To this end a new axiom is necessary, viz.,

Ax. Π_0. $\vdash (x, y)\,(\Pi x \supset xy)$,[29] which expresses as a formula the fact stated as a rule in Rule Π.

It should be noted that, since the transitivity of P_n^* has been proved, we may make use of the theorems in III which depend on that transitivity.

THEOREM 16.

$$\vdash (x, y_1, y_2, \cdots, y_n)\,(\Pi_{m+n}\,x \supset \Pi_m (x y_1\,y_2 \cdots y_n)), \qquad m, n = 0, 1, 2, \cdots.$$

Proof. For $n = 0$, the above formula follows from Theorem 13.

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[28] The suffix attached to the symbol IP throughout the Theorems 7 and 8 of III 7 is one unit too high.

[29] This axiom may also be stated in the equivalent forms

$$\vdash P_2^*\,(K \cdot \Pi)\,\Pi_0,$$
$$\vdash (x, y)\,((z)\,xz \supset xy).$$

For $n = 1$ the proof is as follows:

$$\vdash (x, y)\,(\Pi\,(B\,\Pi_m x) \supset B\,\Pi_m x y) \qquad \left[\text{Subst.} \begin{pmatrix} B\,\Pi_m x, & y \\ x & , y \end{pmatrix} \text{Ax. } \Pi_0\right]$$

$= \vdash (x, y)\,(\Pi_{m+1}\,x \supset \Pi_m\,(x y))$ (III 2 Def. 1; II B 4, Satz 1; Rule B; Th. 4).

The theorem is therefore proved for $n = 1$, m arbitrary.

Next let the theorem be assumed proved for $n = k$. Then it may be proved for $n = k+1$ as follows. By this theorem for $n = k$,

$$\vdash (x, y_1, y_2, \cdots, y_k)\,(\Pi_{m+k+1}\,x \supset \Pi_{m+1}\,(x\,y_1\,y_2 \cdots y_k)).$$

Therefore,

(1) $\quad \vdash (x, y_1, y_2, \cdots, y_{k+1})\,(\Pi_{m+k+1}\,x \supset \Pi_{m+1}\,(x\,y_1\,y_2 \cdots y_k))$ (by Subst.).

By this theorem for $n = 1$ and Subst. $\begin{pmatrix} x\,y_1\,y_2 \cdots y_k, & y_{k+1} \\ x & , y \end{pmatrix}$

(2) $\vdash (x, y_1, y_2, \cdots, y_k, y_{k+1})\,(\Pi_{m+1}\,(x\,y_1\,y_2 \cdots y_k) \supset \Pi_m\,(x\,y_1\,y_2 \cdots y_k\,y_{k+1}))$.

From (1) and (2) the formula for $n = k+1$ follows by Theorem 14.

THEOREM 17. *If $\mathfrak{X}$ is a combination of constants and the variables $t_1, t_2, \cdots, t_p, x_1, x_2, \cdots, x_m$, and if $\mathfrak{Y}$ is the expression obtained from $\mathfrak{X}$ by substituting for each x_i, wherever it occurs, an expression $\mathfrak{z}_i$ which is a combination of the constants $a_1, a_2, \cdots, a_q$ and the variables $y_1, y_2, \cdots, y_n$, then*

$$\vdash (t_1, t_2, \cdots, t_p)\,((x_1, x_2, \cdots, x_m)\,\mathfrak{X} \supset (y_1, y_2, \cdots, y_n)\,\mathfrak{Y}).$$

Proof. Let $\mathfrak{X}' \equiv [x_1, x_2, \cdots, x_m]\,\mathfrak{X}$. ($\mathfrak{X}'$ then contains the t's.) Then

(1) $$\vdash_{t,x} \mathfrak{X}'\,x_1\,x_2 \cdots x_m = \mathfrak{X},$$

(2) $$\vdash_{t,y} \mathfrak{X}'\,\mathfrak{z}_1\,\mathfrak{z}_2 \cdots \mathfrak{z}_m = \mathfrak{Y}.$$

As in the proof of Theorem 7, let U be a combinator such that the expression

$$U\mathfrak{X}'\,a_1\,a_2 \cdots a_q\,y_1\,y_2 \cdots y_n$$

reduces formally to the left side of (2), the symbols $\mathfrak{X}', a_1, a_2, \cdots, a_q, y_1, \cdots, y_n$, being all treated formally. U can be supposed regular and is then of order $n+q$ and degree m. Hence by III 7, Theorem 8,[28]

(3) $$\begin{aligned} &\vdash P_1^*\,\Pi_m\,(\Pi_{n+q} \cdot U) \\ &= \vdash (x)\,(\Pi_m\,x \supset \Pi_{n+q}\,(Ux)) \qquad \text{(Th. 10; II B 4, Satz 1; Th. 4).} \end{aligned}$$

But by Theorem 16

$$\vdash (y, u_1, u_2, \cdots, u_q)\,(\Pi_{n+q}\,y \supset \Pi_n\,(y\,u_1\,u_2 \cdots u_q)).$$

Therefore, by Subst. $\begin{pmatrix} Ux, & a_1, & a_2, & \cdots, & a_q \\ y, & u_1, & u_2, & \cdots, & u_q \end{pmatrix}$,

(4) $$\vdash (x)\,(\Pi_{n+q}(Ux) \supset \Pi_n(Ux\,a_1\,a_2 \cdots a_q)).$$

By Theorem 14 it follows from (3), (4) that

$$\vdash (x)(\Pi_m x \supset \Pi_n(Ux\,a_1\,a_2 \cdots a_q));$$

hence by Subst. $\begin{pmatrix} \mathfrak{X}' \\ x \end{pmatrix}$

(5) $$\vdash (t_1, t_2, \cdots, t_p)\,\Pi_m \mathfrak{X}' \supset \Pi_n(U\mathfrak{X}'\,a_1\,a_2 \cdots a_q).$$

By definition of $\mathfrak{X}'$ and Definition 4,

(6) $$\Pi_m \mathfrak{X}' \equiv (x_1, x_2, \cdots, x_m)\,\mathfrak{X}.$$

Moreover since it follows combinatorially that

$$\vdash_{y,t} U\mathfrak{X}'\,a_1\,a_2 \cdots a_q\,y_1\,y_2 \cdots y_n = \mathfrak{Y},$$

we have, by Theorem 4,

(7) $$\vdash^{C}_{t} \Pi_n(U\mathfrak{X}'\,a_1\,a_2 \cdots a_q) = (y_1, y_2, \cdots, y_n)\,\mathfrak{Y}.$$

Putting (6) and (7) in (5) we have the theorem by Theorem 4.

THEOREM 18. *If $\mathfrak{X}$ and $\mathfrak{Y}$ are combinations of constants and the variables $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n$, then*

$$\vdash (x_1, \cdots, x_m)\,((y_1, \cdots, y_n)(\mathfrak{X} \supset \mathfrak{Y}) \supset ((y_1, \cdots, y_n)\,\mathfrak{X} \supset (y_1, \cdots, y_n)\,\mathfrak{Y})).$$

Proof.

$$\vdash P_2^*\,P_n^*\,(\Psi P \Pi_n) \qquad \text{(III 6, Th. 5d)}$$

(1) $$= \vdash (u, v)\,(P_n^*\,uv \supset P(\Pi_n u)(\Pi_n v)) \qquad \text{(Th. 10; III 4, Th. 3a; Th. 4)}$$

$$= \vdash (u, v)\,(P_n^*\,uv \supset (\Pi_n u \supset \Pi_n v)) \qquad \text{(Def. 3)}$$

To this apply Subst. $\begin{pmatrix} \mathfrak{X}', & \mathfrak{Y}' \\ u, & v \end{pmatrix}$, where

(2) $$\mathfrak{X}' \equiv [y_1, \cdots, y_n]\,\mathfrak{X}, \qquad \mathfrak{Y}' \equiv [y_1, \cdots, y_n]\,\mathfrak{Y}.$$

The result is

(3) $$\vdash (x_1, \cdots, x_m)\,(P_n^*\,\mathfrak{X}'\,\mathfrak{Y}' \supset (\Pi_n \mathfrak{X}' \supset \Pi_n \mathfrak{Y}')).$$

Now by Theorems 10 and 5,

$$\vdash^{C}_{x} P_n^*\,\mathfrak{X}'\,\mathfrak{Y}' = (y_1, \cdots, y_n)\,(\mathfrak{X}'\,y_1\,y_2 \cdots y_n \supset \mathfrak{Y}'\,y_1\,y_2 \cdots y_n)$$

$$= (y_1, \cdots, y_n)\,(\mathfrak{X} \supset \mathfrak{Y}) \qquad \text{[by (2); Ths. 2 and 4]:}$$

also

$$\vdash^{C}_{x} \Pi_n \mathfrak{X}' = (y_1, \cdots, y_n)\, \mathfrak{X},$$
$$\vdash^{C}_{x} \Pi_n \mathfrak{Y}' = (y_1, \cdots, y_n)\, \mathfrak{Y} \qquad \text{(Def. 2)}.$$

Putting these last three results in (3) we have the formula to be proved.

THEOREM 19. *If* 1) $\mathfrak{X}$ *is a combination of constants and the variables* $x_1, x_2, \cdots, x_m$, *and* 2) $\mathfrak{Y}$ *is a combination of constants and the variables* $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n$, *then*

$$\vdash (x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n)\, ((\mathfrak{X} \supset (y_1, y_2, \cdots, y_n)\, \mathfrak{Y}) \supset (\mathfrak{X} \supset \mathfrak{Y})).$$

Proof. Let $\mathfrak{Y}' \equiv [y_1, y_2, \cdots, y_n]\, \mathfrak{Y}$. ($\mathfrak{Y}'$ is then an expression involving the variables $x_1, x_2, \cdots, x_m$.) Then, by Definition 4,

$$(1) \qquad (y_1, y_2, \cdots, y_n)\, \mathfrak{Y} \equiv \Pi_n \mathfrak{Y}'.$$

Now by Theorem 16

$$(2) \qquad \begin{aligned} &\vdash (x_1, \cdots, x_m, y_1, \cdots, y_n)\, (\Pi_n \mathfrak{Y}' \supset \mathfrak{Y}'\, y_1\, y_2 \cdots y_n) \\ &= \vdash (x_1, \cdots, x_m, y_1, \cdots, y_n)\, (\Pi_n \mathfrak{Y}' \supset \mathfrak{Y}) \qquad \text{(Ths. 2 and 4)}. \end{aligned}$$

On the other hand, by Subst. $\begin{pmatrix} \Pi_n \mathfrak{Y}', & \mathfrak{Y}, & \mathfrak{X} \\ x & , y, & z \end{pmatrix}$ Ax. (PB),

$$\vdash (x_1, \cdots, x_m, y_1, \cdots, y_n)\, ((\Pi_n \mathfrak{Y}' \supset \mathfrak{Y}) \supset ((\mathfrak{X} \supset \Pi_n \mathfrak{Y}') \supset (\mathfrak{X} \supset \mathfrak{Y}))).$$

Hence, by (2) and Theorem 12,

$$\vdash (x_1, \cdots, x_m, y_1, \cdots, y_n)\, ((\mathfrak{X} \supset \Pi_n \mathfrak{Y}') \supset (\mathfrak{X} \supset \mathfrak{Y})).$$

In view of (1), this is precisely the formula to be proved.

COROLLARY 1. *Under the same hypotheses*

$$\vdash (x_1, \cdots, x_m)\, ((\mathfrak{X} \supset (y_1, \cdots, y_n)\, \mathfrak{Y}) \supset (y_1, \cdots, y_n)\, (\mathfrak{X} \supset \mathfrak{Y})).$$

Proof. Follows from the theorem by Theorem 15.[30]

COROLLARY 2. (Converse of Theorem 15.) *Under the same hypotheses, if*

$$\vdash (x_1, \cdots, x_m)\, (\mathfrak{X} \supset (y_1, \cdots, y_n)\, \mathfrak{Y}),$$

then

$$\vdash (x_1, \cdots, x_m, y_1, \cdots, y_n)\, (\mathfrak{X} \supset \mathfrak{Y}).$$

Proof. Follows at once from Corollary 1 by Theorems 12 and 6.

[30] $(y_1, \cdots, y_n)\, \mathfrak{Y}$ does not contain the variables y.

THEOREM 20. *If $\mathfrak{X}$ is a combination of constants and the variables $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n$, and $\mathfrak{Y}$ is a combination of constants and the variables $x_1, x_2, \cdots, x_m$, then*

$$\vdash (x_1, \cdots, x_m)\,((y_1, \cdots, y_n)\,(\mathfrak{X} \supset \mathfrak{Y}) \supset ((y_1, \cdots, y_n)\,\mathfrak{X} \supset \mathfrak{Y})).$$

Proof. Let V_n be defined as in proof of Theorem 15. Then

$$\vdash^{C}_{x,y} V_n\,\mathfrak{Y}\,y_1\,y_2 \cdots y_n = \mathfrak{Y}.$$
$$\therefore \vdash^{C}_{x} (y_1, y_2, \cdots, y_n)\,\mathfrak{Y} = \Pi_n (V_n\,\mathfrak{Y}) \qquad \text{(Th. 4, Cor. 1).}$$

By Subst. $\begin{pmatrix} V_n\,\mathfrak{Y}, & y_1, \cdots, & y_n \\ x, & y_1, \cdots, & y_n \end{pmatrix}$, Theorem 16,

$$\vdash (x_1, \cdots, x_m, y_1, \cdots, y_n)\,(\Pi_n (V_n\,\mathfrak{Y}) \supset V_n\,\mathfrak{Y}\,y_1\,y_2 \cdots y_n)$$
$$= \vdash (x_1, \cdots, x_m, y_1, \cdots, y_n)\,((y_1, \cdots, y_n)\,\mathfrak{Y} \supset \mathfrak{Y}) \qquad \text{(Th. 4)}$$
$$(1) \qquad \supset \vdash (x_1, \cdots, x_m)\,((y_1, \cdots, y_n)\,\mathfrak{Y} \supset \mathfrak{Y})^{31} \qquad \text{(Th. 17).}$$

By Subst. $\begin{pmatrix} (y_1, \cdots, y_n)\,\mathfrak{Y}, & \mathfrak{Y}, & (y_1, \cdots, y_n)\,\mathfrak{X} \\ x, & y, & z \end{pmatrix}$, Ax. (PB),

$$\vdash (x_1, x_2, \cdots, x_m)\,(\mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{C})),$$

where

$$\mathfrak{A} \equiv (y_1, \cdots, y_n)\,\mathfrak{Y} \supset \mathfrak{Y},$$
$$\mathfrak{B} \equiv (y_1, \cdots, y_n)\,\mathfrak{X} \supset (y_1, \cdots, y_n)\,\mathfrak{Y},$$
$$\mathfrak{C} \equiv (y_1, \cdots, y_n)\,\mathfrak{X} \supset \mathfrak{Y}.$$

But by (1),

$$\vdash (x_1, \cdots, x_m)\,\mathfrak{A}.$$
$$\therefore \vdash (x_1, \cdots, x_m)\,(\mathfrak{B} \supset \mathfrak{C}) \qquad \text{(Th. 12).}$$

Again,

$$\vdash (x_1, \cdots, x_m)\,((y_1, \cdots, y_n)\,(\mathfrak{X} \supset \mathfrak{Y}) \supset \mathfrak{B}) \qquad \text{(Th. 18)}$$
$$\therefore \vdash (x_1, \cdots, x_m)\,((y_1, \cdots, y_n)\,(\mathfrak{X} \supset \mathfrak{Y}) \supset \mathfrak{C}), \text{ q. e. d.} \qquad \text{(Th. 14).}$$

THEOREM 21. *If $\mathfrak{X}$ and $\mathfrak{Y}$ are expressions as in Theorem 19, and if further*

$$\vdash (x, y, z)\,((x \supset (y \supset z)) \supset (y \supset (x \supset z))),$$

then

$$\vdash (x_1, \cdots, x_m)\,((y_1, \cdots, y_n)\,(\mathfrak{X} \supset \mathfrak{Y}) \supset (\mathfrak{X} \supset (y_1, \cdots, y_n)\,\mathfrak{Y})).$$

Proof. As in Theorem 15, we prove

$$(1) \qquad \vdash (x_1, \cdots, x_m)\,(\mathfrak{X} \supset \Pi_n (V_n\,\mathfrak{X}))$$
$$= \vdash (x_1, \cdots, x_m)\,(\mathfrak{X} \supset (y_1, \cdots, y_n)\,\mathfrak{X}).$$

[31] Since the expression $((y_1, \cdots, y_n)\,\mathfrak{Y} \supset \mathfrak{Y})$ does not contain the y's, we may substitute for the y's anything we please, say the constant B.

By Subst. $\begin{pmatrix} (y_1, \cdots, y_n)\,\mathfrak{X}, & (y_1, \cdots, y_n)\,\mathfrak{Y}, & \mathfrak{X} \\ x & , \quad y & , z \end{pmatrix}$, Ax. (PB), we have

$$\vdash (x_1, \cdots, x_m)\,(\mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{C})),$$

where

$$\mathfrak{A} \equiv (y_1, \cdots, y_n)\,\mathfrak{X} \supset (y_1, \cdots, y_n)\,\mathfrak{Y},$$
$$\mathfrak{B} \equiv \mathfrak{X} \supset (y_1, \cdots, y_n)\,\mathfrak{X},$$
$$\mathfrak{C} \equiv \mathfrak{X} \supset (y_1, \cdots, y_n)\,\mathfrak{Y}.$$

Hence, by Subst. $\begin{pmatrix} \mathfrak{A}, & \mathfrak{B}, & \mathfrak{C} \\ x, & y, & z \end{pmatrix}$, Hp. and Theorem 12,

$$\vdash (x_1, \cdots, x_m)\,(\mathfrak{B} \supset (\mathfrak{A} \supset \mathfrak{C})).$$

But by (1),

$$\vdash (x_1, \cdots, x_m)\,\mathfrak{B}.$$
$$\therefore \vdash (x_1, \cdots, x_m)\,\mathfrak{A} \supset \mathfrak{C} \qquad \text{(Th. 12)}.$$

On the other hand

$$\vdash (x_1, \cdots, x_m)\,((y_1, \cdots, y_n)\,(\mathfrak{X} \supset \mathfrak{Y}) \supset \mathfrak{A}) \qquad \text{(Th. 18)},$$
$$\therefore \vdash (x_1, \cdots, x_m)\,((y_1, \cdots, y_n)\,(\mathfrak{X} \supset \mathfrak{Y}) \supset \mathfrak{C}), \text{ q.e.d.} \qquad \text{(Th. 14)}.$$

UNIVERSITY OF CHICAGO.

CHARACTERISTIC FUNCTIONS AND THE ALGEBRA OF LOGIC.[1]

By Hassler Whitney.[2]

Introduction. The algebra of logic differs from ordinary algebra in that there are two formulas,

$$A \odot A = A, \qquad A \oplus A = A,$$

which have no analogues in ordinary algebra. On this account the rules of the algebra of logic must be learned afresh; this usually involves studying a set of axioms and deducing formulas from them.[3] We shall show in this paper how, by means of "characteristic functions",[4] the formulas of the algebra of logic can be expressed as equations of ordinary algebra; with this representation, the common operations on sets are easily understood.

In Part II "generalized sets" are taken up; these are useful in various mathematical theories.

I. Real sets.

1. **Characteristic functions.** Let R' (the universe of discourse) be any set of objects (elements) $a, b, c, \cdots$, (for instance, the books on a table, or the points of a plane), and let A' be any subset of these elements (say the red books, or the points inside a certain circle). The *characteristic function* $A(x)$ of the set A' is a function which associates either the number 1 or the number 0 with each element x of R': 1 if x is in A', and 0 otherwise. Thus

$$A(x) = \begin{cases} 1 \text{ if } x \text{ is in } A', \\ 0 \text{ if } x \text{ is not in } A'. \end{cases} \tag{1}$$

For instance, if R' consists of two red books a and b and one book c not red, then $A(a) = A(b) = 1$ and $A(c) = 0$.

[1] Received May 31 and July 7, 1932.—Presented to the American Mathematical Society. December 28, 1931.

[2] National Research Fellow.

[3] For a treatment of this algebra, see C. I. Lewis, A Survey of Symbolic Logic, University of California Press, 1918, Ch. II, or D. Hilbert and W. Ackermann, Grundzüge der theoretischen Logik, Springer, Berlin 1928, Ch. I and II.

[4] See de la Vallée Poussin, Intégrales de Lebesgue, Fonctions d'Ensembles, Classes de Baire, Paris 1916, p. 7.

Any set is determined by its characteristic function. Two sets A' and B' are the same if and only if their characteristic functions are identically equal:

(2′) $$A(x) = B(x) \text{ for all } x \text{ in } R'.$$

We shall write this simply

(2) $$A = B.$$

If A' contains all the elements of R', that is, A' is R', then the characteristic function A is identically 1:

(3) $$A = 1.$$

If A' contains no elements, A is 0:

(4) $$A = 0.$$

We must keep in mind that *all equations in characteristic functions are identities, holding true for each element x in R'.* In such equations, 1 or 0 is the function equal to one or zero for each element in R'.

2. **Combinations of sets.** A mathematical operation $\dot{+}$ (read: plus dot) will be useful here; it is defined by the equations

(5) $$1\dot{+}1 = 1\dot{+}0 = 0\dot{+}1 = 1, \qquad 0\dot{+}0 = 0.$$

If we make the general definition, for any two numbers X and Y,

(6) $$X\dot{+}Y = X+Y-XY,$$

the above formulas are seen to hold.

Suppose A' and B' are any two sets, and $A'\odot B'$ denotes that set whose elements are in both A' and B' (the common part of A' and B'). Its characteristic function must be 1 for elements in both A' and B', and 0 otherwise; that is, it must be 1 when both A and B are 1, and 0 when either A or B is 0. But this is just

(7) $$AB,$$

the product of the two characteristic functions A and B.

If $A'\oplus B'$ denotes the set whose elements are either in A' or in B' or in both (the union, or sum, of A' and B'), its characteristic function must be 1 when either A or B is 1, and 0 when both A and B are 0. This is just

(8) $$A\dot{+}B = A+B-AB.$$

Note that

$$A \dot{+} B = A + B \quad \textit{if and only if} \quad AB = 0.$$

If $\tilde{A}'$ is the set whose elements are not in A' (the complementary set of A'), then its characteristic function, which we call $\bar{A}$, is evidently

$$\bar{A} = 1 - A.^5 \tag{9}$$

3. **Operations with sets.** With the above arithmetical representation of sets, we can perform all the common operations on sets with ease. Some examples will illustrate this.

The two most typical relations in the algebra of logic are: $A' \odot A'$ is A', and $A' \oplus A'$ is A', i. e. the elements in both A' and A' are just those in A', as are those in either A' or A'. The first relation, in characteristic functions, is

$$AA = A, \tag{10}$$

which is easily seen to be true. For take any element x of R'; $A(x) \cdot A(x) = A(x)$ because $A(x) =$ either 1 or 0, and $1 \cdot 1 = 1, 0 \cdot 0 = 0$. To prove the second, we have

$$A \dot{+} A = A + A - AA = A + A - A = A. \tag{11}$$

The associative and commutative laws are evident, when we use (8). Some fundamental identities are

$$\begin{gathered} A \cdot 0 = 0, \qquad A \cdot 1 = A, \\ A \dot{+} 0 = A + 0 - A \cdot 0 = A, \qquad A \dot{+} 1 = A + 1 - A = 1, \\ A(B \dot{+} C) = A(B + C - BC) = AB + AC - ABC \\ = AB + AC - AB\,AC = AB \dot{+} AC, \\ (A \dot{+} B)\,(A \dot{+} C) = A \dot{+} BC. \end{gathered} \tag{12}$$

Some identities involving complementary sets are

$$\begin{gathered} \bar{1} = 1 - 1 = 0, \quad \bar{0} = 1 - 0 = 1, \\ \bar{\bar{A}} = 1 - \bar{A} = 1 - (1 - A) = A, \\ A\bar{A} = A(1 - A) = A - A = 0, \\ A \dot{+} \bar{A} = A + \bar{A} - A\bar{A} = A + (1 - A) = 1, \\ \bar{A} \dot{+} \bar{B} = 1 - AB = \bar{A}\bar{B}, \\ \overline{A \dot{+} B} = 1 - (A \dot{+} B) = (1 - A)(1 - B) = \bar{A}\bar{B}. \end{gathered} \tag{13}$$

[5] Note that although equations like this have been used by Boole and others, they were taken in an entirely different sense. It seemed as though the formulas (10) and (11) could not possibly occur in ordinary algebra.

An important relation between sets is that of *inclusion.* A set A' is *contained in* a set B' if every element of A' is an element of B'. (A' may be identical with B'.) In characteristic functions, if $A(x) = 1$ for a given element x, then $B(x) = 1$ also. This is equivalent to stating that for each element x, either $A(x) = 0$ or $\bar{B}(x) = 0$; or,

$$A\bar{B} = 0, \quad \text{or,} \quad \bar{A}\dot{+}B = 1. \tag{14}$$

As $A - AB = A\bar{B}$, and as $(A\dot{+}B) - B = A + B - AB - B = A\bar{B}$, two equivalent equations are

$$AB = A, \quad A\dot{+}B = B. \tag{15}$$

Another way of writing this is

$$A \leqq B. \tag{16}$$

4. **Properties and propositions.** Given a set of elements R', suppose A_1 is a property which each element either has or has not. Corresponding to this property A_1 we can define a set A', the set of elements with the property A_1. The characteristic function corresponding to the elements with both properties A_1 and B_1, or with either property, or without the property A_1, or with any property which holds for all the elements, or with any property which holds for none of the elements, is AB, or $A\dot{+}B$, or $\bar{A}$, or 1, or 0. The statement "every element with the property A_1 has also the property B_1" translates into "A' is contained in B'", or "$A\bar{B} = 0$".

We can reduce the logic of propositions to characteristic functions as follows. Consider a single imaginary element a. If P^* is any proposition, let P be the corresponding characteristic function, putting $P(a) = 1$ if P^* is true, and $P(a) = 0$ if P^* is false. Corresponding to the proposition P^* *and* Q^*, or *either* P^* *or* Q^*, or *not-*P^*, or P^* *implies* Q^*, is the characteristic function PQ, or $P\dot{+}Q$, or $\bar{P}$, or $\bar{P}\dot{+}Q$ (see equation (14)).

5. **First normal form.** We begin by explaining an important process, that of *expanding a characteristic function in terms of other characteristic functions.* To expand A in terms of B, we write

$$A = A \cdot 1 = A(B + \bar{B}) = AB + A\bar{B}. \tag{17}$$

The interpretation is simple: The elements in $A' \odot B'$ together with the elements in $A' \odot \tilde{B}'$ make up the elements in A'. As another example, to expand 1 in terms of A, or in terms of A and B, we write

$$1 = A + \bar{A}, \tag{18}$$

or

$$1 = A(B + \bar{B}) + \bar{A}(B + \bar{B}) = AB + A\bar{B} + \bar{A}B + \bar{A}\bar{B}. \tag{19}$$

In general, 2^n terms are formed in expanding a function in terms of n other functions. We give now the first normal form.

To reduce any function F to the first normal form, we first remove any $\dot{+}$ signs there may be by (8), *next multiply out, next expand each term in terms of all other characteristic functions appearing in F, and finally collect terms.*

For instance,

$$F = A(A \dot{+} B) - A\bar{B} = A(A + B - AB) - A\bar{B}$$
$$= A + AB - AB - A\bar{B} = A - A\bar{B} = (AB + A\bar{B}) - A\bar{B} = AB.$$

As another example,

$$F = A + B = (AB + A\bar{B}) + (AB + \bar{A}B) = 2AB + A\bar{B} + \bar{A}B.$$

Suppose the functions appearing in F were $A_1, A_2, \cdots, A_n$. Then each term in the normal form will be of the form $A_1^* A_2^* \cdots A_n^*$, where each A_k^* is either A_k or $\bar{A}_k$, and will have an integral coefficient, positive, negative or zero. There are 2^n possible terms of this sort, which may be found by expanding 1 in terms of all the functions. Note that *if H and I are any two of these terms, then $HI = 0$*; for for some k, H contains A_k (or $\bar{A}_k$) and I contains $\bar{A}_k$ (or A_k); but $A_k \bar{A}_k = 0$.

If G is F in the first normal form, G is determined by naming the coefficient of each of these 2^n terms (some coefficients may be zero). For an explanation of what values these coefficients can take on, see § 8.

Any characteristic function F has a unique expression in the first normal form (provided, of course, that we make no use of equations we may happen to know).

For suppose G_1 and G_2 were two expressions for F in the normal form, and suppose that for some one of the terms, say $A_1^* A_2^* \cdots A_n^*$, G_1 had the coefficient k_1, G_2 had the coefficient k_2, and $k_1 \neq k_2$. As $G_1 = F = G_2$, $G_1 - G_2 = 0$ for all x in R'. Thus

$$(k_1 - k_2) A_1^* A_2^* \cdots A_n^* + \cdots = 0, \qquad \text{all } x \text{ in } R'.$$

Let us interpret R' as being a set containing but a single element a, such that $A_1^*(a) = A_2^*(a) = \cdots = A_n^*(a) = 1$. This we may do, as the reduction to the normal form is independent of the interpretation of the sets involved. Then, as $\bar{A}_1^*(a) = \bar{A}_2^*(a) = \cdots = \bar{A}_n^*(a) = 0$, the above equation reduces to

$$(k_1 - k_2) A_1^*(a) A_2^*(a) \cdots A_n^*(a) = k_1 - k_2 = 0,$$

contradicting the hypotheses that $k_1 \neq k_2$.

6. **Application of the normal form.** In § 3 we gave a number of identities in characteristic functions. If we reduce each side of an identity

to the first normal form, the coefficient of each term in the result must be the same on both sides, by the uniqueness theorem. This gives a simple method of checking any identity. As an example, each side of the last equation of (12) reduces to $ABC+AB\bar{C}+A\bar{B}C+A\bar{B}\bar{C}+\bar{A}BC$ (or, $A+\bar{A}BC$).

Suppose we know a number of things, which we write in the form of equations, and we wish to deduce consequences. The simplest way to get a view of the equations as a whole is to write each equation $F=G$ in the form $F-G=0$, and expand $F-G$ into the first normal form, expanding in terms of all functions involved. A typical equation is now

$$a_1 H_1 + a_2 H_2 + \cdots + a_m H_m = 0, \tag{20}$$

where each a_i is $\neq 0$. Here, if $A_1, A_2, \cdots, A_n$ are the functions involved, each H_i is of the form $A_1^* A_2^* \cdots A_n^*$, where each A_k^* is either A_k or $\bar{A}_k$. We can then write

$$H_1 = H_2 = \cdots = H_m = 0. \tag{21}$$

For suppose not. Then one of these H_i is not identically 0, that is, there is an element x_1 in R' such that $H_i(x_1) = 1$. But as $H_i H_j = 0$ for $j \neq i$, $H_j(x_1) = 0$ for $j \neq i$. As (20) is true for every x in R' it is true for x_1; but for x_1, it becomes

$$a_i = 0,$$

a contradiction.

Examples. Let us show that "If the proposition P^* is true and the proposition P^* implies Q^* is true, then the proposition Q^* is true". The premises are $P=1$ and $\bar{P}\dot{+}Q=1$; we must show that $Q=1$. The first premise we write in the form $1-P=\bar{P}=0$, or expanding in terms of Q, $\bar{P}Q+\bar{P}\bar{Q}=0$. As $1-(\bar{P}\dot{+}Q)=P\bar{Q}$, the premises become

$$\bar{P}Q = \bar{P}\bar{Q} = P\bar{Q} = 0.$$

To compare Q and 1, we expand them:

$$Q = PQ+\bar{P}Q = PQ,$$
$$1 = PQ+P\bar{Q}+\bar{P}Q+\bar{P}\bar{Q} = PQ.$$

Thus $Q=1$, as required.[6]

Let us solve a problem of Venn.[7] "A certain club has the following rules: (a) The financial committee shall be chosen from the general com-

[6] Of course we could note simply that as $P=1$ and hence $\bar{P}=0$, $\bar{P}\dot{+}Q=1$ only if $Q=1$.

[7] See Lewis, loc. cit., p. 201. We are following the procedure given in this book.

mittee. (b) No one shall be a member of both the general and library committees unless he be also on the financial committee. (c) No member of the library committee shall be on the financial committee. Simplify the rules."

Letting R' be the set of people in the club, and putting F', G' and L' for the financial, general and library committees respectively, the premises become

$$F\bar{G} = 0, \qquad GL\bar{F} = 0, \qquad LF = 0.$$

Expanding, we find that

$$F\bar{G}L = F\bar{G}\bar{L} = \bar{F}GL = FGL = 0.$$

Combining the first two terms and the last two terms, we have the equivalent equations

$$F\bar{G} = 0, \qquad GL = 0.$$

Thus the simplified rules will be (a), together with: (b') No member of the general committee shall be on the library committee.

7. **Relation to the algebra of logic.** The essential difference between the present method and the algebra of logic is that here, one does not lose sight of the elements making up a set—any equation $A = B$ is an equation holding for every element under consideration. When one considers a set as consisting of certain specified elements, one can define the sum of two sets etc. directly, and hence the relations holding between combinations of sets can be deduced immediately; the arithmetic method furnishes a convenient tool for doing this. In the algebra of logic, a set is considered as an abstract quantity; the sum of two sets etc. must be defined abstractly, and one knows nothing about it except what one postulates. For this reason axioms must be proposed, enough so that all ordinary formulas can be derived therefrom.

We can, if we wish, consider a characteristic function as an abstract quantity; in this case, we can deal with sets just as described above, but without thinking of each equation as being an identity for all elements of some basic set. That we can deduce all the standard relations between sets from the relations given above follows from the fact that we can prove a set of axioms for the algebra of logic.[8]

II. Generalized Sets.

8. **Real and generalized sets.** Suppose we associate with each element of a set R' any integer, positive, negative or zero, instead of

[8] For instance those of Lewis, loc. cit. His "elements" are our "characteristic functions".

merely one or zero. The resulting function will not in general be the characteristic function of a real set; but we may consider it as the characteristic function of a *generalized set,* where each element is counted any number of times. (This is done for example with chains in analysis situs.) We shall discuss here the question as to whether any given set is real or generalized, and shall then take up two more normal forms.

Note that formulas previously deduced which depend on (10) or (11) do not hold for generalized sets.

Given any function F, let us expand it into the first normal form—this expansion holds of course whether the corresponding set is real or generalized. We prove now several facts.

If each coefficient in the first normal form is either 1 *or* 0, F' *is real.*

For each element of R' falls into exactly one of the 2^n sets corresponding to the 2^n characteristic functions $A_1^* A_2^* \cdots A_n^*$ described in § 5; F has therefore the value 1 or 0 for each element, and corresponds to a real set.

Conversely, if F' *is real, each coefficient in the first normal form is* 1 *or* 0, *at least after we have dropped out any of the* 2^n *terms which are* 0 (i. e. 0 for all elements in R').

For suppose not; then there is a term $A_1^* A_2^* \cdots A_n^*$ in the first normal form of which the coefficient k is neither 0 nor 1, while $A_1^* A_2^* \cdots A_n^*$ is not 0, that is, for some element x_1, $A_1^*(x_1) A_2^*(x_1) \cdots A_n^*(x_1) = 1$. Now from the normal form we see that the value of F for x_1 is just k; hence the element x_1 is counted in F' a number k of times not one or zero, and F' is not real, a contradiction.

As an example, if we know the function $F = A + B = 2AB + A\bar{B} + \bar{A}B$ to be real, then necessarily $AB = 0$; after we have dropped out this term, each coefficient is 1 or 0.

9. **Second and third normal forms.** *In the second normal form, we remove all* $+$ *signs, replace all characteristic functions with dashes by functions without, multiply out and collect terms.* Thus

$$F = \bar{B} - A\bar{B}C + ABC = (1-B) - A(1-B)C + ABC$$
$$= 1 - B - AC + 2ABC.$$

In the third normal form, we remove all $+$ *signs, replace all functions without dashes by functions with (but we do not alter the numerical term, if there is one), multiply out, and collect terms.* Thus, for the same F,

$$F = \bar{B} - (1-\bar{A})\bar{B}(1-\bar{C}) + (1-\bar{A})(1-\bar{B})(1-\bar{C})$$
$$= 1 - \bar{A} - \bar{B} - \bar{C} + 2\bar{A}\bar{B} + \bar{A}\bar{C} + 2\bar{B}\bar{C} - 2\bar{A}\bar{B}\bar{C}.$$

If there are n functions $A_1, A_2, \cdots, A_n$ involved in F, and G is F in the second (or third) normal form, then G is determined by naming the

coefficient of each of the 2^n terms $A_{i_1} A_{i_2} \cdots A_{i_k}$ (or $\bar{A}_{i_1} \bar{A}_{i_2} \cdots \bar{A}_{i_k}$), where $i_1, i_2, \cdots, i_k$ are any set of distinct numbers chosen from the set of numbers $1, 2, \cdots, n$. In the above example, $n = 3$ and $2^n = 8$.

Any function F has a unique expression in either normal form. We prove this for the second normal form; the third is exactly alike.

Suppose F contains the n characteristic functions $A_1, A_2, \cdots, A_n$. Let G and G' be two expressions for F in the second normal form. If we write $q_1, q_2, \cdots, q_{2^n}$ for the 2^n quantities $A_{i_1} A_{i_2} \cdots A_{i_k}$, and if $k_1, k_2, \cdots, k_{2^n}$, and $k'_1, k'_2, \cdots, k'_{2^n}$ are the coefficients of these terms in G and G' respectively, then as $G = G'$, or $G - G' = 0$, we must have for each x in R',

$$(k_1 - k'_1)\, q_1(x) + (k_2 - k'_2)\, q_2(x) + \cdots + (k_{2^n} - k'_{2^n})\, q_{2^n}(x) = 0.$$

We shall choose a definite interpretation of the functions $A_1, A_2, \cdots, A_n$, by letting R' be a set containing just 2^n elements, one lying in each of the 2^n sets corresponding to the 2^n functions $A_1^* A_2^* \cdots A_n^*$ of § 5. Call these functions $p_1, p_2, \cdots, p_{2^n}$ for short, and let x_i lie in the set corresponding to p_i, $i = 1, 2, \cdots, 2^n$. As then $p_i(x_i) = 1$ and $p_i(x_j) = 0$, $j \neq i$, the 2^n functions $p_i(x)$ are linearly independent.[9] Expanding these into the second normal form, they are expressed in terms of the 2^n functions $q_i(x)$; hence these are linearly independent also. Therefore the determinant of the $q_i(x_j)$ is $\neq 0$, and the above equation is only possible for every x if each coefficient vanishes, i. e. if $k_i = k'_i$, $i = 1, 2, \cdots, 2^n$.

It follows from the uniqueness theorems that *a necessary and sufficient condition that two functions $F(A_1, A_2, \cdots, A_n)$ and $G(A_1, A_2, \cdots, A_n)$ be the same for all determinations of $A_1, A_2, \cdots, A_n$, is that they be identical in any normal form.*

10. **Relation between the second and third normal forms.** Let F be a function and let F_2 and F_3 be F expanded into the second and third normal forms respectively. Consider in F_2 all terms containing i factors (e. g. AB contains two factors, 1 contains none), and let α_i be the sum of the coefficients of these terms. Define β_i similarly for F_3. Thus, in the last example,

$$\alpha_0 = 1, \quad \alpha_1 = -1, \quad \alpha_2 = -1, \quad \alpha_3 = 2;$$
$$\beta_0 = 1, \quad \beta_1 = -3, \quad \beta_2 = 5, \quad \beta_3 = -2.$$

We wish to find a relation between the α_i and the β_i.

To do this, consider any term of the first normal form, such as ABC (taking the coefficient 1), and expand it into the second normal form:

[9] That is, if $c_1 p_1(x) + c_2 p_2(x) + \cdots + c_{2^n} p_{2^n}(x) = 0$ for every x in R', then

$$c_1 = c_2 = \cdots = c_{2^n} = 0.$$

$$ABC = (1-\bar{A})(1-\bar{B})(1-\bar{C})$$
$$= 1-(\bar{A}+\bar{B}+\bar{C})+(\bar{A}\bar{B}+\bar{A}\bar{C}+\bar{B}\bar{C})-\bar{A}\bar{B}\bar{C}.$$

This term contributes $1 = \binom{3}{0}$ to β_0, $-3 = -\binom{3}{1}$ to β_1, $3 = \binom{3}{2}$ to β_2, and $-1 = -\binom{3}{3}$ to β_3. In general, a term (with coefficient 1) in the first normal form with k factors contributes $(-1)^i\binom{k}{i}$ to β_i. Hence the total contributed to β_i by terms containing k factors is $(-1)^i\binom{k}{i}\alpha_k$, and

$$\beta_i = (-1)^i \sum_k \binom{k}{i} \alpha_k. \tag{22}$$

Of course α_i is given by the same formula in terms of the β_k:

$$\alpha_i = (-1)^i \sum_k \binom{k}{i} \beta_k. \tag{23}$$

These formulas are easily verified in the above examples.

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(ADDED IN PROOF). P. Delens has observed (Comptes Rendus, vol. 195 (1932), p. 686–689) that with the help of (8), (9), and (10), the formulas of the algebra of logic take on an arithmetical form. However, he does not employ characteristic functions.

Some recent literature may be noted: C. I. Lewis and C. H. Langford, Symbolic Logic, New York 1932; E. V. Huntington, New sets of independent postulates for the algebra of logic, Transactions of the American Mathematical Society, vol. 35 (1933), pp. 274–304.

PRIME ENDS AND ORDER.[1]

By N. E. Rutt.[2]

I. The Prime Ends of $X+X_a$.

1. **Introduction.** This paper deals with plane point sets Z of the following type; Z is the sum of X, a bounded continuum, and of $[X_\alpha]$, a collection of continua each element of which contains at least one point of X, no two of which have any point in common which is not a point of X, no one of which is disconnected by the omission from it of its subset in X, and no one of which when added to X forms a set whose complement includes a bounded component containing a point of any element whatever of $[X_\alpha]$. The symbol $c(P)$ will hereafter represent the complement of the point set P in whatever space P is being considered, in this paper generally the plane. If P is a bounded set, $c_u(P)$ represents the unbounded component of $c(P)$. If C is a simple closed curve in the plane, $i(C)$ represents its interior and $e(C)$ its exterior, while by a natural combination of the symbols $ce(C)$ represents $c(e(C))$ or $C+i(C)$ and $ci(C)$ represents $c(i(C))$ or $C+e(C)$.

Order may be assigned among the elements of $[X_\alpha]$.[3] Hereafter it will be assumed that this assignment has been made with respect to the element X_r of $[X_\alpha]$ where X_r is chosen so that $X_r \cdot c(X)$ contains the end of a ray R all of whose nonend points belong to $c(Z)$. Of course an element X_r of this type always exists if Z is closed and bounded. Notions of precedence, separation, and series[4] (clockwise and counterclockwise) in connection with the elements of $[X_\alpha]$ thus need no explanation.

A special property of the elements of $[X_\alpha]$ required later is as follows:

LEMMA. *If X_b and X_r separate X_a and X_c, and if K is an arc cut of $c_u(X_a+X+X_c)$ with one end in $c(X)\cdot X_a$, the other in $c(X)\cdot X_c$, and no point in X_r+R, then the set $c_u(X_a+X+X_c)$ consists of an interval of K*

[1] Received March 21, 1932.

[2] Special cases of most of the theorems in this paper were obtained while the author was holding a National Research Fellowship, and were reported to the National Research Council Feb. 21, 1929 and Mar. 4, 1930.

[3] R. L. Moore, *Concerning the sum of a countable number of continua in the plane*, Fundamenta Mathematicae, vol. 6, pp. 189–202; J. H. Roberts, *Concerning collections of continua not all bounded*, American Journal of Mathematics, vol. 52 (1930), pp. 551–562; N. E. Rutt, *On certain types of plane continua*, Transactions of the American Mathematical Society, vol. 33, no. 3, pp. 806–816.

[4] A countable subset $[X_i]$ of $[X_\alpha]$ is called a series if for $i = 2, 3, 4, \cdots$ X_r and X_i separate X_{i-1} and X_{i+1}.

and two connected domains γ_b and γ_r, the bounded one of which, γ_b, contains all the components of $c(X+K)\cdot X_b$ with limit points in X; whereas if X_b and X_r do not separate X_a and X_c then γ_r contains all the components of $c(X+K)\cdot X_b$ with limit points in X.

2. **Notation.** Let X_a be a bounded element of $[X_a]$ not contained in X. Hence $X+X_a$ is a bounded continuum whose complement has the unbounded component $\gamma = c_u(X+X_a)$. Let $\mathfrak{C}$ be the simple closed curve of the prime ends of γ.[5]

3. **Theorem.** *If an element of $\mathfrak{C}$ has a chief point in $c(X)$ it has no points in $c(X_a)$.*

Suppose the statement false with respect to the element $\mathfrak{A}$ of $\mathfrak{C}$ which has a chief point a in $c(X)$ and yet contains a point x of $c(X_a)$. Let $\mathfrak{A}$ be determined by the set of cuts $[A_i]$ whose only limit point is a and by $[\alpha_i]$, the corresponding set of domains. There exists a subscript k such that for $i>k$ both ends of the cut A_i are points of $c(X_a)$. Now the boundary H_i of α_i contains x and thus contains a point x_i of $c(X_a)$ which is end of an arc consisting except for x_i of points of α_i. Let this arc be B_i. As $B_i \supset a$, there is a subscript j such that $A_j \cdot B_i = 0$. Then the boundary H_j of α_j since it contains x contains a point x_j different from x_i which is end of an arc B_j consisting except for x_j of points of α_j. Let B be an arc in α_i irreducibly connecting a point of B_i and one of B_j, and intersecting A_j just once. Thus B_i+B+B_j contains an arc cut A of $c_u(X)$ which it separates into two connected domains σ_1 and σ_2. Having but one point in common with A_j it also separates A_j into two connected sets one clearly contained in σ_1 and the other in σ_2. As the ends of A_j are in $c(X)\cdot X_a$, one of them in σ_1 and the other in σ_2, both σ_1 and σ_2 contain points of $c(X)\cdot X_a$. This is a contradiction for $(X+A)\cdot X_a \cdot c(X) = 0$ and yet $X+A$ separates the connected set $c(X)\cdot X_a$. Thus x of $\mathfrak{A}$ can not exist.

4. **Lemma.** *If $\mathfrak{A}$ is an element of $\mathfrak{C}$, γ contains a ray with the chief points of $\mathfrak{A}$ as limits and no other limits, and $\mathfrak{A}$ is limit of the ray.*[6]

The lemma is an easy consequence of the method used by Carathéodory in proving that, any prime end $\mathfrak{G}$ of $\mathfrak{C}$ being preassigned, γ contains a connected and simply connected subdomain whose boundary has in common with the boundary of γ only the chief points of $\mathfrak{G}$. As a matter of fact,

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[5] The prime ends of a connected and simply connected plane domain whose boundary is compact are defined in C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen 73 (1912), pp. 323–370. See also P. Urysohn, *Über ein Problem von Herrn C. Carathéodory*, Fundamenta Mathematicae, vol. VI, especially pp. 234–235.

[6] Here and, as the context will make clear, sometimes elsewhere the term *ray* is used as an abbreviation for *ray of a domain* and does not accordingly represent, as usual, a ray of space.

if a particular chain of cuts defining $\mathfrak{G}$ is also preassigned, the ray in question may be drawn so as to have in common with each cut of the chain at most one point.

5. **Theorem.** *Two elements of $\mathfrak{C}$ whose chief points are points of X can not separate in $\mathfrak{C}$ two elements of $\mathfrak{C}$ whose points are not all points of X.*

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two elements of $\mathfrak{C}$ whose chief points are all points of X and are limits through $\mathfrak{A}$ and $\mathfrak{B}$ respectively of the rays R_a and R_b of γ. Clearly R_a and R_b may be selected so that $R_a \cdot R_b = 0$. Let C be a simple closed curve enclosing $X + X_a + R_a + R_b$ except for the ends of R_a and R_b which it contains, and let G be a minimal subarc of C containing $C \cdot (R_a + R_b)$. Then $c_u(X) \cdot c(G + R_a + R_b)$ is the sum of two connected domains γ_a and γ_b which between them contain all the elements of $\mathfrak{C}$ except $\mathfrak{A}$ and $\mathfrak{B}$. If both of γ_a and γ_b contained points of X_a then the set $X + R_a + R_b + G$ would separate the connected set $c(X) \cdot X_a$ with which it has no common points, so one of them contains no points of X_a. But if the theorem is not true it is possible to select $\mathfrak{A}$ and $\mathfrak{B}$ so that they separate two elements of $\mathfrak{C}$ neither of which has all its points in X and thus must have some in $c(X) \cdot X_a$. One of these prime ends must be contained in γ_a and the other in γ_b, and as γ_a must contain all the points of any element of $\mathfrak{C}$ which it contains except those which are in X, and as γ_b must contain all similar points of the elements of $\mathfrak{C}$ contained in it, if the theorem is not true both γ_a and γ_b contain points of $c(X) \cdot X_a$, a contradiction.

6. **Theorem.** *If the element $\mathfrak{P}$ of $\mathfrak{C}$ contains chief points in $c(X)$, $\mathfrak{P}$ belongs to an interval of elements of $\mathfrak{C}$ each of which has chief points in $c(X)$.*

In case at most one element of $\mathfrak{C}$ has no chief points in $c(X)$ the theorem is obvious. In other cases the proof follows. Let $\mathfrak{Q}$ be an element of $\mathfrak{C}$ having no chief points in $c(X)$ and let R_q be a ray of γ having as limits the chief points of $\mathfrak{Q}$ through $\mathfrak{Q}$ and no other points of $c(\gamma)$. Let $[P_i]$ be a chain of cuts defining $\mathfrak{P}$ and having as its only limit point a point of $c(X)$; let the corresponding chain of domains be $[\varrho_i]$. It is clear under the circumstances that there exists a definite subscript j such that $\varrho_j + P_j$ contains no point of R_q and that $P_j \cdot (X + X_a) \subset c(X)$. The extremities of P_j are points accessible through the elements $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathfrak{C}$ each of which therefore has points in $c(X)$. Moreover the complement in $\mathfrak{C}$ of $\mathfrak{A} + \mathfrak{B}$ is a pair of intervals $\mathfrak{C}_p$ and $\mathfrak{C}_q$ the first of which includes $\mathfrak{P}$ and the second $\mathfrak{Q}$. Now if $\mathfrak{C}_p$ were to contain an element $\mathfrak{D}$ every chief point of which belonged to X then the separation of $\mathfrak{A}$ and $\mathfrak{B}$ by $\mathfrak{Q}$ and $\mathfrak{D}$ would be a contradiction of 5. Thus $\mathfrak{C}_p$ contains no such element as $\mathfrak{D}$ and is the interval required.

7. **Theorem.** *If two or more elements of* $\mathfrak{C}$ *have no chief points in* $c(X)$ *then those which have chief points in* $c(X)$ *lie upon a certain minimum subinterval of* $\mathfrak{C}$ *whose ends have no chief points in* $c(X)$.

8. **Notation.** The collection of prime ends of γ consisting of all those each of which contains a chief point in $c(X)$ will be represented by $\mathfrak{C}_a$. In all applications to be made later $\mathfrak{C}_a$ will be an interval of a simple closed curve $\mathfrak{C}$ whose elements are the prime ends of some domain of the same sort as γ, and the ends of the segment will be distinct elements $\mathfrak{U}$ and $\mathfrak{V}$ of $\mathfrak{C}$.

9. **Theorem.** *When two or more elements of* $\mathfrak{C}$ *have no chief points in* $c(X)$ *then the subset of* $\mathfrak{C}$ *consisting of those of its elements having points but no chief points in* $c(X)$ *is a subset of* $\mathfrak{U}+\mathfrak{V}$.

10. **Summary and comments.** The relations possible between the set X_a and the elements of $\mathfrak{C}$ are of the following three principal types.

(a) All elements of $\mathfrak{C}$ have chief points in $c(X)\cdot X_a$. In this case $\mathfrak{U}$ and $\mathfrak{V}$ can not exist and $\mathfrak{C}_a$ is identical with $\mathfrak{C}$. Moreover no continuum may contain a point of X and one of $c_u(X)$ without containing a point of $c(X)\cdot X_a$ so that $[X_\alpha]$ consists of the single element X_a, and every point of the boundary of γ is a point of X_a.

(b) One and but one element of $\mathfrak{C}$ has no chief point in $c(X)\cdot X_a$. Here again $\mathfrak{U}$ and $\mathfrak{V}$ can not exist but the specified element may resemble them in containing points of $c(X)\cdot X_a$ which are not chief points of it. It is the unique limit in $\mathfrak{C}$ of every set $c(X)\cdot X_b$ where X_b is an element of $[X_\alpha]$ distinct from X_a, and is the only element of $\mathfrak{C}$ not included by $\mathfrak{C}_a$.

(c) Two or more elements of $\mathfrak{C}$ have no chief points in $c(X)\cdot X_a$. In this case $\mathfrak{U}$ and $\mathfrak{V}$ exist. Complementary to $\mathfrak{U}+\mathfrak{V}$ there are two intervals of $\mathfrak{C}$, the one consisting entirely of elements of $\mathfrak{C}$ whose points all belong to X, the other consisting entirely of elements whose points all belong to X_a and being in fact $\mathfrak{C}_a$.

It is clear that no additional cases can occur, for by supposition X_a is not a subset of X, $c(X)\cdot X_a$ is therefore non-vacuous, and the set of prime ends $\mathfrak{C}_a$ must consequently exist.

II. The Prime Ends of X_a+X+X_r.

1. **Notation and comments.** In the introduction and in the sections following this one X_r is an element of $[X_\alpha]$ selected in a particular way. In this section however it is merely an element of $[X_\alpha]$ different from X_a and not contained in X. The notation employed in connection with it in this section will be used also in later sections where X_r is an element more particularly chosen. In this section and hereafter γ will represent $c_u(X_r+X+X_a)$ and $\mathfrak{C}$ will be the simple closed curve of the prime ends of γ.

2. **Theorem.** *The set* $\mathfrak{C}$ *consists of a non-vacuous subinterval every element of which has a chief point in* $c(X)\cdot X_a$, *a non-vacuous subinterval every element of which has a chief point in* $c(X)\cdot X_r$, *and two segments, possibly reducing to single elements, complementary to these intervals whose nonend elements have no chief points in* $c(X)$ *and whose end elements alone among the elements of* $\mathfrak{C}$ *may contain points of* $c(X)$ *without having chief points in* $c(X)$.

Upon regarding momentarily the set $X+X_r$ as X the existence of the first subinterval required follows easily (I, 7) as it is clear that at least two elements of $\mathfrak{C}$ having no chief points in $c(X+X_r)$ must exist; and it appears also that the elements of this interval are identical with the elements of $\mathfrak{C}_a$ in $c_u(X+X_a)$ since each element of $\mathfrak{C}_a$ may be defined by means of a chain of cuts each one of which belongs to $c_u(X+X_r)$. Similarly upon regarding momentarily $X+X_a$ as X the existence of the other required interval becomes evident; and of its elements it is also seen that they are identical with those of $c_u(X+X_r)$ having chief points in $c(X)\cdot X_r$. The existence of the additional complementary segments, or single elements, is an immediate consequence of the nature of $\mathfrak{C}$ and the fact that the two intervals already identified must be mutually exclusive since all of their elements contain respectively points of either $c(X+X_r)$ or $c(X+X_a)$, but not of both, as chief points (I, 3). Nonend elements of these segments must have all their points in $(X+X_r)\cdot(X+X_a)$, that is in X, while the end elements may have points, but of course no chief points, in $c(X)$ (I, 9), indeed in case either segment reduces to just a single element that element may contain points of both $c(X)\cdot X_a$ and $c(X)\cdot X_r$.

3. **Notation and comments.** The two intervals mentioned above will be represented by $\mathfrak{C}_a$ and $\mathfrak{C}_r$, the two segments related to these by $\mathfrak{C}_u$ and $\mathfrak{C}_v$. It is understood that if $\mathfrak{C}$ contains either of the elements $\mathfrak{U}$ and $\mathfrak{V}$ then $\mathfrak{C}_u$ does not contain $\mathfrak{V}$ and $\mathfrak{C}_v$ does not contain $\mathfrak{U}$, whereas if $\mathfrak{C}$ contains neither $\mathfrak{U}$ nor $\mathfrak{V}$ then neither $\mathfrak{C}_u$ nor $\mathfrak{C}_v$ can exist. Henceforth conventions of separation among the elements of $\mathfrak{C}$ will utilize an arbitrary element of $\mathfrak{C}_r$ as reference element. If $\mathfrak{C}_p$ represents a subcollection of $\mathfrak{C}$ then $\mathfrak{C}_{pg}$ will represent a subcollection of $\mathfrak{C}_p$.

4. **Theorem.** *If* X_b *is an element of* $[X_\alpha]$ *distinct from* X_a *and from* X_r *and not contained by* X, *then the elements of* $\mathfrak{C}$ *which are limits of* $c(X)\cdot X_b$ *are a subset of* $\mathfrak{C}_u+\mathfrak{C}_v$, *and if* $\mathfrak{C}_u$ *contains any of them then* $\mathfrak{C}_v$ *contains none of them and vice versa.*

If $\mathfrak{B}$ is any element of $\mathfrak{C}_a+\mathfrak{C}_r$ then $\mathfrak{B}$ has a chief point b in $c(X)\cdot(X_a+X_r)$. But if $\mathfrak{B}$ is limit of $c(X)\cdot X_b$ then b is also limit of $c(X)\cdot X_b$ and so b is a point of X_b, contradicting the nature of the collection $[X_\alpha]$. Accordingly those elements of $\mathfrak{C}$ which are limits of $c(X)\cdot X_b$ belong to $\mathfrak{C}_u+\mathfrak{C}_v$.

Let $\mathfrak{A}$ be an element of $\mathfrak{C}_a$ containing a point a arcwise accessible from γ through $\mathfrak{A}$ and let $\mathfrak{R}$ be an element of $\mathfrak{C}_r$ containing a point r arcwise accessible from γ through $\mathfrak{R}$. Let R_a and R_b be mutually exclusive rays of space contained except for their ends, respectively the points a and r, in $c(X_a+X_b+X_r+X)$. The continuum $R_a+X_a+X+X_r+R_r$ divides γ into two components γ_u and γ_v, the first containing all the elements of $\mathfrak{C}_u$, the second all those of $\mathfrak{C}_v$. As $c(X)\cdot X_b$ is a connected set without points in common with $R_a+X_a+X+X_r+R_r$, either γ_u contains $c(X)\cdot X_b$ or γ_v contains it. A contradiction would thus arise if it were assumed that some element of $\mathfrak{C}_u$ were limit of $c(X)\cdot X_b$ as well as some one of $\mathfrak{C}_v$.

5. **Theorem.** *If X_c and X_b are distinct elements of $[X_\alpha]$ each of them different from both X_a and X_r and if X_c and X_b do not separate X_a and X_r in $[X_\alpha]$ and $\mathfrak{C}_u$ includes an element which is limit of $c(X)\cdot(X_b+X_c)$ then $\mathfrak{C}_v$ includes no such element, whereas if X_b and X_c separate X_a and X_r in $[X_\alpha]$ and $\mathfrak{C}_u$ contains those elements of $\mathfrak{C}$ which are limits of $c(X)\cdot X_b$ then $\mathfrak{C}_v$ contains those which are limits of $c(X)\cdot X_c$ and vice versa.*

Rays R_a and R_r may be drawn so as to avoid both X_b and X_c and yet still determine the domains γ_u and γ_v above the first of which contains all elements of $\mathfrak{C}_u$ and the second all of $\mathfrak{C}_v$. Now if X_b and X_c do not separate X_a and X_r, it will be possible to construct in $c(X_a+X+X_r)$ an arc with ends belonging respectively to $c(X)\cdot X_b$ and $c(X)\cdot X_c$ but no other points in X_b+X_c which with X_b+X+X_c separates $c_u(X)$ into two domains one of which contains both $c(X)\cdot X_a$ and $c(X)\cdot X_r$, and thus an arc which has in fact no point in R_a+R_r.[7] Such being the case $c(X)\cdot(X_b+X_c)$ must be a subset either of γ_u or of γ_v and thus the first conclusion of the theorem is true. If when X_b and X_c separate X_a and X_r it were possible to draw an exactly similar arc a contradiction of the assumed order conditions would result, hence γ_u contains one of the sets $c(X)\cdot X_b$ and $c(X)\cdot X_c$ while γ_v contains the other. The second conclusion of the theorem now follows easily (4).

6. **Theorem.** *If $[X_i]$ is a series of $[X_\alpha]$ having a limit point in $c(X)\cdot X_a$, $\mathfrak{P}$ is an element of $\mathfrak{C}$ which is for some value of i limit of an element of $[c(X)\cdot X_i]$, and $\mathfrak{Q}$ is another such element of $\mathfrak{C}$, and if $\mathfrak{C}_u$ contains $\mathfrak{P}$ then $\mathfrak{C}_u$ contains $\mathfrak{Q}$; whereas if $\mathfrak{C}_v$ contains $\mathfrak{P}$ then $\mathfrak{C}_v$ contains $\mathfrak{Q}$ also.*

Since X_a and X_r can not separate any two elements of $[X_i]$[8] the proof of the theorem is immediate (4, 5). The theorem would clearly follow in the same way if it were assumed merely that X_a and X_r do not separate any two elements of the series $[X_i]$.

[7] See references to order among $[X_\alpha]$ listed above.

[8] Rutt, loc. cit., p. 813, Corollary II.

7. **Lemma.** *Any point p belonging to the prime end $\mathfrak{P}$ of the interval $\mathfrak{L}$ of the simple closed curve $\mathfrak{C}$ of the prime ends of the domain γ is limit through $\mathfrak{P}$ of points belonging to other elements of $\mathfrak{L}$ and indeed of points arcwise accessible from γ through elements of $\mathfrak{L}$.*

Let $\mathfrak{P}$ be defined by the domains $[\varrho_i]$ separated from γ by $[P_i]$. Let ϱ_j be one of $[\varrho_i]$ such that every element of $\mathfrak{C}$ contained in ϱ_j belongs to $\mathfrak{L}$. Now p, as it is on the boundary of ϱ_j, is limit of points of the boundary which are arcwise accessible from ϱ_j and not contained by P_j. Such points then belong to prime ends of γ contained in ϱ_j, that is to elements of $\mathfrak{L}$.

8. **Remarks and notation.** If p is a point of the element $\mathfrak{P}$ of an arc $\mathfrak{L}$ of the simple closed curve $\mathfrak{C}$ of prime ends, there may be a domain ϱ of the collection $\mathfrak{P}$ and a neighborhood η of p such that $\eta \cdot \varrho$ contains only elements of $\mathfrak{L}$ preceding $\mathfrak{P}$ or only elements of $\mathfrak{L}$ following $\mathfrak{P}$. According as either of these two possibilities or a third one occurs, the points of $\mathfrak{P}$ can be classified as, those which are limits through $\mathfrak{P}$ of points contained in elements of $\mathfrak{L}$ preceding $\mathfrak{P}$ but not through $\mathfrak{P}$ of points in elements of $\mathfrak{L}$ following $\mathfrak{P}$, those which are limits through $\mathfrak{P}$ of points in elements of $\mathfrak{L}$ following $\mathfrak{P}$ but not through $\mathfrak{P}$ of points in elements of $\mathfrak{L}$ preceding $\mathfrak{P}$, and those, including all the chief points of $\mathfrak{P}$, which are limits through $\mathfrak{P}$ both of points in elements of $\mathfrak{L}$ preceding and of points in elements of $\mathfrak{L}$ following $\mathfrak{P}$. When $\mathfrak{P}$ is an element of $\mathfrak{C}_a$ it will be natural to refer to these subsets of $\mathfrak{P}$ respectively as P_u, P_v, and P_c whenever $\mathfrak{U}$ is regarded as preceding $\mathfrak{P}$. Of course P_u and P_v may be vacuous.

9. **Lemma.** *If $\mathfrak{P}$ is an element of the simple closed curve $\mathfrak{C}$ of the prime ends of the domain γ, and $[Q_\alpha]$ is a set of continua contained in γ none of which is contained in ϱ, a certain element of the collection $\mathfrak{P}$, and if $\mathfrak{P}$ is limit of $\sum Q_\alpha$, then the subset of points of $\mathfrak{P}$ made up of all those of its points which are limits of $\sum Q_\alpha$ through $\mathfrak{P}$ is connected and includes all chief points of $\mathfrak{P}$.*

As all but a finite number of the cuts in each chain of cuts defining $\mathfrak{P}$ must contain a point of some element of $[Q_\alpha]$, in order that $\mathfrak{P}$ should be limit of $\sum Q_\alpha$, it is necessary that every chief point of $\mathfrak{P}$ be limit of $\sum Q_\alpha$ through $\mathfrak{P}$.

Consider now p, point but not chief point of $\mathfrak{P}$, which is limit of $\sum Q_\alpha$ through $\mathfrak{P}$. Let $[\varrho_i]$ be a particular chain defining $\mathfrak{P}$. Contained in ϱ_i is a component of $Q_i \cdot \varrho_i$, where Q_i is some element of $[Q_\alpha]$, having a point at a distance from p less than $(\frac{1}{2})^i$. Contained in ϱ_j $(j > i+1)$ is a component of $Q_j \cdot \varrho_j$, where Q_j is an element of $[Q_\alpha]$ different from Q_i, having a point at a distance from p less than $(\frac{1}{2})^j$. The infinite collection of components of $[Q_i \cdot \varrho_i]$ selected when this process is continued indefinitely has no limits in $\mathfrak{P}$ which are not limits through $\mathfrak{P}$, and contains no infinite subcollection

which does not include among its limits the single point of $\mathfrak{P}$ which is limit of the selected chain of cuts defining $\mathfrak{P}$. But p is among the limits of a subcollection of the chosen components of $[Q_i \cdot \varrho_i]$ which has a sequential limiting set[9] necessarily in this case a continuum. Therefore p belongs to a subcontinuum of $\mathfrak{P}$ all whose points are limits of $\sum Q_\alpha$ through $\mathfrak{P}$ and include a chief point of $\mathfrak{P}$. Accordingly, as the set of chief points of $\mathfrak{P}$ is a continuum, the subset of $\mathfrak{P}$ consisting of those points of $\mathfrak{P}$ which are limits of $\sum Q_\alpha$ through $\mathfrak{P}$ is connected and being closed as well is a continuum.

10. **Theorem.** *If the element $\mathfrak{A}$ of $\mathfrak{C}_a$ is limit of $\sum X_i$, a series of $[X_\alpha]$, the subset of points of $\mathfrak{A}$ each of which is limit of $\sum X_i$ through $\mathfrak{A}$ is a continuum including all the chief points of $\mathfrak{A}$.*

If α is an element of the collection $\mathfrak{A}$ of domains, a collection of continua contained in $\alpha \cdot \sum X_i$ among whose limits is every point of $\mathfrak{A}$ which is limit of $\sum X_i$ through $\mathfrak{A}$ may easily be found and then the theorem follows at once (9).

11. **Theorem.** *The subcollection $\mathfrak{M}$ of the prime ends $\mathfrak{C}$ of the domain γ each element of which contains the point p is closed.*

For if $\mathfrak{P}$ is any element of $\mathfrak{C}$ which is limit of a subaggregate of $\mathfrak{M}$ then any domain ϱ of the collection $\mathfrak{P}$ contains one of $\mathfrak{M}$ and thus every domain of the collection $\mathfrak{P}$ has p upon its boundary.

III. Intercepted Sets.

1. **Remarks.** Suppose that Z is bounded and the accessible element X_r exists. Let $\mathfrak{A}$ be an element of $\mathfrak{C}_a$ and suppose it limit of $[X_i]$, a series of $[X_\alpha]$. Then X_a and $[X_i]$ are particularly closely associated with a subset of $c(X)$ now to be identified, which will be called the set intercepted by X_a and $[X_i]$. Suppose that $\mathfrak{A}$ is defined by $[\alpha_i]$, a chain of domains; and from $[\alpha_i]$ and $[X_i]$ select infinite subseries $[\alpha_i']$ and $[X_i']$ which are such that $\alpha_i' \cdot (R + X_r) = 0$, $\alpha_i' \cdot X_i' \neq 0$, and $\alpha_{i+1}' \cdot X_i' = 0$, $(i = 1, 2, 3, \cdots)$. In view of the fact that some of the chief points of $\mathfrak{A}$ belong to $c(X) \cdot X_a$, no element of $[X_\alpha]$ has $\mathfrak{A}$ as limit and so the selection above may be made. Suppose that W_i is an arc cut contained in $\alpha_i' \cdot c(X_i')$ except for its ends which belong to $c(X) \cdot X_i'$ and $c(X) \cdot X_a$. Now $X_i' + W_i + X$ and $X_a + W_i + X$ are continua whose common points consist of precisely two components. Thus their sum contains the boundary of exactly two components of the complement of their sum not already contained entirely in either X or X_a. One of these is unbounded and contains R. It is soon seen that the other is bounded; let it be ω_i.

[9] R. G. Lubben, *Limiting sets in abstract spaces*, Transactions of the American Mathematical Society, vol. 30, No. 4, p. 675.

2. **Definition.** Any point which is contained in an infinite number of $[\omega_i]$ will be said to be *intercepted* by X_a and $[X_i]$. The set of all such points will be called *a set intercepted* by X_a and $[X_i]$.

3. **Remarks.** It will presently be clear that the same set of points is determined no matter what element $\mathfrak{A}$ of $\mathfrak{C}_a$ limit of $[X_i]$, or what chain $[\alpha_i]$ defining $\mathfrak{A}$, or what subseries $[\alpha_i']$ or $[X_i']$ is selected satisfying the conditions stated. It will then be proper to speak of *the* set intercepted by X_a and $[X_i]$. In general intercepted sets are objects of great complexity about which few general statements may be made. But when Z is bounded and closed useful facts may be proved about them as will presently appear. It will be assumed throughout the remainder of this section that Z is bounded and closed. In this and following sections any symbols used in them and also in sections I and II will have the significance that they had in II except X_r which will be supposed selected as in the introduction.

4. **Theorem.** *If S is a set intercepted by X_a and $[X_i]$, and if s is a limit point of S in $c(X+X_a+S)$ or a limit point of $c(X+X_a+S)$ in S, then s is a limit point of $\sum X_i$.*

5. **Theorem.** *A necessary and sufficient condition that a given element X_a not contained in X and a given series $[X_i]$ among whose limits is an element of $\mathfrak{C}_a$ intercept only nonvacuous sets is that there exists an element of $[X_\alpha]$ separating X_a from $[X_i]$ with respect to X_r.*

Suppose that an element X_b of $[X_\alpha]$ occurs between X_a and $[X_i]$. No matter what element of $\mathfrak{C}_a$ included among the limits of $\sum X_i$ is selected as $\mathfrak{A}$ or what chain $[\alpha_i]$ of domains is used to define $\mathfrak{A}$ or what series of cuts is chosen to determine the collection $[\omega_i]$ of domains by means of which an intercepted set in question is identified, since at most a finite number of $[W_i]$ contain any points of X_b all but a finite number of $[\omega_i]$ contain $c(X)\cdot X_b$ (Lemma I, 1). The sufficiency of the condition is consequently demonstrated.

Suppose on the other hand that the intercepted set S is nonvacuous, p being a point of it. Let R_p be a ray of space in $c(X_a+X+X_r+R)$ with end p; R_p contains either a last point of S or a first point of $c(S)$. Suppose that the point q of R_p is the last point of S in R_p. Then q is contained in an infinite subset $[\omega_i']$ of $[\omega_i]$ defining the intercepted set, and being the last such point of R_p must be limit of $\sum \Omega_i$ where Ω_i is the boundary of ω_i for each value of i, and thus indeed must be limit of $\sum X_i$. But Z is closed and $q \subset c(X)$ so q is a point of an element of $[X_\alpha]$. On the other hand if q is the first point of $c(S)\cdot R_p$ the subarc (pq) of R_p contains points of infinitely many of $[\Omega_i]$ some clearly being points of Z. So when nonvacuous S necessarily contains a point of Z that is a point of some element of $[X_\alpha]$. But as no such element can intersect more than

a finite number of the cuts $[W_i]$ used in any determination of S, each such element X_b must except for its points in X be contained in all but a finite number of $[\omega_i]$ and thus separate X_a from $[X_i]$ (Lemma I, 1). That the condition is necessary now follows easily.

6. **Theorem.** *When the element X_a of $[X_\alpha]$ contains in $c(X)$ a limit point of $[X_i]$ a series of $[X_\alpha]$, then every set intercepted by X_a and $[X_i]$ is identical with $\sum c(X)\cdot X_b + \sum c_u(X)\cdot(c(X+X_b) - c_u(X+X_b))$, where the summation extends over all the elements X_b of $[X_\alpha]$ which separate X_a from $[X_i]$.*

It has been proved that if the element X_q of $[X_\alpha]$ is between X_a and $[X_i]$ then S contains $c(X)\cdot X_q$. Since any simply connected domain containing $c(X)\cdot X_q$ must contain a point of any component of $c(X+X_q)$ which has points of $c(X)\cdot X_q$ upon its boundary, each element of $[\omega_i]$ which contains $c(X)\cdot X_q$ contains points of all components of $c(X+X_q)$ which are contained in $c_u(X)$. Therefore it must contain each of them except $c_u(X+X_q)$ entirely, for indeed if the element ω_j of $[\omega_i]$ were to contain only part of the bounded component ϱ of $c(X+X_q)$ while containing $c(X)\cdot X_q$ then ϱ would contain points of the boundary of ω_j, that is ϱ would contain points of X or of X_a+X_j or of W_j. Each of these alternatives is easily seen to be ridiculous for all or for all but a finite number of values of j. Thus S contains the set specified in the theorem.

Now any point of $c_u(X)$ belonging neither to an element of $[X_\alpha]$ nor to a bounded component of $c(X+X_s)$, where X_s is an element of $[X_\alpha]$, belongs to $c_u(Z)$.[10] As any point of $c_u(Z)$ is end of a ray of space in $c(Z)$ no such point can belong to any intercepted set since as has been seen in the proof of (5) any ray in $c_u(X)$ containing a point of an intercepted set must contain also a point of Z.

7. **Remarks.** *The prediction made in paragraph 3 is now fulfilled.* The following theorem repeats it. Hereafter it will be allowable to speak of *the set* intercepted by X_a and $[X_i]$.

8. **Theorem.** *Whenever an element of $\mathfrak{C}_a$ is limit of a series $[X_i]$ of $[X_\alpha]$, the same set of points is determined no matter what element $\mathfrak{A}$ of $\mathfrak{C}_a$ limit of $[X_i]$, or what chain of domains $[\alpha_i]$ defining $\mathfrak{A}$, or what subseries $[\omega_i']$ or $[X_i']$ is selected during the construction of the domains $[\omega_i]$.*

9. **Notation.** Hereafter when it is useful to refer somewhat specificly to the means used in determining a given intercepted set S, $[\omega_i]$ will represent any suitable set of domains determined from an element $\mathfrak{A}$ of $\mathfrak{C}_a$ or from a point of $c(X)\cdot X_a$ by a system of cuts W_i of the type described in (1), whose limiting set is contained in $c(X)$. If $[X_\alpha]\supset X_s$ and $S\cdot X_s \neq 0$, S will be said to contain X_s.

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[10] Rutt, loc. cit., Corollary IV.

10. **Theorem.** *If* $\mathfrak{A}_u$ *and* $\mathfrak{A}_v$ *are two elements of* $\mathfrak{C}_a$ *each of which is limit of* (X_i) *and* $\mathfrak{A}$ *is an element of* $\mathfrak{C}_a$ *separating* $\mathfrak{A}_u$ *from* $\mathfrak{A}_v$ *with respect to* $\mathfrak{C}_r$ *then* $\mathfrak{A}$ *is also limit of* $[X_i]$.

Upon assuming that $\mathfrak{A}$ is not limit of $[X_i]$ a contradiction will be obtained. If $\mathfrak{A}$ is not limit of $[X_i]$ there is an element α of the collection $\mathfrak{A}$ containing no point of $\sum X_i$. Every element of $\mathfrak{C}$ contained in α belongs to $\mathfrak{C}_a$ and separates $\mathfrak{A}_u$ from $\mathfrak{A}_v$. Construct a ray of space in $c(X_a + X + X_r)$ with end in α, and extend it in α as a ray S to a point s of $c(X) \cdot X_a$ accessible from α.

Consider first the possibility that S may be drawn so as to have points in common with at most a finite number of $[X_i]$, and no point in common with R. If this is possible then $R + S$ divides $c_u(X_r + X + X_a)$ into two domains γ_u and γ_v, the first containing all elements of $\mathfrak{C}_u$ and either $\mathfrak{A}_u$ or $\mathfrak{A}_v$, suppose it contains $\mathfrak{A}_u$, and the second containing $\mathfrak{C}_v$ and $\mathfrak{A}_v$. All elements of $\mathfrak{C}$ which are limits of $\sum c(X) \cdot X_i$ are contained wither by $\mathfrak{C}_u$ or by $\mathfrak{C}_v$ so that as $R + S$ has points in common with at most a finite number of $[c(X) \cdot X_i]$ all but a finite number of these are contained entirely by one of γ_u or γ_v. But this implies that either $\mathfrak{A}_u$ or $\mathfrak{A}_v$ is not limit of $\sum X_i$, despite hypothesis.

Thus if $R \cdot S = 0$ then S necessarily intersects infinitely many of $[c(X) \cdot X_i]$. Such being the case either S has a first point t, counting from s, which belongs to $\sum c(X) \cdot X_i$ or a last point t which does not, and in each case t is a point in $c(X)$ of an element X_t of $[X_\alpha]$. The subinterval (st) of S is a cut of $c_u(X_a + X + X_t)$ which it divides into the domains γ_u and γ_v. The ends of (st) evidently belong to different elements of X_α and are in $c(X)$. Now momentarily identify X_t with the X_r of section II. All the conclusions of II apply with the proper adjustments of notation and when $\gamma_u \supset \mathfrak{A}_u$ then $\gamma_v \supset \mathfrak{A}_v$, and as (st) has no points whatever of $\sum c(X) \cdot X_i$ upon it either γ_u contains $\sum c(X) \cdot X_i$ or γ_v contains it. Neither alternative is possible as both $\mathfrak{A}_u$ and $\mathfrak{A}_v$ are limits of $\sum X_i$. Thus the contradiction is general and the theorem proved.

11. **Hypotheses.** In the remaining paragraphs of this section the following hypotheses in addition to those of (3) will be understood in connection with the theorems given. The series $[X_i]$ has in $c(X) \cdot X_a$ a limit point and in $\mathfrak{C}_a$ a limit element and intercepts with X_a a nonvacuous set S. All limit elements in $\mathfrak{C}$ of members of $[c(X) \cdot X_i]$ belong to $\mathfrak{C}_u$.

12. **Theorem.** *If the element* $\mathfrak{A}_v$ *of* $\mathfrak{C}_a$ *is limit of* S *so is every element of* $\mathfrak{C}$ *between* $\mathfrak{A}_v$ *and* $\mathfrak{U}$.

Suppose that $\mathfrak{A}_u$ is an element of $\mathfrak{C}$ between $\mathfrak{A}_v$ and $\mathfrak{C}_u$. Evidently $\mathfrak{C}_a \supset \mathfrak{A}_u$. If $\mathfrak{A}_u$ is not a limit of S there is an element α of the collection $\mathfrak{A}_u$ which contains no point of S. Let $\mathfrak{A}_v$ be defined by the chain of domains

$[\alpha_i]$, this chain and α being clearly determinable so that $\alpha + R + X_r$ has no point in common with any one of α_i. As $\mathfrak{A}_v$ is limit of S it is limit of a subseries $[Y_i]$ of $[X_\alpha]$ whose members are contained in S (6). But then X_a and $[Y_i]$ are situated so that they may intercept a nonvacuous set S_y, and S_y may be identified by means of $\mathfrak{A}_v$, $[\alpha_i]$, and a chain of cuts $[W_i]$ (9), if it is nonvacuous. Now S_y is nonvacuous because each of $[\omega_i]$ contains α; for momentarily regard X as $X + X_r$, think of X_i as X_r, and notice then that elements of $\mathfrak{C}_a$ and $\mathfrak{C}_r$ are limits of the respective ends of W_i so that ω_i contains either $\mathfrak{C}_u$ or $\mathfrak{C}_v$; and finally, returning to the original notation of this section, so that as ω_i can not contain $\mathfrak{C}_r$ it must contain $\mathfrak{A}$ which is separated from $\mathfrak{C}_r$. Thus S_y is nonvacuous and contains α. That is $\mathfrak{A}_u$ is limit of S_y. But $S \supset S_y$ (6), a contradiction. Thus no such domain as α exists and the theorem is proved.

13. **Theorem.** *If the element $\mathfrak{A}_v$ of $\mathfrak{C}_a$ is limit of S and not of $\sum X_i$ every element of $\mathfrak{C}$ between $\mathfrak{A}_v$ and $\mathfrak{U}$ is limit of S and not of $\sum X_i$.*

Let $\mathfrak{A}_u$ be an element of $\mathfrak{C}$ between $\mathfrak{U}$ and $\mathfrak{A}_v$. Now $\mathfrak{A}_u$ is a limit of S. If it were limit of $\sum X_i$ then $\mathfrak{A}_v$ would be also (10). Hence the theorem.

14. **Theorem.** *If the element $\mathfrak{A}_v$ of $\mathfrak{C}_a$ is limit of $\sum X_i$, any element between $\mathfrak{A}_v$ and $\mathfrak{C}_u$ not limit of $\sum X_i$ is limit of S.*

If $\mathfrak{A}_u$ is an element of $\mathfrak{C}$ between $\mathfrak{A}_v$ and $\mathfrak{U}$, upon determining S (9) by means of $\mathfrak{A}_v$ it is seen at once that if $\mathfrak{A}_u$ is not limit of $\sum X_i$ an element in the collection $\mathfrak{A}_u$ is contained in each of an infinite number of $[\omega_i]$. Thus $\mathfrak{A}_u$ is limit of S, as was to be proved.

15. **Summary, notation, and comments.** With the exceptions mentioned below:

(a) The subset $\mathfrak{C}_{ax}$ of $\mathfrak{C}_a$ consisting of all elements which are limits of $\sum X_i$ is an interval of $\mathfrak{C}$.

(b) The subset $\mathfrak{C}_{as}$ of $\mathfrak{C}_a$ consisting of all elements which are limits of S is an interval of $\mathfrak{C}$.

(c) The subset $\mathfrak{C}_{a\overline{x}}$ of $\mathfrak{C}_a$ consisting of all elements which are limits of $\sum X_i$ but not of S is an interval of $\mathfrak{C}$.

(d) The subset $\mathfrak{C}_{a\overline{s}}$ of $\mathfrak{C}_a$ consisting of all elements which are limits of S but not of $\sum X_i$ is an interval of $\mathfrak{C}$.

The exceptions to the items of the summary are these:

(e) If $\mathfrak{A}$ is an end of the interval in (a) which belongs to the interval $\mathfrak{C}_a$ then $\mathfrak{A}$ belongs to $\mathfrak{C}_{ax}$.

(f) If $\mathfrak{A}$ is an end of the interval in (b) which belongs to the interval $\mathfrak{C}_a$ then $\mathfrak{A}$ belongs to $\mathfrak{C}_{as}$.

(g) If $\mathfrak{A}$ is an end of the interval in (c) which belongs to the interval $\mathfrak{C}_a$ and is separated from $\mathfrak{C}_{as}$ by elements of $\mathfrak{C}$ then $\mathfrak{A}$ belongs to $\mathfrak{C}_{a\overline{x}}$.

Item (a) follows from (10); (b) from (12); (d) from (13); (c) from (a), (b), and (14); (e) from (II,7) and (II,8); (f) from (II,7), (II,8), and (6); (g) from (II,7), (II,8), and the fact that $\mathfrak{A}$ under the circumstances can not belong to $\mathfrak{C}_{as}$.

It is to be observed that:

(h) $\mathfrak{C}_{ax} \supset \mathfrak{C}_{\overline{ax}}$ and $\mathfrak{C}_{as} \supset \mathfrak{C}_{\overline{as}}$.

(i) $\mathfrak{C}_a - \mathfrak{C}_{ax}$ and $\mathfrak{C}_a - \mathfrak{C}_{\overline{ax}}$ may be disconnected but that $\mathfrak{C}_a - \mathfrak{C}_{as}$ and $\mathfrak{C}_a - \mathfrak{C}_{\overline{as}}$ can not be disconnected.

(j) $\mathfrak{C}_{ax} \cdot \mathfrak{C}_{as} \neq 0$, $\mathfrak{C}_{ax} \cdot \mathfrak{C}_{\overline{as}} = 0$, $\mathfrak{C}_{\overline{ax}} \cdot \mathfrak{C}_{as} = 0$, and $\mathfrak{C}_{\overline{ax}} \cdot \mathfrak{C}_{\overline{as}} = 0$.

It is supposed throughout the summary above that S is nonvacuous. If this part of the hypotheses be eliminated, all of the statements there remain true, some being vacuously satisfied, except the inequality in (*j*).

IV. Principal Theorems.

1. **Definitions.** If $[X_i]$ and $[Y_i]$ are two series of elements of $[X_\alpha]$ such that given any subscript i_1 there exist subscripts i_2 and i_3, $i_1 < i_2 < i_3$, for which X_{i_2} separates Y_{i_1} from Y_{i_3} and Y_{i_2} separates X_{i_1} from X_{i_3} then $[X_i]$ and $[Y_i]$ are called *concurrent series.*

2. **Remarks.** Throughout this section it is assumed that Z is closed and bounded. Whenever $[X_i]$ and $[Y_i]$ are random series of $[X_\alpha]$ it will be assumed that such elements of $\mathfrak{C}$ as are limits of members of $[c(X) \cdot X_i]$ or $[c(X) \cdot Y_i]$ are contained by $\mathfrak{C}_u$ (II, 6). If $\sum X_i$ or $\sum Y_i$ have limit points in X_a the resulting intercepted sets will be designated by S_x and S_y respectively.

3. **Theorem I.** *If* $[X_i]$ *and* $[Y_i]$ *are two series of* $[X_\alpha]$ *which intercept with respect to* X_a *the same nonvacuous intercepted set then the series are concurrent, while if the series are concurrent and each one intercepts with respect to* X_a *a nonvacuous set then the intercepted sets are identical.*

When $[X_i]$ and $[Y_i]$ intercept the same nonvacuous set S, there must be an element X_s of $[X_\alpha]$ not contained in X such that $S \supset c(X) \cdot X_s$ (III, 5). Consequently X_a can not separate $[X_i]$ from $[Y_i]$ for if it did X_a would separate either X_s from $[X_i]$ or X_s from $[Y_i]$ and either S_x or S_y would contain X_a, a contradiction (III, 6). Suppose then that all elements of $\mathfrak{C}$ which are limits either of members of $[c(X) \cdot X_i]$ or of $[c(X) \cdot Y_i]$ are in $\mathfrak{C}_u$ (II, 5; II, 6). Given then any corresponding pair of elements X_{i_1} and Y_{i_1} of $[X_i]$ and $[Y_i]$ there must exist an element X_m of $[X_i]$ and an element Y_n of $[Y_i]$, $m > i_1$, $n > i_1$, such that X_m separates X_a from Y_{i_1} and Y_n separates X_a from X_{i_1}, for if X_m should fail to exist then S_x would contain Y_{i_1} (III, 6) although S_y does not and $S_x = S_y$, and if Y_n should fail to exist then S_y would contain X_{i_1}, a contradiction in either case. But the existence of both X_m and Y_n demonstrates the existence of a subscript i_2,

$i_1 < i_2$, such that X_{i_2} separates Y_{i_1} from X_a and Y_{i_2} separates X_{i_1} from X_a. The existence of a subscript i_3 having the required qualities follows in the same way.

When $[X_i]$ and $[Y_i]$ are concurrent series of $[X_\alpha]$ intercepting nonvacuous sets with respect to X_a, that $S_x = S_y$ appears at once (III, 6).

4. **Theorem II.** *If $[X_i]$ and $[Y_i]$ are concurrent series of $[X_\alpha]$ and if $\mathfrak{A}$ is an element of $\mathfrak{C}_a$ limit of the nonvacuous set S intercepted with respect to X_a by both $[X_i]$ and $[Y_i]$ but not limit of $\sum X_i$, then it is not limit of $\sum Y_i$; and if s is a point of S which is limit of $\sum X_i$ then it is also limit of $\sum Y_i$; and in both cases vice versa.*

In case $\mathfrak{A}$ is not limit of $\sum X_i$ there is an element α of the collection $\mathfrak{A}$ which contains no points of $\sum X_i$ so, since $\alpha \cdot S \neq 0$, $S_x \supset \alpha$. Now if $\mathfrak{A}$ were limit of $\sum Y_i$ then α would contain points of $\sum Y_i$, and so S_x would contain an element of $[Y_i]$. This contradicts the supposition that $S_x = S_y = S$.

On the other hand if s were limit of $\sum X_i$ but not of $\sum Y_i$ there would be a neighborhood η of s excluding the set $X_a + X + \sum Y_i$. As s is a point of S it is a point of S_y and η contains points of the boundaries of at most a finite number of the domains $[\omega_i]$ defining S_y (III, 9) so $\eta \subset S_y$. But η contains points of $\sum X_i$ so it would appear that S_y must contain some element of $[X_i]$, a contradiction of the fact that $S_x = S_y = S$.

5. **Theorem III.** *If $[X_i]$ and $[Y_i]$ are concurrent series of elements of $[X_\alpha]$ having in $\mathfrak{C}_a$ and $c(X) \cdot X_a$ the nonvacuous limiting sets of prime ends and points $\mathfrak{C}_{ax}$ and L_x, $\mathfrak{C}_{ay}$ and L_y respectively, then*

(1) *either $\mathfrak{C}_{ax} \supset \mathfrak{C}_{ay}$ or $\mathfrak{C}_{ay} \supset \mathfrak{C}_{ax}$ and*

(2) *either $L_x \supset L_y$ or $L_y \supset L_x$.*

Proof of (1). Suppose that $\mathfrak{C}_{ay} \not\subset \mathfrak{C}_{ax}$ and that $\mathfrak{A}$ is an element of $\mathfrak{C}_{ay}$ not included in $\mathfrak{C}_{ax}$. If $\mathfrak{B}$ were an element of $\mathfrak{C}_{ax}$ between $\mathfrak{A}$ and $\mathfrak{B}$ then (2; III, 14) $\mathfrak{U}$ would be an element of $\mathfrak{C}_{as}$ (III, 15, (d)) and this is a contradiction (4). So if $\mathfrak{B}$ is an element of $\mathfrak{C}_{ax}$ it separates $\mathfrak{A}$ and $\mathfrak{U}$. Now any such element if not limit of $\sum Y_i$ is limit of S (III, 14) and is not limit of $\sum X_i$ (4), so $\mathfrak{B}$ if not limit of $\sum Y_i$ can not belong to $\mathfrak{C}_{ax}$. The assertion is consequently proved.

The proof of (2) is difficult to achieve by means of prime ends so the forthcoming less complex method is to be preferred. Assume the theorem untrue, assume, that is, that $c(X) \cdot X_a$ contains points p_x and p_y, the first of which belongs to L_x but not to L_y, the second to L_y but not to L_x. Let C_x and C_y be circles with centers respectively P_x and P_y and radii selected in such a way that $ce(C_x) \cdot (\sum Y_i + X + X_r + R) = 0$ (II, 1; III, 1) and $ce(C_y) \cdot (\sum X_i + X + X_r + R) = 0$. It is clear that infinite subseries of the original given series may be selected so that the three first elements in each series occur alternately in $[X_\alpha]$ and contain points of either $ce(C_x)$

or $ce(C_y)$, and so as it is intended to obtain a contradiction it may as well be assumed that the series $[X_i]$ and $[Y_i]$ as first given possessed these qualifications also. To be precise let Y_1 separate X_1 and X_2, etc.

Accordingly among the components of $c(X+X_1+X_2+X_3+ce(C_x))$ are ϱ_r, ϱ_1, and ϱ_2 of which $\varrho_r \supset R+c(X)\cdot X_r$, $\varrho_1 \supset c(X)\cdot Y_1$, and $\varrho_2 \supset c(X)\cdot Y_2$. If $\varrho_r \cdot \varrho_1 \neq 0$ there is a ray of space R_1, ϱ_r is unbounded, in $\varrho_r+\varrho_1$ with its end point in $c(X)\cdot Y_1$ and no point in $R+X_r$. So owing to the order relations in $[X_i]$ and $[Y_i]$ the connected set $c(X)\cdot(X_1+X_2+ce(C_x))$ is disconnected by the set $R+X_r+X+Y_1+R_1$ with which it has no common points (Lemma, I, 1). The contradiction shows that $\varrho_r \cdot \varrho_1 = 0$ and a similar one shows that $\varrho_r \cdot \varrho_2 = 0$.

If $\varrho_1 \cdot \varrho_2 \neq 0$ then $\varrho_1 = \varrho_2$ and ϱ_1 contains $c(X)\cdot(Y_1+Y_2)$. Thus in ϱ_1 there is an arc K irreducibly connecting a point of Y_1 with one of Y_2, and this arc as a cut of $c(X+Y_1+Y_2)$ separates it into two connected domains, σ_b bounded, and σ_u unbounded. Since $\varrho_1 \supset K$ each of these domains has points in ϱ_1, and σ_u has points in ϱ_r for it contains R. As $\varrho_r \cdot (X+\varrho_1) = 0$, $\varrho_r \cdot (X+Y_1+Y_2+K) = 0$, and so ϱ_r being a connected set containing a point of σ_u and no point of its boundary must be entirely contained in σ_u. Also as ϱ_r has upon its boundary points of $ce(C_x)$, $ce(C_x)+c(X)\cdot(X_1+X_2+X_3) \subset \sigma_u$. Now $\varrho_1 \supset K$, so $\varrho_1 \cdot \sigma_b \neq 0$ and if $\varrho_1 \not\supset \sigma_b$ then $\sigma_b \cdot c(\varrho_1) \neq 0$. As ϱ_1 is a connected set it would then follow that $\sigma_b \cdot (X+X_1+X_2+X_3+ce(C_x)) \neq 0$, a contradiction because $c(X) \supset \sigma_b$ and $\sigma_u \supset ce(C_x)+c(X)\cdot(X_1+X_2+X_3)$, accordingly $\varrho_1 \supset \sigma_b$. But this is a contradiction for $\sigma_b \cdot (R+X_r+X_2) = 0$ and yet X_r and X_2 separate Y_1 and Y_2; that is a situation like the one dismissed as impossible in discussing the supposition $\varrho_r \cdot \varrho_1 \neq 0$ has arisen, so $\varrho_1 \cdot \varrho_2 = 0$.

But $\varrho_1 \supset c(X)\cdot Y_1$ and $\varrho_2 \supset c(X)\cdot Y_2$ so $\varrho_1 \cdot ce(C_y) \neq 0$ and $\varrho_2 \cdot ce(C_y) \neq 0$. As $ce(C_y)\cdot(X+X_1+X_2+X_3+ce(C_x)) = 0$, $\varrho_1 \cdot ce(C_y) \neq 0$ implies $\varrho_1 \supset ce(C_y)$ and $\varrho_2 \cdot ce(C_y) \neq 0$ implies $\varrho_2 \supset ce(C_y)$. Thus $\varrho_1 \cdot \varrho_2 \supset ce(C_y)$, a contradiction of the fact that $\varrho_1 \cdot \varrho_2 = 0$.

So assuming that part (2) is untrue has resulted in an absurdity.

6. **Theorem IV.** *If* $[X_i]$ *and* $[Y_i]$ *are concurrent series of* $[X_\alpha]$ *having in* $\mathfrak{C}_a$ *and* $c(X)\cdot X_a$ *the nonvacuous limiting sets of prime ends and points* $\mathfrak{C}_{ax}$ *and* L_x, $\mathfrak{C}_{ay}$ *and* L_y *respectively, and if* $\mathfrak{C}_{ax} \supset \mathfrak{C}_{ay}$ *but* $L_x \not\supset L_y$ *then* $\mathfrak{C}_{ay} \supset \mathfrak{C}_{ax}$, *and there is an element* $\mathfrak{W}$ *separating all other elements of* $\mathfrak{C}_{ay}$ *from* $\mathfrak{B}$.

When $\mathfrak{C}_{ax} \supset \mathfrak{C}_{ay}$ any point of an element of $\mathfrak{C}_{ay}$ which is not an end element of $\mathfrak{C}_{ax}$ must belong to L_x (II, 7). Thus if $\mathfrak{C}_{ax} \supset \mathfrak{C}_{ay}$ but $L_x \not\supset L_y$ then $L_y - L_x$ must contain a point belonging to a common end element of $\mathfrak{C}_{ax}$ and $\mathfrak{C}_{ay}$. Let $\mathfrak{B}$ be a nonend element of $\mathfrak{C}_{ax}$ if there is such an element, and if there is none the theorem must be true as then $\mathfrak{C}_{ax}$ and

$\mathfrak{C}_{ay}$ consist each of a single element. Let $\mathfrak{A}$ be an end element of $\mathfrak{C}_{ax}$. It will be shown that $\mathfrak{A}$ contains points limit through it of $\sum Y_i$ but not of $\sum X_i$ only if $\mathfrak{A}$ separates $\mathfrak{B}$ from $\mathfrak{B}$ (2).

Suppose that $\mathfrak{A}$ separates $\mathfrak{B}$ from $\mathfrak{U}$, and that z is any point of $\mathfrak{A}$. If $\mathfrak{A}$ is not an element of $\mathfrak{C}_a$ then $\mathfrak{A} = \mathfrak{U}$ (II, 3). As $\mathfrak{A}$ is thus the common end of $\mathfrak{C}_u$ and $\mathfrak{C}_{ax}$ the point z is limit either of points of X contained in elements of $\mathfrak{C}_u$ or of points of $c(X) \cdot X_a$ contained in elements of $\mathfrak{C}_a$ not separated from $\mathfrak{B}$ by $\mathfrak{B}$ (II, 7). But z is in $c(X)$ and so can not be limit of a subset of X, and all points in elements of $\mathfrak{C}_a$ between $\mathfrak{U}$ and $\mathfrak{B}$ are points of L_x (II, 7; II, 10). So if $\mathfrak{A} = \mathfrak{U}$ no point of $\mathfrak{A}$ belongs to $L_y - L_x$. On the other hand if $\mathfrak{A}$ is not $\mathfrak{U}$ then all elements of $\mathfrak{C}$ between $\mathfrak{A}$ and $\mathfrak{U}$ belong to $\mathfrak{C}_{\overline{as}}$ (III, 15, (*d*)). As z is a point of either A_u, A_c, or A_v (II, 8), if it does not belong to L_x it must belong to A_u (II, 7; II, 10). Suppose that z is not limit of $\sum X_i$ through $\mathfrak{A}$. Then there is a neighborhood η of z and an element α of $\mathfrak{A}$ such that $\eta \cdot \alpha$ is a set of components $[\eta_\alpha]$ none of which contains a point of $X + \sum X_i$ and none of which contains an element of $\mathfrak{C}_a$ between $\mathfrak{B}$ and $\mathfrak{B}$. But under these circumstances if any of $[\eta_\alpha]$ contain no point of S between $\mathfrak{U}$ and $\mathfrak{A}$ there would be an element of $\mathfrak{C}_a$ not belonging to $\mathfrak{C}_{as}$ and yet separating an element of $\mathfrak{C}_{as}$, namely $\mathfrak{A}$, from $\mathfrak{U}$, a contradiction (III, 15, (*b*); III, 15, (*i*)). So if η_a is any one of $[\eta_\alpha]$ then $S \cdot \eta_a \neq 0$ and as $S_x = S$ so $S_x \cdot \eta_a \neq 0$ and finally $S_x \supset \eta_a$ since $\sum X_i \cdot \eta_a = 0$. If therefore z were limit of $\sum Y_i$ through $\mathfrak{A}$, some element η_a of $[\eta_\alpha]$ would contain a point of some element of $[Y_i]$ and thus S_x would contain an element of $[Y_i]$. As this would contradict hypotheses no point of $\mathfrak{A}$ can be limit through $\mathfrak{A}$ of $\sum Y_i$ without being limit also through $\mathfrak{A}$ of $\sum X_i$. Thus if z is limit through $\mathfrak{A}$ of $\sum Y_i$ but not of $\sum X_i$, $\mathfrak{A}$ can not separate $\mathfrak{B}$ from $\mathfrak{U}$ and must consequently separate $\mathfrak{B}$ from $\mathfrak{B}$.

Such being the case the end elements of $\mathfrak{C}_{ax}$ and $\mathfrak{C}_{ay}$ must be identical and so $\mathfrak{C}_{ay} \supset \mathfrak{C}_{ax}$, as was to be proved.

7. **Remarks.** In Theorems III and IV it was assumed that all four of the sets $\mathfrak{C}_{ax}$, L_x, $\mathfrak{C}_{ay}$, and L_y were nonvacuous. Most of the interesting possibilities are included under these hypotheses. When one or both of $\mathfrak{C}_{ax}$ and $\mathfrak{C}_{ay}$ are vacuous the only part of the theorems not vacuously satisfied is the second part of III. But the argument employed to demonstrate this fact originally still applies without modification. When either L_x or L_y is vacuous both theorems are vacuously satisfied.

8. **Theorem V.** *If* $Z_x = X + X_r + X_a + \sum X_i$ *and* $Z_y = X + X_r + X_a = \sum Y_i$ *where* Z_x *and* Z_y *are closed bounded sets of type Z and* $[X_i]$ *and* $[Y_i]$ *are series of elements whose members are of the same sort as those of* $[X_\alpha]$ *having respectively in* $c(X) \cdot X_a$ *limit sets of points* L_x *and* L_y *and in* $\mathfrak{C}_a$ *limit sets*

of elements $\mathfrak{C}_{ax}$ *and* $\mathfrak{C}_{ay}$, *and if* $\mathfrak{C}_u$ *contains all elements of* $\mathfrak{C}$ *which are limits of members of* $[c(X)\cdot(X_i+Y_i)]$; *then*

(1) *either* $\mathfrak{C}_{ax}\supset\mathfrak{C}_{ay}$ *or* $\mathfrak{C}_{ay}\supset\mathfrak{C}_{ax}$, *and*

(2) *either* $L_x\supset L_y$ *or* $L_y\supset L_x$, *and*

(3) *if* $\mathfrak{C}_{ax}\supset\mathfrak{C}_{ay}$ *and* $L_x\not\supset L_y$ *then* $\mathfrak{C}_{ay}\supset\mathfrak{C}_{ax}$.

This theorem will be needed later. It resembles closely Theorems III and IV combined but is not a special case of them, for elements of $[c(X)\cdot X_i]$ may have points in common with those of $[c(X)\cdot Y_i]$ and the series $[X_i]$ and $[Y_i]$ are in general not concurrent. As the statement of the theorem definitely excludes the possibility X_a may with either $[X_i]$ or $[Y_i]$ intercept a nonvacuous set generalizations are clearly possible which have been omitted here only because the requirements of the paper do not include them. A condensed proof follows.

Since $[X_i]$ and $[Y_i]$ can not intercept nonvacuous sets with respect to X_a, part (1) of the theorem follows from arguments similar to but simpler than those used in proving part (1) of Theorem III. Also after part (2) has been established (3) follows from reasoning like that in Theorem IV. In proving (2) proceed thus. Suppose that $c(L_y)\cdot L_x$ contains a point x and $c(L_x)\cdot L_y$ contains a point y. Let M and $[M_i]$, N and $[N_i]$ $(i=1,2,3,\cdots)$ be circles with centers respectively x and y, and radii r and $[r_i]$ such that $ce(M)\cdot(R+X_r+X+L_y+\sum Y_i+ce(N_1))=0$, and so is $ce(N)\cdot(R+X_r+X+L_x+\sum X_i+ce(M_1))$, and that $r_1>r>0$ while $r_i=r+(\frac{1}{2})^{i-1}(r_1-r)$. Let X_i^m and Y_i^m $(i=1,2,3,\cdots)$ be subcontinua respectively of X_i and Y_i selected so that $X_i^m\cdot X\neq 0$, $Y_i^n\cdot X\neq 0$, $c(X)\cdot X_i^m$ is a component of $c(X)\cdot ci(M_i)\cdot X_i$, and $c(X)\cdot Y_i^n$ is a component of $c(X)\cdot ci(N_i)\cdot Y_i$. Let $X_b=X_a+M+N$.

Consider the prime ends, a simple closed curve $\mathfrak{G}$, of $\delta=c_u(X_r+X+X_b)$. These are identical with $\mathfrak{C}$ upon $\mathfrak{C}_u$, $\mathfrak{C}_r$, and $\mathfrak{C}_v$ but not upon $\mathfrak{C}_a$. Upon $\mathfrak{C}_b$ are subaggregates $\mathfrak{G}_m$ and $\mathfrak{G}_n$ consisting respectively of those elements of $\mathfrak{G}$ whose points belong to $c(X_a)\cdot M$ and $c(X_a)\cdot N$. Both of these are nonvacuous sets as there are in δ sets which are continua with respect to δ, contained for instance in elements of $[X_i]$ and $[Y_i]$, and which join points of $c(X_a)\cdot M$ or of $c(X_a)\cdot N$ to X. Now no two elements of $\mathfrak{G}_m$ separate two of $\mathfrak{G}_n$ for otherwise an easy contradiction would result upon extending in δ from two representative separating pairs mutually exclusive rays of the plane four in number forming in pairs with the circles associated continua which separate each other and yet have no common points. Thus either $\mathfrak{G}_m$ and $\mathfrak{G}_n$ occupy minimal segments $\mathfrak{F}_m$ and $\mathfrak{F}_n$ of $\mathfrak{U}+\mathfrak{C}_b+\mathfrak{V}$ mutually exclusive except for one common end element, or the minimal subsegment, say $\mathfrak{F}_m$, of $\mathfrak{C}_b$ containing $\mathfrak{G}_m$ is separated from $\mathfrak{C}_r$ by two minimal subsegments $\mathfrak{F}_{nu}$ and $\mathfrak{F}_{nv}$ together containing $\mathfrak{G}_n$ and having in

common with $\mathfrak{F}_m$ at most its end points. Now it is clear from the construction that $\mathfrak{F}_m$ contains a limit element of $\sum X_i^m$ whereas $\mathfrak{G}_n$ does not, and that $\mathfrak{F}_n$ contains a limit element of $\sum Y_i^n$ while $\mathfrak{G}_m$ does not. Also for both $[X_i^m]$ and $[Y_i^n]$ the set intercepted with X_b is vacuous, so that their limit quantities in $c(X) \cdot X_b$ do not include sets $\mathfrak{C}_{bs}$ (III, 15). But in case $\mathfrak{F}_m$ and $\mathfrak{F}_n$ exist either $\mathfrak{C}_{bx} \supset \mathfrak{G}_n$ or $\mathfrak{C}_{by} \supset \mathfrak{G}_m$, a contradiction; while in case $\mathfrak{F}_n$ does not exist then $\mathfrak{C}_{bx} \supset \mathfrak{G}_n \cdot \mathfrak{F}_{nu}$, a contradiction. Other possibilities may be dismissed in the same way after a mere change of notation. Thus supposition of the existence of both x and y involves an absurdity and so (2) is true.

9. **Theorem VI.** *Suppose* (a) *that* $[X_i]$ *is a series of* $[X_\alpha]$ *determining with respect to* X_a *the intercepted set* S *containing the nonvacuous subset* $[X_\alpha]$ *of* $[X_\alpha]$, (b) *that* H_i *and* K_i *are connected subsets of* $c(X) \cdot X_i$ $(i = 1, 2, 3, \cdots)$ *whose frontiers are nonvacuous sets entirely contained by* X, *and* (c) *that in connection with* X_a *the limit quantities of* $[H_i]$ *and* $[K_i]$ *are* L_h, L_k, $\mathfrak{C}_{ah}$, *and* $\mathfrak{C}_{ak}$; *then*

(1) *either* $\mathfrak{C}_{ah} \supset \mathfrak{C}_{ak}$ *or* $\mathfrak{C}_{ak} \supset \mathfrak{C}_{ah}$, *and*

(2) *either* $L_h \supset L_k$ *or* $L_k \supset L_h$, *and*

(3) *if* $\mathfrak{C}_{ah} \supset \mathfrak{C}_{ak}$ *but* $L_h \not\supset L_k$ *then* $\mathfrak{C}_{ak} \supset \mathfrak{C}_{ah}$.

A subset W of Z will first be identified.

(a′) If X_a separates from $[X_i]$ no series $[X_i]_b$ of $[X_\alpha]$ among whose limit points is a point in $c(X)$ of an element X_b of $[X_\alpha]$ separated from $[X_i]_b$ by all the elements of $[X_\sigma]$, then W will be the sum of $X + X_r + X_a$, all of the elements of $[X_\sigma]$, and all the elements of $[X_\alpha]$ separated from $[X_i]$ by X_a.

(b′) If X_a separates from $[X_i]$ a set of series $[[X_i]_\beta]$ of $[X_\alpha]$ each of which has a limit point x_β in $c(X)$ in one of $[X_\beta]$ a subset of $[X_\alpha]$ whose members are separated by all the elements of $[X_\sigma]$ from the specific series of $[[X_i]_\beta]$ whose limit contains x_β, then W will be determined as follows. From the character of the space in question it is clear that $[X_\alpha]$ includes a countable subset $[Y_i]$ having the property that any two elements of $[X_\alpha]$ are separated by one of $[Y_i]$. Into the subclass $[Y_j]$ of $[Y_i]$ put all elements of $[Y_i]$, exclusive of X_r, which are separated from $[X_i]$ by X_a but do not separate all elements of all series of $[[X_i]_\beta]$ from $[X_i]$. In this case W will be the sum of $X + X_r + X_a$, all elements of $[X_\sigma]$, and all elements of $[X_\mu]$, whose members belong to $[X_\alpha]$ and separate X_a from all of $[Y_j]$.

The set W is closed. For suppose that w is a limit point of W in its complement. The point w belongs to an element X_w of $[X_\alpha]$, and is limit of a series $[X_j]$ of $[X_\alpha]$ contained in W. Either X_w is separated from X_a

by $[X_\sigma]$ or is separated from $[X_\sigma]$ by X_a. Suppose that X_w is separated from X_a by $[X_\sigma]$ and consider the possible positions for $[X_j]$. If $[X_\sigma]$ includes infinitely many of $[X_j]$ then w is limit of S, therefore of $\sum X_i$ (III, 3), and this is impossible as X_w must be separated from X_a by infinitely many of $[X_i]$.[11] Hence all but a finite number of $[X_j]$ belong to $[X_\mu]$. This means, however, that $[X_j]$ is a series all elements of which are separated from $[X_i]$ by X_a having a limit point w in an element X_w separated from it by all the elements of $[X_\sigma]$. In short $[X_j]$ is one of $[[X_i]_\beta]$, a ridiculous situation by virtue of the selection of $[X_\mu]$. Suppose then that X_a separates X_w and $[X_\sigma]$. In (a′) W would then contain X_w so only (b′) remains to be discussed. But in (b′) it is clear that there can be no element of $[Y_j]$ which separates X_a from all other elements of $[Y_j]$, and no element of $[X_\alpha]$ which separates all elements of $[Y_j]$ from all of $[X_\mu]$. Thus X_w must be separated from $[X_\sigma]$ and from the series $[X_j]$ by some elements of $[Y_j]$ and indeed by all but a finite number of the elements of some series $[X_i]_b$ of $[[X_i]_\beta]$. Thus the set S_w intercepted by X_w and $[X_j]$ contains all but a finite number of $[X_i]_b$. As the point x_b in $c(X)\cdot X_b$ is limit of $[X_i]_b$ with X_b, member of $[X_\beta]$, separated from $[X_i]_b$ by $[X_j]$, x_b must be limit of $[X_j]$ (III, 3). This results in the same sort of contradiction as above.

If $[T_i]$ is any series whose elements are mutually exclusive and selected from among those of $[H_i]$ and $[K_i]$ then $W+\sum T_i$ is a bounded continuum Z_t of type Z. That it is bounded and of type Z follows from the selection of $[H_i]$ and $[K_i]$ and the fact that Z is bounded. Since any point of $c(X)$ which is limit of $\sum(H_i+K_i)$ is also limit of $\sum X_i$, a fact easily seen, proving $W+\sum X_i$ closed will prove Z_t closed. The proof that $W+\sum X_i$ is closed involves no points not treated in detail during the proof that W is closed.

It is intended to finish the argument by means of Theorem IV but as yet the relation of $[H_i]$ to $[K_i]$ is not suited to its application. Subcollections $[H_j]$ and $[K_j]$ of these to which it may be applied must be selected. It is to be observed that the set $c(W)+c(X)\cdot X_a$ is locally compact and separable and so has the countably distributive property.[12] But this implies that $[H_i]$ contains a convergent subcollection and indeed, owing to the special characteristics of $[H_i]$, a collection converging in $c(W)+c(X)\cdot X_a$ to the set L_h. It is assumed in the next paragraph that $\mathfrak{C}_u$ contains all elements of $\mathfrak{C}$ which are limits of the members of $[c(X)\cdot\sum X_i]$.

Given any element Ω of $\mathfrak{C}_{ah}$ it is clear that there exists a subseries $[H_i]_q$ of $[H_i]$ every infinite subseries of which has Ω among its limits in $\mathfrak{C}_a$,

[11] Rutt, loc. cit., p. 813, Corollary II.

[12] Lubben, loc. cit.

and such a set may be chosen by means of any chain defining $\mathfrak{Q}$. It is to be observed first that from $[H_i]$ may be selected a series $[H_i]_c$ every infinite subseries of which has in $\mathfrak{C}_a$ a limiting subcollection identical to $\mathfrak{C}_{ah}$. If $\mathfrak{C}_{ah}$ has an element $\mathfrak{W}$ (III, 15, (e)) separating no other element of $\mathfrak{C}_{ah}$ from $\mathfrak{U}$, consider any series $[H_i]_w$. Suppose $\mathfrak{Q}$ is an element of $\mathfrak{C}_{ah}$ different from $\mathfrak{W}$. If $\mathfrak{C}_{aw} \supset \mathfrak{Q}$, then $[H_i]_q$ can be selected so as to have no elements whatever in common with $[H_i]_w$, so that $[H_i]_w$ and $[H_i]_q$ are series with contradictory properties (4). Thus $\mathfrak{C}_{aw} = \mathfrak{C}_{ah}$ and $[H_i]_w$ is in this case the set $[H_i]_c$ required. If there is no such element as $\mathfrak{W}$ then no element of $\mathfrak{C}_a$ separates all of $\mathfrak{C}_{ah}$ from $\mathfrak{V}$. In this case it is clear that a subseries $[H_i]_v$ of $[H_i]$ exists with $\mathfrak{V}$ as limit element. Let $\mathfrak{Q}$ be an element of $\mathfrak{C}_{ah}$. Owing to the existence of $[H_i]_q$ if $\mathfrak{Q}$ were not limit of $[H_i]_v$ then from elementary considerations it would follow that $\mathfrak{V}$ was not either. Thus $\mathfrak{C}_{av} = \mathfrak{C}_{ah}$ and in this case $[H_i]_v$ will serve as the required set $[H_i]_c$. Suppose that in $c(X) \cdot X_a$ the set of limit points of $[H_i]_c$ is L_c.

If $\mathfrak{C}_{ah}$ has no last element $\mathfrak{W}$ then from the fact that $\mathfrak{C}_{av} = \mathfrak{C}_{ah}$ follows the fact $L_c = L_h$ (II, 7; II, 10; 6). Whereas if $\mathfrak{W}$ exists and $L_c \neq L_h$ then L_h has points in $\mathfrak{W}$ not contained in any other element of $\mathfrak{C}_{ah}$ (II, 7; II, 10). But in the latter case any subseries of $[H_i]$ converging to L_h necessarily has $\mathfrak{W}$ among its limits and is a series $[H_i]_w$ which may be used as $[H_i]_c$. Let $[K_i]_c$ be a subseries of $[K_i]$ chosen so as to have the same properties with respect to $\mathfrak{C}_{ak}$ and L_k as $[H_i]_c$ has with respect to $\mathfrak{C}_{ah}$ and L_h.

Clearly now infinite subseries $[H_j]$ and $[K_j]$ may be chosen having the same limiting quantities as $[H_i]_c$ and $[K_i]_c$ such that no one of $[X_i]$ contains an element of both, and when $[H_j]$ and $[K_j]$ are added to W a closed bounded continuum of type Z results in which $[H_j]$ and $[K_j]$ are concurrent series and the theorem is true (5; 6; 7).

V. An Application to Order.

1. **Remarks.** In his search for entities definable with respect to the boundary of any connected simply connected and bounded domain which should possess as a collection order relations identical with those of the points of a simple closed curve, Professor Carathéodory was completely successful. The prime ends of such a domain possess cyclic order. It is natural to inquire to what extent the order relations among the prime ends themselves may be usefully extended to the point sets contained in them. The reply to this query is less satisfactory than the character of the question predicts. A natural answer would be this: "Let the notion

of order be referred to a particular point r of the element $\mathfrak{R}$ of $\mathfrak{C}$, the simple closed curve of the prime ends of the domain in question. Then with respect to a particular direction around $\mathfrak{C}$, of two points p_1 and p_2 not in $\mathfrak{R}$ it will be said that p_1 precedes p_2, if the first prime end $\mathfrak{P}_1$ containing p_1 precedes in $\mathfrak{C}$ the first prime end $\mathfrak{P}_2$ containing p_2".

Various objections to this method, some of them serious and some not, soon present themselves. The concluding pages of this paper offer a contribution to the subject of order providing an advance upon the system mentioned in at least one substantial respect. It is clear above that among the points of a given prime end no distinctions with regard to precedence exist unless some of these points are contained also by other prime ends. It will be shown below that this flaw can be removed. The method will consist in assigning to the points of a continuum X_a in question an order depending upon the circumstances of their inclusion within the elements of an aggregate $[L_\sigma]$ whose members are point sets of a sort investigated in IV, where it will be clear that $[L_\sigma]$ is an aggregate simply ordered with respect to the inclusion of its elements one within another. The exposition of this method will be furthered by a simplified method of determining the elements of $[L_\sigma]$. This simplification will be undertaken next.

2. **Hypotheses and forecast.** A class of bounded continua of type Z will be considered, each made up of the three preassigned continua X_a, X, and X_r of the usual sort except that X_a and X_r are mutually separated sets, and an additional arbitrary collection of continua $[X_\alpha]$ also of the usual sort except that no subseries of $[X_\alpha]$ having a limit in $c(X)\cdot X_a$ will have any limit point in $c(X)$ belonging to any element different from X_a. By restricting the problem in these respects no generality valuable to the special aims of this section will be lost.

A series $[Y_i]$ of arcs forming with X_a+X+X_r a continuum of the type in question will be determined such that subarcs $[Y_i]_s$ of $[Y_i]$ exist whose limit quantities in $c(X)\cdot X_a$ are identical with L_s and $\mathfrak{C}_s$, the limit quantities in $c(X)\cdot X_a$ of any series $[X_i]_s$ of any $[X_\alpha]$ in any set Z of the class being considered.

3. **Construction.** (a) Construct a simple closed curve C_1 separating X_a and X_r and having in $c_u(X_a)$ at least one point at a distance less than $\frac{1}{2}$ from each point of the frontier of $c_u(X_r+X+X_a)$ contained in $c(X)\cdot X_a$ aud no point in $c_u(X_r+X+X_a)$ at a distance from X_a greater than 1. For $n = 2, 3, 4, \cdots$ let C_n be a simple closed curve separating C_{n-1} from X_a and having in $c_u(X_a)$ at least one point at a distance less than $(\frac{1}{2})^n$ from each point of the frontier of $c_u(X_r+X+X_a)$ contained in $c(X)\cdot X_a$ and no point in $c_u(X_r+X+X_a)$ at a distance from X_a greater than $(\frac{1}{2})^{n-1}$.

(b) Select from $c_u(X)\cdot C_i$ that component D_i separating in $c_u(X)$ the set $c(X)\cdot X_r$ from the set $c(X)\cdot X_a$ and not separated from $c(X)\cdot X_a$ by any other component of $c_u(X)\cdot C_i$.

(c) The ends of D_i being u_i and v_i where D_i approaches u_i through an element of $\mathfrak{C}_u$ and v_i through an element of $\mathfrak{C}_v$ form from $u_i+D_i+v_i$ the subarcs U_i and V_i where U_i consists of u_i and all of D_i except a subinterval of diameter less than $(\frac{1}{2})^i$ with one end v_i, and V_i consists of v_i and all of D_i except a subinterval of diameter less than $(\frac{1}{2})^i$ with one end u_i.

(d) If $[X_i]_s$ is a series selected at random from the elements $[X_a]$ of any arbitrary member of the class of continua under investigation let $[Y_i]$ be $[U_i]$ if $\mathfrak{C}_u$ contains all members of $\mathfrak{C}$ which are limits of members of $[c(X)\cdot X_i]_s$ otherwise let $[Y_i]$ be $[V_i]$.

4. **Comments.** In (3) various special situations which involve no material difficulties have been left unmentioned; for instance, u_i and v_i may coincide.

It is to be observed that every point p in $c(X)$ of any prime end $\mathfrak{P}$ on $\mathfrak{C}_a$ is limit of every infinite subseries of either $[U_i]$ or $[V_i]$ through $\mathfrak{P}$. For if not let ϱ be any element of the collection $\mathfrak{P}$ containing no point of R and let η be a neighborhood of p such that $\varrho\cdot\eta\cdot\sum Y_i = 0$. Let τ be any component of $\varrho\cdot\eta$ and let T be a ray of space in $c(X+X_a+X_r+R)$ whose end is an accessible point of $c(X)\cdot X_a$ in a prime end $\mathfrak{T}$ contained in τ. If T contains points of infinitely many of $[Y_i]$ a contradiction results, for then a limit point of $\sum Y_i$ exists in $T\cdot c(X+X_a+\tau)$ and at a distance greater than some definite positive constant from X_a, despite construction. However it is obvious that T must intersect all elements of $[D_i]$ and that $\sum D_i\cdot T$ is a point set whose distance from X is likewise greater than some definite positive constant, say ε. Therefore those maximal connected subsets of $\sum D_i - T\cdot\sum D_i$ each of which has at its end a point of $\sum(u_i+v_i)$ must everyone be of diameter greater than ε. Since for each i one of the two components under discussion is part of a subset of D_i omitted from D_i during the construction of Y_i, the fact that all are of diameter greater than ε is a contradiction of the procedure used. The contradiction occurs whether $[Y_i]$ is identical with $[U_i]$ or with $[V_i]$.

5. **Notation.** When $\mathfrak{C}_s$ does not include $\mathfrak{C}_a$ there is an element $\mathfrak{A}$ (III, 15, (e)) of $\mathfrak{C}_s$ which is not separated from $\mathfrak{V}$ by any other element of $\mathfrak{C}_s$. Suppose that $[\alpha_i]$ is a chain of domains defining $\mathfrak{A}$ none of which contains a point of R, the corresponding chain of cuts consisting of arcs whose diameters have limit zero, their end points being contained in $c(X)\cdot X_a$ and belonging to $[\mathfrak{U}_i]$ and $[\mathfrak{V}_i]$ of $\mathfrak{C}_a$ in the order $\mathfrak{U}$, $\mathfrak{U}_i$, $\mathfrak{U}_{i+1}$ $\mathfrak{A}$, $\mathfrak{V}_{i+1}$, $\mathfrak{V}_i$, $\mathfrak{V}$ through which they are limits of the set of cuts. Let P be the subset of $c(L_s+X)$ contained in $\mathfrak{A}$ and let $[p_i]$ be a countable

subset of P such that every point of P is point or limit point of $[p_i]$. Let σ_i be a circular neighborhood of p_i of such radius that $d(\sigma_i, L_s+X)<\frac{1}{2}d(p_i, L_s+X)$.

6. **Construction and discussion.** When $\mathfrak{C}_s \supset \mathfrak{C}_a$ let $[Y_i]_s = [Y_i]$ (4). If $\mathfrak{C}_s \supset\!\!\!\cdot\, \mathfrak{C}_a$ the situation is complicated and treated in detail below.

(a). Let $[Y_{i0}]$ be $[Y_i]$.

For each value of j, $j = 1, 2, 3, \cdots$, let $[Y_{ij}]$ be that collection whose first element, Y_{1j}, is the minimal subarc of $Y_{1,j-1}$ containing a point of X as one end point and a unique component P_{1j} of $Y_{1,j-1}\cdot\alpha_j$ with both ends on A_j, if such a component exists, and if not is identical with $Y_{1,j-1}$; and whose ith element, $i = 2, 3, 4, \cdots$, is the minimal subarc of $Y_{i,j-1}$ having as one end a point of X and containing a unique component P_{ij} of $Y_{i,j-1}\cdot\alpha_j$ whose ends separate upon A_j the ends of $P_{i-1,j}$ from those of A_j provided that such a component exists and if not is identical with $Y_{i,j-1}$.

Because $\mathfrak{A}$ is limit of $[Y_i]$ and is separated from $\mathfrak{B}$ by other elements of $\mathfrak{C}_a$ also limits of $[Y_i]$, it is clear that for any arbitrary value of j at most a finite number of $[Y_{i,j-1}]$ contain no such sets $[P_{ij}]$.

It is to be observed that every point of $c(X)$ in $\mathfrak{A}$ is limit through $\mathfrak{A}$ for each value of j of

(a) $\sum Y_{ij}$,

(b) every infinite subseries of $[Y_{ij}]$,

(c) $\sum P_{ij}$, and

(d) every infinite subseries of $[P_{ij}]$.

For this has already been observed in the case of $[Y_{i0}]$ and an induction may be established. Indeed it will be sufficient to show (d) true for j when (b) is true for $j-1$. Let p be a point of $c(X)$ in $\mathfrak{A}$. As p is limit of every infinite subseries of $[Y_{i,j-1}]$, a random example being $[Y_{i,j-1}]_p$, α_j contains points of infinitely many of $[Y_{i,j-1}]_p$ and indeed as was noted above all but a finite number of $[Y_{i,j-1}]_p$ contain actual arcs $[P_{ij}]_p$ included in $[P_{ij}]$. The limits of the end points of $[P_{ij}]$ are identical with the ends of A_j. If the point p of $\mathfrak{A}$ is not limit of $[P_{ij}]_p$ through $\mathfrak{A}$ there exists a neighborhood η of p such that $\eta\cdot\alpha_j$ contains no points of any element of $[P_{ij}]_p$. Under such circumstances it would be possible to draw in α_j an arc one end of which is a point of $c(X)\cdot X_a$ approached by it through a component of $\eta\cdot\alpha_j$ and the other is a nonend point of A_j. Such an arc would have to intersect infinitely many of $[P_{ij}]_p$ and thus contain in α_j a point which is limit of such intersection points, that is a point of $c(X_a)$ which is limit of $\sum Y_i$. As this is impossible (3 (a)) p must be a limit of $[P_{ij}]_p$, and so by an easy inference p is limit of any infinite subset of $[P_{ij}]_p$. All four statements now follow.

(b). Let the collection $[Z_{i0}]$ consist for each i of $\prod Y_{ij}$ extended over all values of j. It is clear that each element of $[Z_{i0}]$ exists, for indeed Z_{10} is identical with Y_1 unless P_{11} exists. In case P_{11} does exist there is necessarily counting from the end of Y_1 in X, a first point of Y_1 in A_1 and this point as it belongs to Y_{1j} for every value of j must belong to Z_{10}. Similarly Z_{i0} is identical with Y_i unless P_{i1} exists and if it does exist again there is a first point of Y_i in A_1 which must always belong to Y_{ij} and thus to Z_{i0}. Moreover the limit quantities $\mathfrak{C}_0$ and L_0 of $[Z_{i0}]$ in $\mathfrak{C}_a$ and $c(X)\cdot X_a$ have the following relations to $\mathfrak{C}_s$ and L_s, $\mathfrak{C}_0 \supset \mathfrak{C}_s$ and $L_0 \supset L_s$. It is seen without much difficulty that among the limits of $\sum Y_{ij}$ for a particular value of j are all elements of $[\mathfrak{U}_i]$ and all those of $[\mathfrak{V}_i]$ having $i \geqq j$, but not $\mathfrak{V}_{j-1}$; and thus $\mathfrak{C}_0$ must include all of $[\mathfrak{U}_i]$ and none of $[\mathfrak{V}_i]$. That is $\mathfrak{C}_0$ must include all elements of $\mathfrak{C}_a$ between $\mathfrak{A}$ and $\mathfrak{U}$ and must accordingly also include $\mathfrak{A}$ and no elements of $\mathfrak{C}_a$ between $\mathfrak{A}$ and $\mathfrak{V}$; so $\mathfrak{C}_0 = \mathfrak{C}_s$ (IV, 8). If $L_0 \supset L_s$ then $L_s \supset L_0$ (IV, 8) and there is an element $\mathfrak{Q}$ of $\mathfrak{C}_s$ containing a point s which belongs to L_s but not to L_0. But $\mathfrak{Q}$ is not $\mathfrak{A}$, for every point of $c(X)\cdot X_a$ in $\mathfrak{A}$ belongs to L_0 (6 (a)), and it can not be an element different from $\mathfrak{A}$ (II, 7; II, 10); so no such element $\mathfrak{Q}$ exists.

(c) Let $[Z_{i1}]$ be the set of maximal subarcs of $[Z_{i0}]$ each having an end point in X and no point whatever in σ_1. In general for $j = 2, 3, 4, \cdots$ let $[Z_{ij}]$ be the set of maximal subarcs of $[Z_{i,j-1}]$ respectively, each having an end point in X and no point whatever in σ_j.

Let the limit quantities of $[Z_{ij}]$ $(j = 0, 1, 2, \cdots)$ in $\mathfrak{C}_a$ and $c(X)\cdot X_a$ be respectively $\mathfrak{C}_j$ and L_j. It is to be noted that $\mathfrak{C}_j \supset \mathfrak{C}_{j+1} \supset \mathfrak{C}_s$ and $L_j \supset L_{j+1} \supset L_s$. As for $\mathfrak{C}_1$ and L_1 if all but a finite number of $[Z_{i1}]$ are identical with the corresponding members of Z_{i0}, then $\mathfrak{C}_0 = \mathfrak{C}_1$ and $L_0 = L_1$, so that the inequalities noted are true. Whereas in case infinitely many of $[Z_{i1}]$ are proper subsets of the corresponding members of $[Z_{i0}]$, infinitely many of $[Z_{i1}]$ have a point upon the boundary of σ_1, and thus the limit set of these in $c(X)\cdot X_a$ contains a point upon the boundary of σ_1, that is a point of $c(X)\cdot X_a \cdot c(L_s)$. This means that L_1 is not a subset of L_s so that $L_1 \supset L_s$ (IV, 8) and also that $\mathfrak{C}_1 \supset \mathfrak{C}_s$: while at the same time it is obvious that $\mathfrak{C}_0 \supset \mathfrak{C}_1$ and $L_0 \supset L_1$. And as for $\mathfrak{C}_j$ and L_j in general, by the same sort of arguments and the same references the inequalities noted are shown to be true also for these.

(d) Let $[Y_i]_s$ be the collection whose ith element is $\prod Z_{ij}$ where the product is extended over all values of j.

The limit quantities $\mathfrak{C}_z$ and L_z of $[Y_i]_s$ in $\mathfrak{C}_a$ and $c(X)\cdot X_a$ are identical respectively with $\mathfrak{C}_s$ and L_s. This is practically obvious in the case of $\mathfrak{C}_z$ and $\mathfrak{C}_s$ for $\mathfrak{C}_z$ must contain all the elements common to the members

of $[\mathfrak{C}_j]$ and these are all identical to $\mathfrak{C}_s$ (6 (b); 6 (c)). It is clear that $L_0 \supset P + L_s$ (6 (b)) and that $L_0 \cdot L_1 \cdot L_2 \cdot \cdots \cdot L_j \cdot \cdots = L_z$ so that $L_z \supset L_s$. But $L_z \cdot P = 0$ (6 (c)), so $L_z = L_s$.

The construction of $[Y_i]_s$ with the properties required is thus complete.

7. **Summary.** It has now been established that there exists a set Z_a composed of $X_r + X + X_a$ and $[Y_i]$, a set of arcs, such that $[Y_i]$ contains a set of subarcs $[Y_i]_s$ forming with $X_r + X + X_a$ a set of type Z and having in X_a limit quantities identical with the limit quantities $\mathfrak{C}_s$ and L_s of any series $[X_i]_s$ in a set Z of the class being considered.

8. **Introduction of order.** Consider $[L_\sigma]$, the set of all distinct subsets of $c(X) \cdot X_a$ obtainable in Z_a as limits of sets of subarcs of $[Y_i]$ each having an end in X. The set $[L_\sigma]$ has the following properties easily deduced from the references given.

(a) It includes the limit set in $c(X) \cdot X_a$ of every possible series in any set $Z = X_r + X + X_a + \sum X_i$ where the order of $[Y_i]$ and $[X_i]$ are the same (7).

(b) It is a simply ordered aggregate (IV, 9).

(c) Each inner and outer limiting set in $c(X)$ of its elements is included in it ((a)).

(d) Between any two of its elements there is a third ((a)).

(e) If it contains an element L_s containing a point s which is not contained by any element of $[L_\sigma]$ preceding L_s, there exists one of the set $\mathfrak{C}_a$, say $\mathfrak{S}$, which contains s, and if $\mathfrak{S}$ is selected as the first element from $\mathfrak{U}$ which contains s (II, 11) then $\mathfrak{S}$ contains all the points of L_s none of which is contained in any element of $[L_\sigma]$ which precedes it. For indeed if such were not the case it would be possible to prove that between L_s and all those elements of $[L_\sigma]$ which precede L_s there is an element of $[L_\sigma]$.

Of course if $[Y_i]$ is identical with $[V_i]$ instead of with $[U_i]$ an entirely different collection $[L_\sigma]$ arises possessing as a set however the properties listed above.

Associated with each member L_s of $[L_\sigma]$ there is a point set Q_s contained in L_s no point of which is contained in any element of $[L_\sigma]$ preceding L_s. In view of the definition of $[Q_\sigma]$ and the properties of $[L_\sigma]$ listed above it is readily seen that $[Q_\sigma]$ has the following properties.

(a) The elements of $[Q_\sigma]$ are mutually exclusive.

(b) The elements of $[Q_\sigma]$ form a simply ordered aggregate.

(c) Between any two elements of $[Q_\sigma]$ there is a third.

(d) If the element Q_s of $[Q_\sigma]$ contains the point p which is not contained by any element of $[Q_\sigma]$ preceding Q_s, and $\mathfrak{S}$ is the first element of $\mathfrak{C}_a$ (II, 11) counted from $\mathfrak{U}$ containing p, then $L_s \supset S_u + S_c$ (II, 8; II, 9; II, 7; III, 15, (a)).

Accordingly the elements of $[Q_\sigma]$ distinguish between points which are not contained in the same element of $\mathfrak{C}_a$. In addition in a given element $\mathfrak{P}$ of $\mathfrak{C}_a$ they distinguish between the points of P_v and those of $P_c + P_u$ and also to some extent introduce distinctions among the points of P_v, but are unable to distinguish through $\mathfrak{P}$ between points contained in $P_c + P_u$. Of course if $[Y_i]$ is identical with $[V_i]$ instead of with $[U_i]$ then the corresponding set $[L_\sigma]$ distinguishes P_u in $\mathfrak{P}$ from $P_c + P_v$ but is unable to distinguish through $\mathfrak{P}$ between any two points in $P_c + P_v$.

9. **Conclusion.** Let X_a be any plane bounded continuum and r any point of Δ, the boundary of $c_u(X_a)$, contained in the prime end $\mathfrak{R}$ of $c_u(X_a)$. Suppose for the moment that the omission from X_a of the continuum of chief points of $\mathfrak{R}$ does not disconnect X_a. To assign order among the points of Δ with respect to r through $\mathfrak{R}$, let R be a point set composed of two rays with common end point and no other common point, one a ray of space in $c(X_a)$ the other a ray of $c_u(X_a)$ with limit points in X_a identical with the chief points of $\mathfrak{R}$ (I,4). Let R differentiated into suitable subsets play the combined roles of X, X_r, and R in the foregoing sections, and construct $[U_i]$ and $[V_i]$. Use $[U_i]$ to form $[L_\sigma]$ and $[Q_\sigma]$, assign to points in different elements of $[Q_\sigma]$ the order of the elements containing them, and to those in the same element of $[Q_\sigma]$ and thus in a first element $\mathfrak{P}$ of $\mathfrak{C}_a$ (II, 11) assign the reverse order assigned to them through $\mathfrak{P}$ by $[V_i]$ if there are subseries, belonging to the class considered, of $[V_i]$ which distinguish them apart.

When X_a is disconnected by the omission from it of the continuum of chief points of $\mathfrak{R}$, order may first be assigned among the points of Δ in each maximal connected subset separately in the fashion already described and then extended throughout the collection of these subsets by way of the order relations among them.

It appears above as if the assignment of order is being made in Δ rather with respect to the chief points of $\mathfrak{R}$ than with respect to r, and this is indeed the case; but the situation may be corrected after the assignment above has been made by means of adjustments affecting only the points of Δ within $\mathfrak{R}$. Evidently the use of $[V_i]$ in determining $[L_\sigma]$ will give an ordering of the points of Δ in general drastically different in outcome from the one just obtained by means of subarcs of $[U_i]$, but the fact that a given point of Δ may belong to more than one element of $\mathfrak{C}_a$ makes such a result unavoidable.

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ON THE LINKING OF JORDAN CONTINUA IN E_n BY $(n-2)$-CYCLES.[1]

By R. L. Wilder.

The following is a classical theorem[2] in the theory of Jordan continua:[3] *If, in the euclidean plane, two points P and Q are separated*[4] *by a Jordan continuum M, then P and Q are separated by a simple closed curve of M.* The view has been prevalent among set-theorists that this theorem is one of those that are peculiar to the *plane* point set theory; thus, two points may be separated by a Jordan continuum in E_3 without being separated by any simple surface of that continuum. The primary purpose of this paper is to show that this viewpoint is a mistaken one—that the above theorem is only the plane case of a general *linking* theorem to the effect that if an $(n-2)$-cycle links a Jordan continuum M in E_n, then it links a simple closed curve of M. Incidentally, for the case $n=2$, a new proof, essentially combinatorial, is furnished for the above theorem. In addition, I shall extend to E_n certain theorems of R. L. Moore for the plane which concern the set of all points on simple closed curves of M that the $(n-2)$-cycle links; also, it will be shown that given a set of r cycles Γ^{n-2} of the $(n-2)$-basis of $E_n - M$, there exists on M a non-singular linear graph among whose irreducible cycles r link the Γ's.

As all of the results given below hold in euclidean space of n dimensions $(n>1)$, we may assume this understood without explicit mention of the fact hereafter.

Theorem 1. *Let M be a Jordan continuum, and t an arc which has its end-points, and only these, in common with M. Also, let Γ^{n-2} be an $(n-2)$-cycle of $E_n - (M+t)$ that does not link M. Then if Γ^{n-2} links $M+t$, it links every simple closed curve of $M+t$ that contains t.*

Proof. Let J be a simple closed curve of $M+t$ that contains t, and suppose that Γ^{n-2} does not link J. Obviously $J-<t>=h$ is an arc[5] of M. There exist complexes K_1^{n-1} and K_2^{n-1} such that

[1] Received April 28, 1932. Presented to the American Mathematical Society, November 28, 1931.

[2] Cf. R. L. Moore, *Concerning continuous curves in the plane*, Math. Zeitschr., vol. 15 (1922), pp. 254–260; also C. Kuratowski, *Sur les coupures du plan*, Fund. Math., vol. 6 (1924), pp. 130–145.

[3] A *Jordan continuum* (*continuous curve* or *Peano continuum*) is a compact, closed and connected point set which is locally connected.

[4] Two points P and Q are *separated* by a set M in E_n if there exists no continuum in $E_n - M$ which contains both P and Q.

[5] By $<t>$ is meant t without its end-points.

$$K_1^{n-1} \to \Gamma^{n-2}, \qquad (E_n - M),$$
$$K_2^{n-1} \to \Gamma^{n-2}, \qquad (E_n - J).$$

By a theorem of Alexander[6] we also have

$$K_1^{n-1} + K_2^{n-1} \sim 0, \qquad (E_n - h)$$

and as $M \cdot J = h$, it follows from the Alexander Addition Theorem[7] that

$$\Gamma^{n-2} \sim 0, \qquad [E_n - (M+J) = E_n - (M+t)],$$

which contradicts the fact that Γ^{n-2} links $M+t$.

THEOREM 2. *Let M be a Jordan continuum and t an arc. Then a cycle Γ^{n-2} of $E_n - (M+t)$ that does not link M can link $M+t$ only if it links a simple closed curve of $M+t$.*

Proof. Suppose Γ^{n-2} links $M+t$ but links no simple closed curve of $M+t$. As Γ^{n-2} does not link M, and cannot link t, it follows easily from the Alexander Addition Theorem that $M \cdot t$ is a non-vacuous set of points that is not connected.

Let K^{n-1} be a complex bounded by Γ^{n-2} in $E_n - M$ and let d denote the distance $\varrho(K^{n-1}, M)$.

A component of $t - M \cdot t$ is either an arc without one end-point or an arc without both end-points. Of the former type only two arcs are possible, and we may consider these as added to M—their addition does not alter the situation as regards Γ^{n-2} linking M (by use of the Alexander Addition Theorem as these arcs are successively added to M). Then any component of $t - M \cdot t$ may be assumed to be an arc without both end-points. Since only a finite number of such arcs can be of diameter greater than $d/2$, not more than a finite number of them meet K^{n-1}. Denote those that do meet K^{n-1} by $t_1, t_2, \cdots, t_k$, and those that do not meet K^{n-1} by $t_{k+1}, \cdots$. Let $M' = M + \sum_{i=k+1}^{\infty} t_i$. Obviously Γ^{n-2} does not link M', and M' is a Jordan continuum.

Consider $M' + t_1$. As Γ^{n-2} does not link M' and does not link any simple closed curve of $M' + t_1$, it cannot, by Theorem 1, link $M' + t_1$. Similarly Γ^{n-2} does not link $M' + t_1 + t_2$, and continuing in this way we finally have that Γ^{n-2} does not link $M + t \left(= M' + \sum_{i=1}^{k} t_i\right)$, thus contradicting the hypothesis.

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[6] J. W. Alexander, *A proof and extension of the Jordan-Brouwer separation theorem*, Trans. Amer. Math. Soc., vol. 23 (1922), pp. 333–349, Theorem T^i.

[7] J. W. Alexander, loc. cit., Corollary W^i.

The following corollary now follows easily from Theorem 2:

COROLLARY 1. *Let M be a point set consisting of k (finite) arcs a_i and Γ^{n-2} a cycle of $E_n - M$. Then if Γ^{n-2} links M, it links a simple closed curve of M.*

THEOREM 3. *Let M be a Jordan continuum and Γ^{n-2} an $(n-2)$-cycle linking M. Then Γ^{n-2} links a simple closed curve of M.*

Proof. Let d denote the distance $\varrho(M, \Gamma^{n-2})$. Since M is uniformly locally connected,[8] there exists a positive number $\delta < d/4$ such that if P and Q are points of M whose distance apart is $< \delta$, then P and Q are the end-points of an arc of M whose diameter is less than $d/4$. Let T denote a subdivision of E_n into simplexes of diameter $< \delta/4$, and denote by Q that complex which consists of all n-simplexes of T which meet M. Obviously $\Gamma^{n-2} \subset E_n - Q \subset E_n - M$ and Γ^{n-2} links Q. By a theorem of Pontrjagin[9] there exists in Q a 1-cycle Γ^1 which is linked by Γ^{n-2}. We may assume Γ^1 irreducible, and adopting a sense on it, order its cells in a sequence $t_0, t_1, \cdots, t_k$ according to this sense. Denote the vertices of t_i by P_i' and P_{i+1}' (subscripts reduced modulo $k+1$) where P_i' precedes P_{i+1}' in the above sense.

Consider any cell t_i; it belongs to some n-simplex S_i of Q, and S_i contains a point P_i of M. A pair of points P_i, P_{i+1} evidently satisfies the relation $\varrho(P_i, P_{i+1}) < \delta$, so that there is an arc a_i of M with end-points P_i and P_{i+1} such that the diameter of a_i is less than $d/4$. Let $K = \sum_{i=0}^{k} a_i$.

There exists a continuous transformation f of Γ^1 into K defined as follows: For every i, $P_i = f(P_i')$ and $a_i = f(t_i)$, where the transformation of an individual t_i into a_i is topological. This transformation, $K = f(\Gamma^1)$, may be taken as the basis for a rectilinear deformation Δ of Γ^1 into K.[10] It is readily seen that Δ is a d-deformation and that during the course of the deformation Γ^1 does not meet Γ^{n-2}.

By a theorem of Pontrjagin[11] the cycle Γ^{n-2} which links Γ^1 must also link K, and by Corollary 1 it follows that Γ^{n-2} links a simple closed curve of K. As $K \subset M$ the theorem is thereby proved.

We may now state the following corollary, which follows from Theorem 1 and the Alexander Addition Theorem:

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[8] Cf. H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, Wien. Akad. Sitz., vol. 123 (1914), Abt. IIa, pp. 2433–2489.

[9] L. Pontrjagin, *Zum Alexanderschen Dualitätssatz*, Gött. Nach., Math.-Phys. Kl., 1927, pp. 315–322, Satz I.

[10] I. e., by the stipulation that the function $F(P, x)$ ($P \subset M$, $0 \leqq x \leqq 1$) defining Δ carries P along the straight line joining P and $f(P)$ in such a way that $\varrho[P, F(P, x)]/\varrho[P, f(P)] = x$.

[11] Loc. cit., Satz IV.

COROLLARY 2. *If $M_1, M_2, \cdots, M_k$ (k finite) are Jordan continua such that for $i>1$ the set $M_i \cdot (M_1 + M_2 + \cdots + M_{i-1})$ is either vacuous or connected, and Γ^{n-2} is a cycle which links the set $\sum_{i=1}^{k} M_i$, then Γ^{n-2} links a simple closed curve which is contained wholly in one of the sets M_i.*

The following is an extensive generalization of Corollary 1:

COROLLARY 3. *Let $M_1, M_2, \cdots, M_k$ (k finite) be Jordan continua and Γ^{n-2} a cycle which links $\sum_{i=1}^{k} M_i$. Then there exists in $\sum_{i=1}^{k} M_i$ a simple closed curve linked by Γ^{n-2}.*

Proof. The components of the set $\sum_{i=1}^{k} M_i$ form a finite set of Jordan continua $N_1, N_2, \cdots, N_h$ ($h \leqq k$), such that $N_i \cdot N_j = 0$ ($i \neq j$). The conclusion follows from Corollary 2.

As another corollary of Theorem 3 (and the Alexander Duality Theorem) we can state the following characterization of a simple closed curve:

COROLLARY 4. *A necessary and sufficient condition that a Jordan continuum be a simple closed curve is that it be irreducibly linked by an $(n-2)$-cycle.*

From Corollary 4 we can obtain the following interesting result: Suppose K is a locally connected, compact closed point set of dimension 1, such that the Brouwer number[12] $\beta^1(K)>0$, and such that if F is any proper closed subset of K then $\beta^1(F) = 0$. By the Menger imbedding theorem K can be imbedded in E_3. By the duality theorem for closed sets[13] there exists a cycle Γ^1 which links K, and since the first Brouwer number of every proper closed subset of K is zero, Γ^1 links K irreducibly. From the Alexander Addition Theorem it is obvious that K is connected and therefore a Jordan continuum, and by Corollary 4 is a simple closed curve. Thus,

COROLLARY 5. *Every locally connected compact closed point set of dimension one whose first Brouwer number is positive and whose proper closed subsets have first Brouwer numbers zero is a simple closed curve.*

In particular, then, *every closed cantorian manifold*[14] *of dimension* 1 *which is locally connected must be regularly closed*[14] *and, moreover, is a simple closed curve.*

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[12] For the definition of *Brouwer number* see Brouwer, Math. Ann., vol. 72 (1912), pp. 422–425 and L. Vietoris, Math. Ann., vol. 97 (1927), pp. 454–472.

[13] Cf. F. Frankl, Wien. Akad. der Wiss., Math.-Naturw. Kl., Sitz., Abt. IIa, vol. 136 (1927), pp. 689–699; P. Alexandroff, Annals of Math., vol. 30 (1928), pp. 101–187; S. Lefschetz, Annals of Math., vol. 29 (1928), pp. 232–254; and earlier papers referred to in the last two citations.

[14] See P. Alexandroff, loc. cit., p. 176.

We have, finally, the following obvious corollary of Theorem 3:

COROLLARY 6. *If M is an acyclic Jordan continuum,*[15] *then every $(n-2)$-cycle of $E_n - M$ bounds in the complement of M.*

Before proceeding to the case of several independent $(n-2)$-cycles linking a Jordan continuum, we inquire as to the nature of the set of *all* simple closed curves (of a Jordan continuum) which an $(n-2)$-cycle links. To be more specific, if L represents the set of all points that lie on such simple closed curves, what is the relation of L to the Jordan continuum? We shall see that L has a very simple structure,—is, indeed, the sum of a finite number of Jordan continua each of which consists of a finite set of true cyclic elements of M.

In the plane, R. L. Moore has settled a special case of this problem, having shown[16] that if P is a point complementary to a Jordan continuum M in E_2, the set of points on all simple closed curves of M that enclose P is the sum of a finite number of Jordan continua. Our proof for the general case is much simpler than that of Moore for the special case, due partially to the fact that we are able to take advantage of the cyclic element theory of Whyburn which has been developed in the past few years.[17]

LEMMA. *Let J be a simple closed curve and t an arc which has only its end-points, A and B, in common with J. Let J_1 and J_2 denote the arcs into which A and B divide J. Then, if a cycle Γ^{n-2} links J, and $\Gamma^{n-2} \cdot t = 0$, the cycle Γ^{n-2} links either $t+J_1$ or $t+J_2$.*

Proof. If Γ^{n-2} links $t+J_1$ the conclusion of the theorem follows. Suppose Γ^{n-2} does not link $t+J_1$. Now Γ^{n-2} links $(t+J_1)+J_2 = t+J$, since it links J. Then by Theorem 1, Γ^{n-2} links every simple closed curve of $J+t$ that contains J_2; in particular, then, it links $t+J_2$.

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Proof. Let C denote the cyclic element of M determined by J. Let x denote any point of C not on J. Then there is an arc t in C which contains x and has its end-points, A and B, and only these, on J.[18] If

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[15] A Jordan continuum is called acyclic if it contains no simple closed curve.

[16] *Concerning paths that do not separate a given continuous curve,* Proc. Nat. Acad. Sci., vol. 12 (1926), pp. 745–753, Theorems 7 and 8.

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Proof. If Γ^{n-2} links $t+J_1$ the conclusion of the theorem follows. Suppose Γ^{n-2} does not link $t+J_1$. Now Γ^{n-2} links $(t+J_1)+J_2 = t+J$, since it links J. Then by Theorem 1, Γ^{n-2} links every simple closed curve of $J+t$ that contains J_2; in particular, then, it links $t+J_2$.

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Proof. Let C denote the cyclic element of M determined by J. Let x denote any point of C not on J. Then there is an arc t in C which contains x and has its end-points, A and B, and only these, on J.[18] If

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J_1 and J_2 denote the arcs into which A and B divide J, it follows from the Lemma that Γ^{n-2} links either $t+J_1$ or $t+J_2$, and in either case x is on a simple closed curve of M that Γ^{n-2} links.

An interesting corollary of this theorem is:

COROLLARY 7. *If M is a cyclicly connected*[19] *Jordan continuum which is linked by a cycle Γ^{n-2}, then every point of M is on some simple closed curve of M that Γ^{n-2} links.*

THEOREM 5. *Let M be a Jordan continuum linked by a cycle Γ^{n-2}. Then, if K denotes the set of all points that lie on simple closed curves of M that are linked by Γ^{n-2}, the set K is the sum of a finite number of Jordan continua each of which consists of a finite set of true cyclic elements of M.*

Proof. By Theorem 4, if J is a simple closed curve of M linked by Γ^{n-2}, then the cyclic element of M determined by J lies in K. Conversely, if Γ^{n-2} links a cyclic element C of M, then all points of C lie on simple closed curves of C that are linked by Γ^{n-2} (Corollary 7).

Suppose there exists an infinite sequence of distinct cyclic elements of M, viz., $C_1, C_2, C_3, \cdots$, such that for every i, C_i is linked by Γ^{n-2}. In each C_i there is a simple closed curve J_i linked by Γ^{n-2}, by Theorem 3. No two curves J_i have more than one point in common (since no two cyclic elements have more than one point in common). But $\lim_{i\to\infty} \delta(C_i) = 0$, and this is incompatible with the fact that $\varrho(\Gamma^{n-2}, M)$ is positive. Hence Γ^{n-2} can link only a finite number of the cyclic elements of M, and the theorem is established.

Relative to the linking of a Jordan continuum by more than one $(n-2)$-cycle we now state the following theorem:

THEOREM 6. *Let M be a Jordan continuum and Γ_i^{n-2} $(i = 1, 2, \cdots, r)$ be independent cycles of $E_n - M$ (as regards homologies mod 2, m or zero). Then there exists on M a (non-singular) linear graph K among whose irreducible cycles r link the Γ's. More precisely, if γ_j^1 $(j = 1, 2, \cdots, s)$ are the irreducible cycles of K, the matrix of looping coefficients*

$$\| Lc(\gamma_j^1, \Gamma_i^{n-2}) \| \qquad (i = 1, 2, \cdots, r;\ j = 1, 2, \cdots, s)$$

is of rank r.[20]

Proof. Let $d = \varrho\left(M, \sum_{i=1}^{r} \Gamma_i^{n-2}\right)$, and let δ be such that two points of M a distance apart less than δ are joined by an arc of M of diameter $< d/8$. As in the proof of Theorem 3 we assume a subdivision of E_n into simplexes

[19] I. e., every two points of M lie on a simple closed curve of M.

[20] See S. Lefschetz, *Topology*, p. 206. With him we designate by $Lc(\Gamma, \Gamma')$ the looping coefficient of Γ and Γ'.

of diameter $< \delta/4$, and define Q as before. By a theorem of Pontrjagin[21] there exists in the 1-dimensional basis of Q a set of cycles $g_1^1, g_2^1, \cdots, g_r^1$ such that the matrix

$$\| Lc(\Gamma_i^{n-2}, g_j^1) \| \qquad (i, j = 1, 2, \cdots, r)$$

is of rank r.

The set of cycles g_j^1 form a polygonal graph K_1. In what follows we shall denote vertices of K_1 by a_i and corresponding points in M by b_i; thus, b_i is a point of M in the same n-simplex of Q as a_i. We now proceed to define a deformation Δ of K_1 into the required graph K.

Let a_1 and a_2 be vertices of a 1-cell C_1 of K_1. Then b_1 and b_2 are end-points of an arc t_1 of M of diameter $< d/8$ and into which C_1 may be $d/4$-deformed. Let C_2 be a 1-cell of K_1 with vertices a_1 and a_3, and t_2' a corresponding arc of M joining b_1 and b_3. On t_2' let c_1 be the last point of t_1.[22] Then $b_1 c_1$ of t_1 together with $c_1 b_3$ of t_2' forms an arc t_2 of M from b_1 to b_3 into which C_2 may be $d/4$-deformed. If there is a third cell C_3 of K_1 with a_1 as one of its vertices, let the other vertex be a_4, t_3' be the corresponding arc in M from b_1 to b_4, c_2 be the last point of $t_1 + t_2$ on t_3', and suppose c_2 on t_2 for instance. Then $b_1 c_2$ of t_2 together with $c_2 b_4$ of t_3' is an arc t_3 of M into which C_3 may be $d/4$-deformed.

Having dealt with all 1-cells of K_1 that have a_1 as a vertex we have a set of arcs $t_1, t_2, \cdots, t_\nu$ of M which form a non-singular graph K_2 with vertices $b_1, b_2, \cdots, b_{\nu+1}, c_1, c_2, \cdots, c_{\nu-1}$. Denote the 1-cells of K_2 by l_i. Let $C_{\nu+1}$ denote a 1-cell of K_1 with a_2 as a vertex, and denote its other vertex by $a_{\nu+2}$. We get $t_{\nu+1}'$ from b_2 to $b_{\nu+2}$ in M as usual, and if K_2 and $t_{\nu+1}'$ form a non-singular linear graph we may proceed to the next step. If not, we first proceed as follows: Let the last point of $t_{\nu+1}'$ on K_2 be c_ν, and suppose c_ν is on l_{i_1} (if a vertex of K_2, c_ν may also be on another l_i, but this is immaterial). Let $c_{\nu+1}$ be the first point of l_{i_1} on $t_{\nu+1}'$, and replace $c_\nu c_{\nu+1}$ of $t_{\nu+1}'$ by $c_\nu c_{\nu+1}$ of l_{i_1}. We then consider $b_2 c_{\nu+1}$ of $t_{\nu+1}'$; if $c_{\nu+1}$ is a limit point on this arc of points of $K_2 t_{\nu+1}'$, then infinitely many of these points belong, say, to $l_{i_2} (i_2 \neq i_1)$, and we let $c_{\nu+2}$ be the first point of l_{i_2} on $b_2 c_{\nu+1}$ and replace $c_{\nu+1} c_{\nu+2}$ of $t_{\nu+1}'$ by $c_{\nu+1} c_{\nu+2}$ of l_{i_2}. If $c_{\nu+1}$ is not such a limit point, we let $c_{\nu+2}$ be the last point of K_2 on $b_2 c_{\nu+1}>$ of $t_{\nu+1}'$, and let $c_{\nu+3}$ be the first point of l_{i_2} (where $l_{i_2} \supset c_{\nu+2}$) on $b_2 c_{\nu+1}$, then replacing $c_{\nu+2} c_{\nu+3}$ of $t_{\nu+1}'$ by $c_{\nu+2} c_{\nu+3}$ of l_{i_2}. The process is continued in this way, and after the

[21] Loc. cit., Satz II.

[22] In all cases hereafter we shall let the order on an arc be from the vertex of smaller subscript to that of higher subscript. If a and b are points of an arc t, we denote that portion of t from a to b by "ab of t".

replacements are completed we obtain a new arc $t_{\nu+1}$ into which $t'_{\nu+1}$ may be $d/4$-deformed; then the 1-cell $C_{\nu+1}$ of K_1 may be $d/2$-deformed into $t_{\nu+1}$, and the sum of the arcs $t_1, t_2, \cdots, t_{\nu+1}$ forms a non-singular linear graph K_3.

We proceed as indicated in the preceding paragraph through the remainder of the 1-cells of K_1, obtaining finally a non-singular linear graph K on M into which K_1 may be continuously $d/2$-deformed. Call such a deformation Δ.

Consider now any two cycles Γ_i^{n-2} and g_j^1. The cycle g_j^1 undergoes a transformation induced by the deformation Δ into a cycle h_j^1 on K; denote the path of the deformation by[23] $\mathfrak{D} g_j^1$, and let C^2 be any complex bounded by g_j^1. Then $Lc(\Gamma_i^{n-2}, g_j^1) = (\Gamma_i^{n-2} \cdot C^2)$. We have

$$E_n - \Gamma_i^{n-2} \supset \mathfrak{D} g_j^1 \to h_j^1 - g_j^1.$$

Also, since $\mathfrak{D} g_j^1$ does not meet Γ_i^{n-2}, $(\Gamma_i^{n-2} \cdot \mathfrak{D} g_j^1) = 0$. Hence, denoting $C^2 + \mathfrak{D} g_j^1$ by C_1^2, $(C_1^2 \to h_j^1)$,

$$\begin{aligned} Lc(\Gamma_i^{n-2}, h_j^1) &= (\Gamma_i^{n-2} \cdot C_1^2) = (\Gamma_i^{n-2} \cdot C^2) + (\Gamma_i^{n-2} \cdot \mathfrak{D} g_j^1) \\ &= (\Gamma_i^{n-2} \cdot C^2) = Lc(\Gamma_i^{n-2}, g_j^1). \end{aligned}$$

Consequently,

$$\| Lc(h_j^1, \Gamma_i^{n-2}) \| = \| Lc(g_j^1, \Gamma_i^{n-2}) \|. \tag{1}$$

Now h_j^1 is homologous on K to a linear combination of the γ's, the irreducible cycles of K, and if $(\gamma_i)_j$ represents this linear combination, we have

$$\| Lc((\gamma_i)_j, \Gamma_i^{n-2}) \| = \| Lc(h_j^1, \Gamma_i^{n-2}) \|. \tag{2}$$

From (1) and (2) we see that the rank of the matrix $\| Lc((\gamma_i)_j, \Gamma_i^{n-2}) \|$ is r. But the matrix $\| Lc(\gamma_i, \Gamma_i^{n-2}) \|$, after being subjected to suitable elementary transformations, contains the former matrix and is therefore itself of rank r, thus proving the theorem.

Appendix.

Professor Lefschetz has suggested a briefer method of proof for Theorems 3 and 6. We here reproduce the proof of Theorem 3 essentially in his words. Theorem 6 is proved in a similar manner, except that in place of a single cycle we have several and the matrix property of the statement replaces the linking property of the single cycle.

[23] Cf. Lefschetz, loc. cit., pp. 76 ff.

By known duality theorems (see footnote in connection with Corollary 4), Γ^{n-2} links a polygonal Γ^1 whose vertices are on M and whose cells are of diameter $<\varepsilon$ assigned. The cells of Γ^1 make up a linear graph K_1. Since M is uniformly locally connected, given an arbitrary δ we can choose ε such that any two points of M not more than ε apart can be joined by an arc of M of diameter $<\delta$. Applying this to the endpoints of the segments of Γ^1 we replace any segment by an arc on M, the passage from one to the other being obtainable by a 2 δ-deformation in E_n. In particular, if $t_1, \cdots, t_r$ are the new arcs and if $M' = \sum t_i$, K_1 is thus 2 δ-deformed into a cycle $\Gamma_1' \subset M'$.

Let us call for the present the intersection $t_h \cdot t_i$ *regular* if it consists of a finite number of points and arcs.

Writing $M^i = t_1 + \cdots + t_i$, let i be the least index such that M^i includes pairs of arcs intersecting irregularly. In other words every $t_j \cdot t_k\,(j, k<i)$ is regular and at least one $t_h \cdot t_i$, $h<i$, is not. We shall assume that h is the least index for which $t_h \cdot t_i$ is irregular. There is a least arc $\lambda = \alpha\beta$ on t_i such that $\lambda \subset t_h \cdot t_i$. If we replace λ of t_i by the arc of t_h whose endpoints are α, β we obtain a new t_i such that $t_h \cdot t_i$ is regular. The resulting modification in Γ_1' may be obtained by means of a 2 δ-deformation in E_n. We treat in a similar manner the other irregular intersections $t_k \cdot t_i$, $h<k<i$, if any are present, until finally we have a new t_i intersecting all its predecessors regularly, so that the new M^i is a graph and also a subset of the former M^i, and Γ_1' has been deformed into a subcycle of the graph. If the arc t_i of the graph is of diameter $>\delta$, we subdivide it, and Γ_1' accordingly. The new M^i will behave as its predecessor and have all its arcs of diameter $<\delta$. We now replace M^{i+1} by the arc t_{i+1} plus the new M^i and repeat our reasoning, etc. Ultimately, after a finite number of steps we shall have a new M' subset of the former which is a graph K_1^* and Γ_1' will have been reduced to a subcycle Γ_1^* of K_1^* by successive subdivisions and 2 δ-deformations in E_n.

The successive 2 δ-deformations may be chosen rectilinear in E_n, and hence the paths of the points during deformations will all lie within a distance 2 δ of M. Therefore choosing $\delta < \frac{1}{4} d$ (where d is defined as in the proof of Theorem 3 above), the deformations will all take place in $E_n - \Gamma^{n-2}$, and since their product is a continuous deformation of K_1 we have $\Gamma_1 \sim \Gamma_1^*$ in $E_n - \Gamma^{n-2}$. It follows that Γ_1^* links Γ^{n-2}. Since Γ_1^* is a linear combination of the cycles of K_1^* which coincide with its single circuits, Γ^{n-2} links a cycle of the latter type. This proves Theorem 3.

UNIVERSITY OF MICHIGAN.

DIOPHANTINE EQUATIONS FROM ALGEBRAIC INVARIANTS AND COVARIANTS.[1]

By E. T. Bell.

1. Few Diophantine equations of degree higher than the second in more than two indeterminates have been completely solved in integers. As remarked by Dickson,[2] "It is not safe, without re-examination, to place confidence in any claim that a homogeneous equation has been completely solved in integers." To facilitate such re-examination for the equations discussed in this paper I have given the analysis in full, with references in square brackets to preceding results from which any important step follows. A reference like [2, 17] is to statements (2), (17) of the section in which the step occurs; if it is necessary to refer to preceding sections, this is indicated by [3. 2, 1. 17], which cites § 3 (2) and § 1 (17).

As an example of the method described in § 2, I give the *complete solution in integers a, b, c, d of the equation*

$$(1) \qquad (9ad-bc)^2-4(b^2-3ac)(c^2-3bd) = 0.$$

From this the complete solution of

$$(ad-bc)^2-4(b^2-ac)(c^2-bd) = 0$$

can easily be obtained if desired. See also § 7. The solution of (1) is summarized in § 6; it is expressed in terms of polynomials of degree 11 in 9 parameters which take only integer values; the coefficients in the polynomials are integers. This is in accordance with Dickson's (loc. cit., p. 318) "desirable form for the integral solutions of any homogeneous equation, viz., expressibility by one or more sets of formulas involving only integral parameters".

To solve (1) completely, we need the complete solutions of 6 subsidiary equations in § 4. The footnote to § 4 (5) may lead to something new and interesting.

In work of this kind it is a great advantage to have an independent derivation of the critical formulas by another worker. Professor Morgan Ward has kindly done this for me. To his vigilance I owe whatever confidence I have that fallacies have been avoided and that the final solutions are indeed complete.

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[1] Received October 6, 1932.

[2] L. E. Dickson, *Fallacies and Misconceptions in Diophantine Analysis*, Bull. Amer. Math. Soc., vol. 27 (1921), pp. 312–319.

2. In geometric terms, (1) is the equation of the general developable (rational) surface of degree 4. We shall consider the parametric representation of this surface as an instance of what follows.

A *general* quantic is one whose coefficients are constant multiples of independent variables. A necessary and sufficient condition that a general binary quantic have a repeated factor is the vanishing of the discriminant. Expressing the condition directly in terms of the roots and coefficients of the quantic, we get a parametric representation of the locus represented by equating the discriminant to zero. The gist of the application to Diophantine analysis is in what immediately follows.

Each set of values of the parameters gives one and only one point on the locus. Conversely, (it can be shown that) *to each point on the locus corresponds at least one set of values of the parameters.*

Hence, *all integer points* (≡ all points all of whose coördinates are integers) *on the locus are known when all sets of values* (rational and irrational) *of the parameters have been found which give integer points.*

Similarly, the conditions, necessary and sufficient, that a general quantic of given order and degree be a product of linear factors, or that it have a quadratic factor, etc., are expressed by the *identical* vanishing of certain covariants. The conditions are thus equivalent to a set of equations in the coefficients of the quantic. For example, the conditions that the general ternary cubic be completely factorable reduce to 5 (and no fewer) homogeneous equations of degree 4 in the 10 coefficients. Proceeding in any instance as just sketched for discriminants, we write down parametric equations for the corresponding loci by expressing the given condition (complete factorability, etc.) directly, using undetermined coefficients.

It may be obvious that the necessary converse always holds, namely, that to each point on the locus corresponds at least one set of values of the parameters, but for safety *I have verified in each application of the method that the relevant equations actually have a solution.* Equally trivial misapplications of geometric intuition have led to false conclusions.[3] Looked at more closely, the general assertion that the parametric equations obtained by the particular device described above always have at least one solution for an assigned point on the locus does not seem self-evident. Ultimately the question is one in modular systems, and a general proof would be of interest if the converse is universally true; otherwise, a general statement of the conditions under which the converse holds would be of value.

If the quantic be written without multinomial coefficients, the foregoing

[3] For example, in reference to the ternary cubic, the fallacious reasoning in Salmon's Higher Plane Curves, Art. 197, leads to an erroneous result.

leads to parametric representations by polynomials in the parameters with integer coefficients. Thus it is shown at least that the Diophantine system whose equations are those of the locus has an infinity of non-trivial integer solutions.

Once the complete solution has been obtained for a system arising from a quantic written without multinomial coefficients, it is a straightforward exercise in congruences to get the complete solution for the corresponding system derived in the same way from the quantic with multinomial coefficients. Thus the solution of § 1 (1) leads at once to that of

$$(ad - bc)^2 - (b^2 - ac)(c^2 - bd) = 0.$$

As the result is rather long and detailed, it is not reproduced.

3. The complete solution of the equation § 1 (1) is obtainable, quite remarkably, by repeated applications of the following facts (1) — (8) concerning arithmetical divisibility and the G. C. D.

The symbols $x \mid y$, (x, y), $(x \mid y)'$ are defined only when x, y are integers and $xy \neq 0$: $x \mid y$ means "x divides y", and hence, if x, y, z are integers, and $xyz \neq 0$, the statements $x \mid y$, $y = xz$ are equivalent; $(x \mid y)'$ is the negation[4] of $x \mid y$; (x, y) denotes the G. C. D. of x and y.

The letters x, y, z, w in (1) — (8) *denote integers* $\neq 0$.

(1) *If* $(xy, zw) = 1$, *then* $1 = (x, z) = (x, w) = (y, z) = (y, w)$.
(2) *If* $x \mid yz$, *and* $(x, y) = 1$, *then* $x \mid z$.
(3) *If* $(x, y) = w$, *then* $x = x_1 w$, $y = y_1 w$, $(x_1, y_1) = 1$.
(4) *If* $(xw, yw) = 1$, *then* $(x, y) = \mid w \mid = 1$.
(5) *If p is prime,* $1 = (p, xy\cdots)$ *is equivalent to* $1 = (p, x) = (p, y) = \cdots$.
(6) *If* $xy \mid zw$, *and* $1 = (x, z) = (y, w)$, *then* $x \mid w$ *and* $y \mid z$.
(7) *If* $x = y + z$, *and* $w \mid x$, $w \mid y$, *then* $w \mid z$.
(8) $x = y + z$, $w \mid x$, $(w \mid y)'$, $w \mid z$ *are incompatible.*

4. *In this section all letters in formulas denote integers* $\neq 0$. Hence, in this section, *complete solution* means *complete solution in integers* $\neq 0$. It will be seen in § 5 that the critical step in the solution of the equation (1) of § 1 is that *the complete solution of* $xyz^2 = uvw^2$ *is*

$$\begin{aligned} x &= a\,\alpha_1\beta_1\varphi_1^2, & u &= a\,\alpha_2\beta_2\varphi_2^2, \\ y &= b\,\alpha_2\beta_1\psi_1^2, & v &= b\,\alpha_1\beta_2\psi_2^2, \\ z &= c\,\beta_2\varphi_2\psi_2, & w &= c\,\beta_1\varphi_1\psi_1, \end{aligned}$$

[4] The customary sign for $(x \mid y)'$ is not carried by the printer.

where the 11 *integer parameters* $a, b, c, \alpha_1, \alpha_2, \beta_1, \beta_2, \varphi_1, \varphi_2, \psi_1, \psi_2$ *are subject to either of the equivalent sets of conditions*

$$a = (x, u), \quad b = (y, v), \quad c = (z, w);$$
$$1 = (\alpha_1 \beta_1 \varphi_1, \alpha_2 \beta_2 \varphi_2) = (\alpha_1 \beta_2 \psi_2, \alpha_2 \beta_1 \psi_1) = (\beta_1 \varphi_1 \psi_1, \beta_2 \varphi_2 \psi_2).$$

Professor Ward has given a much shorter proof of this than the chain process followed here, but as each link of the chain gives the complete solution of an equation occurring frequently in applications of § 2, the longer proof is presented.

(2) *The complete solution of*

$$xy = uv$$

is

$$x = \delta\varphi, \quad y = \mu\psi, \quad u = \delta\psi, \quad v = \mu\varphi; \quad (\varphi, \psi) = 1.$$

Proof [§ 3].

$$(x, u) = \delta_1; \quad x = \delta_1 x_1, \quad u = \delta_1 u_1; \quad (x_1, u_1) = 1.$$
$$x_1 y = u_1 v: \quad x_1 | v, u_1 | y; \quad y = y_1 u_1, \quad v = v_1 x_1.$$
$$x_1 y_1 u_1 = u_1 v_1 x_1; \quad y_1 = v_1 = \mu_1.$$
$$\delta_1, \mu_1, x_1, u_1 \to \delta, \mu, \varphi, \psi.$$

The arrow indicates a change in notation.

(3) *The complete solution of*

$$xy = u^2$$

is

$$x = \delta\varphi^2, \quad y = \delta\psi^2, \quad u = \delta\varphi\psi; \quad (\varphi, \psi) = 1.$$

Proof [2]. Take $v = u$ in (2), and apply (2) to the resulting (2):

$$u = \delta\psi = \mu\varphi;$$
$$\delta = \delta_1\varphi_1, \quad \psi = \mu_1\psi_1, \quad \mu = \delta_1\psi_1, \quad \varphi = \mu_1\varphi_1; \quad (\varphi_1, \psi_1) = 1.$$
$$x = \delta\varphi = \delta_1\mu_1\varphi_1^2; \quad y = \mu\psi = \delta_1\mu_1\psi_1^2; \quad u = \delta_1\mu_1\varphi_1\psi_1.$$

Here δ_1, μ_1 occur only in the combination $\delta_1\mu_1$. Hence $\delta_1\mu_1$ can be replaced by a single letter.

$$\delta_1\mu_1, \varphi_1, \psi_1 \to \delta, \varphi, \psi.$$

(4) *The complete solution of*

$$xy = uv^2$$

is

$$x = \alpha\beta\sigma^2, \quad y = \beta\gamma\tau^2, \quad u = \gamma\alpha, \quad v = \beta\sigma\tau;$$
$$1 = (\gamma, \beta\sigma) = (\sigma, \tau).$$

Proof [2, 3].

$$x = \delta\varphi, \quad y = \mu\psi, \quad u = \delta\psi, \quad v^2 = \mu\varphi, \quad (\varphi, \psi) = 1;$$
$$v = \lambda\theta\chi, \quad \mu = \lambda\theta^2, \quad \varphi = \lambda\chi^2, \quad (\theta, \chi) = 1;$$
$$\delta, \lambda, \chi, \psi, \theta \to \alpha, \beta, \sigma, \gamma, \tau.$$

(5) *The complete solution*[5] *of*

$$xy^2 = uv^2$$

is

$$x = \lambda\chi^2, \quad y = \nu\theta, \quad u = \lambda\theta^2, \quad v = \nu\chi; \quad (\theta, \chi) = 1.$$

Proof [3.6].

$$x = ax_1, \quad u = au_1, \quad (x_1, u_1) = 1;$$
$$y = by_1, \quad v = bv_1, \quad (y_1, v_1) = 1.$$
$$x_1y_1^2 = u_1v_1^2;$$
$$x_1 | v_1^2, \quad y_1^2 | u_1, \quad u_1 | y_1^2, \quad v_1^2 | x_1;$$
$$x_1 = jv_1^2, \quad u_1 = jy_1^2, \quad |j| = 1;$$
$$x = jav_1^2, \quad u = jay_1^2, \quad y = by_1, \quad v = bv_1.$$
$$ja, b, y_1, v_1 \to \lambda, \nu, \theta, \chi.$$

(6) *The complete solution of the system*

$$x_1y_1 = uv_1^2, \quad x_2y_2 = uv_2^2$$

is

$$x_1 = nl\zeta\theta^2, \quad x_2 = rl\xi\pi^2,$$
$$y_1 = mn\xi\chi^2, \quad y_2 = mr\zeta\varepsilon^2,$$
$$v_1 = n\theta\chi, \quad v_2 = r\pi\varepsilon,$$
$$u = lm\xi\zeta;$$
$$1 = (n\theta, m\xi) = (r\pi, m\zeta) = (\theta, \chi) = (\pi, \varepsilon) = (\xi, \zeta).$$

Proof [4,2].

$$x_1 = \alpha_1\beta_1\sigma_1^2, \quad x_2 = \alpha_2\beta_2\sigma_2^2,$$
$$y_1 = \beta_1\gamma_1\tau_1^2, \quad y_2 = \beta_2\gamma_2\tau_2^2,$$
$$u = \gamma_1\alpha_1, \quad u = \gamma_2\alpha_2,$$
$$v_1 = \beta_1\sigma_1\tau_1, \quad v_2 = \beta_2\sigma_2\tau_2;$$

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[5] The most obvious method of attacking this equation is an application of (3), (4), or of (2), (3), similar to the solution of (4). But this, as can be easily verified, *will not lead to a solution*, since one equation to be solved reappears (in new variables) at two steps. Possibly there is the germ of an interesting invariance here. In a paper on Reciprocal Arrays and Diophantine Analysis, to be printed in the American Journal of Mathematics, I give a uniform method for all such equations in which this curious invariance can not occur. The reappearance of the original equation is like that in the method of descent.

$$1 = (\gamma_1, \beta_1\sigma_1) = (\gamma_2, \beta_2\sigma_2) = (\sigma_1, \tau_1) = (\sigma_2, \tau_2).$$

$$\gamma_1 = \delta\varphi, \quad \alpha_1 = \mu\psi, \quad \gamma_2 = \delta\psi, \quad \alpha_2 = \mu\varphi; \quad (\varphi, \psi) = 1.$$

$$\mu, \delta, \beta_1, \beta_2, \sigma_1, \tau_1, \sigma_2, \tau_2, \varphi, \psi,$$
$$\rightarrow l, m, n, r, \theta, \chi, \pi, \varepsilon, \xi, \zeta.$$

We now prove (1) of this section. Apply (2) to $xyz^2 = uvw^2$. Then

$$xy = \delta\varphi, \quad z^2 = \mu\psi, \quad uv = \delta\psi, \quad w^2 = \mu\varphi; \quad (\varphi, \psi) = 1.$$

Hence [2,3],

$$x = \delta_1\varphi_1, \quad z = \varepsilon_1\varrho_1\zeta_1, \quad u = \delta_2\varphi_2, \quad w = \varepsilon_2\varrho_2\zeta_2,$$
$$y = \mu_1\psi_1, \quad \mu = \varepsilon_1\varrho_1^2, \quad v = \mu_2\psi_2, \quad \mu = \varepsilon_2\varrho_2^2,$$
$$\delta = \delta_1\psi_1, \quad \psi = \varepsilon_1\zeta_1^2, \quad \delta = \delta_2\psi_2, \quad \varphi = \varepsilon_2\zeta_2^2;$$
$$\varphi = \mu_1\varphi_1, \quad \psi = \mu_2\varphi_2,$$
$$1 = (\varphi_1, \psi_1) = (\varphi_2, \psi_2) = (\varrho_1, \zeta_1) = (\varrho_2, \zeta_2).$$

Here apply (2) to the values of δ, (4) to those of φ, ψ, and (5) to those of μ. Then

$$\delta_1 = \delta_3\varphi_3, \quad \mu_1 = \alpha_1\beta_1\sigma_1^2, \quad \varepsilon_1 = \lambda_1\chi_1^2, \quad \mu_2 = \alpha_2\beta_2\sigma_2^2,$$
$$\psi_1 = \mu_3\psi_3, \quad \varphi_1 = \beta_1\gamma_1\tau_1^2, \quad \varrho_1 = \nu_1\theta_1, \quad \varphi_2 = \beta_2\gamma_2\tau_2^2,$$
$$\delta_2 = \delta_3\psi_3, \quad \varepsilon_2 = \gamma_1\alpha_1, \quad \varepsilon_2 = \lambda_1\theta_1^2, \quad \varepsilon_1 = \gamma_2\alpha_2,$$
$$\psi_2 = \mu_3\varphi_3, \quad \zeta_2 = \beta_1\sigma_1\tau_1, \quad \varrho_2 = \nu_1\chi_1, \quad \zeta_1 = \beta_2\sigma_2\tau_2;$$
$$1 = (\gamma_i, \beta_i\sigma_i) = (\sigma_i, \tau_i) \qquad (i = 1, 2),$$
$$1 = (\varphi_3, \psi_3) = (\theta_1, \chi_1).$$

Apply (6) to the values of $\varepsilon_1, \varepsilon_2$:

$$\gamma_2\alpha_2 = \lambda_1\chi_1^2, \quad \gamma_1\alpha_1 = \lambda_1\theta_1^2;$$
$$\gamma_1 = \mu_4\beta_3\psi_4\sigma_3^2, \quad \gamma_2 = \mu_4\beta_4\varphi_4\sigma_4^2,$$
$$\alpha_1 = \beta_3\delta_4\varphi_4\tau_3^2, \quad \alpha_2 = \beta_4\delta_4\psi_4\tau_4^2,$$
$$\theta_1 = \beta_3\sigma_3\tau_3, \quad \chi_1 = \beta_4\sigma_4\tau_4,$$
$$\lambda_1 = \delta_4\mu_4\varphi_4\psi_4;$$
$$1 = (\beta_3\sigma_3, \delta_4\varphi_4) = (\beta_4\sigma_4, \delta_4\psi_4) = (\sigma_3, \tau_3) = (\sigma_4, \tau_4) = (\varphi_4, \psi_4).$$

By substituting back we get the complete solution required. The resulting form can be greatly simplified, as we shall see in a moment, but as the unreduced form will be required in another application of § 2 (to the discriminant of a ternary quadratic) which will be considered elsewhere, we give the crude solution.

(7) *The complete solution (unreduced) of*

$$xyz^2 = uvw^2$$

is

$$\begin{aligned}
x &= \delta_3\,\mu_4\,\varphi_3\,\psi_4\,\beta_1\,\beta_3\,(\sigma_3\,\tau_1)^2,\\
y &= \delta_4\,\mu_3\,\varphi_4\,\psi_3\,\beta_1\,\beta_3\,(\sigma_1\,\tau_3)^2,\\
z &= \delta_4\,\mu_4\,\varphi_4\,\psi_4\,\beta_2\,\beta_3\,\sigma_2\,\sigma_3\,\tau_2\,\tau_3\,\nu_1\,(\beta_4\,\sigma_4\,\tau_4)^2,\\
u &= \delta_3\,\mu_4\,\varphi_4\,\psi_3\,\beta_2\,\beta_4\,(\sigma_4\,\tau_2)^2,\\
v &= \delta_4\,\mu_3\,\varphi_3\,\psi_4\,\beta_2\,\beta_4\,(\sigma_2\,\tau_4)^2,\\
w &= \delta_4\,\mu_4\,\varphi_4\,\psi_4\,\beta_1\,\beta_4\,\sigma_1\,\sigma_4\,\tau_1\,\tau_4\,\nu_1\,(\beta_3\,\sigma_3\,\tau_3)^2,
\end{aligned}$$

where the 21 integer parameters are subject to the conditions

$$\begin{aligned}
1 &= (\beta_1\,\beta_3\,\sigma_1\,\sigma_3\,\tau_1\,\tau_3,\ \beta_2\,\beta_4\,\sigma_2\,\sigma_4\,\tau_2\,\tau_4) = (\beta_1\,\sigma_1,\ \beta_3\,\sigma_3) = (\beta_2\,\sigma_2,\ \beta_4\,\sigma_4),\\
&= (\beta_1\,\sigma_1\,\tau_1,\ \nu_1\,\beta_4\,\sigma_4\,\tau_4) = (\beta_2\,\sigma_2\,\tau_2,\ \nu_1\,\beta_3\,\sigma_3\,\tau_3) = (\beta_3\,\sigma_3\,\tau_3,\ \beta_4\,\sigma_4\,\tau_4),\\
&= (\beta_1\,\beta_3\,\sigma_3\,\tau_1,\ \mu_3\,\psi_3) = (\beta_2\,\beta_4\,\sigma_4\,\tau_2,\ \mu_3\,\varphi_3),\\
&= (\varphi_3,\ \psi_3) = (\sigma_i,\ \tau_i) \qquad (i = 1, \cdots, 4);\\
1 &= |\,\delta_4\,\mu_4\,\varphi_4\,\psi_4\,| \qquad [1\cdot 4].
\end{aligned}$$

Two processes of reduction, *absorption of units*, and *cancellation of G.C.D.'s*, used in all chain solutions, are now applied.

The units $\delta_4, \mu_4, \varphi_4, \psi_4$ are absorbed by regrouping the factors in $x, \cdots, w$. Here the appropriate pairs are

$$\mu_4, \delta_3;\quad \psi_4, \varphi_3;\quad \delta_4, \mu_3;\quad \varphi_4, \psi_3;\quad \delta_4\,\mu_4\,\varphi_4\,\psi_4, \nu_1.$$

Hence $\delta_4, \mu_4, \varphi_4, \psi_4$ may be suppressed in the solution $x, \cdots, w$.

The second reduction can be applied at any link of the chain. If the shortest solution is sought, the reduction is made at the beginning. This sacrifices at least one of the complete solutions obtained incidentally in the entire chain, if the chain has more than two links. If instead of solving $xyz^2 = uvw^2$, we start with

$$\begin{aligned}
&XYZ^2 = UVW^2,\\
&(X, U) = a,\quad X = ax,\quad U = au,\\
&(Y, V) = b,\quad Y = by,\quad V = bv,\\
&(Z, W) = c,\quad Z = cz,\quad W = cw,
\end{aligned}$$

we get

$$xyz^2 = uvw^2;\quad 1 = (x, u) = (y, v) = (z, w).$$

The last 3 conditions applied to the unreduced $x, \cdots, w$ in (7), with the units suppressed by the first reduction, give

$$1 = |\,\delta_3\,| = |\,\mu_3\,| = |\,\beta_3\,\beta_4\,\nu_1\,\sigma_3\,\sigma_4\,\tau_3\,\tau_4\,|.$$

These new units can be absorbed:

$$x, \mu_3; \quad u, \mu_3; \quad y, \delta_3; \quad v, \delta_3;$$
$$\varphi_3, \delta_3\mu_3\beta_3\beta_4; \quad \psi_3, \delta_3\mu_3\beta_3\beta_4; \quad \beta_1, \beta_4; \quad \beta_2, \beta_3;$$
$$\sigma_2, \sigma_3; \quad \sigma_1, \sigma_4; \quad \tau_1, \nu_1\tau_4; \quad \tau_2, \nu_1\tau_3;$$

hence they may be suppressed in (7). The reduced conditions $(x, u) = 1$, etc., and those in (7) are now compared. The set from (7) contains

$$1 = (\sigma_1, \tau_1) = (\sigma_2, \tau_2)$$

as the only conditions not included in the second set which is

$$1 = (\varphi_3\beta_1\tau_1, \psi_3\beta_2\tau_2) = (\varphi_3\beta_2\sigma_2, \psi_3\beta_1\sigma_1) = (\beta_1\sigma_1\tau_1, \beta_2\sigma_2\tau_2).$$

To see that the above pair is superfluous, assume on the contrary that

$$\sigma_1 = \sigma_3\sigma, \quad \tau_1 = \tau_3\sigma; \quad (\sigma_3, \tau_3) = 1, \quad \sigma > 1;$$
$$\sigma_2 = \sigma_4\tau, \quad \tau_2 = \tau_4\tau; \quad (\sigma_4, \tau_4) = 1, \quad \tau > 1,$$

and write $\beta_1' \equiv \sigma^2\beta_1$, $\beta_2' \equiv \tau^2\beta_2$. Then, for the values β_1', β_2' of β_1, β_2 and $\tau_3, \tau_4, \sigma_3, \sigma_4$ of $\tau_1, \tau_2, \sigma_1, \sigma_2$ the two conditions are included in the second set.

Finally, by the change of notation

$$\varphi_3, \psi_3; \quad \sigma_1, \sigma_2; \quad \tau_1, \tau_2,$$
$$\to \alpha_1, \alpha_2; \quad \psi_1, \psi_2; \quad \varphi_1, \varphi_2,$$

we get the conclusion in (1) of this section.

5. Applying the method of § 2 to the cubic without binomial coefficients, we write the identity in t,

$$at^3 + bt^2 + ct + d \equiv f(t+g)(t+h)^2,$$

and get *the parametric representation* (2) *of* (1),

$$(1) \qquad (9ad - bc)^2 - 4(b^2 - 3ac)(c^2 - 3bd) = 0,$$

$$(2) \quad a = f, \quad b = f(g+2h), \quad c = fh(2g+h), \quad d = fgh^2.$$

The integer solutions a, b, c, d *of* (1) *are required.*

If b, c be replaced by $3b, 3c$, the solution of

$$(ad - bc)^2 - 4(b^2 - ac)(c^2 - bd) = 0$$

follows from that of (1) by the method suggested in § 2, end.

To find all integer solutions [§ 2] of (1), it suffices to find all values of f, g, h which make a, b, c, d integers in (2). The solutions with $d = 0$ may be found more readily, as follows.

If $d = 0$ in (1), then

$$c^2(b^2 - 4ac) = 0, \tag{3}$$

and the integer solutions a, b, c of (3) are required. If $c \neq 0$ in (3), then $b^2 - 4ac = 0$, and if in the last $a \neq 0$, then b is even, $b \neq 0$, and hence [4.3]

$$a = \delta\varphi^2, \quad b = 2\delta\varphi\psi, \quad c = \delta\psi^2; \quad (\varphi, \psi) = 1; \quad \delta\varphi\psi \neq 0.$$

But if $c \neq 0$, $a = 0$, then $b = 0$, and c is an arbitrary integer. Next, if $c = 0$ in (3), then a, b are arbitrary integers. This exhausts the possibilities for (1) with $d = 0$. We shall prove next

(4) *If $d \neq 0$ in* (1), *then g, h in* (2) *are rational.*

Proof. $fgh \neq 0$; f integer [1; $d \neq 0$].

$$g + 2h = r, \qquad h(2g + h) = s, \qquad gh^2 = u, \tag{5}$$

$$r, s, u, \delta \equiv r^2 - 3s \text{ rational } [f, a, b, c, d \text{ are integers}]. \tag{6}$$

$$0 = 3g^2 - 2rg + (4s - r^2) = 3h^2 - 2rh + s \quad [5]; \tag{7}$$

$$3g = r \pm 2\delta^{1/2}, \qquad 3h = r \mp \delta^{1/2} \quad [7]; \tag{8}$$

$$27u = r(r^2 - 3\delta) \pm 2\delta \cdot \delta^{1/2} \quad [5,7]. \tag{9}$$

$$\delta^{1/2} \text{ rational} \quad [6,9], \tag{10}$$

$$g, h \text{ rational} \quad [8,10]. \tag{11}$$

The simple conclusion here is no criterion for other equations solvable by the present method.[6]

We now find all integer solutions of (1) with $d \neq 0$. Hence $fgh \neq 0$; *f is an integer* [1,2]; g, h are rational numbers [4]. Hence we may take

$$g = g_1/g_2, \; h = h_1/h_2; \quad g_1, g_2, h_1, h_2 \text{ integers} \neq 0. \tag{12}$$

Hence [2,4.1],

$$\begin{gathered} fg_1h_1^2 = dg_2h_2^2, \\ f = p\alpha_1\beta_1\varphi_1^2, \quad d = p\alpha_2\beta_2\varphi_2^2, \\ g_1 = q\alpha_2\beta_1\psi_1^2, \quad g_2 = q\alpha_1\beta_2\psi_2^2, \\ h_1 = r\beta_2\varphi_2\psi_2, \quad h_2 = r\beta_1\varphi_1\psi_1; \end{gathered} \tag{13}$$

$$1 = (\alpha_1\beta_1\varphi_1, \alpha_2\beta_2\varphi_2) = (\alpha_1\beta_2\psi_2, \alpha_2\beta_1\psi_1) = (\beta_1\varphi_1\psi_1, \beta_2\varphi_2\psi_2);$$

[6] Thus $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ requires 3 quadratic irrationalities among possible values of the associated parameters. At one stage of the solution of this equation over 4000 parameters appear.

all the letters denote integers $\neq 0$. Hence [2, 12, 13],

$$(14)\qquad \begin{aligned} b\beta_2\psi_1\psi_2^2 &= p(2\alpha_1\beta_2^2\varphi_1\varphi_2\psi_2^3+\alpha_2\beta_1^2\varphi_1^2\psi_1^3),\\ c\beta_1\psi_1^2\psi_2 &= p(2\alpha_2\beta_1^2\varphi_1\varphi_2\psi_1^3+\alpha_1\beta_2^2\varphi_2^2\psi_2^3), \end{aligned}$$

and therefore [14, last of 13, 3.7, 3.8],

$$(15)\qquad \beta_2\psi_2^2 \mid p,\quad \psi_1 \mid 2p\varphi_1,\quad \beta_1\psi_1^2 \mid p,\quad \psi_2 \mid 2p\varphi_2.$$

The second and fourth of (15) are redundant. Hence [15, last of 13, 4.2],

$$\begin{gathered} p = p_1\beta_1\psi_1^2,\quad p = p_2\beta_2\psi_2^2;\qquad p_1,\ p_2 \text{ integers} \neq 0;\\ p_1\beta_1\psi_1^2 = p_2\beta_2\psi_2^2,\quad (\beta_1\psi_1,\ \beta_2\psi_2) = 1;\\ p_1 = \delta\varphi,\quad \beta_1\psi_1^2 = \mu\psi,\quad p_2 = \delta\psi,\quad \beta_2\psi_2^2 = \mu\varphi, \end{gathered}$$

where all the letters denote integers $\neq 0$, and

$$1 = |\mu| = (\varphi, \psi).$$

Absorb μ in β_1, β_2. Then

$$(16)\qquad p = \delta\beta_1\beta_2\psi_1^2\psi_2^2 \quad (\delta \text{ integer}, \neq 0).$$

Using (16) in (14) we have the results stated for b, c in (2) of the summary in § 6.

6. *All integer solutions a, b, c, d of*

$$(9ad-bc)^2-4(b^2-3ac)(c^2-3bd) = 0$$

are as follows, in which all letters denote integers.

(1) *Those in which $d = 0$:*

$$a = \delta\varphi^2,\quad b = 2\delta\varphi\psi,\quad c = \delta\psi^2;\qquad (\varphi, \psi) = 1;\qquad \delta\varphi\psi \neq 0;$$

$$a = b = 0,\quad c \text{ arbitrary};$$

$$a, b \text{ arbitrary},\quad c = 0.$$

(2) *Those in which $d \neq 0$:*

$$\begin{aligned} a &= \delta\alpha_1\beta_1^2\beta_2\varphi_1^2\psi_1^2\psi_2^2,\\ b &= \delta\beta_1\psi_1(2\alpha_1\beta_2^2\varphi_1\varphi_2\psi_2^3+\alpha_2\beta_1^2\varphi_1^2\psi_1^3),\\ c &= \delta\beta_2\psi_2(2\alpha_2\beta_1^2\varphi_1\varphi_2\psi_1^3+\alpha_1\beta_2^2\varphi_2^2\psi_2^3),\\ d &= \delta\alpha_2\beta_1\beta_2^2\varphi_2^2\psi_1^2\psi_2^2; \end{aligned}$$

all Greek letters denote integers $\neq 0$.

7. The method mentioned in the footnote to § 5 (4) enables us to obtain the complete solution in integers a, b, c, d of the equation

$$(9a^md^n-bc)^2-4(b^2-3a^mc)(c^2-3bd^n) = 0,$$

where m, n are any constant integers >0, and similarly for

$$(a^m d^n - bc)^2 - 4(b^2 - a^m c)(c^2 - bd^n) = 0,$$

from the solution (not reproduced here) of the equation after § 1 (1). For, by that method, we find in a straightforward way the complete solution in integers $x_1, \cdots, x_r, y_1, \cdots, y_s$ of the equation (among many others)

$$x_1^{\alpha_1} \cdots x_r^{\alpha_r} = y_1^{\beta_1} \cdots y_s^{\beta_s},$$

where the α, β are arbitrary constant integers >0.

This is applied to the values of a, d in § 6,

$$\lambda^m = \delta \alpha_1 \beta_1^2 \beta_2 \varphi_1^2 \psi_1^2 \psi_2^2,$$
$$\mu^n = \delta \alpha_2 \beta_1 \beta_2^2 \varphi_2^2 \psi_1^2 \psi_2^2,$$

where the equations are treated separately, reapplied to the parametric expressions thus obtained for δ, β_1, β_2, ψ_1, ψ_2, and so on, until no parameter appears with distinct parametric expressions. The process terminates in a number of steps which is determined by m, n. The total number of parameters in the final formulas increases with astonishing rapidity as m, n increase.

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FACTORIZATION OF CERTAIN CYCLOTOMIC FUNCTIONS.[1]

By D. H. Lehmer.[2]

Introduction. Two general methods for discovering large primes have been in use for a long time. The older method consists of searching directly or indirectly for a factor of a given number. If no factor is found we conclude that the number is a prime. The last half century has seen the introduction of direct or positive criteria for primality by means of which the given number is shown to be a prime if it divides a certain other number. This latter method has been developed to such an extent that it has outstripped the former method. Nevertheless there are many special numbers whose possible prime factors are so restricted that the first method lends itself more readily than the second. These numbers are values of special numerical functions which are the best substitutes we have for actual formulas for prime numbers. Perhaps the most fundamental functions of this kind are special cases of Δ_n defined by (2). This paper is devoted to a detailed study of this function especially as regards sequences of numbers to which the first method of discovering primes has practical applications.

1. **Definition of Δ_n.** Let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be the roots of the polynomial

$$f(x) = x^r + a_1 x^{r-1} + a_2 x^{r-2} + \cdots + a_r \tag{1}$$

whose coefficients are rational integers. Then Δ_n is defined by

$$\Delta_n = \Delta_n(f) = \prod_{\nu=1}^{r} (\alpha_\nu^n - 1). \tag{2}$$

This function was introduced by T. A. Pierce[3] who has studied the forms of its primitive factors.

2. **Essential and characteristic factors of Δ_n.** Let $Q_m(x)$ be the irreducible cyclotomic function whose roots are the primitive mth roots of unity. Then we define the integer Q_m^* by

$$Q_m^* = Q_m^*(f) = \prod_{\nu=1}^{r} Q_m(\alpha_\nu). \tag{3}$$

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[2] National Research Fellow.

[3] Annals of Mathematics (2), 18, pp. 53–64.

It follows from a similar property[4] of $Q_m(x)$ that

$$\Delta_n = \prod_{\delta|n} Q_\delta^*, \tag{4}$$

where δ, as indicated, ranges over *all* the divisors of n. This gives a partial factorization of Δ_n into integer factors. If we assume that each Δ, whose subscript is a *proper* divisor of n, has been factored, the complete factorization of Δ_n depends only on that of Q_n^*. For this reason we call this latter number the *essential factor* of Δ_n. Under this same assumption we may factor Δ_n completely when we know the prime factors of Δ_n which do *not* divide Δ_d, where d is a proper divisor of n. These factors are called the *characteristic prime factors*[5] of Δ_n. We begin with a few theorems relative to essential and characteristic factors.

THEOREM 1. *The essential factor Q_n^* of Δ_n contains all the characteristic prime factors of Δ_n.*

Proof. By (4) a characteristic prime factor p of Δ_n must divide Q_δ^* for some divisor δ of n. If δ were less than n, Q_δ^*, and hence p would divide Δ_δ, contrary to the definition of p. Therefore $\delta = n$ and the theorem follows.

It is not true that the essential factor of Δ_n is made up exclusively of characteristic prime factors as the following example shows. Consider $f(x) = x^3 - x - 1$. Then $\Delta_{24} = 875$ and $Q_{24} = 25$. Here 5 is not a characteristic factor of Δ_{24} because $\Delta_4 = 5$. In this case Δ_{24} has no characteristic prime factor.

THEOREM 2. *A characteristic prime factor p of Δ_n cannot divide n.*

Proof. If possible, let $n = p\delta$. Then by the multinomial theorem modulo p, we have

$$\begin{aligned} 0 \equiv \Delta_n \equiv \Delta_{p\delta} &\equiv 1 - \Sigma \alpha_1^{p\delta} + \Sigma(\alpha_1\alpha_2)^{p\delta} - + \cdots \\ &\equiv (1 - \Sigma \alpha_1^{\delta} + \Sigma(\alpha_1\alpha_2)^{\delta} - + \cdots)^p \\ &\equiv \Delta_\delta^p \equiv \Delta_\delta \pmod p. \end{aligned}$$

But this contradicts the hypothesis that p is a characteristic factor of Δ_n.

The essential factor Q_n^* may, however, have a factor in common with n. For example, for $f(x) = x^3 + x + 1$, $\Delta_3 = -9$ and $Q_3^* = 3$.

Before going further it is worth noting that if f is reducible in the rational field so that $f_1 \cdot f_2 = f$, then

$$\Delta_n(f) = \Delta_n(f_1)\,\Delta_n(f_2)$$

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[4] Namely $x^n - 1 = \prod_{\delta|n} Q_\delta(x)$.

[5] Pierce, loc. cit., p. 64. In dealing with Lucas' U_n a characteristic prime factor is defined as one which appears as a factor of U_n, but which divides no U_k, where $k < n$. The distinction between the two definitions is essential.

is a factorization into integers. Hence for our purposes we may suppose that f is *irreducible.*

THEOREM 3. *If p^e $(e>0)$ is the highest power of a characteristic prime factor p of $\Delta_n(f)$, where f is irreducible and of degree r, and if ω is the exponent*[6] *to which p belongs modulo n, then $\omega \leqq r$ and e is divisible by ω.*

Proof. Let α be a root of f. We begin by showing that $\alpha^n - 1$ and p have a common non-unit factor in the algebraic number field generated by α. If possible, let p be prime to $\alpha^n - 1$. Then two integers x and y of the field exist such that

$$x(\alpha^n - 1) = yp + 1.$$

Taking the norms of both sides we obtain

$$N(x)N(\alpha^n - 1) = N(x)\Delta_n \equiv 1 \pmod{p}.$$

This is impossible since p divides Δ_n. Therefore $\alpha^n - 1$ and p have a factor in common. Let us write

$$\alpha^n - 1 = \mathfrak{k}\mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_t^{e_t}, \tag{5}$$

where $\mathfrak{k}$ and p are coprime and $\mathfrak{p}_\nu$ $(\nu = 1, 2, \cdots, t)$ are prime ideal divisors of p. Let the degree of $\mathfrak{p}_\nu$ be f_ν. Then

$$f_\nu \leqq r. \tag{6}$$

Since $\alpha^n \equiv 1 \pmod{\mathfrak{p}_\nu}$, α is prime to $\mathfrak{p}_\nu$. Let ε_ν be the exponent to which α belongs $(\mathrm{mod}\,\mathfrak{p}^\nu)$. Then by (5), ε_ν divides n. Writing

$$\alpha^{\varepsilon_\nu} - 1 = \mathfrak{c}\mathfrak{p}_\nu$$

and taking the norm of both sides we have

$$N(\alpha^{\varepsilon_\nu} - 1) = \Delta_{\varepsilon_\nu} \equiv 0 \pmod{p^{f_\nu}}.$$

But because p is a characteristic factor this implies $\varepsilon_\nu = n$.

By Fermat's theorem we have

$$\alpha^{\varphi(\mathfrak{p}_\nu)} \equiv 1 \pmod{\mathfrak{p}_\nu}.$$

Hence $\varepsilon = n$ divides $\varphi(\mathfrak{p}_\nu) = p^{f_\nu} - 1$. That is,

$$p^{f_\nu} \equiv 1 \pmod{n}.$$

Hence ω divides f_ν for every ν, so that the first part of the theorem is a consequence of (6).

[6] By Theorem 2, ω exists.

Writing $f_\nu = \sigma_\nu \omega$ and taking the norm of both sides of (5) we obtain

$$\Delta_n = N(\mathfrak{f}) p^{\tau\omega},$$

where $\tau = \Sigma e_\nu \sigma_\nu$. But $N(\mathfrak{f})$ is prime to p so that $\tau\omega = e$. This completes the proof.

If we think of n as fixed and p unknown, this theorem gives us all the results of Pierce concerning the forms of the characteristic factors of Δ_n. We have altered Pierce's method in order to use only the simplest facts and concepts of the theory of algebraic numbers, and in order to show that the divisors of the discriminant of f are actually not exceptional.

From Theorem 3 we can formulate the following two rules for finding the characteristic factors of $\Delta_n(f)$ for any f of degree r. Make a list of primes and powers of primes up to the rth power. For a given n choose those entries in this list which are of the form $nx+1$. Only these entries need be tried as factors of Δ_n. Or we may first try as divisors all primes of the form $nx+1$ less than $\sqrt{\Delta_n}$. Then we may try all primes less than $\sqrt[4]{\Delta_n}$ of the forms $nx+z_i$, where z_i are the solutions of the congruence $z^2 \equiv 1 \pmod{n}$, excluding of course the solution $z=1$ already considered. Next we may seek for prime factors less than $\sqrt[6]{\Delta_n}$ of the form $nx+w$, where $w^3 \equiv 1 \pmod{n}$. This process may be continued until finally we consider primes less than $\sqrt[2r]{\Delta_n}$ of the form $nx+u$, where $u^r \equiv 1 \pmod{n}$. All prime factors thus discovered must be removed from Δ_n to the highest power to which they occur.

A special case, important in the sequel, is that in which f is symmetric. For this case we can improve the preceding theorem as follows:

THEOREM 3A. *If p^e $(e>0)$ is the highest power of a characteristic prime factor p of $\Delta_n(f)$, where f is irreducible and symmetric and of degree*[7] *$r = 2s$, and if p^η is the least power of p congruent to $+1$ or -1 modulo n, then $\eta \leqq s$ and 2η divides e.*

Proof. Let us consider the field K and its subfield K_1 generated respectively by α and $\beta = \alpha + \alpha^{-1}$. Then the number

$$\alpha^{-\varphi(n)/2} Q_n(\alpha) = X(\beta),$$

being a rational integral function of β, is an integer in K_1. Taking the norm in K_1 we have

$$\{N(X(\beta))\}^2 = \prod_{\nu=1}^{2s} Q_n(\alpha_\nu) = Q_n^*(f) \equiv 0 \pmod{p}.$$

[7] If the degree r of $f(x)$ were odd, then -1 would be a root of $f(x)$.

From this we can conclude, as in the first part of the proof of the preceding theorem, that p and $X(\beta)$ have a common factor in K_1. Let $\mathfrak{p}$ be any prime ideal dividing this common factor, and let its degree be k. Then by Fermat's theorem

$$\beta^{p^k} \equiv \beta \pmod{\mathfrak{p}}.$$

From the identity

$$(x+y)^{p^k} \equiv x^{p^k} + y^{p^k} \pmod{p}$$

we have

$$\alpha^{p^k} + \alpha^{-p^k} \equiv \alpha + \alpha^{-1} \pmod{\mathfrak{p}}.$$

Let us now examine this congruence in the field K, taking as modulus any prime ideal factor $\mathfrak{q}$ of the ideal corresponding to $\mathfrak{p}$. We have then

$$\alpha^{-p^k}(\alpha^{p^k} - \alpha)(\alpha^{p^k} - \alpha^{-1}) \equiv 0 \pmod{\mathfrak{q}}.$$

Now α is prime to $\mathfrak{q}$ so that

$$\alpha^{p^k \pm 1} - 1 \equiv 0 \pmod{\mathfrak{q}}.$$

But $\mathfrak{q}$ is a factor of $Q_n(\alpha)$ which divides $\alpha^n - 1$. Hence, by the reasoning used in the last part of the proof of Theorem 3, n is the exponent to which α belongs (mod $\mathfrak{q}$). The last congruence then gives us

$$p^k \equiv \pm 1 \pmod{n}.$$

Hence k is a multiple of η and is $\leqq s$, the degree of K_1. Therefore $\eta \leqq s$. Since η divides the degree of every prime ideal factor common to $X(\beta)$ and p, we have, as in the preceding theorem,

$$N(X(\beta)) = cp^{w\eta},$$

where c is an integer prime to p. Since $\{N(X(\beta))\}^2 = Q_n^*$,

$$e = 2w\eta.$$

This proves the theorem.[8]

3. **Schur's function.** We consider next the forms of the primes dividing the essential factor. If we eliminate x from the pair of equations

$$y = f(x) \tag{7}$$

and

$$0 = Q_n(x) \tag{8}$$

we obtain a polynomial $\Phi(y)$ given by

[8] We are indebted to Professor Uspensky for helpful suggestions concerning the proofs of Theorems 3 and 3A.

$$\Phi(y) = \prod_{\nu=1}^{\varphi(n)} (y - f(\varepsilon_\nu)), \tag{9}$$

where $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{\varphi}$ are the distinct primitive nth roots of unity. This polynomial is the Tschirnhaus transform of Q_n by f and is a special case of a function introduced by Schur.[9] For this case Schur's result is as follows:

THEOREM 4. *If for some integer a, $\Phi_n(a)$ is divisible by the odd prime p then either*

1) *p divides n, or*
2) *p divides the discriminant $D(\Phi_n)$ of Φ_n, or*
3) *$p = nx+1$.*

It is possible to eliminate case 1) of this theorem by proving

THEOREM 5. *In Theorem* 4, 1) *implies* 2).

We need the following Lemma:

LEMMA 1. *Every odd prime factor of n divides the discriminant $D(Q_n)$ of $Q_n(x)$.*

Proof. Let $p_1, p_2, \cdots, p_t$ be the distinct prime factors of n. Then $D(Q_n)$ has the formula[10]

$$D(Q_n) = \frac{n^{\varphi(n)}}{\prod_{\nu=1}^{t} p_\nu^{\varphi(n)/(p_\nu-1)}}.$$

Hence if p^α is the highest power of a prime p dividing n, then $D(Q_n)$ will contain p^β, where

$$\beta = \alpha\varphi(n) - \frac{\varphi(n)}{p-1} = \frac{\varphi(n)}{p-1}(\alpha(p-1)-1).$$

If p is odd,[11] $\alpha(p-1)-1$ is positive so that p divides $D(Q_n)$.

Theorem 5 now follows easily. In fact it is sufficient to observe that

$$\frac{D(\Phi_n)}{D(Q_n)} = \prod_{i \neq j=1}^{\varphi(n)} \left(\frac{f(\varepsilon_i) - f(\varepsilon_j)}{\varepsilon_i - \varepsilon_j}\right)$$

is an integral symmetric function of the roots of Q_n. Hence $D(\Phi_n)$ is divisible by $D(Q_n)$ which in turn is divisible by every prime factor of n. This proves Theorem 5.

In order to show the connection between $\Phi_n(y)$ and $Q_n^*(f)$ we prove

THEOREM 6. $\Phi_n(0) = \pm Q_n^*(f)$.

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[9] Sitzungsberichte der Berliner Mathematischen Gesellschaft 11 (1912), p. 41.

[10] E. T. Lehmer, Bulletin of the American Mathematical Society 36, p. 296.

[11] Incidentally, if n is even, $n/2$ and $D(Q_n)$ are of the same parity.

Proof. By (9)

$$|\Phi_n(0)| = \left| \prod_{\nu=1}^{\varphi} f(\varepsilon_\nu) \right| = \prod_{\nu=1}^{\varphi} \prod_{i=1}^{r} |\alpha_i - \varepsilon_\nu|$$

$$= \left| \prod_{i=1}^{r} Q_n(\alpha_i) \right| = |Q_n^*(f)|.$$

Combining Theorems 1, 2, 4 and 6 we get as a substitute for Theorem 3

THEOREM 7. *The odd characteristic factors of Δ_n which do not divide $D(\Phi_n)$ are of the form $nx+1$.*

To illustrate both methods of factorization consider the example in which $f(x) = x^3 + 21x + 7$. It is found that

$$\Delta_1 = 29,$$
$$\Delta_2 = 435 = 29 \cdot (3 \cdot 5),$$
$$\Delta_3 = 9773 = 29 \cdot (337),$$
$$\Delta_4 = 195315 = 29 \cdot 3 \cdot 5 \cdot (449),$$
$$\Delta_6 = 88396785 = 29 \cdot 3^2 \cdot 5 \cdot 337 \cdot (301),$$

where the characteristic prime factors are enclosed in parentheses. To factor

$$\Delta_{12} = 78315\ 42603\ 204405$$

we first remove from Δ_{12} the characteristic prime factors of Δ_1, Δ_2, Δ_3, Δ_4, and Δ_6 to the powers to which they occur. We are left with 197317. Applying Theorem 3, we seek for prime factors p of this number such that p, p^2, or p^3 is of the form $12x+1$. Thus we find that $197317 = 23^2 \cdot 373$ where for the prime 23, $\omega = 2$ in Theorem 3. On the other hand we may write[12]

$$Q_{12}^* = \frac{\Delta_{12}\,\Delta_2}{\Delta_6\,\Delta_4} = 197317.$$

Moreover

$$Q_{12}(x) = x^4 - x^2 + 1. \tag{10}$$

Setting

$$y = f(x) = x^3 + 21x + 7 \tag{11}$$

and eliminating x from (10) and (11) we find

$$\Phi_{12}(y) = y^4 - 28y^3 - 103y^2 + 4186y + 197317$$

with discriminant

$$D(\Phi_{12}) = (1296153684)^2.$$

[12] It happens that Q_{12}^* contains only the characteristic prime factors of Δ_{12}. This is not true for Q_6^* which is equal to $903 = 3 \cdot 301$.

Taking the G. C. D. of this number and $\Phi_{12}(0) = Q_{12}^*$ we find $529 = 23^2$. The residual factor 373 of Q_n is seen to be a prime by Theorem 7. Comparing the two methods of factorization we see that the second has the advantage of decomposing the essential factor of Δ_n, in a nontentative way, into two factors one of which contains all the prime factors of Q_n^* which are not of the form $nx+1$. But without the first method we have no information regarding the prime factors of this divisor of Q_n^*. For much larger values of n, the calculation of $\Phi_n(y)$ and its discriminant becomes too laborious to recommend the application of Theorem 7. On the other hand the class of primes for which $\omega > 1$ in Theorem 3 becomes reasonably small as n increases.

Theorem 6 may lead the reader to think that the polynomial $\Phi_n(y)$ represents a larger class of numbers than $Q_n^*(f)$. As f varies, however, both functions cover the same class of numbers. To show this we need to put the variable f in evidence and write

$$\Phi_n(y) = \Phi_n(y; f), \quad Q_n^* = Q_n^*(f).$$

With n fixed we let y and f run through their possible values. Let C_1 and C_2 be the classes of values assumed by $\Phi_n(y; f)$ and $Q_n^*(f)$ respectively. By Theorem 6 each member of C_2 is in C_1. To show that each member of C_1 is in C_2 we prove

THEOREM 8. *If* $g(x) = f(x) - y$, *then*

$$\Phi_n(y; f) = Q_n^*(g).$$

Proof. By (9)

$$\Phi_n(y; f) = \prod_{\nu=1}^{\varphi} (y - f(\varepsilon_\nu)) = \prod_{\nu=1}^{\varphi} (0 - g(\varepsilon_\nu))$$
$$= \Phi_n(0; g) = Q_n^*(g).$$

4. **Poulet's functions.** P. Poulet[13] has introduced a function $P_n = P_n(f)$ defined below by (12) which bears the same relation to Δ_n that Lucas'[14] function U_n does to $x^n - 1$. Empirical evidence led Poulet to discover the forms of the factors of P_n, but he was not able to find a proof except in the case in which f is quadratic.[15] With the help of a recent extension of Lucas' theory[16] we show here that as far as factorization is concerned Poulet's function may be regarded as an interesting special case of Δ_n and

[13] L'Intermédiaire des Mathématiciens, vol. 27, pp. 86–87; (2), vol. 1, p. 47. (2), vol. 3, p. 61.

[14] Dickson's History of the Theory of Numbers, vol. 1, chap. 17.

[15] If f is linear, P_n reduces to Lucas' U. Poulet's proof is included in a manuscript communicated to the present writer.

[16] Annals of Mathematics, vol. 31, pp. 419–448.

that Poulet's empirical results are thereby readily explained. We take as definition of P_n the following explicit expression:

$$P_n = P_n(f) = \frac{2^r}{\sqrt{f(y)}} \prod_{\nu=1}^{r} \sin n \arccos(\sqrt{\alpha_\nu}/2), \tag{12}$$

where α_ν are the roots of $f(x) = 0$. For $\alpha_\nu = \varrho e^{i\theta}$ we interpret $\sqrt{\alpha_\nu} = +\sqrt{\varrho}\, e^{i\theta/2}$.

5. **Connection of P_n with Lucas' U_n.** In Lucas' function

$$U_n = \frac{a^n - b^n}{a - b}$$

set $a = e^{i\theta} = b^{-1}$, where θ may be complex. Then

$$U_n = \frac{\sin n\theta}{\sin \theta}. \tag{13}$$

If for convenience we set

$$4\cos^2\theta = x \tag{14}$$

we find that

$$U_n = U_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n-k-1}{k} x^{(n-2k-1)/2}. \tag{15}$$

This polynomial replaces $x^n - 1$ in Pierce's theory. From (13) and (14) it follows that

$$\prod_{\nu=1}^{r} U_n(\alpha_\nu) = \prod_{\nu=1}^{r} \frac{\sin n \arccos(\sqrt{\alpha_\nu}/2)}{\sin \arccos(\sqrt{\alpha_\nu}/2)}$$

$$= \prod_{\nu=1}^{r} 2 \frac{\sin n \arccos(\sqrt{\alpha_\nu}/2)}{\sqrt{4 - \alpha_\nu}} = P_n.$$

From (15) we see that $U_n(x)$ is either a polynomial in x with integer coefficients or $\sqrt{x}$ times such a polynomial according as n is odd or even. Hence $P_n = \prod_{\nu=1}^{r} U_n(\alpha_\nu)$ is an integer or an integer multiple of $\sqrt{(-1)^r a_r}$ according as n is odd or even.

6. **Algebraic factorization of P_n.** The polynomial $G_n(x)$ defined[17] by

$$G_1(x) = 1,$$
$$G_n(x + x^{-1} + 2) = Q_n(x)\, x^{-\varphi(n)/2} \tag{16}$$

[17] Equivalent definitions are $G_n(x) = \prod_{\delta|n} U_\delta(x)^{\mu(n/\delta)}$ (where μ is Möbius inversion function) and $G_n(x) = \prod_k (x - 4\cos^2(\pi k/n))$ where k runs over the positive integers less than $n/2$ and prime to n, $n > 2$.

takes the place of the Q function of § 2. We need mention only the following simple properties of $G_n(x)$:

$$G_2(x) = \sqrt{x}. \tag{17}$$

(18) $G_n(x)$ for $n>2$ has integer coefficients.

$$\prod_{\delta/n} G_\delta(x) = U_n(x). \tag{19}$$

If we introduce the function G_n^* defined by

$$G_n^* = G_n^*(f) = \prod_{\nu=1}^{r} G_n(\alpha_\nu) \tag{20}$$

we see that $G_1^* = 1$, $G_2^* = \sqrt{(-1)^r a_r}$, and that G_n^* for $n>2$ is an integer. Moreover from (19) and (20) we have

$$P_n = \prod_{\nu=1}^{r} U_n(\alpha_\nu) = \prod_{\delta/n} G_\delta^*. \tag{21}$$

This gives us a partial factorization of P_n similar to (4) and throws the burden of factorization on G_n^*. This latter number we call the *essential factor of* P_n.

7. **Relations between Pierce's and Poulet's functions.** Consider the polynomial $F(x)$ defined by

$$F(x) = \prod_{\nu=1}^{r} (x - (\alpha_\nu + \alpha_\nu^{-1} + 2))$$

where α_ν are the roots of $f(x)$. From (16) we have

$$\alpha_\nu^{\varphi(n)/2} G_n(\alpha_\nu + \alpha_\nu^{-1} + 2) = Q_n(\alpha_\nu),$$

or

$$(-1)^{r\varphi(n)/2} a_r^{\varphi(n)/2} G_n^*(F) = Q_n^*(f). \tag{22}$$

It is easy to see that $F(x)$ has coefficients which are integer multiples of $1/a_r$. Hence $G_n(f)$ is a rational number of the form $q/a_r^{\varphi(n)/2}$ where q is an integer. It thus appears that if we agree to include Poulet's functions of polynomials whose coefficients are rational numbers we can represent the essential factor of $\Delta_n(f)$, where f has integer coefficients, as the essential factor of an appropriately chosen Poulet's function.

In order to show that the essential factor of P_n is included in that of Δ_n we give the following

THEOREM 9. *Let* $f(x) = x^r + a_1 x^{r-1} + \cdots + a_r$ *be any polynomial with integer coefficients and let*

$$\psi(x) = x^r f(x + x^{-1} + 2).$$

Then

$$(G_n^*(f))^2 = Q_n^*(\psi).$$

Proof. Let $\beta_1, \beta_2, \cdots, \beta_{2r}$ be the roots of $\psi(x)$. Since these roots occur in reciprocal pairs the polynomial

$$F(x) = \prod_{\nu=1}^{2r} (x - (\beta_\nu + \beta_\nu^{-1} + 2))$$

is a perfect square. In fact we see that

$$F(x) = (f(x))^2.$$

Hence (22) becomes

$$G_n^*(F) = (G_n^*(f))^2 = Q_n^*(\psi).$$

From this theorem we have at once three others.

THEOREM 10. *Under the hypothesis of Theorem 9*

$$P_n^2(f) = \Delta_n(\psi)/\Delta_1(\psi).$$

THEOREM 11. *If f is a symmetric polynomial, then $Q_n(f)$ is a perfect square.*

THEOREM 12. *If $p^e (e > 0)$ is the highest power of a characteristic prime factor p of $P_n(f)$, where f is of degree r, and if p^η is the least power of p congruent to $+1$ or -1 modulo n, then $\eta \leqq r$ and η divides e.*

This theorem follows from Theorem 10 and Theorem 3A applied to $\Delta_n(\psi)$, where ψ is of degree $2r$ and with the e of Theorem 3A replaced by $2e$.

Considering n as fixed, and p unknown this theorem establishes rigorously the facts conjectured by Poulet, namely, that if p is a characteristic prime factor of P_n, then some power of p not greater than the rth power is of the form $nx \pm 1$. It shows, moreover, that the highest power of p dividing P_n is itself of this form. We can therefore formulate from Theorem 12 rules for the factorization of P_n similar to those given for Δ_n following Theorem 3.

8. **The recurring series for Δ_n.** In order to render the factorization of Δ_n practical, it is first necessary to have a simple method of calculating its actual value. This is done with the help of a polynomial $M(x)$ uniquely determined by $f(x)$ in the following manner. Let

$$f_0(x) = x-1,$$
$$f_1(x) = f(x) = \prod_{i=1}^{r}(x-\alpha_i),$$
$$f_2(x) = \prod_{i>j=1}^{r-1}(x-\alpha_i\alpha_j),$$
$$\cdots\cdots\cdots\cdots$$
$$f_r(x) = x-\alpha_1\alpha_2\cdots\alpha_r = x-(-1)^r a_r.$$

In general, if $0<\nu\leqq r$, $f_\nu(x)$ has for roots the products of roots of f taken ν at a time. $M(x)$ is defined as the least common multiple of these f's. That is,

$$M(x) = [f_0(x), f_1(x), \cdots, f_r(x)]$$
$$= x^m + A_1 x^{m-1} + A_2 x^{m-2} + \cdots + A_m.$$

Then we have

THEOREM 13. *The numbers*

$$\Delta_0, \Delta_1, \Delta_2, \Delta_3, \cdots$$

form a recurring series whose scale is $M(x)$. That is, for every $n\geqq 0$,

$$\Delta_{n+m} + A_1\Delta_{n+m-1} + \cdots + A_{m-1}\Delta_{n+1} + A_m\Delta_n = 0. \tag{23}$$

Proof. If we perform the indicated multiplication in

$$\Delta_n = \prod_{\nu=1}^{r}(\alpha_\nu^n - 1),$$

we see that Δ_n is a linear combination (with integer coefficients) of the nth powers of the roots of $M(x)$. Theorem 13 follows at once from the theorem of Lagrange.[18]

The polynomial $M(x)$ may of course be obtained[19] rationally in terms of the coefficients of f. Also, it is not necessary to use $M(x)$. In fact any common multiple of $f_0, f_1, f_2, \cdots, f_r$, for instance their product, may replace $M(x)$. In any case we are led to a recurrence formula whose order does not exceed 2^r from which the successive Δ's may be readily calculated. It is hardly more difficult to compute the polynomials $f_1, f_2, \cdots, f_r$ for a general f than to make the calculation for a single particular case. The general results for $r<6$ are given, once and for all, below.

$r = 2$ $\quad f_1(x) = x^2 + a_1 x + a_2,$
$\quad\quad\quad f_2(x) = x - a_2.$

[18] Lucas, *Théorie des Nombres*, p. 304.

[19] In some cases when the degree of f is large it is easier to use approximate values for the roots of f to find the exact integer values of the sums of powers of the roots of $M(x)$ from which its coefficients can be obtained.

$$r=3 \quad \begin{aligned} f_1(x) &= x^3+a_1x^2+a_2x+a_3,\\ f_2(x) &= x^3-a_2x^2+a_1a_3x-a_3^2,\\ f_3(x) &= x+a_3. \end{aligned}$$

$$r=4 \quad \begin{aligned} f_1(x) &= x^4+a_1x^3+a_2x^2+a_3x+a_4,\\ f_2(x) &= x^6-a_2x^5+(a_1a_3-a_4)x^4-(a_3^2-2a_2a_4+a_1^2a_4)x^3\\ &\quad +(a_1a_3a_4-a_4^2)x^2-a_2a_4^2x+a_4^3,\\ f_3(x) &= x^4+a_3x^3+a_2a_4x^2+a_1a_4^2x+a_4^3,\\ f_4(x) &= x-a_4. \end{aligned}$$

$$r=5 \quad \begin{aligned} f_1(x) &= x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5,\\ f_2(x) &= x^{10}-a_2x^9+(a_1a_3-a_4)x^8+(a_1a_5-a_1^2a_4-a_3^2+2a_2a_4)x^7\\ &\quad +(a_1^3a_5-3a_1a_2a_5+a_1a_3a_4-a_4^2+a_3a_5)x^6\\ &\quad +(2a_1a_4a_5-a_1^2a_3a_5-2a_5^2-a_2a_4^2+2a_2a_3a_5)x^5\\ &\quad +(a_1a_2a_4a_5-a_1^2a_5^2+a_2a_5^2+a_4^3-3a_3a_4a_5)x^4\\ &\quad +(2a_1a_3a_5^2-a_2^2a_5^2+a_4a_5^2-a_1a_4^2a_5)x^3\\ &\quad +(a_2a_4a_5^2-a_1a_5^3)x^2-a_3a_5^3x+a_5^4,\\ f_3(x) &= x^{10}+a_3x^9+(a_2a_4-a_1a_5)x^8\\ &\quad -(2a_1a_3a_5-a_2^2a_5+a_4a_5-a_1a_4^2)x^7\\ &\quad +(a_1a_2a_4a_5-a_1^2a_5^2+a_2a_5^2+a_4^3-3a_3a_4a_5)x^6\\ &\quad -(2a_1a_4a_5^2-a_1^2a_3a_5^2-2a_5^3-a_2a_4^2a_5+2a_2a_3a_5^2)x^5\\ &\quad +(a_1^3a_5^3-3a_1a_2a_5^3+a_1a_3a_4a_5^2-a_4^2a_5^2+a_3a_5^3)x^4\\ &\quad -(a_1a_5^4-a_1^2a_4a_5^3-a_3^2a_5^3+2a_2a_4a_5^3)x^3\\ &\quad +(a_1a_3a_5^4-a_4a_5^4)x^2+a_2a_5^5x+a_5^6,\\ f_4(x) &= x^5-a_4x^4+a_3a_5x^3-a_2a_5^2x^2+a_1a_5^2x-a_5^4,\\ f_5(x) &= x+a_5. \end{aligned}$$

9. **Computation of Δ_n from s_n.** The degree of $M(x)$ may be as high as 2^r. Thus it is desirable to have a convenient method of calculating

$$\Delta_0, \Delta_1, \Delta_2, \cdots, \Delta_{m-1}$$

in order to apply the recurrence (23). This may be done by using the sums s_n of the nth powers[20] of the roots of f. In terms of s_n, Δ_n has the expression

$$\Delta_n = 1-\frac{c_1}{1!}+\frac{c_2}{2!}-\frac{c_3}{3!}+\cdots+\frac{(-1)^r c_r}{r!},$$

[20] These numbers satisfy a recurrence whose scale is f so that its order is r instead of 2^r.

where

$$
\begin{aligned}
c_1 &= s_n, \\
c_2 &= s_n^2 - s_{2n}, \\
c_3 &= s_n^3 - 3s_n s_{2n} + 2s_{3n}, \\
c_4 &= s_n^4 - 6s_n^2 s_{2n} + 8s_n s_{3n} + 3s_{2n}^2 - 6s_{4n}, \\
c_5 &= s_n^5 - 10s_n^3 s_{2n} + 20s_n^2 s_{3n} + 15s_n s_{2n}^2 - 30s_n s_{4n}, \\
c_6 &= s_n^6 - 15s_n^4 s_{2n} + 40s_n^3 s_{3n} + 45s_n^2 s_{2n}^2 - 90s_n^2 s_{4n} + 40s_{3n}^2 - 120s_{6n} \\
&\quad - 120s_n s_{2n} s_{3n} + 144s_n s_{5n} - 15s_{2n}^3 + 90s_{2n} s_{4n}, \\
&\cdots\cdots\cdots

of unity. Then since f is irreducible all its roots are primitive kth roots of unity, so that $f(x) = Q_k(x)$, where k is some divisor of τ. But Δ_n is of period k, for if n is any integer $\geqq 0$,

$$\Delta_{n+k} = \prod_\nu (\alpha_\nu^{n+k} - 1) = \prod_\nu (\alpha_\nu^n - 1) = \Delta_n, \tag{24}$$

where α_ν are the primitive kth roots of unity. Hence τ is a divisor of k. Therefore $k = \tau$.

Proof of sufficiency. This is now obvious from (24) with $k = \tau$.

THEOREM 15. *If Δ_n is periodic with proper period τ, then*

$$\Delta_n = \begin{cases} 0 & n \equiv 0 \pmod{\tau}, \\ p^{\varphi(\tau) p^{1-\lambda}(p-1)^{-1}} & \text{if } \tau = p^\lambda (n, \tau),\ p \text{ a prime}, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 14, $f(x) = Q_\tau(x)$. Let $(n, \tau) = \delta$ and write $\tau = \tau' \delta$. Then the polynomial whose roots are the nth powers of the roots of $Q_\tau(x)$ is seen to be

$$\{ Q_{\tau'}(x) \}^{\varphi(\tau)/\varphi(\tau')}.$$

Now Δ_n is the value of this polynomial for $x = 1$. From a theorem of Trudi[21]

$$Q_{\tau'}(1) = \begin{cases} 0 \text{ if } \tau' = 1 \\ p \text{ if } \tau' = p^\lambda \\ 1 \text{ otherwise} \end{cases} \qquad (p \text{ a prime}).$$

From this the theorem follows at once[22].

Having discussed the case in which Δ_n is periodic, we shall confine ourselves in what follows to non-periodic sequences Δ_n. This amounts to supposing that no root of f is a root of unity.

12. **Rate of increase of Δ_n.** We have seen that if Δ_n is non-periodic, the absolute value of Δ_n, as n increases, is unbounded. If we are interested in obtaining large primes from the factorization of Δ_n, it is to our advantage to choose $f(x)$ so that the sequence $\Delta_1, \Delta_2, \Delta_3, \cdots$ does not increase very rapidly. In fact, the restriction on the characteristic prime factors of Δ_n as offered by Theorem 3 increases with n. If we can choose a large n for which Δ_n is not unreasonably large, we may hope to make

[21] Trudi, Atti Accademia Napoli, vol. 3 (1866-8), pp. 20-29.

[22] Since $\Delta_1, \Delta_2, \Delta_3, \cdots$ is a recurring series, it is either unbounded or purely periodic. This gives a very simple proof of the following theorem of Kronecker: If the roots of f lie on the unit circle, then they are roots of unity. In fact under this hypothesis Δ_n is bounded being $\leqq 2^r$ in absolute value. Hence Δ_n is periodic and the theorem follows from Theorem 14.

a complete examination of the possible prime factors of Δ_n without undue effort.

In order to anticipate the rate of increase of the absolute value of $\Delta_n(f)$ we need to consider the ratio

$$\frac{\alpha^{n+1}-1}{\alpha^n-1}. \tag{25}$$

It is clear that if $|\alpha|<1$, this ratio tends to 1 as $n\to\infty$, while if $|\alpha|>1$, it tends to α. Hence we have

THEOREM 16. *Let $\Omega=\Omega(f)$ be the absolute value of the product of those roots of f which lie outside the unit circle. Then if no root of f lies on the unit circle,*

$$\lim_{n=\infty}|\Delta_{n+1}/\Delta_n| = \Omega.$$

It may happen that f has a root α on the unit circle. For $|\alpha|=1$, (25) contributes an oscillating factor which, although it never vanishes or becomes infinite (since f is not a cyclotomic function), cannot be estimated readily. For lack of something better we use Ω to measure the rate of increase of the sequence

$$\Delta_1, \Delta_2, \Delta_3, \cdots$$

even when some of the roots of f lie on the unit circle.

13. **A problem in the theory of equations.** The following problem arises immediately. *If ε is a positive quantity, to find a polynomial of the form*

$$f(x) = x^r + a_1 x^{r-1} + \cdots + a_r$$

where the a's are integers, such that the absolute value of the product of those roots of f which lie outside the unit circle, lies between 1 and $1+\varepsilon$.

This problem, of interest in itself, is especially important for our purposes. Whether or not the problem has a solution for $\varepsilon<0.176$ we do not know.

In the first place, in seeking for $f(x)$ such that $\Omega(f)$ is small we may exclude the cases $\Omega(f)\geqq 2$ since the sequence of Mersenne numbers 2^n-1 has an increase $\Omega=\Omega(x-2)=2$. If $\Omega<2$, then the product of all the roots of f is ± 1. There are now three transformations of $f(x)$ which leave Ω unaltered. These replace $f(x)$ by $(-1)^r f(-x)$, $x^r f(1/x)$, and $f(x^k)$, and replace the roots by their negatives, reciprocals, and kth roots respectively. These three transformations do not alter the value of $|\Delta_n|$. Having found an $f(x)$ we need not consider the polynomials resulting from these transformations of $f(x)$. In particular we can suppose that $f(x)$ is a polynomial in x but not in any higher power of x.

The best linear function f is of course $x-2=0$.
The best quadratic is x^2-x-1, for which $\Omega = 1.61803399$.
The best cubic is x^3-x-1, for which $\Omega = 1.32471796$.
The best quartic is x^4-x-1, for which $\Omega = 1.380277569$.

The examination of all quintic equations would require much time. The above results would all suggest that good examples might be found among trinomial equations. We have accordingly examined all trinomials of degrees 5, 6, and 7 with the following results:

The best trinomial quintic is x^5-x^3-1, $\Omega = 1.3625986$.
The best trinomial sextic is x^6-x-1, $\Omega = 1.3707$.
The best trinomial septic is x^7-x^3-1, $\Omega = 1.3797$.

None of the equations given above have a root on the unit circle. Another important class of polynomials are those which are symmetric. The best polynomials of this type have all their roots but two on the unit circle. The best polynomials of degree 2, 4, 6, and 8 are

$$\begin{aligned} &x^2-3x+1, && \Omega = 2.6180339885.\\ &x^4-x^3-x^2-x+1, && \Omega = 1.722083806.\\ &x^6-x^4-x^3-x^2+1, && \Omega = 1.401268369.\\ &x^8-x^5-x^4-x^3+1, && \Omega = 1.280638157. \end{aligned}$$

We have not made an examination of all 10th degree symmetric polynomials but a rather intensive search has failed to reveal a better polynomial than

$$x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1, \quad \Omega = 1.176280821.$$

All efforts to find a better equation of degree 12 and 14 have been unsuccessful. Poulet has made a similar investigation of symmetric polynomials with practically the same results.

14. **Δ series which increase slowly.** In order to illustrate the behaviour of $\Delta_n(f)$ where $\Omega(f)$ is small we give the following example. Suppose $f(x) = x^3-x-1$. Referring to § 8 we have

$$f_2(x) = x^3+x^2-1, \quad f_3(x) = x-1 = f_0(x).$$

Hence

$$M(x) = [f_0(x), f_1(x), f_2(x), f_3(x)] = x^7-2x^5-2x^4+2x^3+2x^2-1$$

and

$$\Delta_{n+1} = 2\Delta_{n+5}+2\Delta_{n+4}-2\Delta_{n+3}-2\Delta_{n+2}+\Delta_n.$$

To illustrate the application of Theorem 3 consider

$$\Delta_{107} = 11\,673\,647\,136\,223.$$

We first test directly for prime factors of the form[23] $107x+1$. The fifth prime of this form, namely 3853, is a factor of Δ_{107}. There are no more prime factors of this form less than 5000. We have as a residual factor

$$\Delta_{107}/3853 = 3\,029\,755\,291 = N = pq.$$

By Theorem 3, this number is a prime or a product of two numbers both $< N/5000$ and of the form $107x+1$. Writing $p = 107k+1$, $q = 107l+1$, then

$$a^2-b^2 = N = pq \equiv 1+107(k+l) \equiv 1+60\cdot 107 \pmod{107^2}.$$

Hence

$$2a = 107(k+l)+2 \equiv 60\cdot 107+2 \pmod{107^2}.$$

Therefore

$$a \equiv 3211 \pmod{11449}.$$

Since

$$N \equiv 11 \pmod{2^4}, \qquad a \equiv \pm 2 \pmod{2^3}$$

and

$$N \equiv 7 \pmod{3^2}, \qquad a \equiv \pm 4 \pmod{3^2}.$$

But

$$a < N/10000 < 306000.$$

The only possible values of a satisfying the above congruences are

$$a = 83354 \text{ and } 174946.$$

Using the first value of a we find

$$(83354)^2 - N = (62595)^2.$$

Hence $N = 20759\cdot 145949$.

By the same method it is found that

$$\Delta_{113} = 63\ 088\ 004\ 325\ 217$$

and

$$\Delta_{127} = 3\ 233\ 514\ 251\ 032\ 733$$

are both prime numbers.

In case $f(x)$ is symmetric we may use $\sqrt{\Omega}$ as the effective rate of increase of Δ_n, as far as factorization is concerned, since the essential factors are perfect squares. For the equation

$$f(x) = x^8-x^5-x^4-x^3+1, \tag{26}$$

[23] Obviously there are no numbers of the form $107x-1 < \sqrt{5000}$, etc., so that the above examination is complete to 5000.

for example, the series of square roots of $\Delta_n(f)$, or what is the same thing, the series $P_n(x^4-8x^3+20x^2-17x+3)$ has 1.131652843 for its rate of increase, and satisfies a recurrence whose scale is

$$x^{16}-\sqrt{3}x^{15}+x^{14}+\sqrt{3}x^{13}-5x^{12}+3\sqrt{3}x^{11}-x^{10}-3\sqrt{3}x^9+9x^8$$
$$-3\sqrt{3}x^7-x^6+3\sqrt{3}x^5-5x^4+\sqrt{3}x^3+x^2-\sqrt{3}x+1.$$

Values of this function of n fluctuate considerably. For example one of the six roots of (26) which lie on the unit circle is $e^{2\pi i\theta}$ where

$$\theta = .218250120 = (12/55)+.000068302.$$

Hence we can expect that Δ_{55} will be small in comparison with Δ_{56} and Δ_{54}. In fact

$$\Delta_{54} = 6143283,$$
$$\Delta_{55} = 529,$$
$$\Delta_{56} = 5250987.$$

The 200th term of the series of square roots is larger than any preceding term, and is

$$\sqrt{\Delta_{200}} = P_{200} = 161\ 862\ 935\ 017\ \sqrt{3}$$
$$= \sqrt{3}\cdot 19\cdot 101\cdot 151\cdot 199\cdot 401,$$

the last two being characteristic factors of P_{200}.

The series $\sqrt{\Delta_n(f)}$ for

$$f(x) = x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$$

has a rate of increase of only 1.084564807. Poulet has calculated this series as far as the 379th term, namely,

$$\sqrt{\Delta_{379}} = 1\ 794\ 327\ 140\ 357,$$

which is a prime number.

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THEORY OF NON-COMMUTATIVE POLYNOMIALS.*

By Oystein Ore.

In the present paper I have tried to give the principal results of a general non-commutative polynomial theory. The polynomials considered have coefficients in an arbitrary commutative or non-commutative field, while the multiplication of polynomials is so restricted that the degree of a product is equal to the sum of the degrees of the factors. In this form the theory contains not only the equations studied in non-commutative algebras but also most of the linear operational equations of analysis as, for instance, linear differential and difference equations.

One could have deduced this theory using the theory of moduli studied by Noether and Schmeidler.[1] One would then have to study the residue-classes of non-commutative polynomials; these form a generalized Abelian group according to the terminology introduced by Krull,[2] and there exists a correspondence between the structure of this generalized Abelian group and the corresponding polynomial such that the properties of a polynomial can be deduced from the general theorems on generalized Abelian groups. I have preferred, however, to build up the theory directly, that is, to use only the properties of the polynomials themselves as, for instance, in the ordinary polynomial theory. This seems preferable for various reasons. It makes the theory independent of the more general theory and it brings out more clearly some of the specific properties of polynomials.

In Chapter I, section 1, one finds the principal properties of the two operations *conjugation* and *differentiation* defined by means of the rules of multiplication. In section 2 the Euclid algorithm is introduced; of particular interest is the formula for the least common multiple for non-commutative multiplication; one also finds a construction of a quotient-field for general polynomials. Sections 4 and 5 are devoted to the transformation of polynomials, a very important notion which is characteristic for the non-commutative polynomials. In section 6 the connection between left-hand and right-hand divisibility properties is considered. The general theorems about the structure of non-commutative polynomials then follow in Chapter II.

* Received December 7, 1932.

[1] E. Noether and Schmeidler, *Moduln in nichtkommutativen Bereichen, insbesondere aus Differential- und Differenzenausdrücken*, Math. Zeitschrift 8 (1920), pp. 1–35.

[2] W. Krull, *Über verallgemeinerte endliche Abelsche Gruppen*, Math. Zeitschrift 23 (1925), pp. 161–196. *Theorie und Anwendung der verallgemeinerten Abelschen Gruppen.* Sitz.-Ber. Heidelberger Akademie 1926.

CHAPTER I.

Fundamental Properties.

1. **Multiplication.** In the following let K denote an arbitrary commutative or non-commutative field with an arbitrary characteristic p in the sense of Steinitz. The objects of our investigations are then the formal polynomials

$$F(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n, \tag{1}$$

where the coefficients a belong to K while x is a formal variable or symbol. Let

$$G(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m \tag{2}$$

be a second polynomial of the same kind; the sum and difference $F(x) \pm G(x)$ is defined as the polynomial one obtains from (1) and (2) by adding or subtracting corresponding coefficients. The polynomial $c_0 F(x)$, where c_0 is an arbitrary element of K, is the polynomial one obtains from $F(x)$ by multiplying all coefficients on the left with c_0. The polynomials (1) therefore form an additive Abelian group with K as domain of multipliers. When $a_0 \neq 0$ the number n is said to be the *degree* of $F(x)$. $F(x)$ is said to be reduced when $a_0 = 1$.

We shall now define multiplication for the additive group formed by the polynomials (1) so that the *group* is made a *ring*. We assume that the multiplication of polynomials shall be associative and both-sided distributive. One can obtain a definition satisfying these conditions in an infinite number of ways, but we shall here further limit the possibilities by means of the following postulate:

The degree of a product shall be equal to the sum of the degrees of the factors.

It is clear that, due to the distributive property, it suffices to define the product of two monomials $bx^r \cdot ax^s$, or even more specifically, to define the product $x \cdot a$. According to our assumption one must have

$$x \cdot a = \bar{a} x + a', \tag{3}$$

where $\bar{a}$ and a' are elements of K. We shall call $\bar{a}$ the *conjugate* and a' the *derivative* of a. It follows from our postulate that $\bar{a} = 0$ only when $a = 0$. We can introduce higher conjugates and derivatives and we shall use the notation

$$\bar{a} = a^{[1]}, \ \bar{\bar{a}} = a^{[2]}, \cdots, a^{[n]}, \cdots, \qquad a' = a', \ a'' = a^{(2)}, \cdots, a^{(n)}, \cdots.$$

From (3) one easily obtains

$$x(a+b) = (\bar{a}+\bar{b})x + a' + b',$$
$$x(ab) = \bar{a}\bar{b}x + \bar{a}b' + a'b.$$

This leads to the following rules for the conjugates and derivatives of a sum and a product:

$$(4) \qquad \overline{(a+b)} = \bar{a}+\bar{b}, \qquad \overline{ab} = \bar{a}\,\bar{b},$$

$$(5) \qquad (a+b)' = a'+b', \qquad (ab)' = \bar{a}\,b'+a'b.$$

One also obtains

$$(6) \qquad \overline{a^{-1}} = \bar{a}^{-1}, \qquad (a^{-1})' = -\bar{a}^{-1}a'\,a^{-1}.$$

It may not be superfluous to mention a few special cases which often appear in the applications. In the common non-commutative theory of polynomials one assumes that the variable x is permutable with the coefficients with the result that

$$\bar{a} = a, \qquad a' = 0.$$

In the theory of linear differential equations K is commutative and $\bar{a} = a$ so that

$$(7) \qquad (a+b)' = a'+b', \qquad (ab)' = ab'+a'b,$$

and the derivative has the ordinary properties of a derivative. It is therefore possible to introduce the notion of a derivative into non-commutative fields, letting it satisfy the relations (5). We mention finally that in the theory of difference equations one puts $a' = 0$, though $\bar{a}$ usually is different from a.

The special properties of the operations $\bar{a}$ and a' will not be discussed further; this can be done by the methods of the abstract theory of fields. Only a few almost trivial facts will be mentioned.

THEOREM 1. *Through the correspondence $a \to \bar{a}$ one obtains a homeomorphism in K, the elements $\bar{a}$ form a subfield $\bar{K}$ of K.*

This follows immediately from (4), (5), and (6). The correspondence between K and the conjugate field $\bar{K}$ is a one-to-one correspondence, for from $\bar{a} = \bar{b}$, where $a \neq b$, it follows that $x(a-b)$ has the degree zero.

One also observes, using (4), (5), and (6), that those elements for which $\bar{a} = a$ form a new field $K^{(1)}$ which we shall call the invariant field of K. When a is an element of $K^{(1)}$ one obtains

$$xa = ax+a',$$

and for the elements of the invariant field one has the same rules of operation as for the ordinary linear differential polynomials.

THEOREM 2. *Those elements a for which $a' = 0$ form a field $K^{(0)}$ which we shall call the constant field of K.*

For if $a' = b' = 0$, then

$$(a \pm b)' = 0, \quad (ab)' = 0, \quad (a^{-1})' = 0.$$

For an element of the constant field, $xa = \bar{a}x$. This leads to

THEOREM 3. *The elements permutable with x form a field which is the greatest common subfield of the invariant field and the constant field.*

If one wants to determine those elements of K which are permutable with all polynomials $F(x)$ one easily finds that they form a field which is the greatest common subfield of $K^{(0)}$, $K^{(1)}$ and the center Z of K.

When one repeats the multiplication (3) one obtains

$$x^2a = \overline{\overline{a}}x^2 + (\overline{a}' + \overline{a'})x + a'',$$

$$x^3a = \overline{\overline{\overline{a}}}x^3 + (\overline{\overline{a}}' + \overline{\overline{a}'} + \overline{\overline{a'}})x^2 + (\overline{a}'' + \overline{a}'' + \overline{a''})x + a''',$$

and, in general,

$$x^n a = S_{n,0}(a)x^n + S_{n,1}(a)x^{n-1} + \cdots + S_{n,n}(a), \tag{8}$$

where

$$S_{n,0}(a) = a^{[n]},\ S_{n,n}(a) = a^{(n)}, \tag{9}$$

and

$$S_{n,i}(a) = (a^{[n-i]})^{(i)} + \cdots + (a^{(i)})^{[n-i]}. \tag{10}$$

$S_{n,i}(a)$ denotes the sum of the $\binom{n}{i}$ elements which one obtains when a is conjugated $n-i$-times and differentiated i-times in an arbitrary order.

For an arbitrary product one obtains finally

$$G(x)\,F(x) = c_0x^{n+m} + c_1x^{n+m-1} + \cdots + c_{n+m}, \tag{11}$$

where

$$c_0 = b_0\,a_0^{[m]} \tag{12}$$

when $G(x)$ and $F(x)$ are defined by (1) and (2), and, in general,

$$c_i = \sum_{\alpha=0}^{i} b_\alpha \sum_{\beta=0}^{i-\alpha} S_{m-\alpha,\,i-\alpha-\beta}(a_\beta). \tag{13}$$

2. The Euclid algorithm. When $F(x) = D_1(x)\,D_2(x)$ we shall call $D_2(x)$ a *right-hand* and $D_1(x)$ a *left-hand divisor* of $F(x)$. The congruence

$$F(x) \equiv G(x) \pmod{M(x)}$$

denotes that $F(x) - G(x)$ is divisible by $M(x)$ on the right, while

$$(\mathrm{mod}\ M(x))\quad F(x) \equiv G(x)$$

indicates that this difference is divisible by $M(x)$ on the left.

When $F(x)$ and $G(x)$ are given by (1) and (2), and $n \geqq m$, the difference

$$F_1(x) = F(x) - a_0(b_0^{-1})^{[n-m]}x^{n-m}\,G(x) \tag{14}$$

is of degree lower than n. It follows that one can always perform a right-hand *division*

$$F(x) = Q(x)\,G(x) + R(x),$$

where $Q(x)$ is of degree $n-m$ and the degree of $R(x)$ does not exceed $m-1$. One furthermore concludes that a Euclid Algorithm

$$
\begin{aligned}
F_1(x) &= Q_1(x)\ F_2(x) + F_3(x),\\
F_2(x) &= Q_2(x)\ F_3(x) + F_4(x),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
F_{n-2}(x) &= Q_{n-2}(x)\ F_{n-1}(x) + F_n(x),\\
F_{n-1}(x) &= Q_{n-1}(x)\ F_n(x)
\end{aligned}
\tag{15}
$$

exists for two arbitrary polynomials $F_1(x)$ and $F_2(x)$.

It follows from the Euclid Algorithm that there exists a unique, reduced, greatest common right-hand divisor or *cross-cut*

$$D(x) = F_n(x) = (F_1(x),\ F_2(x)),$$

where of course $F_n(x)$ is assumed to be reduced. A further consequence is

THEOREM 4. *When $C(x)$ is divisible by the cross-cut $D(x)$ of $F_1(x)$ and $F_2(x)$ one can determine polynomials $A_1(x)$ and $A_2(x)$ such that*

$$A_2(x)\,F_1(x) + A_1(x)\,F_2(x) = C(x). \tag{16}$$

When $D(x) = 1$ we say that $F_1(x)$ is relatively prime to $F_2(x)$. From Theorem 4 one obtains

THEOREM 5. *When $F_1(x)$ is relatively prime to $F_2(x)$ the congruence*

$$Y(x)\,F_1(x) \equiv C(x) \pmod{F_2(x)} \tag{17}$$

always has a solution $Y(x)$ for an arbitrary $C(x)$.

We have here considered only right-hand divisibility; it should be observed in the following that when the type of divisibility is not specified, right-hand divisibility is always implied.

Left-hand division differs from right-hand division in that it cannot always be performed. If one wants to determine, as in (14), an element k of K such that

$$F_1(x) = F(x) - G(x) \cdot k x^{n-m}$$

is of degree less than n, then one obtains for the determination of k, according to (12), $a_0 = b_0 k^{[m]}$. From this condition it follows that one can determine k in general only when each element a of K is also an element $\bar{a}$ of $\bar{K}$.

THEOREM 6. *In general, left-hand division can be performed in K only if the correspondence $a \to \bar{a}$ is an automorphism of K.*

When this condition is satisfied there exist a left-hand Euclid Algorithm, a unique left-hand cross-cut, and theorems analogous to Theorems 4 and 5.

We point out that every polynomial

$$H(x) = x^n b_0 + x^{n-1} b_1 + \cdots + x\,b_{n-1} + b_n$$

with the coefficients as right-hand multipliers can be written according to the definition of multiplication as

$$H(x) = B_0 x^n + B_1 x^{n-1} + \cdots + B_{n-1} x + B_n$$

with left-hand coefficients. According to (8) one obtains the following expressions for these coefficients:

$$B_i = S_{n,i}(b_0) + S_{n-1,i-1}(b_1) + \cdots + S_{n-i,0}(b_i). \tag{18}$$

One can ask, however, whether inversely every polynomial with left-hand coefficients can always be represented as a polynomial with right-hand coefficients. One directly obtains

THEOREM 7. *The necessary and sufficient condition that every right-hand polynomial be also a left-hand polynomial and every left-hand polynomial be also a right-hand polynomial is that* $K = \bar{K}$.

If one denotes the inverse conjugate of a by $a^{[-1]}$ it follows that $\overline{a^{[-1]}} = a$,

$$ax = xa^{[-1]} - (a^{[-1]})', \tag{19}$$

and, in general,

$$ax^n = x^n T_{n,0}(a) - x^{n-1} T_{n-1,1}(a) + \cdots + (-1)^n T_{0,n}(a),$$

where the $T_{n,i}(a)$ are constructed from the inverse conjugates and derivatives in a way similar to that by which the $S_{n,i}(a)$ were obtained from the conjugates and derivatives of a.

3. **Union and quotient field.** Let $A(x)$ and $B(x)$ be arbitrary polynomials; the reduced polynomial $M(x)$ of lowest degree which is right-hand divisible by both $A(x)$ and $B(x)$ will be called the right-hand union of $A(x)$ and $B(x)$ and will be denoted by

$$M(x) = [A(x), B(x)] = A_1(x) \cdot B(x) = B_1(x) \cdot A(x). \tag{20}$$

It is obvious that if the union exists it must be uniquely determined.

We shall now show that the existence of a Euclid algorithm implies the existence of not only the cross-cut but also the union of two polynomials. We shall also show that the Euclid algorithm gives an explicit formula for the union; this result holds for arbitrary non-commutative domains with a Euclid algorithm, but the formula for the union does not seem to have been observed even in special cases. We want to prove

THEOREM 8. *When a Euclid algorithm of* $F_1(x)$ *and* $F_2(x)$ *of the form* (15) *is given. then the union of these polynomials is given by the formula*

$$\begin{aligned}[F_1(x), F_2(x)] = a F_{n-1}(x) F_n(x)^{-1} F_{n-2}(x) F_{n-1}(x)^{-1} \cdots \\ \cdots F_3(x) F_4(x)^{-1} F_2(x) F_3(x)^{-1} F_1(x).\end{aligned} \tag{21}$$

The constant a must be chosen so that the union is reduced.

In order to prove Theorem 8 we first deduce a necessary property of the union.

THEOREM 9. *When*

$$A(x) \equiv B(x) \pmod{C(x)} \tag{22}$$

then

$$[A(x),\ C(x)] = k[B(x),\ C(x)] \cdot B(x)^{-1} \cdot A(x). \tag{23}$$

If the $H(x)$ of $A(x)$ and $C(x)$ exists, then one must have $H(x) = C_1(x)A(x)$, and $C_1(x)$ is the polynomial of lowest degree such that this product is also divisible by $C(x)$. According to (22) one has

$$A(x) = B(x) + Q(x)\,C(x)$$

and one obtains

$$H(x) = C_1(x)\,A(x) = C_1(x)\,B(x) + C_1(x)\,Q(x)\,C(x),$$

and the product $C_1(x)\,B(x)$ must be divisible by $C(x)$.

The lowest degree of $C_1(x)$ is therefore obtained when

$$C_1(x)\,B(x) = [B(x),\ C(x)],$$

and this relation obviously gives (22).

When the formula (23) is repeatedly applied to the union $[F_1(x),\ F_2(x)]$, it follows according to (15) that if the union exists it must have the form (21). It therefore remains only to show that the right-hand side of (21) is a polynomial which is divisible by both $F_1(x)$ and $F_2(x)$.

We prove this by induction; let

$$\varphi_i(x) = F_{n-1}(x)\,F_n(x)^{-1}\,F_{n-2}(x) \cdots F_i(x).$$

It follows from the last equation (15) that $\varphi_{n-1}(x) = F_{n-1}(x)$ is integral and divisible by both $F_{n-1}(x)$ and $F_n(x)$. Let us therefore assume it has already been shown that $\varphi_{i+1}(x)$ is integral and divisible by $F_{i+1}(x)$ and $F_{i+2}(x)$. It is then obvious that

$$\varphi_i(x) = \varphi_{i+1}(x)\,F_{i+2}(x)^{-1}\,F_i(x)$$

is integral and divisible by $F_i(x)$. In order to prove the divisibility by $F_{i+1}(x)$ we observe that

$$F_i(x) = Q_i(x)\,F_{i+1}(x) + F_{i+2}(x)$$

and consequently that

$$\varphi_i(x) = \varphi_{i+1}(x)\,F_{i+2}(x)^{-1}\,Q_i(x)\,F_{i+1}(x) + \varphi_{i+1}(x).$$

This shows the divisibility by $F_{i+1}(x)$.

From (21) it follows furthermore that when the degrees of $F_1(x)$ and $F_2(x)$ are n_1 and n_2 respectively, the cross-cut being of degree d, the union must be of degree $n_1 + n_2 - d$.

In the following let $\delta_{A,B}$ denote the degree of the cross-cut $(A(x),\ B(x))$ and let $\mu_{A,B}$ denote the degree of the union; we have shown then that

$$\alpha + \beta = \mu_{A,B} + \delta_{A,B}, \tag{24}$$

where α and β are the degrees of $A(x)$ and $B(x)$.

For an arbitrary number of polynomials one defines the union

$$[A(x), B(x), C(x), \cdots]$$

in a corresponding manner. Among the more important properties we observe that

$$[A(x), [B(x), C(x)]] = [[A(x), B(x)], C(x)] \tag{25}$$

and that

$$[A(x), B(x), D(x)] = [A(x), B(x)], \tag{26}$$

where $D(x)$ is a divisor of $[A(x), B(x)]$. Further,

$$d_0^{[\mu_{A,B}]}[A(x)D(x), B(x)D(x)] = [A(x), B(x)]D(x), \tag{27}$$

where d_0 is the highest coefficient in $D(x)$ and $\mu_{A,B}$ is the degree of $[A(x), B(x)]$.

When left-hand division can be performed, that is, when $K = \bar{K}$ in accordance with Theorem 6, it follows in the same way that also a left-hand union exists and has analogous properties.

It should be mentioned that it is possible to enlarge the ring of polynomials considered here to a non-commutative field through the introduction of formal quotients. This quotient field then corresponds in the commutative case to the field of all rational functions with coefficients in the field K.

The existence of the quotient-field is a consequence of the existence of a union.[3] If we define the quotient as

$$Q(x) = \frac{B(x)}{A(x)} = A(x)^{-1}B(x),$$

we can define the sum of two quotients as

$$\frac{B(x)}{A(x)} + \frac{D(x)}{C(x)} = \frac{C_1(x)B(x) + A_1(x)D(x)}{M(x)},$$

where

$$M(x) = [A(x), C(x)] = A_1(x)C(x) = C_1(x)A(x)$$

is the right-hand union of $A(x)$ and $C(x)$. We define the product by

$$\frac{B(x)}{A(x)} \cdot \frac{D(x)}{C(x)} = \frac{B_1(x)D(x)}{C_1(x)A(x)},$$

where

$$[B(x), C(x)] = B_1(x)C(x) = C_1(x)B(x).$$

Finally,

$$\left(\frac{B(x)}{A(x)}\right)^{-1} = \frac{A(x)}{B(x)}.$$

One easily shows that addition is commutative and multiplication is associative and distributive on both sides.

[3] O. Ore, *Linear equations in non-commutative fields*, Annals of Mathematics 32 (1931), pp. 463–477.

4. **Transformation.** Having established the existence of a union we can proceed to introduce the fundamental notion of *transformation* of a polynomial.

The polynomial

$$A_1(x) = a_0 b_0^{[\alpha-\delta_{A,B}]}\,[A(x),\ B(x)]\cdot B(x)^{-1} \tag{28}$$

is called the transform of $A(x)$ *by* $B(x)$ *and will be denoted by*

$$A_1(x) = BA(x)\,B^{-1}. \tag{29}$$

Here α denotes the degree of $A(x)$. The transform $A_1(x)$ is of degree $\alpha - \delta_{A,B}$; consequently, when $A(x)$ and $B(x)$ are relatively prime, $A(x)$ and $A_1(x)$ are of the same degree. The multiplicative constant in (28) is chosen so that the highest coefficient in $A_1(x)$ is equal to a_0. It might have been possible to choose this constant in a different way or even to omit it, but after several trials I have found it most satisfactory to use the form given here.

When $A(x)$ is relatively prime to $B(x)$, and $A_1(x)$ is consequently of the same degree as $A(x)$, we shall call the transformation (29) a special transformation and $A_1(x)$ is said to be of *the same kind* or *similar* to $A(x)$. When, however, $A(x)$ and $B(x)$ have a common factor, we shall call the transformation *general*.[4] In both cases one obtains, according to (28) and (29),

$$BA(x)\,B^{-1}\,B(x) = a_0 b_0^{[\alpha-\delta_{A,B}]}\,[A(x),\ B(x)]. \tag{30}$$

We shall now deduce various properties of the transform, and we commence with the following theorem:

THEOREM 10. *When*

$$A(x) \equiv B(x) \pmod{C(x)}$$

then

$$AC(x)\,A^{-1} = BC(x)\,B^{-1}.$$

In order to prove this theorem we have to go back to the identity (23) in Theorem 9. Since here both sides are reduced, one easily obtains for the factor k the value

$$k = (a_0^{-1})^{[\gamma-\delta_{B,C}]}\ b_0^{[\gamma-\delta_{B,C}]}$$

and (23) takes the form

$$a_0^{[\gamma-\delta_{B,C}]}\,[A(x), C(x)]\cdot A(x)^{-1} = b_0^{[\gamma-\delta_{B,C}]}\,[B(x), C(x)]\ B(x)^{-1}.$$

Since $\delta_{B,C} = \delta_{A,C}$, according to assumption, our theorem follows.

The next theorem is particularly important in its applications.

[4] The general transformation can always be reduced to the special. When $D(x)$ is the cross-cut of $A(x) = A_1(x)\,D(x)$ and $B(x) = B_1(x)\,D(x)$, then it follows from the definition of the transformation that

$$AB(x)\,A^{-1} = A_1 B_1(x)\,A_1^{-1}.$$

THEOREM 11. *When a product $A(x)\,B(x)$ is divisible by $C(x)$, then $A(x)$ is divisible by $BC(x)\,B^{-1}$.*

Since $A(x)\,B(x)$ is divisible both by $B(x)$ and $C(x)$, the product must also be divisible by the union of these polynomials, and it is therefore possible to find a polynomial $K(x)$ such that

$$A(x)\,B(x) = K(x)\,c_0\,b_0^{[\gamma-\delta_{C,B}]}\,[B(x), C(x)].$$

Division by $B(x)$ gives

$$A(x) = K(x)\,BC(x)\,B^{-1}.$$

We shall usually apply Theorem 11, as in the commutative theory of polynomials, to the case where $C(x)$ is relatively prime to $B(x)$. The more general formulation 11 seems, however, to command some interest.

THEOREM 12. *When the transformer is a product, one can transform by the factors in order from right to left, that is,*

$$(CB)\,A(x)\,(CB)^{-1} = C(BA(x)\,B^{-1})\,C^{-1}. \tag{31}$$

From the identity

$$[A(x), C(x)\,B(x)] = [A(x), B(x), C(x)\,B(x)]$$

one obtains, applying (27),

$$(CB)\,A(x)\,(CB)^{-1} = a_0\,c_0^{[\alpha-\delta_{A,CB}]}\,[BA(x)\,B^{-1}, C(x)]\cdot C(x)^{-1},$$

and since

$$\delta_{A,CB} = \delta_{A,B} + \delta_{BAB^{-1},C},$$

one has obtained the theorem. The associative law for the transformation can also easily be shown to hold.

The symmetry of the notion of similarity follows from

THEOREM 13. *When $A_1(x)$ is similar to $A(x)$, then $A(x)$ is similar to $A_1(x)$, that is, when*

$$A_1(x) = BA(x)\,B^{-1},$$

where $B(x)$ is relatively prime to $A(x)$, then one can determine $B_1(x)$ so that

$$A(x) = B_1\,A_1(x)\,B_1^{-1}.$$

According to Theorems 10 and 12 it is sufficient to choose $B_1(x)$ such that

$$B_1(x)\,B(x) \equiv 1 \pmod{A(x)},$$

and this always possible according to Theorem 5.

We also see that Theorem 5 can be stated in the following more complete form:

THEOREM 14. *When the congruence*

$$X(x)\,B(x) \equiv C(x) \pmod{A(x)},$$

has a solution $X_0(x)$, the most general solution has the form

$$X(x) = X_0(x) + K(x) B A(x) B^{-1},$$

where $K(x)$ is an arbitrary polynomial.

It follows that

$$(X(x) - X_0(x)) B(x) \equiv 0 \pmod{A(x)},$$

and the theorem follows from Theorem 11.

THEOREM 15. *The transform of a union is equal to the union of the transforms of the components, that is,*

$$C[A(x), B(x)] \cdot C^{-1} = [CA(x) C^{-1}, CB(x) C^{-1}]. \tag{32}$$

The theorem follows immediately when one multiplies both sides by $C(x)$.

Finally it should be mentioned that the corresponding theorem does not hold for the cross-cut. One sees, however, that in any case

$$(CA(x) C^{-1}, CB(x) C^{-1}) \equiv 0 \pmod{C(A(x), B(x)) C^{-1}}, \tag{33}$$

and one concludes that, when $C(x)$ is relatively prime to the union $[A(x), B(x)]$, then

$$(CA(x) C^{-1}, CB(x) C^{-1}) = C(A(x), B(x)) C^{-1}. \tag{34}$$

5. **Transformation of a product.** We shall finally examine the transformation af a product; the results are in this case not quite as simple as for the preceding theorems on transformation. The complications are due chiefly to the multiplicative constant in the definition (28) of the transform. It may be possible to choose this constant in a different way, making the transformation of a product more symmetric, but after various attempts I have found that the former theorems then become correspondingly more complicated.

Since one has for an arbitrary product

$$B(x) A(x) = [B(x) A(x), A(x)]$$

one obtains according to Theorem 5

$$C(B(x) A(x)) C^{-1} \equiv 0 \pmod{CA(x) C^{-1}}. \tag{35}$$

It should be observed that the congruence (33) is a consequence of (35). According to (35) one can always write

$$C(B(x) A(x)) C^{-1} = K(x) CA(x) C^{-1}, \tag{36}$$

and the problem is to determine $K(x)$. We shall simplify the formulas by assuming that the transformation is a special one in that $C(x)$ is relatively prime to $B(x) A(x)$. The same method is, however, applicable even in the most general case.

From (36) it follows by multiplication with $C(x)$ that

$$b_0\, a_0^{[\beta]}\, c_0^{[\alpha+\beta]}\, [B(x)\, A(x), C(x)] = K(x)\, a_0\, c_0^{[\alpha]}\, [A(x), C(x)].$$

Division by $A(x)$ gives

$$b_0\, a_0^{[\beta]}\, c_0^{[\alpha+\beta]}\, (a_0^{-1})^{[\beta+\gamma]} [B(x), A\,C(x)\,A^{-1}] = K(x)\, a_0\, c_0^{[\alpha]}\, (a_0^{-1})^{[\gamma]}\, c_0^{-1}\, A\,C(x)\,A^{-1}.$$

Another division by $C_1(x) = A\,C(x)\,A^{-1}$ gives

$$(37)\quad K(x) = b_0\, a_0^{[\beta]}\, c_0^{[\alpha+\beta]}\, (a_0^{-1})^{[\beta+\gamma]}\, (c_0^{-1})^{[\beta]}\, b_0^{-1}\, C_1 B(x) C_1^{-1}\, c_0\, a_0^{[\gamma]}\, (c_0^{-1})^{[\alpha]}\, a_0^{-1}.$$

One should now observe that, when an arbitrary polynomial $E(x)$ of degree ε with highest coefficient e_0 is transformed by an element k_0 in K, one obtains the polynomial

$$e_0\, k_0^{[\varepsilon]}\, e_0^{-1}\, E(x)\, k_0^{-1}.$$

It is then seen that (37) can be written in the simpler form

$$(38)\qquad K(x) = C_2\, B(x)\, C_2^{-1},$$

where

$$(39)\qquad C_2(x) = a_0\, c_0^{[\alpha]}\, (a_0^{-1})^{[\gamma]}\, c_0^{-1}\, A\,C(x)\,A^{-1}.$$

This proves

THEOREM 16. *When $C(x)$ is relatively prime to $B(x)\,A(x)$ then*

$$(40)\qquad C(B(x)\,A(x))\,C^{-1} = C_2\, B(x)\, C_2^{-1} \cdot C\,A(x)\,C^{-1},$$

where $C_2(x)$ is determined by (39).

The theorem can be extended to an arbitrary number of factors and then gives

$$(41)\quad C(A_n(x) \cdots A_1(x))\,C^{-1} = C_n\, A_n(x)\, C_n^{-1} \cdots C_2 A_2(x)\, C_2^{-1}\, C A_1(x)\, C^{-1},$$

where the transformation, as before, is a special one. *The factors of the transform are similar to the factors of the original product.*

If one assumes that $A(x)$ and $B(x)$ are reduced, or in general, that all $A(x)$ in (41) are reduced, then similar formulas hold also for a general transformation.

6. **Left-hand transformation.** Having thus deduced the main properties of the right-hand transformation, we shall briefly discuss the left-hand properties. It has already been mentioned in section 3 that if $\bar{K}$ is identical with K, there exists a left-hand Euclid algorithm and consequently a left-hand union

$$H_l(x) = [A(x), B(x)]_l$$

for two arbitrary polynomials $A(x)$ and $B(x)$. The left-hand union is of course defined as the reduced polynomial of lowest degree which is left-hand divisible by both $A(x)$ and $B(x)$. The degree of $H_l(x)$ is $\alpha + \beta - \delta_{A,B}$,

where $\delta_{A,B}$ is the degree of the left-hand cross-cut $(A(x), B(x))_l$ of $A(x)$ and $B(x)$.

The left-hand transformation can now be defined by putting

$$B^{-1} A(x)_l B = B(x)^{-1} \cdot [A(x), B(x)]_l \cdot b_0^{[-\alpha-\beta+\delta_{A,B}]} a_0^{[-\alpha-\delta_{A,B}]},$$

where the constants are chosen so that the transform of $A(x)$ has the highest coefficient a_0. One then proves without great difficulty the corresponding theorems for left-hand transformation as before for right-hand transformation.

In many cases however, the left-hand notions can be replaced by the corresponding right-hand notions. We first prove

THEOREM 17. *When $A_1(x)$ and $B_1(x)$ are left-hand relatively prime, then the left-hand union of these polynomials can also be considered as a right-hand union*

$$[A_1(x), B_1(x)]_l = [A(x), B(x)] \tag{42}$$

and this can be done in such a way that $A_1(x)$ is right-hand similar to $A(x)$ and $B_1(x)$ right-hand similar to $B(x)$.

According to the definition of the left-hand union

$$[A_1(x), B_1(x)]_l = A_1(x) B_2(x) = B_1(x) A_2(x), \tag{43}$$

and here $A_2(x)$ is right-hand relatively prime to $B_2(x)$, for if they had a common factor, one could find a polynomial of lower degree that would be left-hand divisible by both $A_1(x)$ and $B_1(x)$. From Theorem 11 and (43) it follows that

$$A_1(x) = K(x) \cdot B_2 A_2(x) B_2^{-1}, \qquad B_1(x) = L(x) \cdot A_2 B_2(x) A_2^{-1}. \tag{44}$$

If α is the degree of $A_1(x)$ and β the degree of $B_1(x)$, then according to (43) $A_2(x)$ is of degree α and $B_2(x)$ is of degree β; one finds furthermore for the highest coefficients in $A_2(x)$ and $B_2(x)$

$$a_2 = (b_1^{-1})^{[-\beta]}, \qquad b_2 = (a_1^{-1})^{[-\alpha]},$$

where a_1 and b_1 are the highest coefficients of $A_1(x)$ and $B_1(x)$. If one substitutes the expressions (44) into (43), one can divide on the right-hand side by $[A_2(x), B_2(x)]$, getting

$$K(x) (b_1^{-1})^{[-\beta]} a_1^{-1} = L(x) (a_1^{-1})^{[-\alpha]} b_1^{-1}.$$

Since (44) shows that $K(x)$ and $L(x)$ are left-hand relatively prime, this identity can hold only when $K(x)$ and $L(x)$ are both of degree zero. It follows then from (44) that

$$K(x) = a_1 b_1^{[-\beta]}, \qquad L(x) = b_1 a_1^{[-\alpha]}.$$

The relation (44) also gives

$$(45)\qquad \begin{aligned} A_1(x) &= a_1 b_1^{[-\beta]} B_2 A_2(x) B_2^{-1} = B_2 A(x) B_2^{-1}, \\ B_1(x) &= b_1 a_1^{[-\alpha]} A_2 B_2(x) A_2^{-1} = A_2 B(x) A_2^{-1}, \end{aligned}$$

where one has put

$$(46)\qquad A(x) = a_1 b_1^{[-\beta]} A_2(x), \qquad B(x) = b_1 a_1^{[-\alpha]} B_2(x).$$

Then one has, however, according to (43)

$$[A_1(x), B_1(x)]_l = [A_2(x), B_2(x)] = [A(x), B(x)],$$

and the theorem is demonstrated.

From the definition of left-hand transform one deduces the notion of *left-hand similarity*: Two polynomials $A_1(x)$ and $A(x)$ are said to be left-hand similar if $A_1(x)$ is obtained from $A(x)$ by transformation by some polynomial $B(x)$ left-hand relatively prime to $A(x)$. Left-hand similarity offers, however, little new, as the following theorem shows:

THEOREM 18. *When two polynomials are left-hand similar they are also right-hand similar.*

One obtains, according to (43) for the left-hand transform of an arbitrary polynomial $A(x)$ by another $B_1(x)$ left-hand relatively prime to $A_1(x)$,

$$(47)\qquad \begin{aligned} H(x) = B_1^{-1} A_1(x)_l B_1 &= A_2(x) b_1^{[-\alpha-\beta]} a_1^{[-\alpha]} \\ &= (b_1^{-1})^{[-\beta]} a_1^{-1} A(x) b_1^{[-\alpha-\beta]} a_1^{[-\alpha]}, \end{aligned}$$

where the notation is as above. The theorem to be proved then asserts that $H(x)$ can also be obtained from $A_1(x)$ through right-hand transformation. According to (45) $A_1(x)$ is right-hand similar to $A(x)$ and one obtains $H(x)$ from $A(x)$ by right-hand transformation with the element

$$(a_1^{-1})^{[-\alpha]} \ (b_1^{-1})^{[-\alpha-\beta]}.$$

CHAPTER II.

The Theorems of Decomposition.

1. **First theorem.** A reduced polynomial $P(x)$ is called a *prime* polynomial when $P(x)$ has no reduced factors aside from constants and $P(x)$ itself.

Among the properties of prime polynomials we observe that

Every polynomial which is similar to a prime polynomial is a prime polynomial.

This follows from Theorems 13 and 16. One deduces furthermore from Theorem 11 the following:

When a product $A(x)\,B(x)$ is right-hand divisible by a prime polynomial $P(x)$, and $B(x)$ is not divisible by $P(x)$, then $A(x)$ is divisible by the prime polynomial $BP(x)\,B^{-1}$.

Let us now consider the decomposition of an arbitrary polynomial into prime factors. To describe the following results more completely we shall need the notion of *interchangeability:*

A polynomial $A(x)$ is said to be interchangeable with a second polynomial $B(x)$ if one can determine a polynomial $A_1(x)$ similar to $A(x)$ such that

$$A(x) = BA_1(x)\,B^{-1}. \tag{1}$$

It is hardly necessary to mention that the notion of interchangeability is not symmetric.

When the identity (1) holds, where $A(x)$ and $B(x)$ are reduced, then the product $A(x)\,B(x)$ is a union

$$A(x)\,B(x) = [A_1(x), B(x)] = A_1\,B(x)\,A_1^{-1}\cdot A_1(x). \tag{2}$$

One has therefore in this case two different decompositions of the product $A(x)\,B(x)$ wherein the factors are similar but occur in inverse order. We shall say that the second decomposition (2) is obtained from the first through *interchange of factors.*

The first decomposition theorem then takes the following form:

THEOREM 1. *Every reduced polynomial has a representation as the product of prime factors. Two different decompositions of the same polynomial have the same number of prime factors and the factors are similar in pairs. One decomposition can be obtained from the other through interchanges of factors.*

It is obvious that every polynomial possesses at least one decomposition into prime factors. Let

$$F(x) = P_r(x)\cdots P_2(x)\,P_1(x) = Q_s(x)\cdots Q_2(x)\,Q_1(x) \tag{3}$$

be two different decompositions. The prime polynomial $Q_1(x)$ then divides the left-hand side of (3). If $P_1(x) = Q_1(x)$, this factor can be cancelled. If $P_1(x) \neq Q_1(x)$, let k be the first number such that the product $P_k(x)\cdots P_1(x)$ is divisible by $Q_1(x)$. The product $P_{k-1}(x)\cdots P_1(x)$ is then relatively prime to $Q_1(x)$ and according to Theorem 11, Chapter 1, $P_k(x)$ is then divisible by the prime polynomial $Q_1'(x)$ obtained from $Q_1(x)$ by transformation with $P_{k-1}(x)\cdots P_1(x)$. Since $P_k(x)$ is a prime polynomial one obtains

$$P_k(x) = (P_{k-1}\cdots P_1)\,Q_1(x)\,(P_{k-1}\cdots P_1)^{-1}. \tag{4}$$

Then, however, according to the definition (1), $P_k(x)$ is interchangeable with $P_{k-1}(x)\cdots P_1(x)$ and one obtains according to (2) and (4)

$$P_k(x)\,P_{k-1}(x)\cdots P_1(x) = Q_1(P_{k-1}(x)\cdots P_1(x))\;Q_1^{-1}\cdot Q_1(x). \tag{5}$$

We have shown by interchange of factors that also the first decomposition (3) contains $Q_1(x)$ as a right-hand factor, so that this factor may be cancelled. $P_k(x)$ and $Q_1(x)$ must be similar according to (4), and when one performs the transformation

$$Q_1 (P_{k-1}(x) \cdots P_1(x)) Q_1^{-1}$$

one obtains, according to Theorem 16, Chapter 1, a product of prime factors which are similar to the $P_{k-1}(x), \cdots, P_1(x)$. After division by $Q_1(x)$ the remaining products in (3) can be treated in the same way and the theorem follows.

The number of prime factors which occur in an arbitrary prime factor decomposition of a polynomial $F(x)$ will sometimes be called the *length* of $F(x)$.

We point out as a consequence of Theorem 16, Chapter 1, that similar polynomials must have decompositions of the same length and with similar factors.

It may not be superfluous to mention that the number of factors of a polynomial is in general not limited by the degree of the polynomial; it may even happen, as for instance in the theory of linear differential polynomials, that the number of divisors is infinite.

2. **Completely reducible polynomials.** A polynomial is said to be *completely reducible* when it is representable as the union of a finite or infinite number of prime polynomials.

THEOREM 2. *A completely reducible polynomial $F(x)$ can always be represented in the form*

$$F(x) = [P_1(x), P_2(x), \cdots, P_r(x)], \tag{6}$$

where the $P_i(x)$ are prime polynomials such that none of them divides the union of the others.

Let $P_1(x)$ be an arbitrary prime divisor of $F(x)$; if then $F(x) \neq P_1(x)$, let $P_2(x)$ be a second prime divisor. The union $[P_1(x), P_2(x)]$ is then a divisor of $F(x)$, and if it is not equal to $F(x)$, let $P_3(x)$ be a prime divisor of $F(x)$ not dividing $[P_1(x), P_2(x)]$. Then $[P_1(x), P_2(x), P_3(x)]$ is a divisor of $F(x)$. This procedure can be continued, and one must finally obtain a representation (6) since the degree of $F(x)$ is finite.

When a completely reducible polynomial (6) is transformed by an arbitrary polynomial $H(x)$, one obtains according to Theorem 15, Chapter 1,

$$HF(x)H^{-1} = [HP_1(x)H^{-1}, \cdots, HP_r(x)H^{-1}]. \tag{7}$$

The transform of a completely reducible polynomial is again completely reducible.

For two similar completely reducible polynomials one can choose the basis representations so that the prime polynomials of the bases are similar in pairs.

One obtains the prime factor representation of a completely reducible polynomial (6) by writing

$$F(x) = [P_1 P_2(x) P_1^{-1}, \cdots, P_1 P_r(x) P_1^{-1}] P_1(x)$$

and repeatedly applying the same process to the perfectly reducible factor

$$[P_1 P_2(x) P_1^{-1}, \cdots, P_1 P_r(x) P_1^{-1}].$$

It is obvious that the length of the decomposition is r and that the factors are similar to the $P_i(x)$.

One can also characterize the completely reducible polynomials by:

THEOREM 3. *The necessary and sufficient condition that a polynomial be completely reducible is that two arbitrary prime factors in an arbitrary decomposition be interchangeable.*

A completely reducible polynomial may therefore also be termed a completely interchangeable polynomial.

We prove first that a completely interchangeable polynomial is also completely reducible. This is obvious when the length of the decomposition does not exceed 2, for when

$$F(x) = P_1(x) P_2(x)$$

then, according to assumption, $P_1(x) = P_2 \bar{P}_1(x) P_2^{-1}$, where $\bar{P}_1(x)$ also denotes a prime polynomial. Hence

$$F(x) = [\bar{P}_1(x), P_2(x)].$$

The general proof follows by induction. We write

$$F(x) = P_r(x) \cdots P_2(x) P_1(x) = F_1(x) P_1(x), \tag{8}$$

where $F_1(x)$ contains a smaller number of prime factors and, since it is completely interchangeable, it is also completely reducible according to assumption. Let

$$F_1(x) = [\bar{P}_r(x), \cdots, \bar{P}_2(x)]. \tag{9}$$

Since all prime polynomials $\bar{P}_i(x)$ $(i = 2, \cdots, r)$ may appear as right-hand factors of $F_1(x)$, they are all interchangeable with $P_1(x)$, so that

$$\bar{P}_i(x) = P_1 Q_i(x) P_1^{-1}, \qquad (i = 2, \cdots, r),$$

where $Q_i(x)$ denote prime polynomials. From (7) and (9) one obtains

$$F_1(x) = P_1[Q_r(x), \cdots, Q_2(x)]\, P_1^{-1}, \tag{10}$$

and from (8),

$$F(x) = [Q_r(x), \cdots, Q_2(x), P_1(x)]. \tag{11}$$

The converse part of the theorem also follows by induction. When $F(x) = P_1(x)\, P_2(x)$ is completely reducible, then $F(x)$ must be divisible by a second prime function $Q_1(x)$ other than $P_1(x)$, and one deduces from Theorem 11, Chapter 1, that $P_1(x) = P_2 Q_1(x)\, P_2^{-1}$, so that $P_1(x)$ is interchangeable with $P_2(x)$. Now let (8) be a representation of a completely reducible polynomial as a product of prime polynomials. Then $F_1(x)$ has the form (10), it is completely reducible, and therefore, by assumption, it is completely interchangeable. It remains only to show that $P_2(x)$ is always interchangeable with $P_1(x)$. This follows easily from the fact that every prime divisor of $F_1(x)$ must be of the form $P_1\, Q(x)\, P_1^{-1}$.

From Theorem 3 one derives:

THEOREM 4. *Every divisor of a completely reducible polynomial is completely reducible, and the basis of a divisor can be completed to a basis for* $F(x)$.

The second part of the theorem follows by the method of construction of a basis. One can also show somewhat more generally that every factor occurring anywhere in a product decomposition of a completely reducible polynomial must also be completely reducible.

Let us say that a completely reducible polynomial $F(x)$ is *uniform* when it is the union of similar prime polynomials so that it may be expressed as

$$F(x) = [A_1\, P(x)\, A_1^{-1}, \cdots, A_r\, P(x)\, A_r^{-1}].$$

Every prime divisor of $F(x)$ is then similar to $P(x)$. The union of all similar prime polynomials which divide a given completely reducible polynomial (6) shall be called a *maximal uniform* divisor. One can then prove

THEOREM 5. *Every completely reducible polynomial is uniquely representable as the union of its maximal uniform divisors.*

It should be mentioned finally that the theory of left-hand completely reducible polynomials is quite analogous. A polynomial is said to be *left-hand completely reducible* when it is the *left union* of prime polynomials. One easily proves

THEOREM 6. *Every left-hand completely reducible polynomial is also right-hand completely reducible and conversely.*

One also finds that the left-hand and the right-hand basis representations contain the same number of prime polynomials and that these are similar in pairs.

3. Second decomposition theorem.[5] Let $F(x)$ be an arbitrary polynomial; the union $H_1(x)$ of all prime functions $P(x)$ dividing $F(x)$ from the right shall be called the *maximal completely reducible* divisor of $F(x)$. This name is justified by the fact that every completely reducible divisor of $F(x)$ must also be a divisor of $H_1(x)$. $H_1(x)$ is uniquely determined and one can write $F(x) = F_1(x)\, H_1(x)$. $F_1(x)$ has then also a unique maximal completely reducible divisor $H_1(x)$, etc.

THEOREM 7. *Every polynomial has a unique representation as the product of maximal completely reducible factors*

$$F(x) = a_0\, H_r(x) \cdots H_2(x)\, H_1(x). \tag{12}$$

There exists a certain relation between the decomposition of $F(x)$ and any divisor of $F(x)$ which we shall express in the following theorem:

THEOREM 8. *Let $F_1(x)$ be a divisor of $F(x)$ and let*

$$F_1(x) = b_0\, G_s(x) \cdots G_2(x)\, G_1(x) \tag{13}$$

be any decomposition of $F_1(x)$ into completely reducible factors, while the decomposition of $F(x)$ into maximal completely reducible factors is given by (12). *Every product $G_i(x) \cdots G_1(x)$ $(i = 1, \cdots, s)$ is then a divisor of $H_i(x) \cdots H_1(x)$.*

According to the definition of a maximal completely reducible factor, $H_1(x)$ is always divisible by $G_1(x)$. The theorem will be proved by induction. Let us suppose that it has been shown that

$$H_{i-1}(x) \cdots H_1(x) = K_{i-1}(x)\, G_{i-1}(x) \cdots G_1(x). \tag{14}$$

Since $F_1(x)$ is a divisor of $F(x)$ one can write $F(x) = L(x)\, F_1(x)$; from (14) one also obtains, when multiplying on the left by $a_0\, H_r(x) \cdots H_i(x)$,

$$F(x) = a_0\, H_r(x) \cdots H_i(x)\, K_{i-1}(x)\, G_{i-1}(x) \cdots G_1(x).$$

Consequently, when one divides on the right by $G_{i-1}(x) \cdots G_1(x)$,

$$L(x)\, b_0\, G_s(x) \cdots G_i(x) = a_0\, H_r(x) \cdots H_i(x)\, K_{i-1}(x). \tag{15}$$

According to Theorem 11 of Chapter 1, and (15), the product $a_0\, H_r(x) \cdots H_i(x)$ is divisible by $K_{i-1}\, G_i(x)\, K_{i-1}^{-1}$. Since $G_i(x)$ is completely reducible, while $H_i(x)$ is the maximal completely reducible factor of the product $a_0\, H_r(x) \cdots H_i(x)$, one concludes that $H_i(x)$ is divisible by $K_{i-1}\, G_i(x)\, K_{i-1}^{-1}$. Consequently

$$H_i(x) = T(x)\, K_{i-1}\, G_i(x)\, K_{i-1}^{-1}.$$

We multiply this relation right-hand by $K_{i-1}(x)$ and, since all the polynomials are reduced, one obtains

[5] Compare Krull, Heidelberger Akademie 1926.

$$H_i(x)\,K_{i-1}(x) = T(x)\cdot[G_i(x), K_{i-1}(x)] = K_i(x)\cdot G_i(x). \tag{16}$$

When finally both sides of (16) are multiplied right-hand by $G_{i-1}(x)\cdots G_1(x)$, one obtains according to (14)

$$H_i(x)\cdots H_1(x) = K_i(x)\,G_i(x)\cdots G_1(x).$$

Q. E. D.

As an application of Theorem 8 one can prove

THEOREM 9. *An arbitrary representation of a polynomial as a product of completely reducible factors never contains a smaller number of factors than the decomposition in maximal completely reducible factors.*

The proof follows when one applies Theorem 8 to the special case $F_1(x) = F(x)$. If one had $s > r$ in (12) and (13), then one would obtain

$$H_s(x)\cdots H_1(x) = K_s(x)\,G_s(x)\cdots G_1(x) = K(x)\,F(x)$$

and a divisor of $F(x)$ would be divisible by $F(x)$ itself.

Every decomposition of a polynomial in completely reducible factors which contains the same number of factors as the decomposition in maximal completely reducible factors may therefore be called a *shortest completely reducible decomposition.*

We have, up till the present time, considered only right-hand completely reducible representations; it follows immediately, however, that corresponding theorems hold also for left-hand maximal completely reducible decompositions. It follows also from Theorem 6 that every left-hand completely reducible decomposition is also a right-hand completely reducible decomposition and it is therefore natural to investigate the connection between the two types.

Let us suppose that $F(x)$ is reduced; the left-hand decomposition of $F(x)$ in maximal completely reducible factors is also a right-hand (not maximal) completely reducible decomposition and, according to Theorem 9, it does not contain fewer factors than the right-hand maximal decomposition. The analogous reasoning holds for the right-hand maximal decomposition, and it follows that the right-hand and left-hand maximal decompositions must contain the same number of factors. From Theorem 8 there follows

THEOREM 10. *The decompositions*

$$F(x) = H_r(x)\cdots H_2(x)\,H_1(x), \tag{17}$$

$$F(x) = L_r(x)\cdots L_2(x)\,L_1(x) \tag{18}$$

of a reduced polynomial $F(x)$ into right-hand and left-hand maximal completely reducible factors are both shortest completely reducible decompositions, and, for every $i = 1, \cdots, r$, $H_i(x)\cdots H_1(x)$ is right-hand divisible by $L_i(x)\cdots L_1(x)$ and $L_r(x)\cdots L_i(x)$ is left-hand divisible by $H_r(x)\cdots H_i(x)$.

We shall finally prove a general property of the shortest completely reducible decompositions which gives a clearer picture of the particular properties of the special shortest decompositions (17) and (18).

THEOREM 11. *Let*

$$F(x) = S_r(x) \cdots S_2(x) S_1(x)$$

be an arbitrary shortest completely reducible decomposition of $F(x)$ *while the two maximal completely reducible decompositions are given in* (17) *and* (18). *Then, for every* $i = 1, \cdots, r$,

$$H_i(x) \cdots H_1(x) \equiv 0 \pmod{S_i(x) \cdots S_1(x)}, \tag{19}$$

$$S_i(x) \cdots S_1(x) \equiv 0 \pmod{L_i(x) \cdots L_1(x)}, \tag{20}$$

and similarly for left-hand divisibility,

$$\pmod{S_r(x) \cdots S_i(x)} \quad 0 \equiv L_r(x) \cdots L_i(x), \tag{21}$$

$$\pmod{H_r(x) \cdots H_i(x)} \quad 0 \equiv S_r(x) \cdots S_i(x). \tag{22}$$

The congruence (19) is a consequence of Theorem 8. The congruence (20) is correct for $i = r$ and we prove it by induction for all smaller i. Let us suppose, therefore, that it has been shown that

$$S_{i+1}(x) \cdots S_1(x) = K_{i+1}(x) L_{i+1}(x) \cdots L_1(x). \tag{23}$$

Since the same theorems will hold for left-hand divisibility as for right-hand divisibility (assuming $K = \bar{K}$), one can conclude that the product $L_{i+1}(x) \cdots L_1(x)$ is left-hand divisible by the left-hand transform of $S_{i+1}(x)$ by $K_{i+1}(x)$. The polynomial $K_{i+1}^{-1} S_{i+1}(x)_l K_{i+1}$ is completely reducible and, since it divides $L_{i+1}(x) \cdots L_1(x)$, it is also a left-hand divisor of $L_{i+1}(x)$. Consequently

$$L_{i+1}(x) = K_{i+1}^{-1} S_{i+1}(\mathbf{x})_l K_{i+1} T(x)$$

and

$$K_{i+1}(x) L_{i+1}(x) = [S_{i+1}(x), K_{i+1}(x)]_l T(x) = S_{i+1}(x) K_i(x).$$

When this expression is substituted in (23) one can divide on the left by $S_{i+1}(x)$ and one obtains the congruence (20). The congruences (21) and (22) can be obtained in a similar way.

Theorem 11 shows that right-hand maximal completely reducible decomposition is uniquely characterized among the shortest completely reducible decompositions by the fact that $H_i(x) \cdots H_1(x)$ has a maximal degree for each i, while for the left-hand maximal completely reducible decomposition the products $L_i(x) \cdots L_1(x)$ have a minimal degree.

4. The third decomposition theorem. A reduced polynomial is said to be *decomposible* when it can be represented as the union of reduced polynomials

$$F(x) = [A(x), B(x)], \tag{24}$$

where $A(x)$ and $B(x)$ are relatively prime; when no such representation exists, $F(x)$ is said to be *indecomposible*. When $A(x)$ or $B(x)$ in (24) can be decomposed further, we may continue the process and obtain the result that *every polynomial is representable as the union*

$$F(x) = [A_1(x), \cdots, A_r(x)]$$

of mutually prime indecomposible polynomials. The term *mutually prime* applied to a system of polynomials $A_i(x)$ will indicate that every polynomial is relatively prime to the union of the remaining polynomials.

In the proof of the principal theorem about such decompositions we shall need an auxiliary theorem which we now deduce.

THEOREM 12. *When $F(x)$ is decomposible and has the decomposition* (24), *then every divisor $F_1(x)$ of $F(x)$ which is divisible by $A(x)$ is also decomposible and has the representation*

$$F_1(x) = [A(x), (B(x), F_1(x))]. \tag{25}$$

It is obvious that the right-hand side of (25) is a divisor of the left-hand side; to prove the converse, let

$$F_1(x) = Q_1(x)\, A(x), \qquad F(x) = Q_2(x)\, F_1(x).$$

Dividing (24) right-hand by $A(x)$ one obtains

$$AB(x)\, A^{-1} = Q_2(x)\, Q_1(x),$$

and $Q_1(x)$ is consequently of the form $AD(x)A^{-1}$. Hence $F_1(x) = [A(x), D(x)]$, where $D(x)$ is a divisor of $B(x)$.

We can now prove the *third* decomposition theorem.[6]

THEOREM 13. *Every polynomial has a representation as the union*

$$F(x) = [A_1(x), \cdots, A_r(x)] \tag{26}$$

of mutually prime indecomposible polynomials. If a second such representation

$$F(x) = [B_1(x), \cdots, B_s(x)] \tag{27}$$

exists, then the number of components in both must be the same and they are similar in pairs. An arbitrary component $A_i(x)$ in (26) *can always be replaced by a suitably chosen component $B_j(x)$ in* (27), *and every $B_j(x)$ can be used for some such replacement.*

[6] By means of the method used by Krull, Math. Zeitschr. 8, one could have deduced the fact that the components are similar. The present proof seems somewhat simpler since the "Zurückleitungsgruppen" of Krull are avoided. This proof also yields the fact that the components are replaceable.

It has already been shown that there exists such a decomposition. We shall prove the general theorem by induction and assume it to be true for all polynomials of degree lower than that of $F(x)$.

We show first: *If two decompositions* (26) *and* (27) *exist, then an* $A_i(x)$ *(or a* $B_i(x)$*) can replace a* $B_j(x)$ *(or an* $A_j(x)$*).* Let $A_1(x)$ be the component of highest degree in the two representations. If then $A_1(x)$ is relatively prime to the union $[B_2(x), \cdots, B_s(x)]$, one must have

$$F(x) = [A_1(x),\ B_2(x), \cdots, B_s(x)],$$

and the conjecture is proven. Let $P(x)$ be a common prime factor of $A_1(x)$ and $[B_2(x), \cdots, B_s(x)]$. Then

$$B(x) = [B_2(x), \cdots, B_s(x)] = \bar{B}(x)\, P(x) \tag{28}$$

and $A_1(x) = \bar{A}_1(x)\, P(x)$. We divide both sides of (26) and (27) by $P(x)$ and obtain

$$[\bar{A}_1(x),\ PA_2(x)\, P^{-1}, \cdots, PA_r(x)\, P^{-1}] = [PB_1(x)\, P^{-1}, \bar{B}(x)]. \tag{29}$$

According to our assumption the theorem holds for the polynomial (29). The indecomposible component $PB_1(x)\, P^{-1}$ can therefore replace a component on the left-hand side, that is, $PB_1(x)\, P^{-1}$ can replace a $PA_i(x)\, P^{-1}$ $(i = 2, \cdots, r)$ or a component of $\bar{A}_1(x)$.

In the first case let $PB_1(x)\, P^{-1}$ replace $PA_2(x)\, P^{-1}$, giving

$$F(x)\, P(x)^{-1} = [\bar{A}_1(x), PB_1(x)\, P^{-1}, PA_3(x)\, P^{-1}, \cdots, PA_r(x)\, P^{-1}].$$

Multiplication with $P(x)$ then gives

$$F(x) = [A_1(x),\ B_1(x),\ A_3(x), \cdots, A_r(x)].$$

In the second case one obtains

$$F(x) \cdot P(x)^{-1} = [PB_1(x)\, P^{-1}, \bar{\bar{A}}_1(x), PA_2(x)\, P^{-1}, \cdots, PA_r(x)\, P^{-1}],$$

and consequently

$$F(x) = [B_1(x), \bar{\bar{A}}_1(x)\, P(x), A_2(x), \cdots, A_r(x)]. \tag{30}$$

This is, however, impossible; if one puts

$$A(x) = [A_2(x), \cdots, A_r(x)], \tag{31}$$

one concludes from (26) and (30), when dividing by $A(x)$, that

$$AA_1(x)\, A^{-1} = [AB_1(x)\, A^{-1}, A\,(\bar{\bar{A}}(x)\, P(x))\, A^{-1}],$$

and it would follow that $A_1(x)$ is decomposible, contrary to assumption.

We have consequently shown that, with a suitable notation, one always has a identity of the form

$$[A_1(x), \cdots, A_r(x)] = [B_1(x), A_2(x), \cdots, A_r(x)]. \tag{32}$$

It then follows easily that the two decompositions (26) and (27) have the same number of components similar in pairs. If one divides (32) by $A(x)$ one obtains

$$A A_1(x) A^{-1} = A B_1(x) A^{-1},$$

and $A_1(x)$ and $B_1(x)$ are similar. From the identity

$$[B_1(x), A_2(x), \cdots, A_r(x)] = [B_1(x), B_2(x), \cdots, B_s(x)]$$

there follows, after division by $B_1(x)$,

$$[B_1 A_2(x) B_1^{-1}, \cdots, B_1 A_r(x) B_1^{-1}] = [B_1 B_2(x) B_1^{-1}, \cdots, B_1 B_s(x) B_1^{-1}], \tag{33}$$

and, since the theorem holds for this polynomial, one must have $r = s$ and the $A_i(x)$ are similar to the $B_i(x)$ in pairs.

It now remains only to show that all components in (27) can replace an $A_i(x)$ and that all $A_i(x)$ can be thus replaced, a similar relation holding for the $B_i(x)$. Since this is true for (33), one may replace any $B_1 B_i(x) B_1^{-1}$ by a $B_1 A_i(x) B_1^{-1}$, and when one multiplies afterward by $B_1(x)$ one obtains

$$F(x) = [B_1(x), B_2(x), \cdots, A_i(x), \cdots, B_r(x)] \qquad (i = 2, \cdots, r), \tag{34}$$

and all $B_i(x)$, with the possible exception of $B_1(x)$, can be replaced. From (34) one concludes, as from (32), that every $A_j(x)$, with the possible exception of $A_i(x)$, may be replaced by a $B_j(x)$ and every $B_j(x)$, except possibly $B_i(x)$, may be used for such a replacement. When $r \geqq 3$, one might have used a different index $j \neq i$ in (34), and the exceptions are thereby avoided. It follows then in the same way that all $B_j(x)$ may be replaced by $A_j(x)$ and that all $A_j(x)$ may be used for replacements.

This leaves only the special case

$$F(x) = [A_1(x), A_2(x)] = [B_1(x), A_2(x)] = [B_1(x), B_2(x)] \tag{35}$$

unaccounted for. We shall have to show that $A_2(x)$ may also be replaced by $B_1(x)$ or $B_2(x)$, that is, that at least one of the relations

$$F(x) = [A_1(x), B_1(x)], \qquad F(x) = [A_1(x), B_2(x)] \tag{36}$$

holds. Let us suppose, for instance, that $F_1(x) = [A_1(x), B_2(x)]$ is only a divisor of $F(x)$; then one has, according to Theorem 12,

$$F_1(x) = [A_1(x), (A_2(x), F_1(x))] = [(F_1(x), B_1(x)), B_2(x)]. \tag{37}$$

The theorem holds for $F_1(x)$; from (35) it follows that $A_2(x)$ and $B_2(x)$ are similar and consequently of the same degree, while the degree of $(A_2(x), F_1(x))$ must be less than the degree of $B_2(x)$. In (37), therefore, $A_1(x)$ can replace only $B_2(x)$, giving

$$F_1(x) = [(F_1(x), B_1(x)), A_1(x)],$$

so that

$$F(x) = [B_1(x), F_1(x)] = [B_1(x), (F_1(x), B_1(x)), A_1(x)] = [A_1(x), B_1(x)].$$

In the same way one shows all the other replacements.

5. **Connection between representations of the second and third kind.** One may ask for the connection between the three main types of representations which we have discussed up till now. It is fairly obvious how one can deduce the prime function decomposition of a polynomial when its representation as a product of maximal completely reducible factors or as a union of indecomposible components is given. We shall study here the connection between the two last types of representations.

Let

$$F(x) = [A(x), B(x)], \quad (A(x), B(x)) = 1, \tag{38}$$

and let

$$A(x) = A_r(x) \cdots A_2(x) A_1(x), \quad B(x) = B_s(x) \cdots B_2(x) B_1(x) \tag{39}$$

be the maximal completely reducible representations of $A(x)$ and $B(x)$. It is easily seen that every prime divisor of $F(x)$ also divides $F_1(x) = [A_1(x), B_1(x)]$ and that $F_1(x)$ is the maximal completely reducible factor of $F(x)$. One also finds that $F(x)$ can be represented in the form

$$F(x) = [\bar{A}(x), \bar{B}(x)] F_1(x),$$

where $\bar{A}(x)$ is similar to $A_r(x) \cdots A_2(x)$ and $\bar{B}(x)$ similar to $B_s(x) \cdots B_2(x)$. When the same conclusions are drawn for $[\bar{A}(x), \bar{B}(x)]$, one finds that the maximal completely reducible factor of this polynomial is $F_2(x) = [\bar{A}_2(x), \bar{B}_2(x)]$, where $\bar{A}_2(x)$ is similar to $A_2(x)$ and $\bar{B}_2(x)$ to $B_2(x)$ and so on.

THEOREM 14. *Let* $[A(x), B(x)]$ *be a decomposible polynomial and let* (39) *be the maximal completely reducible representations of* $A(x)$ *and* $B(x)$. *The maximal completely reducible representation of* $F(x)$ *is then*

$$F(x) = F_r(x) \cdots F_1(x), \quad F_i(x) = [\bar{A}_i(x), \bar{B}_i(x)] \quad (i = 1, \cdots, r),$$

where $\bar{A}_i(x)$ *is similar to* $A_i(x)$ *and* $\bar{B}_i(x)$ *similar to* $B_i(x)$.

When $r > s$ one obviously must put $\bar{B}_i(x) = 1$ for $i > s$. For an arbitrary representation

$$F(x) = [F_1(x), \cdots, F_r(x)],$$

where

$$F_i(x) = \Phi_{s_i}^{(i)}(x) \cdots \Phi_1^{(i)}(x)$$

is the representation of each component as product of maximal completely reducible factors, one finds for the corresponding representation of $F(x)$,

$$F(x) = A_s(x) \cdots A_1(x),$$

where

$$A_j(x) = [\overline{\Phi}_j^{(1)}(x), \cdots, \overline{\Phi}_j^{(r)}(x)]$$

and $\overline{\Phi}_j^{(i)}(x)$ is similar to $\Phi_j^{(i)}(x)$. It is also obvious that s is equal to the greatest of the numbers s_i.

6. **The fourth decomposition theorem.** In the preceding discussion we have deduced some of the more important structure theorems for noncommutative polynomials. They depend on various notions of indecomposibility, each notion leading to its own type of decomposition theorems. There are various other possible definitions of indecomposibility which lead to a number of decomposition theorems. We shall discuss only one of these possibilities.

We shall say that a polynomial is *distributive* if there exists a representation

$$F(x) = [A(x), B(x)]$$

in which $A(x)$ and $B(x)$ are *proper* divisors of $F(x)$; the polynomials $A(x)$ and $B(x)$ do not have to be relatively prime as in the case of decomposible polynomials. Every decomposible polynomial is distributive, but not conversely.

It is easily seen that every reduced polynomial has a representation as the union of non-distributive polynomials

$$F(x) = [A_1(x), \cdots, A_r(x)], \tag{40}$$

where all $A_i(x)$ are supposed to be reduced. In the following we shall always assume that in a representation (40) we have omitted the superfluous components, that is, the polynomials which are divisors of the union of the remaining polynomials. We shall then say that the representation is *minimal.*

We shall first deduce a characteristic property of the non-distributive polynomials. A non-distributive polynomial cannot be divisible on the left by two different prime polynomials, for otherwise one would obtain a representation

$$A(x) = [P_1(x), P_2(x)]_l K(x) \tag{41}$$

and, according to Theorem 17, Chapter 1, the left-hand union could also be considered as a right-hand union

$$[P_1(x),\ P_2(x)]_l = [Q_1(x),\ Q_2(x)],$$

where the prime functions $Q_1(x)$ and $Q_2(x)$ are similar to $P_1(x)$ and $P_2(x)$. From (41) it would then follow that $A(x)$ must be distributive, that is,

$$A(x) = [Q_1(x)\,K(x),\ Q_2(x)\,K(x)].$$

We can also show the converse, namely, that when $A(x)$ is divisible by only a single prime function on the left, that is, when the maximal left-hand completely reducible factor is a prime polynomial, then $A(x)$ is non-distributive. If one had in fact

$$A(x) = [A_1(x),\ A_2(x)],$$

then $A_1(x)$ and $A_2(x)$ would have some right-hand cross-cut $D(x)$ which would have to be a proper divisor of both polynomials. Let us put

$$A_1(x) = \bar{A}_1(x)\,D(x), \qquad A_2(x) = \bar{A}_2(x)\,D(x), \qquad A(x) = \bar{A}(x)\,D(x).$$

Consequently

$$\bar{A}(x) = [\bar{A}_1(x),\ \bar{A}_2(x)],$$

where $\bar{A}_1(x)$ is relatively prime to $\bar{A}_2(x)$. From Theorem 17, Chapter 1, one concludes, however, that $\bar{A}(x)$ must then also be the left-hand union of two relatively prime factors, and $A(x)$ would be left-hand divisible by at least two different prime functions.

THEOREM 15. *The necessary and sufficient condition that a polynomial be non-distributive is that it be left-hand divisible by only a single prime polynomial.*

To every non-distributive polynomial $A(x)$ there exists a unique prime polynomial $P(x)$ which divides it on the left. We shall say that $A(x)$ *belongs to* $P(x)$. An immediate consequence of Theorem 15 is

THEOREM 16. *Every left-hand divisor of a non-distributive polynomial is again non-distributive.*

It is clear that a left-hand divisor of $A(x)$ can be only left-hand divisible by $P(x)$. We shall need also the following result:

THEOREM 17. *The transform of a non-distributive polynomial is non-distributive and belongs to a similar prime polynomial.*

Let us put $A_1(x) = CA(x)C^{-1}$, where we first suppose that the transformation is special, that is, that $(C(x),\ A(x)) = 1$. If then

$$A_1(x) = [L(x),\ M(x)],$$

one obtains

$$A(x) = [C_1 L(x)\,C_1^{-1},\ C_1 M(x)\,C_1^{-1}],$$

where $C_1(x)$ is determined such that $C_1(x)\,C(x) \equiv 1 \pmod{A(x)}$. $A(x)$ would then be distributive. Now let $A(x)$ and $C(x)$ have a greatest common factor $D(x)$, and let

$$A_1(x) = \bar{A}(x)\,D(x), \qquad C(x) = \bar{C}(x)\,D(x).$$

Here $\bar{A}(x)$ is non-distributive, and, since

$$A_1(x) = CA(x)\,C^{-1} = \bar{C}\,\bar{A}(x)\bar{C}^{-1},$$

this case has been reduced to the first. The second part of the theorem follows from the rule of transformation of a product.

We are now able to prove the following fourth decomposition theorem:

THEOREM 18. *Let $F(x)$ be an arbitrary polynomial and let*

$$F_1(x) = [P_1(x), \cdots, P_r(x)]$$

be the maximal left-hand completely reducible factor of $F(x)$. Every minimal representation of $F(x)$ as a union of non-distributive polynomials has then the form

$$F(x) = [A_1(x), \cdots, A_r(x)], \tag{42}$$

where every non-distributive polynomial $A_i(x)$ belongs to a prime polynomial which is similar to a $P_i(x)$.

The theorem is obviously correct for every completely reducible polynomial. We prove it by induction, assuming it to be true for every left-hand divisor of $F(x)$ which is left-hand divisible by $F_1(x)$. We can then assume that $F(x)$ is not completely reducible, so that

$$F(x) = F_1(x) \cdot Q(x).$$

Let $P(x)$ be a right-hand prime divisor of $F(x)$ and $Q(x)$ such that $F(x)\,P(x)^{-1}$ is left-hand divisible by $F_1(x)$.

Let us first suppose that $F(x)$ has a decomposition of the form (42) with $s \neq r$ components, and let us further suppose that $P(x)$ does not divide any $A_i(x)$. One then obtains

$$F(x) \cdot P(x)^{-1} = [PA_1(x)\,P^{-1}, \cdots, PA_s(x)\,P^{-1}], \tag{43}$$

where every component $PA_i(x)\,P^{-1}$ is non-distributive according to Theorem 17. We wish to show that this representation (43) is minimal. If namely $PA_1(x)\,P^{-1}$ were a divisor of the union of the other $PA_i(x)\,P^{-1}$, then one would obtain, when multiplying by $P(x)$,

$$F(x) = [P(x), A_2(x), \cdots, A_s(x)].$$

But this is not possible, since $F(x)$ would be decomposible into

$$F(x) = [P(x), H(x)]; \qquad H(x) = [A_2(x), \cdots, A_s(x)],$$

and the results of section 5, Chapter 2, would show that the maximal left-hand completely reducible factor of $F(x)\,P(x)^{-1}$ would be of lower degree than $F_1(x)$. The representation (43) is then minimal, giving $r = s$, and the components $PA_i(x)\,P^{-1}$, and therefore also the $A_i(x)$, belong to prime polynomials similar to the $P_i(x)$.

Let us assume in the second place that the first t polynomials $A_i(x)$ in (42) are divisible by $P(x)$,

$$A_1(x) = \bar{A}_1(x)\,P(x), \cdots, A_t(x) = A_t(x)\,P(x).$$

It follows from the choice of $P(x)$ as above that the possibility $A_i(x) = P(x)$ is excluded; the polynomials $\bar{A}_i(x)$ are consequently non-distributive according to Theorem 17 and belong to the same prime polynomial as $A_i(x)$.

When (42) is right-hand divided by $P(x)$, one obtains

$$(44)\quad F(x)\,P(x)^{-1} = [\bar{A}_1(x), \cdots, \bar{A}_t(x), PA_{t+1}(x)\,P^{-1}, \cdots, PA_s(x)\,P^{-1}].$$

This non-distributive representation is also minimal. If, for instance, $PA_s(x)\,P^{-1}$ were a divisor of the union of the remaining polynomials, then it would follow from (44) that the original representation was not minimal. If, for instance, $\bar{A}_1(x)$ were a divisor of the union of the remaining, one would obtain for $t > 1$, that the original representation was not reduced and for $t = 1$

$$F(x) = [P(x), A_2(x), \cdots, A_s(x)].$$

This is shown to be impossible as before.

Since our theorem holds for (44), it follows that $s = r$ and that $A_i(x)$ belongs to a prime polynomial similar to $P_i(x)$.

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DIFFERENTIAL FIELDS AND IDEALS OF DIFFERENTIAL FORMS.[1]

By H. W. Raudenbush, Jr.

In an abstract treatment of ordinary algebraic differential equations, a system of elements, closed with respect to the three operations of addition, multiplication and differentiation, will hold a central position. Ascribing to addition and multiplication the usual properties (including the commutative property of multiplication), such a system will be a commutative ring, a domain of integrity or a commutative field. The operation of differentiation is linked with the operations of addition and multiplication by the formulas for the derivatives of a sum and of a product. Such a system will be called a *differential ring*, a *differential domain of integrity* or a *differential field*, as the case may be.

Differential fields have been treated by Baer[2] in a paper in which he obtains necessary and sufficient conditions for the existence of a differential field which is a given commutative algebraic field and has, as the set of elements with zero for derivatives, a given subfield. Baer also shows that a given differential field may be extended to a field each element of which is the derivative of some element; that is, to a differential field closed with respect to integration. Ore[3] has used differential fields in an abstract theory of linear differential operators.

Any given extension H of a given algebraic field K which is not an algebraic extension of K was shown by Steinitz[4] to be the result of a purely transcendental extension followed by an algebraic extension. A purely transcendental extension is obtained by the adjunction of a set of elements that satisfy no algebraic equation with coefficients in the original field not all zero. Each element of an algebraic extension satisfies an algebraic equation with coefficients in the field extended not all zero. We call attention to a lemma used in obtaining this result. An element t of an extension H of a field K which is algebraic with respect to the field $K(M)$, obtained by adjoining to K a set M of elements of H algebraic with respect to K, is itself algebraic with respect to K. In the first part

[1] Received May 17, 1932.

[2] R. Baer, *Algebraische Theorie der differentierbaren Funktionenkörper* I, Heidelberger Akad. der Wiss. Sitz., Math. Natur. Klasse, 1927-28.

[3] O. Ore, *Formale Theorie der linearen Differentialgleichungen* I, Journal für Mathematic, vol. 167 (1931), p. 221.

[4] E. Steinitz, *Algebraische Theorie der Körper*, Journal für Mathematik, vol. 137 (1910), pp. 167-309. This paper has appeared in book form under the editorship of R. Baer and H. Hasse.

of this paper, we establish, by a necessarily more complicated elimination, an analogue, for differential fields and algebraic differential equations, to this lemma. Principally by means of this analogue we establish the result: Any given extension H of a differential field K which is not an *algebraically transcendental extension* of K may be obtained by a *purely hypertranscendental extension* followed by an algebraically transcendental extension. We define a purely hypertranscendental[5] extension of a differential field as an extension obtained by the adjunction of a set of elements which satisfy no algebraic differential equation with coefficients in the original field not all zero. We define an algebraically transcendental extension of a differential field as an extension each of whose elements satisfy an algebraic differential equation with coefficients in the field extended not all zero. Following the analogy, we are able to define degree of hypertranscendency and prove that it is additive. Where proofs of the algebraic analogues of our theorems may be converted easily to our use, references to them are given.

In the second part, we treat prime differential ideals of forms. By a form is meant a polynomial in indeterminates and their derivatives with coefficients in a given differential field K.[6] The set of forms of K that vanish when certain elements of an extension H of K are substituted for the indeterminates is shown to be a prime ideal in the differential ring of all forms of K and since it includes with each form its derivative, we say that it is a prime differential ideal. Results are obtained corresponding to a part of van der Waerden's[7] work on the zero field of a prime ideal of polynomials.

I. Differential Fields.

1. A commutative algebraic field is said to be a *differential field* if to each element a there corresponds one and only one element a_1, which we call the derivative of a, in such a way that for any two elements a and b with derivatives a_1 and b_1, respectively, the derivatives $(a+b)_1$ and $(ab)_1$ of $a+b$ and ab, respectively, are given by the formulas:

$$(1) \qquad (a+b)_1 = a_1+b_1,$$

$$(2) \qquad (ab)_1 = ab_1+a_1b.$$

If a differential field H has as a subset a differential field K in such a way that sums, products and derivatives of elements of K are the same

[5] The term "hypertranscendental" is due to A. Bloch. Cf. Journal de Mathematiques, ser. 9, vol. 10 (1931), p. 290.

[6] This use of the word "form" is due to J. F. Ritt. Cf. Transactions Amer. Math. Soc., vol. 32 (1930), p. 569.

[7] B. L. van der Waerden, *Zur Nullstellentheorie der Polynomideale*, Mathematische Annalen, vol. 96 (1926), p. 183.

in both H and K, then H is said to be an *extension* of K and K is said to be a *subfield* of H.

Let M be a set of elements of an extension H of a differential field K. We shall call a polynomial in the elements of M and their derivatives with coefficients in the differential field K a *form* in M with coefficients in K. Consider the elements of H that can be represented as the quotient of two forms in M with coefficients in K, the denominator not zero. These elements form a differential field which contains the elements of K and the elements of M. We shall call this differential field the differential field obtained by the *adjunction* of M to K and denote it by $K(M)$. $K(M)$ is the smallest subfield of H that contains the elements of K and of M. If a set N contains the set M, then $K(N)$ contains $K(M)$.

Let H be an extension of a differential field K. If an element t of H satisfies an algebraic differential equation with coefficients in K not all zero, then we shall say that t is *algebraically transcendental* with respect to K. An element t of H which is not algebraically transcendental we shall call *hypertranscendental*. If every element of H is algebraically transcendental, then H is said to be an *algebraically transcendental extension* of K.

2. LEMMA. *If t is an element of an extension H of a differential field K and is algebraically transcendental with respect to a differential field $K(M)$ obtained by the adjunction of a set M of elements of H, each element of M algebraically transcendental with respect to K, then t is algebraically transcendental with respect to K.*

Proof. There is at least one form in t with coefficients in $K(M)$ not all zero that vanishes. Let r be the least of the orders in t of all such forms. From among such forms of order r let $F(t)$ be a form of lowest degree in the rth derivative of t.

We may and shall assume $F(t)$ to be a form in t and the elements of M with coefficients in K. Only a finite number of elements of M are *effectively* in $F(t)$; that is, only a finite number of elements of M are represented by themselves or by their derivatives in terms with non-zero coefficients. Let these be denoted by m_i $(i = 1, \cdots, p)$. Let the nth derivative of t and m_i be denoted by t_n and m_{in} respectively.

Let $S = \partial F/\partial t_r$. S is either of lower degree than F in t_r or of lower order than F in t; hence S is not zero. By differentiating $F(t) = 0$, we may write

$$t_{r+1} = A_1/S, \tag{3}$$

where A_1 is a form in t and $m_1, \cdots, m_p$ of order r or less in t with coefficients in K. Differentiating (3), we obtain $t_{r+2} = Q/S^2$, where Q

is a form in t and $m_1, \cdots, m_p$ of order $r+1$ or less in t with coefficients in K. Using (3), we may write $t_{r+2} = A_2/S^3$ with A_2 a form in t and $m_1, \cdots, m_p$ of order r or less in t with coefficients in K. Proceeding in this manner we may write for any positive integer ϱ,

$$t_{r+\varrho} = A_\varrho / S^{\nu_\varrho}, \tag{4}$$

where ν_ϱ is a positive integer and A is a form in t and $m_1, \cdots, m_p$ of order r or less in t with coefficients in K.

There is at least one form in each m_i $(i = 1, \cdots, p)$, with coefficients in K not all zero, that vanishes. Let r_i be the least of the orders in m_i of all such forms in m_i. Choose from among such forms of order r_i in m_i a form of lowest degree in m_{ir_i} and denote it by G_i. We may write, for any positive integer σ,

$$m_i(r_i+\sigma) = B_\sigma / S_i^{\eta_\sigma}, \tag{5}$$

where $S_i = \partial G_i / \partial m_{ir_i} \neq 0$, η_σ is a sufficiently large integer and B_σ is a form in m_i of order r_i or less in m_i with coefficients in K.

Using the relations (5), we may write, in place of (4),

$$t_{r+\varrho} = C_\varrho / P_\varrho, \tag{6}$$

where C_ϱ and P_ϱ are forms in t and $m_1, \cdots, m_p$, with coefficients in K, of order r or less in t and r_i or less in m_i $(i = 1, \cdots, p)$; we have $P_\varrho \neq 0$. In the C_ϱ and P_ϱ there are

$$h = 1 + p + r + r_1 + \cdots + r_p$$

letters $t, m_1, \cdots, m_p$ and their derivatives.[8] The first $h+1$ relations (6) may be written

$$t_{r+\varrho} = D_\varrho / P, \qquad (\varrho = 1, \cdots, h+1), \tag{7}$$

where $P \neq 0$ is the product of the P_ϱ $(\varrho = 1, \cdots, h+1)$. Regard D_ϱ and P as polynomials in the h letters $t, m_1, \cdots, m_p$ and their derivatives, and let n be the maximum of their total degrees.

Let α represent a positive integer to be fixed later. Form all power products V_j of the derivatives $t_{r+\varrho}$ $(\varrho = 1, \cdots, h+1)$ of degree α or less. We may write

$$V_j = E_j / P^\alpha,$$

where the degree of E_j regarded as a polynomial in the h letters $t, m_1, \cdots, m_p$ is at most $n\alpha$. We regard the E_j's as linear functions of products of powers of $t, m_1, \cdots, m_p$ and their derivatives. There are at most[9] $((n\alpha+h)\cdots(n\alpha+1))/h!$ such products of powers in the E_j.

[8] It does not matter in the following that some of these letters may represent the same element of H.

[9] O. Perron, *Algebra*, p. 67.

There are exactly $((\alpha+h+1)\cdots(\alpha+1))/(h+1)!$ distinct products of powers V_j and hence the same number of linear functions E_j. Hence we may choose α sufficiently large so that the number of the E_j's exceeds the number of the terms in the E_j's. Since the coefficients in the E_j's are in K, there exists a linear combination of the E_j's, with coefficients in K, not all zero that vanishes. The same linear combination of the V_j's vanishes, since $P \neq 0$. This is the form in t whose existence we need in order to prove the lemma.

3. An element t of an extension H of a differential field K is said to *depend* on a set M of elements of H (with respect to K) if t satisfies an algebraic differential equation with coefficients in $K(M)$ not all zero. We give five lemmas on dependence.

LEMMA 1. *t depends on t.*

LEMMA 2. *If t depends on M and N is a subset of H that includes M, then t depends on N.*

LEMMA 3. *If t depends on M, t depends on a finite subset M' of M.*

LEMMA 4. *In the notation of Lemma 3 where M' is taken as small as possible, each element of M' depends on the set formed from M' by replacing that element by t.*

LEMMA 5. *If t depends on M and each element of M depends on a set N, then t depends on N.*

We shall prove only the fifth lemma. Let $K(N) = K_1$. $K_1(M)$ includes $K(M)$, since K_1 includes K. Hence t is algebraically transcendental with respect to $K_1(M)$. But each element of M is algebraically transcendental with respect to K_1, hence by the lemma of the preceding paragraph t depends on N with respect to K.

A set M will be said to depend on a set N (with respect to K) if each element of M depends on N with respect to K. Two sets will be called *equivalent* if each depends on the other. A set will be said to be *irreducible* if it is not equivalent with any proper subset of itself, otherwise it will be said to be *reducible.*

By exactly the argument used by van der Waerden[10] to prove Steinitz' theorem, we may prove the analogue:

THEOREM I. *Every set contains an equivalent irreducible subset. In particular, an extension H of a differential field K which is not an algebraically transcendental extension of K may be obtained by the adjunction of an irreducible set followed by an algebraically transcendental extension.*

The extension obtained by the adjunction of an irreducible set will be said to be a *purely hypertranscendental extension.*

[10] B. L. van der Waerden, *Moderne Algebra*, 1930, vol. 1, p. 206.

4. We give two lemmas depending on Lemmas 1–5 which will be used to prove Theorem II.

LEMMA 6. *If an irreducible set S becomes reducible by the addition of an element a, then a is dependent on S.*

This obvious lemma is given in order that we may refer to Steinitz' paper for the proof of

LEMMA 7. *If S is an irreducible set (with respect to K) and if the element a is hypertranscendental with respect to K but dependent on S, then S contains a uniquely determined finite subsystem T with the following properties: Each subsystem of S, on which a depends, contains the set T; if any element of T is replaced by a, then S becomes an irreducible set equivalent to S; no other element of S has this property.*[11]

This lemma can be used to prove

THEOREM II. *If an extension H is equivalent to two irreducible subsets M_1 and M_2, then M_1 and M_2 are sets of the same potency.*

The proof by R. Baer and H. Hasse[12] of the algebraic analogue may be parallelled.

The potency of an equivalent irreducible subset of an extension H of a differential field K may be called the *degree of hypertranscendency* of H with respect to K.

We also have, as in the algebraic theory,

THEOREM III. *If H depends on the irreducible set M with respect to J and J on the irreducible set N with respect to K, then H depends on the irreducible set $M+N$ with respect to K.*[13]

This says that the degree of hypertranscendency of H with respect to K is $m+n$ if m is the degree of hypertranscendency of H with respect to J and n that of J with respect to K.

II. Prime Differential Ideals of Differential Forms.

5. A commutative algebraic ring will be said to be a *differential ring* if each element has a unique derivative and these derivatives satisfy the rules (1) and (2). A *differential domain of integrity* is defined similarly.

In Part I, forms in M with coefficients in K were considered equal if the elements of H which they represented were equal. Thus two forms in M with coefficients in K might be equal though not identical. In this Part we wish to consider two forms as equal if and only if they are identical, that is, if they have equal coefficients for corresponding terms. Hence we replace the set M by a set X of indeterminates x_i with deri-

[11] Steinitz, loc. cit., p. 290, or in the reprint, p. 115.

[12] R. Baer and H. Hasse, in the *Erläuterungen* with the reprint of Steinitz' paper, p. 24.

[13] Steinitz, loc. cit., p. 299, or in the reprint, p. 124.

vatives x_{ir}. The indeterminates may be described as symbols operated with as though they were elements of some differential field containing K but not satisfying any algebraic differential equation with coefficients in K not all zero.[14] If we wish to consider forms in elements of a set M not a set of indeterminates, we shall use the language of substitution, and say that a certain form has certain properties when the elements of M are substituted for those of X.

The set of all forms in X with coefficients in K is a differential ring and indeed a differential domain of integrity.

Let D be a differential domain of integrity. We form the set of all ratios a/b $(b \neq 0)$ of elements of D. We call two ratios a/b $(b \neq 0)$ and c/d $(d \neq 0)$ equal if and only if

$$ad = bc. \tag{8}$$

We devide all ratios into mutually exclusive classes of equal ratios and call each class a *quotient*. We consider the set of quotients. Let a/b $(b \neq 0)$ and c/d $(d \neq 0)$ be ratios defining two quotients. By the sum of these quotients we mean the quotient represented by

$$(ad+bc)/ad. \tag{9}$$

The sum exists and is unique. By the product of these two quotients we mean the quotient represented by

$$ac/bd. \tag{10}$$

The product exists and is unique. Furthermore the set of quotients is known to be an algebraic commutative field for these definitions of sum and product. We define the derivative of the quotient represented by a/b $(b \neq 0)$ as the quotient represented by

$$(a_1 b - a b_1)/b^2. \tag{11}$$

The derivative exists since $b^2 \neq 0$ and $(a_1 b - a b_1)$ are elements of D. To prove that it is unique, suppose that c/d $(d \neq 0)$ also represents the quotient represented by a/b, that is, that $a/b = c/d$ or that the relation (8) holds. We wish to show that $(a_1 b - a b_1)/b^2 = (c_1 d - c d_1)/d^2$ or that

$$(a_1 b - a b_1) d^2 = (c_1 d - c d_1) b^2. \tag{12}$$

[14] A formal treatment of forms in indeterminates may be given. Cf. B. L. van der Waerden, *Moderne Algebra*, vol. I, p. 50, foot-note.

Differentiating (8) we obtain

$$a_1 d - b_1 c = b c_1 - a d_1. \tag{13}$$

Multiplying (13) by bd and using (8) we obtain (12). Similarly, using (9), (10), and (11), we can show that the quotients and their derivatives satisfy (1) and (2). Hence the set of quotients under these definitions of sum, product and derivative form a differential field. We call this differential field the *differential quotient field of* D.

Let K be a given differential field and X a given set of indeterminates. The differential ring of forms in X with coefficients in K will be denoted by $K[X]$. Since $K[X]$ is also a differential domain of integrity, we may form its differential quotient field which we shall denote by $K(X)$.

Two differential fields (or differential rings, differential domains of integrity) are said to be *isomorphic* if their elements can be set into one-to-one correspondence in such a way that sums, products and derivatives of corresponding elements correspond. For our purposes, isomorphic differential fields may be regarded as the same. Hence an extension H of a differential field K_1 that is isomorphic with a differential field K will be regarded as an extension of K. $K(X)$ is an extension of K. $K(X)$ may be said to be a purely hypertranscendental extension of K since X is irreducible with respect to K. We have

THEOREM IV. *For any given differential field K there exists a purely hypertranscendental extension of any degree of hypertranscendency. Furthermore, any two such extensions of the same degree of hypertranscendency of the same differential field K are isomorphic.*

6. A subset I of a differential ring D will be called a *prime differential ideal* if it is a prime ideal of D in the algebraic sense and if together with each of its elements it contains the derivative of that element.

Let H be an extension of K. Let $(a_1, \cdots, a_n)$ be elements of H. We consider the differential ring $K[X]$ of forms in n indeterminates $x_1, \cdots, x_n$ and inquire what elements of $K[X]$ vanish when a_i is substituted for x_i. The set of elements of $K[X]$ that vanish form a prime ideal I in $K[X]$ regarded as an algebraic ring of polynomials with coefficients in K. The prime ideal I is a prime differential ideal since, if it includes a form $f(x_1, \cdots, x_n)$, it must include $f_1(x_1, \cdots, x_n)$. This follows from the fact that $f_1(x_1, \cdots, x_n)$ has the same coefficients as $f_1(a_1, \cdots, a_n)$, where by $f_1(a_1, \cdots, a_n)$ is meant the derivative of the form $f(a_1, \cdots, a_n)$ as an element of H. Also $f_1(a_1, \cdots, a_n) = 0$ since $0_1 = 0$.

We now consider the following question. Suppose we are given a differential field K and a set X of n indeterminates $x_1, \cdots, x_n$. Let I be a prime differential ideal in $K[X]$ which is not $K[X]$ itself. Can we

construct an extension of K by the adjunction of a set of n elements $a_1, \cdots, a_n$ in such a way that the set of forms of $K[X]$ that vanish when a_i is substituted for $x_i (i = 1, \cdots, n)$ is exactly I? Since I is a prime ideal of $K[X]$ the remainder class ring $K[X]/I$ is a domain of integrity. Furthermore, it may be made a differential domain of integrity by defining the derivative of a remainder class as the class containing the derivative of any representative form of that class. This definition is unique. Each form of $K[X]$ that consists of a single term k, where k is an element of K, determines a distinct remainder class in $K[X]/I$. For if k_1 and $k_2 \neq k_1$ were in I, then I would have to be $K[X]$, since $(k_1 - k_2)/(k_1 - k_2)$ would be in I and hence every form. We may then associate with the elements of K those elements of $K[X]/I$ which these forms determine in a one-to-one correspondence and the subset of $K[X]/I$ thus defined is a differential field isomorphic with K.

Now form the differential quotient field H of $K[X]/I$. Let the elements of H represented by the ratios $x_1/\varepsilon, \cdots, x_n/\varepsilon$, where ε is the unity element of K, be designated by $a_1, \cdots, a_n$. H can be regarded as the field obtained by adjoining the set $a_1, \cdots, a_n$ to K. Furthermore, from the manner in which it was constructed, the condition is satisfied that the forms of I and no others of $K[X]$ vanish when the a's are substituted for the x's. This field H may be called the *solution field of* K.

THEOREM V. *For any given differential field K and any prime differential ideal I of forms in the set X of n indeterminates $x_1, \cdots, x_n$ with coefficients in K, there exists an extension $K(a_1, \cdots, a_n)$, obtained from K by adjoining a set $a_1, \cdots, a_n$, such that I is the set of forms that vanish when $a_1, \cdots, a_n$ are substituted for the indeterminates.*

We consider next the degree of hypertranscendency of the solution fields H_1 and H_2 of the differential prime ideals I_1 and I_2, where I_1 contains I_2.

THEOREM VI. *The degree of hypertranscendency of H_2 cannot be less than the degree of hypertranscendency of H_1.*

We need only consider the elements adjoined to K to produce H_1 and H_2. Let these be $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ respectively. Since every form of I_2 is in I_1, every form annulled by the b's is annulled by the a's.

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JORDAN MEASURE AND RIEMANN INTEGRATION.[1]

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In this paper it is shown that the Riemann integral can be defined in terms of Jordan measure in about the same way that the Lebesgue integral is defined in terms of Lebesgue measure. This result is not surprising, but it does not seem to be in the literature. There is a paper on the subject by J. Ridder, "Over de Integraldefinities van Riemann en Lebesgue", Christiann Huygens, vol. 4, (1925–6) pp. 346–350, but the results stated in this paper are incorrect. Ridder attempts to show that a necessary and sufficient condition that a bounded function defined over a closed interval be integrable Riemann is that the function be "Jordan measurable", that is, for every pair of numbers k and l, the set of points x of I for which $k \leqq f(x) < l$, and the set for which $k < f(x) \leqq l$, are measurable Jordan.

But it is easily seen that not even all continuous functions are "measurable Jordan" in this sense. For example, let C be a closed set which is not measurable Jordan (such as a nowhere dense closed set of positive Lebesgue measure) contained in the closed interval I. Let $f(x)$ be the distance of a point x of I from the set C. Then $f(x)$ is continuous, but the set of points of I for which $-1 < f(x) \leqq 0$ is the set C, which is not measurable Jordan. The situation is more complicated than Ridder supposed, as can be seen from the theorems below, where several necessary and sufficient conditions for Riemann integrability in terms of Jordan measure are derived. Some consequences of these results, believed to be new, are also given; for example, it is shown that a bounded function integrable Riemann can be uniformly approximated by functions which take on only a finite number of values, at sets measurable Jordan.

We shall confine our attention to bounded functions defined over a closed interval (not necessarily 1-dimensional). This is sufficiently general, since a function defined over only a subset of the interval may be considered to have the value zero elsewhere. Improper integrals will not be considered. The following notation will be used. If A is any bounded set of points, then by $\overline{A}$, the *closure* of A, we shall mean the set A plus its limit points, and by $\underline{A}$, the *interior* of A, we shall mean the set of all interior points of A. Also $b(A) = \overline{A} - \underline{A}$ is the *boundary* of A, and $f(A) = A - \underline{A}$ is the *frontier* of A. Then the set $\overline{A} - A$ is the frontier of the complement of A.

[1] Received January 23, 1932 and, in revised form, March 29, 1932.

If A is measurable Lebesgue we shall denote its measure by mA. The word *measure*, unless otherwise qualified, will always mean Lebesgue measure. The *outer content* of A is defined to be $m\bar{A}$, and the *inner content* of A is defined to be $m\underline{A}$. Since the sets $\bar{A}$ and $\underline{A}$ are closed and open respectively, they are measurable. If the outer and inner contents of A are equal, their common value is cA, the *content* or *Jordan measure* of A, and A is said to be *measurable Jordan*. It follows immediately that a set is measurable Jordan if and only if its boundary is of measure zero. If $mf(A) = 0$, we say, following Carathéodory, that A is *Jordan measurable from within*, or JMI, and if $\bar{A}-A$, the frontier of the complement of A, is of measure zero, we say A is *Jordan measurable from without*, or JMO.

With the aid of some simple point-set theory the following consequences of these definitions can be established. The logical sum of a countable number of sets JMI is JMI. The logical product of a countable number of sets JMO is JMO. The complement of a set JMI is JMO, and conversely. A set is measurable Jordan if and only if it is both JMI and JMO. A closed interval is measurable Jordan. If A and B are measurable Jordan, then their logical sum $A+B$, their logical product AB, and their logical difference $A-B$, are all measurable Jordan.

It is assumed that the reader is familiar with the ordinary theory of Riemann and Lebesgue integration, and in particular with the theorem that a function defined and bounded on an interval is integrable Riemann if and only if its points of discontinuity form a set of measure zero. Given a function $f(x)$ defined over a closed interval I, we shall denote by G_k, E_k, L_k, GE_k, and LE_k the sets of points of I for which $f(x) > k$, $= k$, $< k$, $\geqq k$, or $\leqq k$ respectively.

THEOREM 1. *If $f(x)$ is defined, bounded, and integrable Riemann on the closed interval I, then for all except a countable number of values of k the set of points G_k for which $f(x) > k$ is measurable Jordan.*

Proof. Since $f(x)$ is integrable Lebesgue, the set of points E_k for which $f(x) = k$ is measurable Lebesgue, and for all except a countable number of values of k is of measure zero. For, no two distinct sets E_k have a point in common, and all are contained in I, hence only a countable number of them can have positive measure.

Suppose now that k is so chosen that $mE_k = 0$, and consider the set G_k of points where $f(x) > k$. What we wish to prove is that G_k is measurable Jordan, that is, that its boundary is of measure zero. Suppose p is any point of the boundary of G_k. Three cases arise. I. If $f(p) > k$, then p is both a member of and a boundary point of G_k, and is therefore a limit point of points where $f(x) \leqq k$. Hence p must be a point of discontinuity of $f(x)$. II. If $f(p) = k$, then p is a point of E_k. III. If

$f(p)<k$, then since p is a boundary point but not a member of G_k, it must be a limit point of G_k, that is of points where $f(x)>k$. Then it is a point of discontinuity of $f(x)$. Hence in any case a boundary point of G_k is a member of either E_k, which is of measure zero, or of the set D of discontinuities of $f(x)$, which is also of measure zero since $f(x)$ is integrable Riemann. Since it is a subset of a set of measure zero, the boundary of G_k is of measure zero, hence G_k is measurable Jordan.

THEOREM 2. *If $f(x)$ is defined, bounded, and integrable Riemann on the closed interval I, then for all except a countable number of values of k, and in fact whenever $mE_k=0$, the sets G_k, E_k, L_k, GE_k, and LE_k are all measurable Jordan.*

Proof. This follows from Theorem 1, from the fact that $-f(x)$ is integrable Riemann, and from the fact that the difference of two sets measurable Jordan is measurable Jordan. In fact, L_k is the set of points where $-f(x)>-k$, $GE_k=I-L_k$, $LE_k=I-G_k$, $E_k=GE_k-G_k$.

THEOREM 3. *If $f(x)$ is defined, bounded, and integrable Riemann on the closed interval I, then for every k the sets G_k and L_k are each the sum of a countable number of sets measurable Jordan, and hence are JMI, and the sets GE_k, LE_k, and E_k are each the logical product of a countable number of sets measurable Jordan and hence are JMO.*

Proof. For any number k we can find a sequence of numbers $k_1, k_2, \cdots, k_n, \cdots$ approaching k from above and such that for every n we have $mE_{k_n}=0$, and a sequence of numbers $l_1, l_2, \cdots, l_n, \cdots$ approaching k from below and such that for every n we have $mE_{l_n}=0$, since all except a countable number of numbers have this latter property. Then $G_k=\sum_{n=1}^{\infty} G_{k_n}$, since in the first place every x which is a member of G_k is a member of every G_{k_n} for which $f(x)>k_n$, and in the second place every point of G_{k_n} is also a point of G_k. Similar reasoning shows us that

$$L_k=\sum_{n=1}^{\infty} L_{l_n}, \qquad GE_k=\prod_{n=1}^{\infty} G_{l_n}=I-L_k, \qquad LE_k=\prod_{n=1}^{\infty} L_{k_n}=I-G_k,$$

$$E_k=GE_k\cdot LE_k=\prod_{n=1}^{\infty} G_{l_n}\cdot\prod_{n=1}^{\infty} L_{k_n}.$$

But by Theorem 2 the sets G_{k_n}, L_{k_n}, G_{l_n}, L_{l_n} are all measurable Jordan. This completes the proof of Theorem 3.

It should be noted that a set can be JMI without being the sum of a countable number of sets measurable Jordan. An example is any set of the second category of measure zero. For, every set of measure zero is JMI; but since a set of content zero is nowhere dense, the logical sum of a countable number of sets of content zero is necessarily of the

first category. Similarly, the complement with respect to I of a set which is of measure zero and of the second category is an example of a set which is JMO without being the logical product of a countable number of sets measurable Jordan.

It follows from Theorem 2 that if $mE_k = mE_l = 0$, the set of points of I for which $k \leqq f(x) < l$ is measurable Jordan. We are now in a position to define the Riemann integral in terms of Jordan measure in the same way in which the Lebesgue integral is defined in terms of Lebesgue measure.

Let $f(x)$ be a function defined, bounded, and integrable Riemann on the closed interval I, and let m be a number less than the greatest lower bound of $f(x)$ and M be a number greater than the least upper bound of $f(x)$. Consider a partition of the interval from m to M, that is, a finite number of values $y_0, \cdots, y_n$ such that $m = y_0 < y_1 < y_2 < \cdots < y_k < \cdots < y_n = M$, and subject to the condition K that $mE_{y_k} = 0$ for every k. Let δ, the norm of the partition, be the maximum of $y_k - y_{k-1}$ for all values of k. Let A_k be the set of points of I for which $y_{k-1} \leqq f(x) < y_k$. Then it follows from Theorem 2 that A_k is measurable Jordan. Let cA_k denote the Jordan measure of A_k, which of course is equal to its Lebesgue measure. Then we can define the Riemann integral of $f(x)$ over the interval I by the relation

$$\int_I f(x)\,dx = \lim_{\delta \to 0} \sum_{k=1}^{n} y_k\, cA_k.$$

This limit exists in the sense that there is a number L such that for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for every partition $y_0, \cdots, y_n$ subject to condition K, and whose norm δ is less than η, we have $\left| L - \sum_{k=1}^{n} y_k\, cA_k \right| < \varepsilon$. The existence of this limit L is proved in the same way as for the Lebesgue integral, and of course it is equal to the Lebesgue integral. That there exist partitions of arbitrarily small norm subject to condition K follows from the fact that $mE_{y_k} = 0$ for all except a countable number of values of y_k.

Having obtained some necessary conditions for the Riemann integrability of bounded functions in terms of Jordan measure, we now look for sufficient conditions of the same type. As we shall prove, the properties stated in the conclusions of Theorems 2 and 3 are each sufficient for Riemann integrability. We shall first show that a condition apparently weaker than either of these is also sufficient.

THEOREM 4. *If the function $f(x)$ is defined and bounded on the closed interval I, and its bounds are m and M, and if for a set of values of k everywhere dense on the interval from m to M the sets G_k and L_k of points*

of I for which $f(x)>k$ and $f(x)<k$ respectively are JMI, then $f(x)$ is integrable Riemann.

Proof. It is sufficient to prove that the set D of discontinuities of $f(x)$ is of measure zero. Note that from the hypothesis of our theorem it follows that for *every* k, G_k and L_k are JMI. For, as before, we can find a sequence of values k_n approaching k from above such that every G_{k_n} is JMI, and a sequence of values l_n approaching k from below such that every L_{l_n} is JMI. Then $G_k = \sum_{n=1}^{\infty} G_{k_n}$ and $L_k = \sum_{n=1}^{\infty} L_{l_n}$. Hence both G_k and L_k are JMI, since each is the sum of a countable number of sets JMI.

Now consider the set O_n of those discontinuities of $f(x)$ at which the oscillation of $f(x)$ is $>1/n$, suppose p is any point of O_n, and let $f(p)=k$. If r and s are rational numbers such that $r<k<s$ and $(s-r)<1/5n$, then p is a point of G_r and also of L_s. But, since the oscillation of $f(x)$ is $>1/n$ at p, p is a limit point either of points where $f(x)<r$ or of points where $f(x)>s$. Then p is a frontier point of either G_r or L_s, since it is either a point of G_r and a limit point of points not in G_r, or a point of L_s and a limit point of points not in L_s. Since the sets G_r and L_s are JMI, their frontiers $f(G_r)$ and $f(L_s)$ are of measure zero. All points of O_n, then, are included in the set $\sum_{i=1}^{\infty} f(G_{r_i}) + \sum_{i=1}^{\infty} f(L_{r_i})$ where the sequence $r_1, r_2, \cdots, r_i, \cdots$ is some enumeration of the rational numbers. Since this set is of measure zero, so also is O_n. But the set of discontinuities of $f(x)$ is $D = \sum_{n=1}^{\infty} O_n$. Hence D is of measure zero and $f(x)$ is integrable Riemann. We have used throughout the fact that a countable number of sets of measure zero has as its logical sum a set of measure zero.

By combining Theorems 2, 3, and 4, a large number of different necessary and sufficient conditions for the Riemann integrability of bounded functions can be obtained. Some of these are found in

THEOREM 5. *Each of the following ten properties is by itself a necessary and sufficient condition for the Riemann integrability of a function $f(x)$, defined and bounded on the closed interval I, and whose bounds are m and M:*

a) *For all except a countable number of values of k the sets G_k and L_k are measurable Jordan.*

b) *For all except a countable number of values of k the sets GE_k and LE_k are measurable Jordan.*

c) *For a set of values of k everywhere dense on the interval from m to M the sets G_k and L_k are measurable Jordan.*

d) *For a set of values of k everywhere dense on the interval from m to M the sets GE_k and LE_k are measurable Jordan.*

e) *For all values of k the sets G_k and L_k are each the logical sum of a countable number of sets measurable Jordan.*

f) *For all values of k the sets GE_k and LE_k are each the logical product of a countable number of sets measurable Jordan.*

g) *For all values of k the sets G_k and L_k are JMI.*

h) *For all values of k the sets GE_k and LE_k are JMO.*

i) *For a set of values of k everywhere dense on the interval from m to M the sets G_k and L_k are JMI.*

j) *For a set of values of k everywhere dense on the interval from m to M the sets GE_k and LE_k are JMO.*

It should be noted that if for a set of values of k everywhere dense on the interval from m to M the sets GE_k and LE_k are JMO, then for the same values of k the sets $G_k = I - LE_k$ and $L_k = I - GE_k$ are JMI, since the complement with respect to I of a set JMO is JMI, and hence $f(x)$ is integrable Riemann by Theorem 4. This is the only fact needed in Theorem 5 not already covered by Theorems 2, 3, and 4.

THEOREM 6. *If $f(x)$ and $g(x)$ are defined, bounded, and integrable Riemann on the closed interval I, then the set of points of I where $f(x) > g(x)$, and the set of points of I where $f(x) \neq g(x)$ are each the logical sum of a countable number of sets measurable Jordan, and the set of points where $f(x) = g(x)$, and the set of points where $f(x) \geqq g(x)$ are each the logical product of a countable number of sets measurable Jordan.*

Proof. This follows from the fact that if $f(x)$ and $g(x)$ are integrable Riemann, so is their difference $d(x) = f(x) - g(x)$. The four sets of points in question are the points of I at which $d(x) > 0$, $\neq 0$, $= 0$, $\geqq 0$, respectively. The conclusion then follows from Theorem 3.

THEOREM 7. *If $f(x)$ and $g(x)$ are defined, bounded, and integrable Riemann on the closed interval I, then the necessary and sufficient condition that $\int_I |f(x) - g(x)|\, dx = 0$ is that the set of points U at which $f(x) \neq g(x)$ be the logical sum of a countable number of sets of content zero.*

Proof. Since the Riemann and Lebesgue integrals are equal when both exist, it follows from the corresponding theorem for Lebesgue integrals that the necessary and sufficient condition for the vanishing of the integral in question is that $m\,U = 0$. But since $f(x)$ and $g(x)$ are integrable Riemann, it follows from Theorem 6 that U is the sum of a countable number of sets measurable Jordan. Since it is also of measure zero, it must be the logical sum of a countable number of sets of content zero.

It should be noted that Theorem 7, and in fact a somewhat sharper result, can also be deduced from a theorem to be found in the first edition of Hobson's "Theory of Functions of a Real Variable", Cambridge 1907, page 347. This theorem states that a necessary and sufficient condition that $\int_I f(x)\,dx$ exist in the sense of Riemann and be zero for every subinterval of I is that for every $k>0$ the set of points of I for which $|f(x)| \geqq k$ is of content zero. From this it follows that under the hypothesis of Theorem 7 a necessary and sufficient condition that $\int_I |f(x)-g(x)|\,dx = 0$ is that for every $k>0$, the set of points of I for which $|f(x)-g(x)| \geqq k$ is of content zero. This implies Theorem 7.

THEOREM 8. *If $f(x)$ is defined, bounded, and integrable Riemann on the closed interval I, and U is any subset of I which is the sum of a countable number of sets of content zero, then there exists a function $g(x)$, also defined, bounded and integrable Riemann over I, such that U is the set of points at which $f(x) \neq g(x)$.*

Proof. We define $g(x)$ to be $f(x)+u(x)$, where $u(x)$ is integrable Riemann and differs from zero only at points of U, as follows. We are given that $U = \sum_{n=1}^{\infty} C_n$, where each C_n is of content zero. If we let $D_1 = C_1$, $D_2 = C_2 - C_1$, $\cdots$, $D_n = C_n - (C_1 + C_2 + \cdots + C_{n-1})$, then $U = \sum_{n=1}^{\infty} D_n$, where each D_n is of content zero, and no two distinct sets D_n have a point in common. This follows from the fact that the logical sum and difference of two sets of content zero are sets of content zero. Now let $u(x)$ be the function which for all values of n is equal to $1/n$ for each x which is a member of D_n, and is equal to zero for each x which is a member of I but not of U. Then $u(x)$ is integrable Riemann by Theorem 5a), since for all values of k except zero the sets G_k and L_k for the function $u(x)$ are measurable Jordan. Hence the function $g(x)$ as defined above is bounded and integrable Riemann and differs from $f(x)$ only at points of U, which was to be proved.

Theorems 7 and 8 are important in connection with those function spaces for which the "distance" of two functions is defined in terms of the Riemann integral of some power of their absolute difference, and in other connections. Now every set which is the logical sum of a countable number of sets of content zero is both of the first category and of measure zero. Is the converse true? It is not, as the following rather complicated example will show. It seems to be even more difficult to construct a similar example for linear sets.

Let a Jordan arc AB in the plane be of superficial (two-dimensional) measure unity, and let it be in one to one continuous correspondence with

a straight line segment OX of unit length in such a way that, if S is any subset of OX and S' is its image in AB, and if one of the sets S, S' is measurable, then the other is also, and the linear measure of S is equal to the superficial measure of S'. It is well known that such a correspondence can exist. Now let S be a subset of OX of linear measure zero, whose complement with respect to OX is of the first category, and let S' be the image of S in AB. Of course S', being a subset of an arc, is nowhere dense in the plane and hence of the first category, and also it is of measure zero. Nevertheless, S' is not the sum of a countable number of sets of (two-dimensional) content zero. For, if it were, it would be a subset of an F_σ of measure zero, and in fact, of an F_σ of measure zero contained in AB, since AB is a closed set. Call the image of this F_σ on OX the set K. Now we have that K is also an F_σ of measure zero, and is thus of the first category. But K contains S, which is of the second category, and this involves a contradiction.

THEOREM 9. *If the function $f(x)$, defined over the closed interval I, takes on only a finite number of values $y_1, y_2, \cdots, y_n$, at the sets $E_1, E_2, \cdots, E_n$ respectively, then the necessary and sufficient condition that $f(x)$ be integrable Riemann is that each of the sets E_i be measurable Jordan, and the integral in this case is equal to* $\sum_{i=1}^{n} y_i \, cE_i$.

Proof. In the first place if each set E_i is measurable Jordan, then by Theorem 5 $f(x)$ is integrable Riemann, since it is bounded and the sets G_k and L_k are measurable Jordan for all values of k. Conversely, if $f(x)$ is integrable Riemann, then the set D of its points of discontinuity is of measure zero. Now if p is a boundary point of one of the sets E_i, then it is at the same time a point of some set E_r and a limit point of points not in E_r, which implies that it is a point of discontinuity of $f(x)$. Hence the boundaries of the sets E_i are all contained in the set D, which is of measure zero. But then these boundaries are of measure zero, and hence the sets E_i are measurable Jordan, which was to be proved. That the integral in this case is equal to $\sum_{i=1}^{n} y_i \, cE_i$ follows from our discussion after Theorem 3, where a definition of the Riemann integral was given similar to that of the Lebesgue integral.

THEOREM 10. *The necessary and sufficient condition that a function $f(x)$, defined and bounded on the closed interval I, be integrable Riemann is that $f(x)$ be the limit of a uniformly convergent sequence of functions each of which assumes only a finite number of functional values over sets measurable Jordan.*

Proof. If $f(x)$ is the limit of such a sequence, then by Theorem 9 each function of the sequence is integrable Riemann. Then by a well

known theorem, $f(x)$, being the limit of a uniformly convergent sequence of functions integrable Riemann, is itself integrable Riemann.

Suppose now on the other hand $f(x)$ is given to be integrable Riemann. We wish to prove that for every integer n there exists a function $g_n(x)$, taking on only a finite number of values over sets measurable Jordan, and such that $|f(x)-g_n(x)|<1/n$. Suppose m is a number less than the lower bound of $f(x)$, and M is a number greater than the upper bound of $f(x)$. In view of Theorem 2 we can select a finite number of values $m = y_0<y_1<\cdots<y_i<\cdots<y_r = M$ on the interval from m to M such that $y_i-y_{i-1}<1/2n$ and such that the set of points A_i for which $y_{i-1}\leqq f(x)<y_i$ is measurable Jordan for all values of i from 1 to r. Define $g_n(x)$ to be the function which for all values of i is equal to y_i for every x which is a member of A_i. Then $|f(x)-g_n(x)|<1/n$, and $g_n(x)$ takes on only a finite number of values y_i at sets A_i measurable Jordan, which was to be proved.

Theorem 10 gives us another method of defining the Riemann integral in terms of Jordan measure, which however is not essentially different from the method previously described. It should be remarked that the theorems of this paper are not restricted to functions defined over a linear interval; the proofs hold equally well for functions defined over a closed interval I of any number of dimensions.

SERIES OF ORTHOGONAL POLYNOMIALS.[1]

By Dunham Jackson.

1. **Introduction.** The theory of systems of polynomials orthogonal over an interval with respect to a given weight function of more or less arbitrary character, and of the convergence of expansions in series of such polynomials, has undergone rapid development during recent years, and the essential facts of the theory are now well known.[2] In view of the increasing attention which has been given to the subject, however, and the diversity of possible applications, there may be room for a brief introductory account bringing out some of the more conspicuous facts with a degree of simplicity and comprehensiveness comparable with that found in existing elementary treatments of Fourier series. The purpose of this article is to give at least a part of such an exposition.

2. **Fundamental formulas.** For a considerable part of the formal theory, the range of the independent variable may be either finite or infinite in extent. In the present paper, however, the interval will be thought of as finite, and may without loss of generality be taken as that from -1 to $+1$. Let $\rho(x)$ be a function which is integrable over this interval (in the elementary sense or in the sense of Lebesgue),[3] nowhere negative, and actually positive at least for such a part of the interval that

$$\int_{-1}^{1} \rho(x)\,dx > 0.$$

Let $p_0(x)$, $p_1(x)$, $p_2(x)$, $\cdots$ be the corresponding sequence of normalized orthogonal polynomials, of degrees indicated by the subscripts respectively, satisfying the conditions that

[1] Received November 16, 1932. Presented to the Mathematical Association of America at Los Angeles, August 30, 1932.

[2] Among outstanding contributions to the extensive literature, beyond the pioneer work of Tchebychef, the following may be mentioned: G. Darboux, *Mémoire sur l'approximation des fonctions de très-grands nombres, et sur une classe étendue de développements en série*, Journal de mathématiques pures et appliquées, (3), vol. 4 (1878), pp. 5–56, 377–416; a series of publications by W. Stekloff, one of which will be referred to specifically later; G. Szegö, *Über die Entwicklung einer willkürlichen Funktion nach den Polynomen eines Orthogonalsystems*, Mathematische Zeitschrift, vol. 12 (1922), pp. 61–94; G. Szegö, *Über den asymptotischen Ausdruck von Polynomen, die durch eine Orthogonalitätseigenschaft definiert sind*, Mathematische Annalen, vol. 86 (1922), pp. 114–139; S. Bernstein, *Sur les polynomes orthogonaux relatifs à un segment fini*, Journal de mathématiques pures et appliquées, (9), vol. 9 (1930), pp. 127–177, and vol. 10 (1931), pp. 219–286.

[3] For a formulation in terms of Stieltjes integrals see J. Shohat, *Stieltjes integrals in mathematical statistics*, Annals of Mathematical Statistics, vol. 1 (1930), pp. 73–94.

$$\int_{-1}^{1} \varrho(x)\, p_m(x)\, p_n(x)\, dx = 0 \qquad (m \neq n),$$
$$\int_{-1}^{1} \varrho(x)\, p_n^2(x)\, dx = 1. \tag{1}$$

It is perhaps not necessary to reproduce here a proof that for any $\varrho(x)$ of the character specified such a sequence of polynomials exists.[4] For $\varrho(x) \equiv 1$ the polynomials are those of Legendre, except for constant factors, and the convergence proof to be given below will be incidentally a proof of convergence of Legendre series. For $\varrho(x) = (1-x^2)^{-1/2}$ the p's are essentially the "trigonometric polynomials" by which the cosine of n times an angle is expressed in terms of the angle itself: $p_n(x) = C_n \cos(n \text{ arc cos } x)$. Both are particular cases of Jacobi polynomials, defined by the specification $\varrho(x) = (x+1)^\alpha (1-x)^\beta$, $\alpha > -1$, $\beta > -1$. Other special cases will be referred to later.[5]

An "arbitrary" function $f(x)$ can be formally expanded in a series of the form

$$c_0 p_0(x) + c_1 p_1(x) + c_2 p_2(x) + \cdots \tag{2}$$

with coefficients determined as in the case of Fourier series, on the basis of the property of orthogonality: if the series (2) is supposed uniformly convergent to the value $f(x)$, multiplication by $\varrho(x)\, p_k(x)$ and integration from -1 to $+1$ with the use of (1) gives

$$c_k = \int_{-1}^{1} \varrho(x) f(x)\, p_k(x)\, dx. \tag{3}$$

If $f(x)$ is any function such that $\varrho(x)\, f(x)$ is integrable, coefficients c_k can be defined by (3) and used to form a series (2), and it may then be inquired whether this series does in fact converge and have $f(x)$ for its sum.

For any such function $f(x)$, the notation $s_n(x)$ will be used to represent the sum of the first $n+1$ terms of the series:

$$s_n(x) = \sum_{k=0}^{n} c_k\, p_k(x),$$

the coefficients c_k being given by (3). It follows at once from the definition that

$$s_n(x) = \int_{-1}^{1} \varrho(t) f(t)\, K_n(x, t)\, dt, \tag{4}$$

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[4] The sequence may be constructed by what is commonly referred to as "Schmidt's process of orthogonalization"; see, for example, the writer's *Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. XI (hereafter referred to as Colloquium), New York, 1930, pp. 89–90, 95.

[5] For the most important cases of orthogonal polynomials see Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. II, Berlin 1925, pp. 75–76, 91–94, 266–267, 287–295.

where

$$K_n(x,t) = \sum_{k=0}^{n} p_k(x)\, p_k(t). \tag{5}$$

Since each $p_k(x)$ is actually of the degree indicated by its subscript, the identities expressing the p's as combinations of powers of x can be solved for the powers of x successively, and x^n, or any polynomial of the nth degree in x, can be expressed as a linear combination of $p_0(x), \cdots, p_n(x)$. If $q(x)$ is any polynomial whatever of degree lower than the nth,

$$\int_{-1}^{1} \varrho(x)\, p_n(x)\, q(x)\, dx = 0 \tag{6}$$

in consequence of (1), since $q(x)$ is a combination of p's with indices not greater than $n-1$.

3. **The Christoffel-Darboux identity.** On the basis of these fundamental properties of the p's, a formula for $K_n(x,t)$, to which the name of Christoffel is attached in the case of Legendre polynomials, can be obtained for general $\varrho(x)$ by a method due to Darboux.[6]

Let a_k denote the coefficient of x^k in $p_k(x)$. The sign of a_k, which is not determined by (1), will be taken as positive. The product $xp_n(x)$, being a polynomial of degree $n+1$, can be linearly expressed in terms of $p_0(x), \cdots, p_{n+1}(x)$, and by comparison of the terms of highest degree can be written in the form

$$xp_n(x) = \frac{a_n}{a_{n+1}} p_{n+1}(x) + C_{nn}\, p_n(x) + C_{n,n-1}\, p_{n-1}(x) + \cdots + C_{n0}\, p_0(x), \tag{7}$$

where the C's are some constants. Let this identity be multiplied by $\varrho(x)\, p_k(x)$ and integrated from -1 to $+1$, k being one of the numbers $0, 1, \cdots, n-1$. By (1) the result reduces to

$$\int_{-1}^{1} x\varrho(x)\, p_n(x)\, p_k(x)\, dx = C_{nk}. \tag{8}$$

For $k \leqq n-2$, $xp_k(x)$ is a polynomial of degree not higher than $n-1$, which may be identified with $q(x)$ in (6), and it is seen that $C_{nk} = 0$ for these values of k. For $k = n-1$, let n be replaced by $n-1$ in (7):

$$xp_{n-1}(x) = \frac{a_{n-1}}{a_n} p_n(x) + q_{n-1}(x),$$

where $q_{n-1}(x)$ is a polynomial of degree $n-1$ at most. Substitution of this in (8) gives

$$C_{n,n-1} = \frac{a_{n-1}}{a_n} \int_{-1}^{1} \varrho(x)\, p_n^2(x)\, dx = \frac{a_{n-1}}{a_n}.$$

[6] Loc. cit., p. 411. The notation is somewhat different, since Darboux takes the leading coefficient as unity.

Thus the p's are found to be subject to the recursion formula

$$xp_n(x) = \frac{a_n}{a_{n+1}}p_{n+1}(x) + C_{nn}p_n(x) + \frac{a_{n-1}}{a_n}p_{n-1}(x), \tag{9}$$

the coefficient C_{nn} being left unspecified, as far as the present calculation is concerned.

With n replaced by k, (9) reads

$$xp_k(x) = \frac{a_k}{a_{k+1}}p_{k+1}(x) + C_{kk}p_k(x) + \frac{a_{k-1}}{a_k}p_{k-1}(x).$$

Let this be multiplied by $p_k(t)$, and the corresponding identity in t,

$$tp_k(t) = \frac{a_k}{a_{k+1}}p_{k+1}(t) + C_{kk}p_k(t) + \frac{a_{k-1}}{a_k}p_{k-1}(t),$$

by $p_k(x)$, and one subtracted from the other; the result is

$$\begin{aligned}(t-x)p_k(t)p_k(x) = {} & \frac{a_k}{a_{k+1}}[p_{k+1}(t)p_k(x) - p_k(t)p_{k+1}(x)] \\ & - \frac{a_{k-1}}{a_k}[p_k(t)p_{k-1}(x) - p_{k-1}(t)p_k(x)].\end{aligned} \tag{10}$$

This holds for any value of $k \geqq 1$. For $k = 0$, $p_0(x) \equiv a_0$, while $p_1(x)$ is of the form $a_1x + b$, so that

$$xp_0(x) = \left(\frac{a_0}{a_1}\right)p_1(x) - \left(\frac{a_0 b}{a_1}\right),$$

and it follows immediately that

$$(t-x)p_0(t)p_0(x) = \frac{a_0}{a_1}[p_1(t)p_0(x) - p_0(t)p_1(x)]. \tag{11}$$

If (10) is written for $k = 1, 2, \cdots, n$ successively, addition of (11) and the n identities (10) yields[7]

$$(t-x)[p_0(t)p_0(x) + \cdots + p_n(t)p_n(x)] = \frac{a_n}{a_{n+1}}[p_{n+1}(t)p_n(x) - p_n(t)p_{n+1}(x)],$$

$$K_n(x, t) = \frac{a_n}{a_{n+1}} \cdot \frac{p_{n+1}(t)p_n(x) - p_n(t)p_{n+1}(x)}{t-x}.$$

[7] A more compact derivation of the formula for K_n, without intermediate use of the recursion formula, is given by Shohat, *On Stieltjes continued fractions*, American Journal of Mathematics, vol. 54 (1932), pp. 79–84.

4. **First convergence theorem.** Since 1 is a polynomial of degree zero, it follows from (6) that

$$\int_{-1}^{1} \varrho(t) p_k(t)\, dt = \int_{-1}^{1} \varrho(t) p_k(t) \cdot 1\, dt = 0$$

for $k \geqq 1$, while

$$p_0(x) \equiv a_0 \equiv p_0(t), \qquad \int_{-1}^{1} \varrho(t) p_0(x) p_0(t)\, dt = \int_{-1}^{1} \varrho(t) p_0^2(t)\, dt = 1.$$

Consequently, by direct substitution of the expression (5) for $K_n(x, t)$,

$$1 \equiv \int_{-1}^{1} \varrho(t) K_n(x, t)\, dt,$$

a special case of (4) for $f(t) \equiv 1$. As $f(x)$ is a constant with respect to the variable of integration, the result of multiplying this identity by $f(x)$ may be written in the form

$$f(x) = \int_{-1}^{1} \varrho(t) f(x) K_n(x, t)\, dt,$$

and subtraction of this from (4) gives

$$\begin{aligned} s_n(x) - f(x) &= \int_{-1}^{1} \varrho(t)[f(t) - f(x)] K_n(x, t)\, dt \\ &= \frac{a_n}{a_{n+1}} \int_{-1}^{1} \varrho(t)[f(t) - f(x)] \frac{p_{n+1}(t) p_n(x) - p_n(t) p_{n+1}(x)}{t - x}\, dt. \end{aligned} \tag{12}$$

The problem of convergence is to show that (12) approaches zero as n becomes infinite.

As a first step, it may be pointed out that[8] $a_n/a_{n+1} < 1$. Let a polynomial $\pi_{n-1}(x)$ be determined among all polynomials of degree $n-1$ by the requirement that the integral

$$\int_{-1}^{1} \varrho(x)[x^n - \pi_{n-1}(x)]^2\, dx$$

shall be a minimum, and let γ_n be the corresponding minimum value of the integral. If $\pi_{n-1}(x)$ is expressed in terms of $p_0(x), \cdots, p_{n-1}(x)$, it is readily seen that a necessary condition for the minimum is that

$$\int_{-1}^{1} \varrho(x)[x^n - \pi_{n-1}(x)] p_k(x)\, dx = 0$$

for $k = 0, 1, \cdots, n-1$, and hence that

[8] Cf. J. Chokhatte [J. Shohat], *Sur le développement de l'intégrale ... et sur les polynomes de Tchebycheff*, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 25–46, pp. 31–33. A treatment of convergence can be based directly on the general least-square property, of which a special case is utilized here; see e. g. Colloquium, pp. 92–101.

Note $\frac{a_m}{a_{m+1}} = \int_{-1}^{+1} x\, p_m(x)\, p_{m+1}(x)\, \rho(x)\, dx \leq \left(\int_{-1}^{+1} x^2 p_m^2(x)\, \rho(x)\, dx\right)^{1/2} \cdot \left(\int_{-1}^{+1} p_{m+1}^2(x)\, \rho(x)\, dx\right)^{1/2} < 1.$

$$\int_{-1}^{1} \varrho(x)\,[x^n - \pi_{n-1}(x)]\,q(x)\,dx = 0$$

for every polynomial $q(x)$ of degree lower than the nth. Let this relation be multiplied by a_n and subtracted from (6); if the combination $p_n(x) - a_n[x^n - \pi_{n-1}(x)]$ is denoted by $r(x)$,

$$\int_{-1}^{1} \varrho(x)\,r(x)\,q(x)\,dx = 0.$$

But $r(x)$ is of degree lower than the nth, since the terms in x^n cancel; it is possible in particular to take $q(x) \equiv r(x)$, so that

$$\int_{-1}^{1} \varrho(x)\,[r(x)]^2\,dx = 0,$$

from which it follows that $r(x) \equiv 0$, $p_n(x) \equiv a_n[x^n - \pi_{n-1}(x)]$. Hence

$$\gamma_n = \frac{1}{a_n^2}\int_{-1}^{1} \varrho(x)\,p_n^2(x)\,dx = \frac{1}{a_n^2}, \qquad a_n = \gamma_n^{-1/2}.$$

Similarly, $a_{n+1} = \gamma_{n+1}^{-1/2}$, where γ_{n+1} is the minimum of

$$\int_{-1}^{1} \varrho(x)\,[x^{n+1} - \pi_n(x)]^2\,dx$$

as $\pi_n(x)$ ranges over all polynomials of the nth degree. But a particular polynomial of the nth degree is $x\pi_{n-1}(x)$, and hence

$$\gamma_{n+1} \leqq \int_{-1}^{1} \varrho(x)\,[x^{n+1} - x\pi_{n-1}(x)]^2\,dx = \int_{-1}^{1} x^2\varrho(x)\,[x^n - \pi_{n-1}(x)]^2\,dx.$$

As $x^2 < 1$ throughout the interior of the interval, the last integral is smaller than that defining γ_n, and $\gamma_{n+1} < \gamma_n$, $a_{n+1} > a_n$.

Let $f(x)$ be a function such that both $\varrho(x)f(x)$ and $\varrho(x)\,[f(x)]^2$ are integrable over $(-1, 1)$. By use of the relations (1) and (3) it is seen that

$$\int_{-1}^{1} \varrho(x)\,[f(x) - s_n(x)]^2\,dx = \int_{-1}^{1} \varrho(x)[\,f(x)]^2\,dx - \sum_{k=0}^{n} c_k^2.$$

Since the left-hand member is positive or zero,

$$\sum_{k=0}^{n} c_k^2 \leqq \int_{-1}^{1} \varrho(x)\,[f(x)]^2\,dx.$$

As this holds for all values of n, while the integral on the right is independent of n, the series $\sum_0^\infty c_k^2$ is convergent, and

$$\lim_{k\to\infty} c_k = 0.$$

For the application to be made presently, this result is needed, not for the function $f(x)$ itself, but for another function related to it; for convenience of reference it may be restated in slightly different notation as follows:

LEMMA. *If $\varphi(t)$ is a function such that $\varrho(t)\varphi(t)$ and $\varrho(t)[\varphi(t)]^2$ are integrable over $(-1, 1)$,*

$$\lim_{n\to\infty}\int_{-1}^{1}\varrho(t)\varphi(t)p_n(t)\,dt = 0.$$

Now let the expression (12) for the remainder of the series be recalled. For a fixed value of x, let $\varphi(t)$ stand for the difference quotient $[f(t)-f(x)]/(t-x)$. Then the remainder may be written in the form

$$s_n(x)-f(x) = \frac{a_n}{a_{n+1}}\left[p_n(x)\int_{-1}^{1}\varrho(t)\varphi(t)p_{n+1}(t)\,dt - p_{n+1}(x)\int_{-1}^{1}\varrho(t)\varphi(t)p_n(t)\,dt\right]. \tag{13}$$

For simplicity in a first convergence theorem, to be reconsidered later in the interest of greater generality, let it be supposed that $f(t)$ is continuous except for a finite number of finite jumps, and that it has for $t = x$ a right-hand and a left-hand derivative, not necessarily equal to each other. Then $\varphi(t)$ is continuous except for a finite number of finite jumps throughout $(-1, 1)$, and the integrals of $\varrho\varphi$ and $\varrho\varphi^2$ certainly exist. By the preceding paragraph, therefore, *the integrals in* (13) *approach zero for* $n\to\infty$. It has been shown that $a_n/a_{n+1} < 1$. It will be seen presently that *in many important cases the polynomials $p_n(x)$ remain bounded as n increases, for a fixed value of x, in the interior of the interval at least,* and in fact are uniformly bounded throughout any closed interval interior to $(-1, 1)$. In such cases the whole right-hand member of (13) approaches zero when x is an interior point (and at the ends of the interval also if the values of $p_n(\pm 1)$ are bounded). The present stage of progress may be summarized in

THEOREM I. *If the weight function $\varrho(x)$ is such that $p_n(x)$ remains bounded as n increases, for a specified value of x in $(-1, 1)$, and if $f(t)$ is continuous throughout the interval except for a finite number of finite jumps, the series* (2) *will converge to the value $f(x)$ if $f(t)$ has a right-hand and a left-hand derivative at the point $t = x$.*

In the case of convergence for $t = +1$, of course, only the left-hand derivative is involved, and similarly at the other end of the interval.

The trigonometric polynomials $\cos(n \arccos x)$, with or without the normalizing factor $(2/\pi)^{1/2}$, are uniformly bounded throughout the whole closed interval $(-1, 1)$. The normalized Legendre polynomials $[(2n+1)/2]^{1/2}X_n(x)$ are uniformly bounded throughout any closed interval interior to $(-1, 1)$, since $|X_n(x)|$ itself remains inferior to a constant multiple of $n^{-1/2}$ throughout such an interval;[9] and Theorem I thus applies

[9] See e. g. L. Fejér, *Abschätzungen für die Legendreschen und verwandte Polynome*, Mathematische Zeitschrift, vol. 24 (1926), pp. 285–298; Colloquium, pp. 27–28.

directly to Legendre series. The fact that Jacobi polynomials in general possess the property in question is a consequence of an asymptotic formula due to Darboux,[10] and the requisite information for broad classes of weight functions is obtainable from the memoirs of Szegö and Bernstein. While it would be far outside the scope of the present "elementary" treatment to reproduce the arguments of these authors, it is easy to give an immediate extension of the range of applicability of Theorem I somewhat beyond the fundamental cases of cosine series and Legendre series.

Let $q_0(x)$, $q_1(x)$, $q_2(x)$, $\cdots$ be the normalized orthogonal polynomials corresponding to a weight function $\varrho_1(x)$, and let $p_0(x)$, $p_1(x)$, $p_2(x)$, $\cdots$ be those corresponding to the weight function $\varrho(x) = \Pi(x)\,\varrho_1(x)$, where $\Pi(x)$ is a polynomial which is non-negative for $-1 \leqq x \leqq 1$. Let $\varrho_1(x)$ be such that the polynomials $q_k(x)$ are uniformly bounded over a point set E contained in the interval $(-1, 1)$, say $|q_k(x)| \leqq H$ for all points in E and for all values of k. For the present general formulation, the set E may consist of one point or several points or an infinity of points, and may or may not include one or both of the points ± 1. Let it be supposed that $\Pi(x)$ remains different from zero and has a positive lower bound, say $\Pi(x) \geqq g > 0$, for all values of x in E. The purpose is to show that the property of boundedness in E carries over from the q's to the p's.

Let the degree of $\Pi(x)$ be m. Since $\Pi(x)\,p_n(x)$ is a polynomial of degree $n+m$, it can be written in the form[11]

$$\Pi(x)\,p_n(x) = c_1\,q_1(x) + c_2\,q_2(x) + \cdots + c_{n+m}\,q_{n+m}(x);$$

the c's of course vary with n, but it is not necessary to complicate the notation for the sake of making this explicitly apparent. *In each case* $c_k = 0$ *for* $k < n$, since

$$c_k = \int_{-1}^{1} \varrho_1(x) \cdot \Pi(x)\,p_n(x) \cdot q_k(x)\,dx,$$

and $p_n(x)$ is orthogonal to every polynomial of lower degree with respect to the weight function $\Pi(x)\,\varrho_1(x)$. Consequently

$$\Pi(x)\,p_n(x) = c_n\,q_n(x) + \cdots + c_{n+m}\,q_{n+m}(x),$$

[10] Cf. Szegö, Mathematische Zeitschrift, vol. 12, loc. cit., p. 89.

[11] Cf. Szegö, *Über die Entwickelung einer analytischen Funktion nach den Polynomen eines Orthogonalsystems*, Mathematische Annalen, vol. 82 (1921), pp. 188–212, footnote on pp. 190–191; Bernstein, Journal de mathématiques, vol. 9, loc. cit., pp. 138–139; J. A. Shohat, *On the development of continuous functions in series of Tchebycheff polynomials*, Transactions of the American Mathematical Society, vol. 27 (1925), pp. 537–550, p. 542.

the number of non-vanishing coefficients on the right being at most $m+1$, *and so remaining finite as n increases.* If x belongs to the set E,

$$|\Pi(x)\, p_n(x)| \leqq H(|c_n| + \cdots + |c_{n+m}|).$$

Let G be the maximum of $\Pi(x)$ for $-1 \leqq x \leqq 1$. For x in E,

$$|p_n(x)| \leqq (H/g)\,(|c_n| + \cdots + |c_{n+m}|).$$

For any value of k from n to $n+m$, obviously,

$$|c_k|^2 = c_k^2 \leqq c_n^2 + \cdots + c_{n+m}^2, \qquad |c_k| \leqq (c_n^2 + \cdots + c_{n+m}^2)^{1/2},$$

and[12]

$$|c_n| + \cdots + |c_{n+m}| \leqq (m+1)\,(c_n^2 + \cdots + c_{n+m}^2)^{1/2}.$$

But inasmuch as the q's are orthogonal and normalized for the weight function $\varrho_1(x)$, while the p's have the same property with respect to the weight function $\Pi(x)\,\varrho_1(x)$, and $\Pi(x) \leqq G$,

$$\begin{aligned} c_n^2 + \cdots + c_{n+m}^2 &= \int_{-1}^{1} \varrho_1(x)\,(c_n q_n + \cdots + c_{n+m} q_{n+m})^2\, dx \\ &= \int_{-1}^{1} \varrho_1(x)\,[\Pi(x)\, p_n(x)]^2\, dx \\ &\leqq G \int_{-1}^{1} \Pi(x)\, \varrho_1(x)\,[p_n(x)]^2\, dx = G. \end{aligned}$$

It follows that $|c_n| + \cdots + |c_{n+m}| \leqq (m+1)\, G^{1/2}$, and

$$|p_n(x)| \leqq (m+1)\, H G^{1/2}/g$$

for x in E, the right-hand member being independent of n.

Without the restriction that $\Pi(x)$ shall not vanish in E, the same reasoning shows that

$$|p_n(x)| \leqq (m+1)\, H G^{1/2}/\Pi(x)$$

at any point of E where $\Pi(x) \neq 0$; the right-hand member, though now involving x, is still independent of n.

Thus Theorem I is applicable whenever $\varrho(x)$ satisfies the conditions of the last paragraphs. The property of boundedness in the interior of the interval for the normalized sequence can be carried over, for example,

[12] By Schwarz's inequality,

$$\begin{aligned} (|c_n| + |c_{n+1}| + \cdots + |c_{n+m}|)^2 &= (|c_n| \cdot 1 + |c_{n+1}| \cdot 1 + \cdots + |c_{n+m}| \cdot 1)^2 \\ &\leqq (c_n^2 + c_{n+1}^2 + \cdots + c_{n+m}^2)\,(1^2 + 1^2 + \cdots + 1^2), \\ |c_n| + \cdots + |c_{n+m}| &\leqq (m+1)^{1/2}\,(c_n^2 + \cdots + c_{n+m}^2)^{1/2}. \end{aligned}$$

But the cruder inequality in the text serves the present purpose equally well.

from the Legendre polynomials $X_n(x)$ to their qth derivatives[13] $X_n^{(q)}(x)$, which are orthogonal with weight function $(1-x^2)^q$, so that a convergence theorem is obtained for developments in terms of the "associated Legendre functions" $(1-x^2)^{q/2} X_n^{(q)}(x)$; or from the polynomials $X_n(x)$ to the polynomials[14]

$$S_n(x) = \sum_0^n \frac{1}{2}(2k+1) X_k(x)$$

corresponding to the weight function $1-x$; or from the "trigonometric polynomials" $\cos(n \arccos x)$, with weight function $(1-x^2)^{-1/2}$, to the polynomials[14] $\sin(n+1)\theta/\sin\theta$, $\theta = \arccos x$, with weight function $(1-x^2)^{1/2}$.

5. **Generalization of the convergence theorem.** At the expense of resorting to somewhat less elementary considerations, the generality of Theorem I can be considerably increased by lightening the unnecessarily severe restrictions imposed on the difference quotient $\varphi(t)$ in the application of the Lemma, and still further by re-examining the hypothesis to which the function $\varphi(t)$ was subjected in the Lemma itself. It was supposed in the Lemma that $\varrho(t)[\varphi(t)]^2$ is integrable from -1 to 1. *Under the assumption that the polynomials $p_n(t)$ are uniformly bounded throughout an interval* $-1+\eta \leqq t \leqq 1-\eta$, let it be demanded of $\varphi(t)$ merely that it be defined throughout $(-1, 1)$, except possibly for a set of measure zero, that $\varrho(t)\varphi(t)$ be summable in the sense of Lebesgue over the whole interval $(-1, 1)$, and that $\varrho(t)[\varphi(t)]^2$ be summable from -1 to $-1+\eta$ and from $1-\eta$ to 1. It is understood that $\varrho(t)$ itself is summable from -1 to 1. *The conclusion of the Lemma is still valid.*

For any positive N, let E_N be the point set consisting of the intervals $(-1, -1+\eta)$ and $(1-\eta, 1)$, together with those points of $(-1+\eta, 1-\eta)$ where $|\varphi(t)| \leqq N$, and let CE_N be the complementary set, made up of the points of $(-1+\eta, 1-\eta)$ where $|\varphi(t)| > N$. Let $\varphi_N(t)$ be defined as equal to $\varphi(t)$ on E_N, and equal to 0 on CE_N. As this function is bounded from $-1+\eta$ to $1-\eta$, $\varrho(t)[\varphi_N(t)]^2$ is summable from -1 to 1. Let $|p_n(t)| \leqq H$ for $-1+\eta \leqq t \leqq 1-\eta$. Let ε be any positive quantity. From the fact that $\varphi(t)$ is defined almost everywhere it follows that the measure of CE_N approaches zero as N increases without limit. Hence, by the summability of $\varrho(t)\varphi(t)$,

$$\lim_{N\to\infty} \int_{CE_N} \varrho(t)|\varphi(t)|\,dt = 0.$$

[13] See e. g. Courant-Hilbert, *Methoden der Mathematischen Physik*, vol. I (second edition), Berlin 1931, p. 282.

[14] See Pólya and Szegö, *Aufgaben und Lehrsätze*, locc. citt.

Let N be taken so large that the value of the last integral is less than $\epsilon/(2H)$, and then held fast. It is thus assured that

$$\left|\int_{CE_N} \varrho(t)\,\varphi(t)\,p_n(t)\,dt\right| \leqq H\int_{CE_N} \varrho(t)\,|\varphi(t)|\,dt < \frac{\epsilon}{2}$$

for all values of n. In the resolution

$$\begin{aligned}\int_{-1}^{1} \varrho(t)\,\varphi(t)\,p_n(t)\,dt &= \int_{E_N} \varrho(t)\,\varphi(t)\,p_n(t)\,dt + \int_{CE_N} \varrho(t)\,\varphi(t)\,p_n(t)\,dt \\ &= \int_{E_N} \varrho(t)\,\varphi_N(t)\,p_n(t)\,dt + \int_{CE_N} \varrho(t)\,\varphi(t)\,p_n(t)\,dt \\ &= \int_{-1}^{1} \varrho(t)\,\varphi_N(t)\,p_n(t)\,dt + \int_{CE_N} \varrho(t)\,\varphi(t)\,p_n(t)\,dt,\end{aligned}$$

the next to the last integral approaches zero as n becomes infinite, while N is held fast, because $\varrho(t)\,[\varphi_N(t)]^2$ is summable. For n sufficiently large, therefore, this integral also is inferior to $\epsilon/2$ in absolute value, and

$$\left|\int_{-1}^{1} \varrho(t)\,\varphi(t)\,p_n(t)\,dt\right| < \epsilon,$$

which is the substance of the conclusion to be proved.

In the corresponding adaptation of the proof of Theorem I for convergence at a specified interior point of the interval, the properties in question carry over directly from $f(t)$ to $\varphi(t)$, except in the neighborhood of the point $t = x$. On the assumption that the polynomials $p_n(x)$ are uniformly bounded throughout any closed interval interior to $(-1, 1)$, *it is sufficient for convergence of* $s_n(x)$ *toward* $f(x)$ *that* $\varrho(t)\,[f(t)]^2$ *be summable over an interval extending to the right from* -1 *and over an interval extending to the left from* $+1$, *that* $\varrho(t)\,f(t)$ *be summable throughout* $(-1, 1)$, *and that for the difference quotient* $\varphi(t) = [f(t) - f(x)]/(t - x)$ *there exist an interval about the point* x *over which* $\varrho(t)\,\varphi(t)$ *is summable.* The last condition can be further interpreted as is customary in the theory of Fourier series.[15]

Another important fact can be brought out in this connection. The requirement as to summability of $\varrho(t)\,\varphi(t)$ will certainly be met if $f(t)$ vanishes for $t = x$ and throughout an interval containing the point x in its interior, and if the other conditions are fulfilled $s_n(x)$ will converge to the value zero. Let this observation be applied to the series for the difference $g(t)$ of two functions $f_1(t)$ and $f_2(t)$, both satisfying the hypotheses of summability previously imposed on $f(t)$, and identical throughout an interval surrounding the point x. The partial sum $s_n(x)$ of the series for $g(x)$ is the difference of the partial sums for $f_1(x)$ and $f_2(x)$ separately.

[15] Cf. e. g. H. Lebesgue, *Leçons sur les séries trigonométriques*, Paris, 1906, Chapter III.

Since $s_n(x)$ approaches zero, it follows that if the series for $f_1(x)$ converges the series for $f_2(x)$ will converge to the same value; such difference as there may be between the functions outside the interval of identity about the point x has no effect on the convergence at that point. Under the hypothesis still that the polynomials $p_n(x)$ are bounded as before in the interior of $(-1, 1)$, it is seen that *beyond the requirement that* $\varrho(t)f(t)$ *be summable from* -1 *to* 1 *and that* $\varrho(t)[f(t)]^2$ *be summable near the ends of the interval, the convergence of the series at a specified interior point depends only on the behavior of* $f(t)$ *in the neighborhood of the point in question.*

6. **More general weight function; second convergence theorem.** While the proof of uniform boundedness of the polynomials $p_n(x)$ in the interior of the interval has been given only under restrictive hypotheses as to the nature of the weight function, it is possible to assign an upper bound to their order of magnitude under very general conditions.[16]

Let $\varrho(x)$ be any function which is summable over $(-1, 1)$ and has a positive lower bound throughout the interval, $\varrho(x) \geqq v > 0$. Let $p_n(x)$ be any polynomial of the nth degree such that

$$\int_{-1}^{1} \varrho(x)[p_n(x)]^2\,dx = 1;$$

the argument is not restricted for the moment to the particular polynomial which belongs to the normalized orthogonal sequence. Since $1 \leqq \varrho(x)/v$,

$$\int_{-1}^{1} [p_n(x)]^2\,dx \leqq (1/v)\int_{-1}^{1} \varrho(x)[p_n(x)]^2\,dx = 1/v.$$

Let $Q(x)$ denote the polynomial

$$Q(x) = \int_{-1}^{x} [p_n(x)]^2\,dx,$$

of degree $2n+1$. Inasmuch as the integrand is positive throughout the interval, $0 \leqq Q(x) \leqq Q(1) \leqq 1/v$ for $-1 \leqq x \leqq 1$. Hence, by the theorems of Markoff and Bernstein[17] (absolute value signs being superfluous as long

[16] For this fact and its application to the theory of convergence of the series, cf. e. g. W. Stekloff, *Sur les problèmes de représentation des fonctions à l'aide des polynomes, etc.*, Proceedings of the International Mathematical Congress held in Toronto, August 11–16, 1924, vol. I, pp. 631–640; Shohat, Transactions, vol. 27, loc. cit.

[17] The theorems in question may be stated as follows: If $P_n(x)$ is a polynomial of the nth degree such that $|P_n(x)| \leqq L$ for $-1 \leqq x \leqq 1$, then $|P_n'(x)| \leqq n^2 L$ throughout the same interval, and $|P_n'(x)| \leqq n\,L/(1-x^2)^{1/2}$ throughout the interior of the interval. See e. g. Marcel Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368; Colloquium, pp. 80–82, 92.

as the quantities written down are positive), the polynomial $Q'(x) = [p_n(x)]^2$ has the upper bound $(2n+1)^2/v$ for $-1 \leqq x \leqq 1$, and the upper bound $(2n+1)/[v(1-x^2)^{1/2}]$ for $-1 < x < 1$. It follows that[18] $|p_n(x)|$ *has an upper bound of the order of magnitude of* n *for* $-1 \leqq x \leqq 1$, *and an upper bound of the order of* $n^{1/2}$ *throughout any closed interval interior to* $(-1, 1)$.

The proof just given depends essentially on the hypothesis that the weight function stays away from zero. Let this requirement now be relaxed, and let it be supposed instead that $[\varrho(x)]^{-1}$ as well as $\varrho(x)$ is summable over $(-1, 1)$. By Schwarz's inequality,

$$\left[\int_{-1}^{1} |p_n(x)|\, dx\right]^2 = \left[\int_{-1}^{1} [\varrho(x)]^{-1/2} [\varrho(x)]^{1/2} |p_n(x)|\, dx\right]^2$$
$$\leqq \int_{-1}^{1} [\varrho(x)]^{-1}\, dx \int_{-1}^{1} \varrho(x)\, [p_n(x)]^2\, dx$$
$$= \int_{-1}^{1} [\varrho(x)]^{-1}\, dx.$$

If the square root of the value of the last integral is denoted by I,

$$\left|\int_{-1}^{x} p_n(x)\, dx\right| \leqq \int_{-1}^{x} |p_n(x)|\, dx \leqq \int_{-1}^{1} |p_n(x)|\, dx \leqq I,$$

and as the first integral in this chain of inequalities is a polynomial of degree $n+1$, with $p_n(x)$ for its derivative, $|p_n(x)|$ has an upper bound of the order of n^2 for $-1 \leqq x \leqq 1$, and an upper bound of the order of n throughout any closed interior interval.

It would be possible to pursue this line of inquiry further, but the results already obtained will be allowed to suffice for the present.

To make use of these facts in a proof of convergence, it is important not only to know that the general coefficient in the expansion of an arbitrary function approaches zero, but also to be able to specify conditions under which it approaches zero with a preassigned degree of rapidity.

Let $f(x)$ be an arbitrary function in the interval $(-1, 1)$, subject to conditions presently to be imposed, and let c_n be the coefficient of $p_n(x)$ in its expansion. There is occasion now to use the least-square property of the development, according to which the partial sum $s_n(x)$ of the series is distinguished among all polynomials of the nth degree by the fact that

$$\gamma_n = \int_{-1}^{1} \varrho(x)\, [f(x) - s_n(x)]^2\, dx$$

[18] For the method of proof cf. J. Shohat, *On the polynomial and trigonometric approximation of measurable bounded functions on a finite interval*, Mathematische Annalen, vol. 102 (1930), pp. 157–175, pp. 165–166.

is a minimum.[19] It has been noted that

$$\gamma_n = \int_{-1}^{1} \varrho(x)\,[f(x)]^2\,dx - \sum_{k=0}^{n} c_k^2.$$

Comparison of this with the corresponding formula for γ_{n-1} shows that $c_n^2 = \gamma_{n-1} - \gamma_n$. As $\gamma_n \geqq 0$, this implies that $c_n^2 \leqq \gamma_{n-1}$, $|c_n| \leqq \gamma_{n-1}^{1/2}$. Information can be obtained with regard to the magnitude of γ_n, in view of the least-square property, from general theorems on approximation by means of polynomials. If $P_n(x)$ is a polynomial of the nth degree such that $|f(x) - P_n(x)| \leqq \varepsilon_n$ for $-1 \leqq x \leqq 1$,

$$\gamma_n \leqq \int_{-1}^{1} \varrho(x)\,[f(x) - P_n(x)]^2\,dx \leqq \varepsilon_n^2 \int_{-1}^{1} \varrho(x)\,dx.$$

If approximating polynomials $P_n(x)$ are constructed for the various values of n successively, with ε_n as an upper bound for the error in each case, $|c_n| \leqq \mu\,\varepsilon_{n-1}$, where μ is independent of n. So theorems on the order of magnitude attainable for ε_n can be translated at once into theorems on the magnitude of c_n.

In particular, it is possible to make $\lim_{n\to\infty} n^{1/2}\varepsilon_n = 0$, if $f(x)$ is such that $\lim_{\delta\to 0} \omega(\delta)/\delta^{1/2} = 0$, where $\omega(\delta)$ is the maximum of $|f(x_2) - f(x_1)|$ for $|x_2 - x_1| \leqq \delta$. This condition will be more than satisfied if $f(x)$ satisfies the Lipschitz condition $|f(x_2) - f(x_1)| \leqq \lambda\,|x_2 - x_1|$. If $f(x)$ has a continuous derivative, $n\varepsilon_n$ can be made to approach zero, and if $f''(x)$ is continuous throughout the interval, $n^2\varepsilon_n$ can be made to approach zero.[20] It follows that $n^{1/2}c_n \to 0$ or $n c_n \to 0$ or $n^2 c_n \to 0$, in the various cases respectively.

For application in (13) as a means to a convergence proof, hypotheses corresponding to those just described are to be imposed on the difference quotient $\varphi(t) = [f(t) - f(x)]/(t - x)$. It is not difficult to show, by appropriate use of the mean value theorem, that $\varphi(t)$ (defined with the value $f'(x)$ for $t = x$) will have a continuous derivative[21] if $f''(t)$ is defined throughout an interval containing x in its interior (or having x

[19] See e. g. Colloquium, pp. 77–80, 95.

[20] See Colloquium, Chapter I, Theorem VI and Corollary of Theorem VIII.

[21] Since

$$f(t) - f(x) = (t-x)f'(x) + \tfrac{1}{2}(t-x)^2 f''(\xi),$$

it is seen that

$$\varphi(t) = f'(x) + \tfrac{1}{2}(t-x)f''(\xi),$$

and

$$[\varphi(t) - \varphi(x)]/(t-x) = \tfrac{1}{2}f''(\xi),$$

which approaches $\frac{1}{2}f''(x)$ as t approaches x, so that $\varphi(t)$ has a derivative equal to $\frac{1}{2}f''(x)$ for $t = x$. For $t \neq x$, $\varphi(t)$ has the continuous derivative

as an end point, if $x = \pm 1$) and continuous for $t = x$, and $f'(t)$ continuous throughout $(-1, 1)$; and $\varphi(t)$ will have a continuous second derivative if $f''(t)$ and $f'(t)$ in the statement just formulated are replaced by $f'''(t)$ and $f''(t)$ respectively. For simplicity, the case of the factor $n^{1/2}$ may be covered by supposing here also that $f(t)$ satisfies the conditions under which $\varphi(t)$ has a continuous derivative.[22] From (13) it is possible then to read off

$$\varphi'(t) = \frac{(t-x)f'(t) - [f(t) - f(x)]}{(t-x)^2}.$$

By an alternative application of the mean value theorem,

$$f(x) = f(t) + (x-t)f'(t) + \tfrac{1}{2}(x-t)^2 f''(\eta),$$

so that the numerator in the expression for $\varphi'(t)$ reduces to $\frac{1}{2}(t-x)^2 f''(\eta)$; hence

$$\varphi'(t) = \tfrac{1}{2} f''(\eta),$$

which approaches $\frac{1}{2}f''(x)$ when t approaches x, and $\varphi'(t)$ is continuous for $t = x$ as well as elsewhere. Similar reasoning shows the existence and continuity of $\varphi''(t)$ when $f''(t)$ and $f'''(t)$ satisfy the appropriate conditions.

[22] With the use of properties of the "modulus of continuity" $\omega(\delta)$ (see, for example, C. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, Paris, 1919, pp. 7–8; E. L. Mickelson, *On the approximate representation of a function of two variables*, Transactions of the American Mathematical Society, vol. 33 (1931), pp. 759–781, pp. 761–762) it can be seen that if $f'(t)$ has a modulus of continuity $\omega(\delta)$, $\varphi(t)$ will have a modulus of continuity not exceeding $2\omega(\delta)$. Since $\varphi(t) = f'(\xi)$, with a value of ξ between x and t, it is recognized that if $|t-x| \leqq \delta$,

$$|\varphi(t) - \varphi(x)| = |f'(\xi) - f'(x)| \leqq \omega(|\xi - x|) \leqq \omega(\delta).$$

If t_1 and t_2 are distinct from x and include x between them,

$$|\varphi(t_2) - \varphi(t_1)| \leqq |\varphi(t_2) - \varphi(x)| + |\varphi(t_1) - \varphi(x)| \leqq \omega(|t_2 - x|) + \omega(|t_1 - x|) \leqq 2\omega(|t_2 - t_1|).$$

If t_1 and t_2 are on the same side of x, and $|t_2 - t_1| \leqq \delta$,

$$\begin{aligned}
\varphi(t_2) - \varphi(t_1) &= (t_2 - t_1)\varphi'(\xi) \quad (\xi \text{ between } t_1 \text{ and } t_2) \\
&= (t_2 - t_1)\frac{(\xi - x)f'(\xi) - [f(\xi) - f(x)]}{(\xi - x)^2} \quad \text{(see preceding footnote)} \\
&= (t_2 - t_1)\frac{(\xi - x)f'(\xi) - (\xi - x)f'(\eta)}{(\xi - x)^2} \quad (\eta \text{ between } x \text{ and } \xi) \\
&= \frac{t_2 - t_1}{\xi - x}[f'(\xi) - f'(\eta)],
\end{aligned}$$

$$|\varphi(t_2) - \varphi(t_1)| \leqq \left|\frac{t_2 - t_1}{\xi - x}\right| \omega(|\xi - \eta|) \leqq \left|\frac{t_2 - t_1}{\xi - x}\right| \omega(|\xi - x|);$$

if $|\xi - x| \geqq \delta$,

$$\frac{\omega(|\xi - x|)}{|\xi - x|} \leqq 2\frac{\omega(\delta)}{\delta}, \qquad |\varphi(t_2) - \varphi(t_1)| \leqq 2\omega(\delta),$$

while if $|\xi - x| < \delta$ it must be that $|t_1 - x| < 2\delta$, $|t_2 - x| < 2\delta$,

$$|\varphi(t_2) - \varphi(t_1)| = |f'(\xi_2) - f'(\xi_1)| \leqq \omega(|\xi_2 - \xi_1|) \leqq \omega(2\delta) \leqq 2\omega(\delta),$$

the points ξ_1 and ξ_2 being on the same side of x and distant from x by less than 2δ.

For the convergence proof it is sufficient that a suitable condition with regard to modulus of continuity be satisfied by $f'(t)$ in the neighborhood of $t = x$, and by $f(t)$ itself throughout the rest of the interval.

THEOREM II. *If the weight function $\varrho(x)$ is summable and has a positive lower bound, and if $f(t)$ has a continuous derivative throughout $(-1, 1)$, the series* (2) *will converge to the value $f(x)$ at any point x of the closed interval in the neighborhood of which $f(t)$ has a second derivative, continuous for $t = x$; if $\varrho(x)$ is any function such that $\varrho(x)$ and $[\varrho(x)]^{-1}$ are summable, the same conditions on $f(t)$ will be sufficient for convergence at an interior point of the interval, and the series will converge to the right value for $x = \pm 1$ also if $f''(t)$ is continuous throughout the interval and $f'''(t)$ defined near and continuous at the point in question.*

It would not be profitable to attempt refinement of the conditions for convergence by further elaboration of details, as more general results than those expressed in the first part of the theorem can be obtained by reasoning along other lines.[23] The interest lies in the extension of the method of proof of Theorem I to more general weight functions, and in the intermediate results which make the extension possible. It is to be noted also that the latter part of Theorem II gives convergence, within the hypotheses stated, even at points where the weight function vanishes.

7. Uniform convergence. The discussion that has been outlined above is admittedly incomplete in important respects. It has not given any information as to convergence at a point of discontinuity—though in Theorem I the function may have discontinuities at points other than the one where convergence is being investigated—and it has not said anything about uniformity of convergence. For the former question, as well as for a thoroughgoing treatment of the latter, reference must be made to other presentations. Certain simple facts with regard to uniform convergence, however, are immediately obtainable from the results already noted.

If the series for a continuous function $f(x)$ converges uniformly throughout the closed interval, it must converge to the value $f(x)$, under very general hypotheses as to the weight function $\varrho(x)$, by an argument which is well known and, in its essentials, frequently encountered in connection with still more general problems. The proof may be given as follows.

Being uniformly convergent, the series represents some function $g(x)$ which is continuous for $-1 \leqq x \leqq 1$. Multiplication of the series by $\varrho(x)\, p_k(x)$ and integration over the interval, with due regard for the properties of the orthogonal polynomials, gives

$$\int_{-1}^{1} \varrho(x)\, g(x)\, p_k(x)\, dx = c_k.$$

If the difference $f(x) - g(x)$ is denoted by $\psi(x)$, comparison of the last equation with the formula (3) defining c_k shows that

[23] See e. g. Colloquium, pp. 92–101.

$$\int_{-1}^{1} \rho(x)\, \psi(x)\, p_k(x)\, dx = 0$$

for all values of k, and consequently, as any polynomial can be expressed linearly in terms of the p's,

$$\int_{-1}^{1} \rho(x)\, \psi(x)\, q(x)\, dx = 0$$

for any polynomial $q(x)$ whatever. By Weierstrass's theorem, the continuous function $\psi(x)$ can be uniformly approximated by a polynomial throughout the interval with any assigned degree of accuracy, or, as an equivalent statement, can be represented by a uniformly convergent series of polynomials. (For, if $q_1, q_2, q_3, \cdots$ are a succession of approximating polynomials, q_n can be regarded as the sum of the first n terms of the series $q_1+(q_2-q_1)+(q_3-q_2)+\cdots$.) If this series is multiplied by $\rho(x)\,\psi(x)$ and integrated term by term from -1 to 1, the result of the integration is 0 for each term, and

$$\int_{-1}^{1} \rho(x)\, [\psi(x)]^2\, dx = 0.$$

Let it be assumed that $\rho(x)$, if it vanishes at all, vanishes only at a set of points of measure zero, as must necessarily be the case, for example, if $[\rho(x)]^{-1}$ is summable; or, more generally, that it is positive over a set of positive measure in every subinterval of $(-1, 1)$. Then, if $\psi(x)$ were not identically zero, $\rho(x)\,[\psi(x)]^2$ would be positive over a set of positive measure, and the value of the integral could not be zero. So it must be that $\psi(x) \equiv 0$, and $g(x) \equiv f(x)$. It is obvious that the theorem can not be proved without some restriction on the vanishing of $\rho(x)$; if $\rho(x)$ were to be identically zero throughout a subinterval of $(-1, 1)$, the values of $f(x)$ in the subinterval would have no influence on the determination of the coefficients, and there would be an infinity of different functions having the same formal expansion.

If $\rho(x)$ has a positive lower bound, let $f(x)$ be assumed to have a continuous second derivative satisfying a Lipschitz condition of order α,

$$|f''(x_2) - f''(x_1)| \leqq \lambda |x_2 - x_1|^{\alpha}, \qquad \alpha > 0.$$

Then[24] it is possible to construct approximating polynomials so that the upper bound of error, denoted above by ϵ_n, does not exceed a constant multiple of $1/n^{2+\alpha}$. It follows that $|c_n|$ has an upper bound of the same order, while $|p_n(x)|$ has an upper bound of the order of n, so that $|c_n\, p_n(x)|$ is uniformly less than a constant multiple of $1/n^{1+\alpha}$ for $-1 \leqq x \leqq 1$,

[24] See Colloquium, Chapter I, Theorem VIII.

and the series is uniformly convergent throughout the interval. If the restriction on the weight function is merely that ϱ and $1/\varrho$ be summable, uniform convergence will be assured if $f(x)$ has a third derivative satisfying a Lipschitz condition of positive order α.

The reasoning of the last few paragraphs has not presupposed the conclusions of Theorems I and II, or required the use of the Christoffel-Darboux formula, on which the proofs of those theorems depended. On the basis of what is known as to the magnitude of the polynomials $p_n(x)$ at interior points, uniform convergence throughout any closed interval interior to $(-1, 1)$ can be deduced with less severe restrictions on $f(x)$, if reference is then made to the theorems for assurance that the series converges to the right value.

8. **The problem of convergence with respect to the representation of statistical frequencies.** One other point is worth noting in this rapid summary. It has been proposed[25] that expansions in Jacobi series be used for the representation of a certain class of frequency functions in statistics. The question then is not primarily that of approximating to the given function point by point, but of approximating to the value of its definite integral over an arbitrary subinterval of the range under consideration. Convergence in this sense is readily proved under fairly general hypotheses.

Let the range once more be the interval $(-1, 1)$, and let $\varrho(x)$ be a function summable over this interval—not necessarily of the particular form leading to the Jacobi polynomials. Let $F(x)$ be another function defined over the interval. It will ordinarily be enough in the way of generality if both $\varrho(x)$ and $F(x)$ are thought of as continuous, except that they may become infinite at isolated points, subject to conditions of integrability. The function $F(x)$ is to be developed in a series of the form

$$c_0 p_0(x) \varrho(x) + c_1 p_1(x) \varrho(x) + c_2 p_2(x) \varrho(x) + \cdots,$$

or, if $F(x)/\varrho(x)$ is denoted by $f(x)$, the latter function is to be expanded in a series of the form (2). *Let $f(x)$ be supposed continuous for $-1 \leqq x \leqq 1$,* after elimination of removable singularities. By Weierstrass's theorem it is possible to define a sequence of polynomials $q_n(x)$, which may be thought of as having the degree indicated by the subscript in each case, so that $f(x) - q_n(x)$ approaches zero uniformly throughout the interval. If $|f(x) - q_n(x)| \leqq \varepsilon_n$,

[25] See e. g. V. Romanovsky, *Generalisation of some types of the frequency curves of Professor Pearson,* Biometrika, vol. 16 (1924), pp. 106–116; E. H. Hildebrandt, *Systems of polynomials connected with the Charlier expansions and the Pearson differential and difference equations,* Annals of Mathematical Statistics, vol. 2 (1931), pp. 379–439.

$$\int_{-1}^{1} \varrho(x)\,[f(x) - q_n(x)]^2\,dx \leqq \varepsilon_n^2 \int_{-1}^{1} \varrho(x)\,dx,$$

and this approaches zero as n becomes infinite when ε_n is made to approach zero. By the least-square property, the integral on the left is greater than or equal to that obtained if $q_n(x)$ is replaced by $s_n(x)$, the partial sum of the series, and consequently, if γ_n denotes the integral

$$\int_{-1}^{1} \varrho(x)\,[f(x) - s_n(x)]^2\,dx,$$

$\lim_{n\to\infty} \gamma_n = 0$. This conclusion holds under much more general conditions, but the greater generality is of secondary interest for the interpretation in terms of frequency functions. For an application of Schwarz's inequality, let

$$u = [\varrho(x)]^{-1/2}\,|F(x) - \varrho(x)\,s_n(x)| = [\varrho(x)]^{1/2}\,|f(x) - s_n(x)|,$$

$$v = [\varrho(x)]^{1/2},$$

$$\delta_n = \int_{-1}^{1} |F(x) - \varrho(x)\,s_n(x)|\,dx = \int_{-1}^{1} u\,v\,dx.$$

Then

$$\delta_n^2 \leqq \int_{-1}^{1} u^2\,dx \int_{-1}^{1} v^2\,dx = \gamma_n \int_{-1}^{1} \varrho(x)\,dx,$$

and as γ_n approaches zero, δ_n approaches zero likewise. If x_1 and $x_2 (> x_1)$ are any two points in the interval,

$$\left| \int_{x_1}^{x_2} [F(x) - \varrho(x)\,s_n(x)]\,dx \right| \leqq \int_{x_1}^{x_2} |F(x) - \varrho(x)\,s_n(x)|\,dx \leqq \delta_n,$$

and from the approach of δ_n to zero it follows that

$$\lim_{n\to\infty} \int_{x_1}^{x_2} \varrho(x)\,s_n(x)\,dx = \int_{x_1}^{x_2} F(x)\,dx,$$

which expresses the desired property of convergence.

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DIE DIFFERENTIALE IN DER ALLGEMEINEN ANALYSIS.

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1. Einleitung. Die vorliegende Arbeit ist zwei Problemen über die Differentiale der Operationen in Banachschen Räumen gewidmet. In dieser Hinsicht kann sie als ein Kapitel der allgemeinen Analysis betrachtet werden.

Das erste Problem besteht in der Aufstellung der Bedingungen, welchen ein Ausdruck genügen muß, um das Differential einer Operation zu sein. Es verallgemeinert das klassische Problem über die vollständigen Differentiale. Das zweite Problem ist die Verallgemeinerung des Satzes von Stokes auf die regulären Hyperflächen des Banachschen Raumes.

Hat insbesondere der Raum endlich viele Dimensionen, so kann man die vorliegende Arbeit als eine natürliche, Koordinaten vermeidende Auffassung der klassischen Probleme betrachten.

Die Möglichkeit der Lösungen der vorgeschlagenen Probleme ist der Einführung des Integrals in die allgemeine Analysis zu verdanken[2]. Dieses ist nur für die Operationen definiert, die einer reellen Zahl ein Element eines beliebigen Banachschen Raumes zuordnen. Doch läßt es sich in derselben Weise für Operationen erklären, die einer endlichen Anzahl reeller Zahlen ein Element zuordnen. Für diese werden wir oft den Ausdruck „Funktion" gebrauchen. Wir benutzen einige Eigenschaften der Integrale, die in den zitierten Arbeiten nicht enthalten sind. Doch lassen sich diese leicht nach der dort entwickelten Methode beweisen. Es ist zu bemerken, daß in dieser Arbeit alle Integrationsbereiche ganz regulär und alle zu integrierenden Funktionen stetig sind.

Das erste Kapitel beschäftigt sich hauptsächlich mit den multilinearen symmetrischen Operationen, das zweite mit den schiefsymmetrischen. Wir setzen voraus, daß der Raum in jedem Falle eine hinreichende Dimensionenzahl besitzt, damit die Untersuchungsgegenstände existieren. So muß im ersten Kapitel der Raum mindestens eine Ebene sein; in dem zweiten muß er je nach dem Paragraphen n- oder $(n+1)$-dimensional sein.

Die kleinen Buchstaben (verschiedener Alphabete) bezeichnen im folgenden immer reelle Zahlen, die großen — die Elemente (Punkte) der Banachschen Räume (insbesondere des Euklidischen Raumes).

Das Verzeichnis der zitierten Arbeiten befindet sich am Schluß der Arbeit.

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[1] Received March 4, 1932.

[2] Siehe [3] und [8] des am Schluß angebrachten Literaturverzeichnisses.

ERSTES KAPITEL.

Das vollständige Differential.

2. **Die Differentiale.** Im folgenden werden wir mit Operationen zu tun haben, die den Elementen $P, X, \cdots$ eines *Banachschen* (*linearen, normierten* und *vollständigen*) Raumes[3] die Elemente eines anderen ebensolchen oder auch desselben Raumes zuordnen.

Wir erinnern an die Definition des Differentials[4]:

Gibt es eine lineare Operation $\Psi(X)$ derart, daß

$$\frac{\| F(P+X) - F(P) - \Psi(X) \|}{\| X \|}$$

mit $\| X \|$ gegen Null strebt, so heißt $\Psi(X)$ das Differential (*die Differentialoperation*) *der Operation $F(P)$ für P.*

Wir bezeichnen sie mit

$$dF(P; X),$$

indem wir die Abhängigkeit von P ins Licht setzen.

Wir werden allgemeiner jede Operation

$$\Phi(P, X), \tag{1}$$

die von P in beliebiger Weise und von X linear abhängt, ein Differential nennen. Dabei sollen P und X demselben Raume angehören. Ist (1) in einem Bereiche ein Differential einer Operation $F(P)$, so wird es zum Unterschied ein *vollständiges Differential* in diesem Bereiche genannt.

Ein Differential (1) kann selbst ein Differential besitzen. Darunter verstehen wir das partielle Differential von $\Phi(P,X)$ in bezug auf P:

$$d\Phi(P, X; Y). \tag{2}$$

Dies ist eine in bezug auf X, Y lineare Operation. Ist $\Phi(P, X)$ ein Differential von $F(P)$, so nennt man (2) das *zweite Differential* von $F(P)$ und schreibt:

$$d\Phi(P, X; Y) = d^2F(P; X; Y).$$

Ebenso kann man schrittweise Differentiale beliebiger Ordnung definieren.

Wir werden allgemein jede Operation

$$\Phi(P, X_1, X_2, \cdots, X_n),$$

die in bezug auf $X_1, X_2, \cdots, X_n$ n-linear ist, ein *n-faches*[5] *Differential* nennen. Ist $\Phi(P, X_1, X_2, \cdots, X_n)$ das n-te Differential einer Operation

[3] Siehe [1].

[4] Siehe [2].

[5] Wir gebrauchen den Ausdruck „n-fach“ und nicht „n-ter Ordnung“ mit Rücksicht auf andere Anwendungen des Begriffes des Differentials, die wir im zweiten Kapitel kennenlernen werden.

$F(P)$, so nennen wir es ein *vollständiges Differential n-ter Ordnung.* Es ist klar, daß das Differential eines vollständigen Differentials selbst vollständig ist.

Ist das Differential $\Phi(P, X_1, X_2, \cdots, X_n)$ in bezug auf $X_1, X_2, \cdots, X_n$ symmetrisch (schiefsymmetrisch), so heißt es kurz ein *symmetrisches (schiefsymmetrisches) Differential.* Ist es in bezug auf P stetig, so heißt es ein *stetiges Differential.* Dieses ist auch in bezug auf alle Veränderlichen zusammen stetig, wie es aus dem Hilfssatz 1 des § 2a folgt.

Die Eigenschaften und Anwendungen der Differentiale sind in vielen Arbeiten enthalten[6]. In diesem Kapitel wollen wir nur die notwendigen und hinreichenden Bedingungen aufstellen, damit ein Differential $\Phi(P, X)$ vollständig sei. Dabei werden wir voraussetzen, daß $\Phi(P, X)$ selbst ein stetiges Differential besitzt.

2a. Zwei Hilfssätze.

Hilfssatz 1. *Ist eine Operation $\Phi(P, X_1, X_2, \cdots, X_n)$ in bezug auf $X_1, X_2, \cdots, X_n$ n-linear und in bezug auf P für $P = P_0$ stetig, so ist sie für $P = P_0$ und für alle $X_1, X_2, \cdots, X_n$ in bezug auf alle Veränderlichen zusammen stetig.*

Beweis. Der Satz gilt für $n = 1$[7]. Wir setzen voraus, daß er für ein bestimmtes $n-1$ gilt, und wollen ihn für n beweisen.

Wir können $\Phi(P, X_1, X_2, \cdots, X_n)$ als eine Operation betrachten, die von zwei Elementen, $(P, X_1, X_2, \cdots, X_{n-1})$ und X_n, abhängt. Für $P = P_0$ und für beliebige $X_1, X_2, \cdots, X_{n-1}, X_n$ ist die von diesem Standpunkte betrachtete Operation in bezug auf $(P, X_1, X_2, \cdots, X_{n-1})$ stetig, was aus der Gültigkeit des Satzes für $n-1$ folgt. Also ist sie auch in bezug auf $(P, X_1, X_2, \cdots, X_{n-1})$ und X_n zusammen für $P = P_0$ und alle $X_1, X_2, \cdots, X_{n-1}, X_n$ stetig. Dies bedeutet aber, daß $\Phi(P, X_1, X_2, \cdots, X_n)$ in bezug auf alle Veränderlichen zusammen für $P = P_0$ und beliebige $X_1, X_2, \cdots, X_n$ stetig ist, was zu beweisen war.

Hilfssatz 2. *Unter den Voraussetzungen des Hilfssatzes 1 gibt es eine positive Zahl m und eine Umgebung von P_0 derart, daß für jedes P dieser Umgebung und für alle $X_1, X_2, \cdots, X_n$ die Ungleichung gilt:*

$$\| \Phi(P, X_1, X_2, \cdots, X_n) \| \leqq m \cdot \| X_1 \| \cdot \| X_2 \| \cdot \cdots \cdot \| X_n \|.$$

Beweis. Da die Voraussetzungen des Hilfssatzes 1 erfüllt sind, so ist $\Phi(P, X_1, X_2, \cdots, X_n)$ für $P = P_0$, $X_1 = 0$, $X_2 = 0, \cdots, X_n = 0$ in bezug auf alle Veränderlichen zusammen stetig. Also gibt es eine positive Zahl a und eine Umgebung von P_0 derart, daß für P, die dieser Umgebung angehören, und für

[6] Vgl. u. a. [2], [3] und [8].

[7] Siehe [5] S. 159 (Satz 3).

$$\|Y_1\| \leqq a, \quad \|Y_2\| \leqq a, \cdots, \|Y_n\| \leqq a$$

die Ungleichung gilt:

$$\|\Phi(P, Y_1, Y_2, \cdots, Y_n)\| \leqq 1.$$

Es gehöre P der erwähnten Umgebung von P_0 an, und es seien X_1, $X_2, \cdots, X_n$ beliebige Elemente des Raumes. Setzt man

$$Y_1 = \frac{aX_1}{\|X_1\|}, \; Y_2 = \frac{aX_2}{\|X_2\|}, \cdots, \; Y_n = \frac{aX_n}{\|X_n\|},$$

so sind alle obigen Ungleichungen erfüllt, und es ist:

$$\left\| \Phi\left(P, \frac{aX_1}{\|X_1\|}, \frac{aX_2}{\|X_2\|}, \cdots, \frac{aX_n}{\|X_n\|}\right) \right\| \leqq 1.$$

Der Linearität in bezug auf $X_1, X_2, \cdots, X_n$ wegen folgt daraus:

$$\|\Phi(P, X_1, X_2, \cdots, X_n)\| \leqq \frac{1}{a^n} \cdot \|X_1\| \cdot \|X_2\| \cdot \cdots \cdot \|X_n\|.$$

Setzt man $\frac{1}{a^n} = m$, so wird der Satz bewiesen.

3. Die Symmetrie von $d^2F(P; X; Y)$.

Satz 1. *Besitzt $F(P)$ in einer Umgebung von P_0 ein stetiges zweites Differential $d^2F(P; X; Y)$, so ist dieses für $P = P_0$ symmetrisch*[8]:

$$d^2F(P_0; X; Y) = d^2F(P_0; Y; X). \tag{3}$$

Beweis. Seien a und b zwei positive Zahlen derart, daß für $|u| \leqq a$ und $|v| \leqq b$ alle Elemente $P_0 + uX + vY$ (bei gegebenen X und Y) der erwähnten Umgebung von P_0 angehören. Die Funktion

$$\frac{d}{dv}[dF(P_0 + uX + vY; X)] = d^2F(P_0 + uX + vY; X; Y)$$

ist für $|u| \leqq a$ und $|v| \leqq b$ in bezug auf v stetig. Also ist für $|u| \leqq a$

$$\begin{aligned} &dF(P_0 + uX + bY; X) - dF(P_0 + uX; X) \\ &= \int_0^b d^2F(P_0 + uX + vY; X; Y)\, dv. \end{aligned} \tag{4}$$

Ebenso ist die Funktion

$$\begin{aligned} &\frac{d}{du}[F(P_0 + uX + bY) - F(P_0 + uX)] \\ &= dF(P_0 + uX + bY; X) - dF(P_0 + uX; X) \end{aligned}$$

für $|u| \leqq a$ in bezug auf u stetig, woraus:

$$\begin{aligned} &F(P_0 + aX + bY) - F(P_0 + aX) - F(P_0 + bY) + F(P_0) \\ &= \int_0^a [dF(P_0 + uX + bY; X) - dF(P_0 + uX; X]\, du. \end{aligned} \tag{5}$$

[8] Vgl. auch [3], S. 176.

Setzen wir (4) in (5) ein, so folgt:

$$F(P_0+aX+bY)-F(P_0+aX)-F(P_0+bY)+F(P_0)$$
$$=\int_0^a du\int_0^b d^2F(P_0+uX+vY;\,X;\,Y)\,dv.$$

Da der Integrand in bezug auf u, v stetig ist, so tauschen wir die Integrationsfolge im letzten Integral um:

$$F(P_0+aX+bY)-F(P_0+aX)-F(P_0+bY)+F(P_0)$$
$$=\int_0^b dv\int_0^a d^2F(P_0+uX+vY;\,X;\,Y)\,du.$$

Differenzieren wir diese Beziehung zuerst nach b und setzen $b=0$,

$$dF(P_0+aX;\,Y)-dF(P_0;\,Y)=\int_0^a d^2F(P_0+uX;\,X;\,Y)\,du,$$

und dann nach a und setzen $a=0$,

$$d^2F(P_0;\,Y;\,X)=d^2F(P_0;\,X;\,Y),$$

so wird der Satz bewiesen.

Den Satz 1 kann man auch folgendermaßen aussprechen:

Satz 1′. *Besitzt das Differential $\Phi(P,X)$ in einem Bereiche $\mathfrak{B}$ ein stetiges Differential $d\Phi(P,X;\,Y)$, so ist die Symmetrie des letzten,*

$$d\Phi(P,X;\,Y)=d\Phi(P,Y;\,X), \tag{6}$$

eine notwendige Bedingung dafür, daß $\Phi(P,X)$ in $\mathfrak{B}$ vollständig sei.

4. **Das invariante Kurvenintegral.** Ein homöomorphes Bild eines Intervalls (a,b) nennen wir, wie üblich, einen *einfachen Bogen*. Er wird durch eine stetige Funktion $P(u)$ dargestellt, wo u die Werte $a\leqq u\leqq b$ durchläuft, und die für verschiedene u keine gleichen Werte annimmt. Wir werden die Klasse der einfachen Bogen kurz mit $(\mathfrak{J}_1)$ bezeichnen.

Läßt die Funktion $P(u)$ eine stetige und von Null verschiedene Ableitung $\frac{dP(u)}{du}=P_u(u)$ zu, so heißt $(\mathfrak{J}_1)$ ein *regulärer Bogen* und wird mit $(\mathfrak{L}_1)$ bezeichnet.

Es sei $\Phi(P,X)$ ein in einem Bereiche $\mathfrak{B}$ stetiges Differential und $\mathfrak{C}$ ein $(\mathfrak{L}_1)$ in $\mathfrak{B}$. Bestimmt $P(u)$ für $a\leqq u\leqq b$ den regulären Bogen $\mathfrak{C}$, so hat das Integral

$$\int_a^b \Phi[P(u),\,P_u(u)]\,du \tag{7}$$

einen Sinn. Denn nach § 2 ist Φ in bezug auf P und X stetig. Setzt man $u=u(v)$, wo $u(v)$ eine stetige positive Ableitung $\frac{du}{dv}$ besitzt und das Intervall (a,b) auf ein Intervall (c,d) abbildet, so kann man $\mathfrak{C}$ auch durch

$$Q(v) = P[u(v)]$$

erklären, wo $c \leqq v \leqq d$ und

$$Q_v(v) = \frac{du}{dv} P_u(u).$$

Formt man das dem Integral (7) gleichgebildete Integral

$$\int_c^d \Phi[Q(v), Q_v(v)]\, dv = \int_c^d \Phi\left\{P[u(v)], \frac{du}{dv} P_u[u(v)]\right\} dv$$

$$= \int_c^d \Phi\{P[u(v)], P_u[u(v)]\} \frac{du}{dv}\, dv = \int_a^b \Phi[P(u), P_u(u)]\, du,$$

um, so überzeugt man sich, daß (7) gegenüber der ausgeführten Parametertransformation invariant bleibt. Es hängt nur von dem Bogen ab und kann auch mit

$$\int_{\mathfrak{C}} \Phi(P, dP)$$

bezeichnet werden.

Allgemeiner kann man beweisen, daß die notwendige und hinreichende Bedingung, daß (7) invariant sei, darin besteht, daß $\Phi(P, X)$ in bezug auf X nicht notwendig linear, sondern *positiv-homogen vom Grade* 1 sei, das heißt, daß für $k>1$

$$\Phi(P, kX) = k\Phi(P, X).$$

Der Beweis unterscheidet sich nicht von dem für Funktionale gegebenen[9]. Auch der Eulersche Satz für homogene und positiv-homogene Operationen $\Phi(X)$ vom Grade n,

$$d\Phi(X; X) = n\Phi(X),$$

läßt sich wie für Funktionale beweisen[10].

5. Integration eines Differentials in einer Kugel.

Satz 2. *Besitzt das Differential $\Phi(P, X)$ in der Kugel $\mathfrak{K}(P_0, a)$ ein stetiges und symmetrisches Differential $d\Phi(P, X; Y)$, so gibt es in der Kugel genau eine Operation $F(P)$, die für P_0 einen bestimmten Wert F_0 annimmt, und deren Differential gleich $\Phi(P, X)$ ist:*

$$dF(P; X) = \Phi(P, X). \tag{8}$$

Beweis. Wir setzen voraus, daß (6) erfüllt ist. Gibt es eine Operation $F(P)$, welche die Bedingungen des Satzes erfüllt, so kann man sie folgendermaßen eindeutig bestimmen. Man verbinde ein Element P der Kugel mit P_0 durch eine gerade Strecke $P_0 + t(P - P_0)$, wo t das Intervall (0,1) durchläuft. Dann ist

$$\frac{dF[P_0 + t(P - P_0)]}{dt} = \Phi[P_0 + t(P - P_0), P - P_0].$$

[9] Vgl. [7] S. 186–187.
[10] Vgl. [6] S. 150.

Also muß

$$F(P) = F_0 + \int_0^1 \Phi[P_0 + t(P-P_0),\, P-P_0]\, dt \tag{9}$$

sein.

Wir wollen beweisen, daß (9) wirklich die Bedingungen des Satzes erfüllt. Da $F(P)$ für $P = P_0$ gleich F_0 ist, so bleibt nur noch ihr Differential zu bestimmen.

Es erhalte P einen Zuwachs X derart, daß $P+X$ noch in der Kugel $\mathfrak{K}(P_0, a)$ liegt. Die Operation $F(P)$ hat dann den Wert

$$F(P+X) = F_0 + \int_0^1 \Phi[P_0 + t(P-P_0+X),\, P-P_0+X]\, dt.$$

Nach Subtraktion der Gleichung (9) folgt:

$$\begin{aligned} &F(P+X)-F(P) \\ &\quad = \int_0^1 \{\Phi[P_0 + t(P-P_0) + tX,\, (P-P_0)+X] \\ &\qquad\qquad - \Phi[P_0 + t(P-P_0),\, P-P_0]\}\, dt, \end{aligned}$$

oder

$$\begin{aligned} &F(P+X)-F(P) \\ &\quad = \int_0^1 \Phi[P_0 + t(P-P_0) + tX,\, X]\, dt \\ &\quad + \int_0^1 \{\Phi[P_0 + t(P-P_0) + tX,\, P-P_0] \\ &\qquad\qquad - \Phi[P_0 + t(P-P_0),\, P-P_0]\}\, dt. \end{aligned} \tag{10}$$

Wir formen das zweite Integral um:

$$\int_0^1 dt \int_0^t d\Phi[P_0 + t(P-P_0) + sX,\, P-P_0;\, X]\, ds.$$

Nach (6) wird es gleich

$$\int_0^1 dt \int_0^t d\Phi[P_0 + t(P-P_0) + sX,\, X;\, P-P_0]\, ds,$$

und nach Dirichletscher Vertauschung der Integrationsfolge[11]:

$$\begin{aligned} &\int_0^1 ds \int_s^1 d\Phi[P_0 + t(P-P_0) + sX,\, X;\, P-P_0]\, dt \\ &\quad = \int_0^1 \{\Phi[P_0 + (P-P_0) + sX,\, X] - \Phi[P_0 + s(P-P_0) + sX,\, X]\}\, ds. \end{aligned}$$

Setzen wir das an Stelle des zweiten Integrals in (10) ein:

$$F(P+X)-F(P) = \int_0^1 \Phi(P+sX,\, X)\, ds,$$

woraus

$$F(P+X)-F(P)-\Phi(P,X) = \int_0^1 [\Phi(P+sX,\, X) - \Phi(P,X)]\, ds,$$

[11] Der Beweis stimmt mit dem üblichen überein.

oder

$$(11)\quad F(P+X)-F(P)-\Phi(P,X)=\int_0^1 ds\int_0^s d\Phi(P+tX,X;X)\,dt.$$

Die bilineare Operation $d\Phi(P,X;Y)$ ist nach Voraussetzung in bezug auf P für P stetig. Nach dem Hilfssatze 2 des § 2a kann man eine Umgebung von P und eine positive Zahl m finden derart, daß für alle Q, die dieser Umgebung angehören, und für alle X, Y die Ungleichung

$$\|d\Phi(Q,X;Y)\|\leqq m\cdot\|X\|\cdot\|Y\|$$

gilt. Ist $\|X\|$ hinreichend klein, damit $P+tX$ für $|t|\leqq 1$ dieser Umgebung angehöre, so schließt man daraus, daß in (11)

$$\|d\Phi(P+tX,X;X)\|\leqq m\cdot\|X\|^2,$$

und folglich

$$\frac{\|F(P+X)-F(P)-\Phi(P,X)\|}{\|X\|}\leqq\frac{1}{2}m\cdot\|X\|.$$

Also strebt die linke Seite mit $\|X\|$ gegen Null, woraus nach Definition des Differentials die Beziehung (8) folgt.

Satz 2 kann man unter Berücksichtigung des Satzes 1′ auch so aussprechen:

Satz 2′. *Besitzt das Differential $\Phi(P,X)$ in der Kugel $\mathfrak{K}(P_0,a)$ ein stetiges Differential $d\Phi(P,X;Y)$, so ist die Symmetrie* (6) *des letzten eine notwendige und hinreichende Bedingung, damit $\Phi(P,X)$ in dieser Kugel vollständig sei.*

6. **Das Integral eines vollständigen Differentials längs eines einfachen Bogens.** Betrachten wir zuerst den Fall, in welchem das Differential in einer Kugel die Bedingungen des Satzes 2 erfüllt. Es seien A und B zwei Elemente der Kugel und $\mathfrak{C}$ ein $(\mathfrak{L}_1)$, der A mit B verbindet und mittels der Funktion $P(u)$ auf das Intervall $(0,1)$ abgebildet wird. Es sei weiter $F(P)$ die im Satze 2 bestimmte Operation. Dann gilt die Beziehung:

$$\frac{dF[P(u)]}{du}=dF[P(u);P_u(u)]=\Phi[P(u),P_u(u)],$$

und da sich rechts eine stetige Funktion befindet,

$$F[P(1)]-F[P(0)]=\int_0^1\Phi[P(u),P_u(u)]\,du,$$

oder

$$(12)\qquad F(B)-F(A)=\int_{\mathfrak{C}}\Phi(P,dP).$$

Also hängt das Integral rechts nur von den Endpunkten A, B des regulären Bogens $\mathfrak{C}$ ab; es ist eine Funktion des geordneten Elementepaares, die wir mit

$$F(A,B)=F(B)-F(A)$$

bezeichnen. Sie ist additiv,

$$F(A, B)+F(B, C) = F(A, C), \tag{13}$$

und eindeutig durch $\Phi(P, X)$ bestimmt.

Ist $\mathfrak{C}$ kein regulärer, sondern ein einfacher Bogen, so kann man formell das Integral längs $\mathfrak{C}$ folgendermaßen definieren:

$$\int_{\mathfrak{C}} \Phi(P, dP) = F(A, B), \tag{14}$$

wo A, B Endpunkte des Bogens sind.

Es sei allgemein $\mathfrak{B}$ ein beliebiger Bereich (eine offene und zusammenhängende Menge) und $\mathfrak{C}$ ein beliebiger ($\mathfrak{J}_1$) darin. Es seien A und B die Endpunkte von $\mathfrak{C}$. Die Funktion $P(u)$ bilde $\mathfrak{C}$ auf das Intervall $(0, 1)$ ab. Das Differential $\Phi(P, X)$ erfülle in $\mathfrak{B}$ alle Voraussetzungen des Satzes 2.

Die Menge der Punkte von $\mathfrak{C}$ und die Komplementarmenge von $\mathfrak{B}$ sind elementenfremd, abgeschlossen und die erste ist kompakt. Also ist die untere Schranke der Entfernungen von Punkten der beiden Mengen von Null verschieden. Sei a eine noch kleinere positive Zahl.

Die Funktion $P(u)$ ist gleichmäßig stetig. Wir zerlegen das Intervall $(0, 1)$ in Teilintervalle

$$(0, \tau_1), (\tau_1, \tau_2), \cdots, (\tau_{n-1}, 1) \tag{15}$$

so, daß in jedem Teilintervalle (τ_{i-1}, τ_i) die Oszillation von $P(u)$ kleiner als a sei. Dann bilden wir die Kugeln, deren Radien gleich a sind, und deren Mittelpunkte in $P(0)$, $P(\tau_1)$, $P(\tau_2)$, $\cdots$, $P(\tau_{n-1})$ liegen. Selbstverständlich liegt der Bogen, der dem Teilintervalle (τ_{i-1}, τ_i) entspricht, in der Kugel vom Mittelpunkt $P(\tau_{i-1})$, die sich ganz in $\mathfrak{B}$ befindet. Diesem Teilbogen entspricht ein bestimmter Wert des Integrals, der durch (14) definiert wird. Wir definieren *das Integral längs* $\mathfrak{C}$ als *die Summe der Integrale längs aller Teilbogen.*

Es ist noch die Eindeutigkeit der Definition zu beweisen. Man betrachte dazu zwei Zerlegungen vom Typus (15), welche die Voraussetzung über die Oszillation von $P(u)$ erfüllen. Führt man eine dritte Zerlegung ein, die durch Aufeinanderlegen der beiden ersten entsteht, so läßt sich leicht mittels der Beziehung (13) beweisen, daß sie denselben Wert des Integrals liefert, wie jede der beiden ersten Zerlegungen. Diese liefern also auch gleiche Werte und das Integral hängt von der Zerlegung nicht ab.

7. Integration des Differentials in einem einfach zusammenhängenden Bereiche. Jetzt sind wir imstande, den Satz 2 auf einen beliebigen einfach zusammenhängenden Bereich zu übertragen. Darunter

soll man einen Bereich verstehen, in welchem jede zwei einfachen Bogen, die dieselben zwei Punkte verbinden, sich stetig ineinander überführen lassen. Dieser klassische Begriff des einfachen Zusammenhanges, genauer des einfachen Linienzusammenhanges, fordert eine präzise Formulierung, die in der folgenden Definition enthalten ist:[12]

1° Zwei einfache Bogen $\mathfrak{C}$ und $\mathfrak{C}'$ des Bereiches $\mathfrak{B}$, die zwei Elemente A und B verbinden, heißen *äquivalent im Bereiche* $\mathfrak{B}$, wenn es Teilbogen $\mathfrak{D}$ und $\mathfrak{D}'$ beziehungsweise von $\mathfrak{C}$ und $\mathfrak{C}'$ gibt, die sich ganz in einer in $\mathfrak{B}$ liegenden Kugel befinden, und von denen $\mathfrak{D}$ alle nicht zu $\mathfrak{C}'$ gehörenden Elemente von $\mathfrak{C}$, und $\mathfrak{D}'$ alle nicht zu $\mathfrak{C}$ gehörenden Elemente von $\mathfrak{C}'$ enthält.

2° Zwei einfache Bogen $\mathfrak{C}$ und $\mathfrak{C}'$ des Bereiches $\mathfrak{B}$, die zwei Elemente A und B verbinden, heißen *ineinander im Bereiche* $\mathfrak{B}$ *überführbar*, wenn es eine endliche Folge von einfachen, A mit B verbindenden Bogen $\mathfrak{C}, \mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_{n-1}, \mathfrak{C}'$ gibt, in welcher jede zwei Nachbarbogen in $\mathfrak{B}$ äquivalent sind.

3° Ein Bereich $\mathfrak{B}$ heißt *einfach zusammenhängend*, wenn jede zwei einfachen Bogen, die zwei Elemente verbinden, in $\mathfrak{B}$ ineinander überführbar sind.

Satz 3. *Besitzt das Differential* $\Phi(P, X)$ *in einem einfach zusammenhängenden Bereiche* $\mathfrak{B}$ *ein stetiges und symmetrisches Differential* $d\Phi(P, X; Y)$, *so gibt es in* $\mathfrak{B}$ *genau eine Operation* $F(P)$, *die für ein bestimmtes Element* P_0 *einen bestimmten Wert* F_0 *annimmt, und deren Differential gleich* $\Phi(P, X)$ *ist.*

Beweis. Betrachten wir ein beliebiges Element P von $\mathfrak{B}$ und verbinden es mit P_0 durch zwei einfache Bogen $\mathfrak{C}$ und $\mathfrak{C}'$. Wir untersuchen die Integrale vom Typus (14), die von P_0 bis P längs $\mathfrak{C}$ und $\mathfrak{C}'$ erstreckt sind.

Da $\mathfrak{B}$ einfach zusammenhängend ist, so sind $\mathfrak{C}$ und $\mathfrak{C}'$ ineinander in $\mathfrak{B}$ überführbar. Sei $\mathfrak{C}, \mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_{n-1}, \mathfrak{C}'$ eine Folge von $(\mathfrak{I}_1)$, welche die unter 2° ausgedrückten Bedingungen erfüllt. Greifen wir zwei Nachbarbogen aus der Folge heraus. Die Restbogen, die nach der Ausschaltung der unter 1° mit $\mathfrak{D}$ und $\mathfrak{D}'$ bezeichneten Teilbogen übrig bleiben, fallen zusammen und liefern gleiche Bestandteile des Integrals. Es bleiben nur die Teilbogen $\mathfrak{D}$ und $\mathfrak{D}'$ zum Vergleich. Diese aber verbinden dieselben zwei Punkte und liegen in einer Kugel, welche sich ganz in $\mathfrak{B}$ befindet, und in welcher die Voraussetzungen des Satzes 2 erfüllt sind. Nach § 6 liefern auch diese Teilbogen gleiche Bestandteile des Integrals (14). Also nimmt das Integral längs der Nachbarbogen gleiche Werte an. Daraus folgt, daß auch die Bogen $\mathfrak{C}$ und $\mathfrak{C}'$ denselben Wert des Integrals (14) liefern.

Gibt es eine Operation $F(P)$, welche die Bedingungen des Satzes erfüllt, so kann man sie folgendermaßen eindeutig bestimmen. Ist P ein Element

[12] Ich verdanke diese Definition einer mündlichen Mitteilung von Herrn Nikodym.

von $\mathfrak{B}$, so verbinde man ihn mit P_0 durch einen regulären Bogen $\mathfrak{C}$, der durch die Funktion $P(u)$ auf das Intervall $(0, 1)$ abgebildet wird. Dann ist

$$\frac{dF[P(u)]}{du} = \Phi[P(u), P_u(u)].$$

Also muß

$$F[P(1)] - F[P(0)] = \int_0^1 \Phi[P(u), P_u(u)]\, du,$$

oder

$$F(P) = F_0 + \int_{\mathfrak{C}} \Phi(P, dP) \tag{16}$$

sein. Nach dem oben Gesagten ist diese Operation durch das Element P nicht nur für reguläre, sondern auch für einfache Bogen $\mathfrak{C}$ *eindeutig* bestimmt.

Wir wollen beweisen, daß (16) wirklich die Bedingungen des Satzes erfüllt. Selbstverständlich ist für $P = P_0$ das Integral gleich Null und $F(P_0) = F_0$.

Was das Differential von $F(P)$ betrifft, so betrachte man ein beliebiges Element P_1 von $\mathfrak{B}$, umgebe es mit einer ganz in $\mathfrak{B}$ gelegenen Kugel, verbinde P_0 mit P_1 mit einem bestimmten einfachen Bogen $\mathfrak{C}$ und führe von P_1 nach allen Elementen der Kugel gerade Strecken. Dadurch wird P_0 mit jedem Element P der Kugel durch einen Bogen verbunden, der aus $\mathfrak{C}$ und der Strecke P_1P besteht. Dann ist für jedes P der Kugel

$$F(P) = F_0 + \int_{\mathfrak{C}} \Phi(P, dP) + \int_{P_1P} \Phi(P, dP) = F(P_1) + \int_{P_1P} \Phi(P, dP).$$

Dies ist aber eine zu (9) analog gebildete Operation, die in der Kugel das Differential $\Phi(P, X)$ besitzt. Also ist in einer Umgebung jedes Elementes P_1 von $\mathfrak{B}$ das Differential von $F(P)$ gleich $\Phi(P, X)$, womit der Satz bewiesen ist.

Den Satz 3 kann man unter Berücksichtigung des Satzes 1′ auch so aussprechen:

Satz 3′. *Besitzt das Differential $\Phi(P, X)$ in einem einfach zusammenhängenden Bereiche $\mathfrak{B}$ ein stetiges Differential $d\Phi(P, X; Y)$, so ist die Symmetrie* (6) *des letzten eine notwendige und hinreichende Bedingung dafür, daß $\Phi(P, X)$ in $\mathfrak{B}$ vollständig sei.*

8. **Vielfache Differentiale.** Der Satz 1 läßt sich folgendermaßen verallgemeinern:

Satz 4. *Besitzt $F(P)$ in einem Bereiche $\mathfrak{B}$ ein stetiges n-tes Differential $d^nF(P; X_1; X_2; \cdots; X_n)$, so ist dieses symmetrisch.*

Beweis. Die Behauptung gilt nach Satz 1 für $n = 2$. Ist sie für $n = m$ richtig, so ist $d^mF(P; X_1; X_2; \cdots; X_m)$ symmetrisch. Also

ist das Differential $d^{m+1}F(P; X_1; X_2; \cdots; X_m; X_{m+1})$ in bezug auf $X_1, X_2, \cdots, X_m$ symmetrisch. Andererseits ist es, als zweites Differential von $d^{m-1}F(P; X_1; X_2; \cdots; X_{m-1})$, wo $X_1, X_2, \cdots, X_{m-1}$ als konstant betrachtet werden sollen, in bezug auf X_m, X_{m+1} symmetrisch (wenn für $n = m+1$ die Voraussetzungen des Satzes noch erfüllt sind). Also ist $d^{m+1}F(P; X_1; X_2; \cdots; X_m; X_{m+1})$ in bezug auf $X_1, X_2, \cdots, X_m, X_{m+1}$ symmetrisch. Damit ist der Satz durch vollständige Induktion bewiesen worden.

Jetzt wollen wir den Satz 3 verallgemeinern:

Satz 5. *Besitzt das n-fache Differential $\Phi(P, X_1, X_2, \cdots, X_n)$ in einem einfach zusammenhängenden Bereiche $\mathfrak{B}$ ein stetiges Differential $d\Phi(P, X_1, X_2, \cdots, X_n; X_{n+1})$, so ist die Symmetrie des letzten die notwendige und hinreichende Bedingung dafür, daß $\Phi(P, X_1, X_2, \cdots, X_n)$ in $\mathfrak{B}$ vollständig sei.*

Beweis. Die Notwendigkeit folgt aus dem Satze 4, wo man $n+1$ anstatt n einsetzt, wenn auch der Bereich nicht einfach zusammenhängend ist.

Was die Hinlänglichkeit der Bedingung betrifft, so ist diese für $n = 1$ durch den Satz 3 bestätigt. Setzen wir voraus, daß sie für $n = m$ bewiesen worden ist. Es sei $\Phi(P, X_1, X_2, \cdots, X_m, X_{m+1})$ ein $(m+1)$-faches Differential, welches die Bedingungen des Satzes (für $n = m+1$), die Symmetrie des Differentials $d\Phi(P, X_1, X_2, \cdots, X_m, X_{m+1}; X_{m+2})$ eingeschlossen, erfüllt. Wir wollen seine Vollständigkeit beweisen.

Man betrachte $X_1, X_2, \cdots, X_m$ als konstant. Dann ist $\Phi(P, X_1, X_2, \cdots, X_m, X_{m+1})$, als Operation von P und X_{m+1}, ein vollständiges einfaches Differential (weil ihr Differential symmetrisch in bezug auf X_{m+1}, X_{m+2} ist). Sei $\Psi(P, X_1, X_2, \cdots, X_m)$ eine Operation, deren Differential gleich $\Phi(P, X_1, X_2, \cdots, X_m, X_{m+1})$ ist. Andererseits folgt aus der Symmetrie von $d\Phi(P, X_1, X_2, \cdots, X_m, X_{m+1}; X_{m+2})$ in bezug auf $X_1, X_2, \cdots, X_m, X_{m+1}$ auch die Symmetrie von $\Phi(P, X_1, X_2, \cdots, X_m, X_{m+1})$. Also erfüllt $\Psi(P, X_1, X_2, \cdots, X_m)$ die Voraussetzungen des Satzes für $n = m$ und ist folglich vollständig. Nach § 2 folgt daraus, daß auch sein Differential $\Phi(P, X_1, X_2, \cdots, X_m, X_{m+1})$ vollständig ist, womit der Satz bewiesen worden ist.

Erfüllt $\Phi(P, X_1, X_2, \cdots, X_n)$ die Voraussetzungen des Satzes 5, so gibt es eine Operation $F(P)$, für die

$$d^n F(P; X_1; X_2; \cdots; X_n) = \Phi(P, X_1, X_2, \cdots, X_n).$$

Diese ist aber nur bis auf eine Operation vom Grade $n-1$ genau bestimmt[13].

[13] Vgl. [2], S. 306. Herr Fréchet gebraucht den Ausdruck „ordre“ (Ordnung). Wir meinen, daß der Ausdruck „Grad“ (degré) besser demselben Begriffe entspricht.

ZWEITES KAPITEL.

Der Satz von Stokes.

9. **Multilineare Operationen.** Eine n-lineare Operation $F(X_1, X_2, \cdots, X_n)$ heißt *symmetrisch*, wenn sie bei jeder Permutation von $X_1, X_2, \cdots, X_n$ ihren Wert beibehält. Sie heißt *schiefsymmetrisch*, wenn sie bei einer geraden Permutation ihren Wert beibehält, bei einer ungeraden nur ihr Zeichen wechselt. Wir werden auch *jede* lineare Operation ($n = 1$) zu den schiefsymmetrischen zählen.

Ist eine n-lineare Operation $F(X_1, X_2, \cdots, X_n)$ vorgegeben, so führen wir eine neue Operation

$$F\left(\begin{vmatrix} X_1, & X_2, & \cdots, & X_n \\ X_1, & X_2, & \cdots, & X_n \\ \cdot & \cdot & \cdots & \cdot \\ X_1, & X_2, & \cdots, & X_n \end{vmatrix}\right)$$

oder kürzer

$$F(|\, X_1, \; X_2, \; \cdots, \; X_n \,|) \tag{17}$$

ein, die wir eine *Operationsdeterminante* nennen, und die folgendermaßen gebildet wird: man entwickle formell die Determinante, man ordne die Faktoren jedes Gliedes nach Zeilen, man ersetze die Multiplikation durch die Operation F[14]. Bezeichnet $\Xi_1, \Xi_2, \cdots, \Xi_n$ eine beliebige Permutation von $X_1, X_2, \cdots, X_n$, so ist:

$$F(|\, X_1, X_2, \cdots, X_n \,|) = \sum \pm F(\Xi_1, \Xi_2, \cdots, \Xi_n),$$

wo man $+$ oder $-$ setzt, je nachdem die Permutation gerade oder ungerade ist. Ist beispielsweise $F(X, Y, Z)$ eine trilineare Operation, so folgt:

$$\begin{aligned} F(|\, X, Y, Z \,|) = {} & F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) \\ & - F(Z, Y, X) - F(Y, X, Z) - F(X, Z, Y). \end{aligned}$$

Die Operationsdeterminante ist immer *schiefsymmetrisch*.

Setzt man

$$Y_i = \sum_{j=1}^{n} a_{ij} X_j \qquad (i = 1, 2, \cdots, n), \tag{18}$$

und bezeichnet man mit a die von a_{ij} gebildete Determinante, so läßt sich leicht beweisen, daß

$$F(|\, Y_1, Y_2, \cdots, Y_n \,|) = a F(|\, X_1, X_2, \cdots, X_n \,|). \tag{19}$$

Ist $F(X_1, X_2, \cdots, X_n)$ symmetrisch, so ist (17) gleich Null. Ist dagegen $F(X_1, X_2, \cdots, X_n)$ schiefsymmetrisch, so gilt die Beziehung:

$$F(|\, X_1, X_2, \cdots, X_n \,|) = n!\, F(X_1, X_2, \cdots, X_n). \tag{20}$$

[14] Man kann auch die Operationsdeterminante für verschiedene Zeilen definieren, doch ist das für unsere Zwecke unnötig.

Daraus folgt, daß die Formel (19) für die schiefsymmetrischen Operationen gilt,

$$F(Y_1, Y_2, \cdots, Y_n) = aF(X_1, X_2, \cdots, X_n), \tag{21}$$

wenn zwischen $X_1, X_2, \cdots, X_n$ und $Y_1, Y_2, \cdots, Y_n$ die Beziehung (18) vorausgesetzt wird.

Im folgenden werden die multilinearen Operationen noch von einem Element P stetig abhängen. Auch in diesem Falle kann man die Operationsdeterminante betrachten. Für $F(P, X_1, X_2, \cdots, X_n)$ soll sie mit

$$F(P, |X_1, X_2, \cdots, X_n|)$$

bezeichnet werden. Die Operation selbst, wie auch die Determinante, sind hier n-fache Differentiale. Am wichtigsten ist der Fall, wo $F(P, X_1, X_2, \cdots, X_n)$ selbst schiefsymmetrisch ist. Dann gelten die Formeln (20) und (21), wo man beiderseits P eingeführt hat.

Ist $F(X_1, X_2, \cdots, X_n)$ ein Funktional, so kann man es als einen n-fach kovarianten Tensor betrachten. Doch ist allgemein die Bestimmung seiner Komponenten nicht möglich. Ist $F(X_1, X_2, \cdots, X_n)$ selbst ein Element desselben Raumes wie $X_1, X_2, \cdots, X_n$, so kann man es als einen einfach kontravarianten und n-fach kovarianten Tensor betrachten. Die Bildung mehrfach kontravarianter Tensoren ist nach meiner Meinung in allgemeinen Banachschen Räumen nicht möglich.

Hängt $F(P, X_1, X_2, \cdots, X_n)$ noch von P ab, so haben wir es mit einem Tensorfeld zu tun. Ist $n = 1$, so bestimmt das Funktional $F(P, X)$ ein Vektorfeld.

10. **Das Rotational.** Jetzt wollen wir einen mit dem Differential eng verknüpften Differentialoperator einführen, den wir das *Rotational* nennen.

Es sei $F(P, X_1, X_2, \cdots, X_n)$ von jetzt ab ein n-faches schiefsymmetrisches Differential, das selbst ein Differential $dF(P, X_1, X_2, \cdots, X_n; X_{n+1})$ besitzt. Der Bequemlichkeit halber schließen wir den Fall $n = 0$ nicht aus, indem wir $F(P)$ als eine 0-lineare schiefsymmetrische Operation oder ein nullfaches schiefsymmetrisches Differential bezeichnen (vergleiche die Bemerkung am Anfang von § 9 für $n = 1$). Das Differential $dF(P, X_1, X_2, \cdots, X_n; X_{n+1})$ ist in bezug auf $X_1, X_2, \cdots, X_n$ schiefsymmetrisch.

Mit $\Xi_1, \Xi_2, \cdots, \Xi_n, \Xi_{n+1}$ bezeichnen wir, wie oben, eine beliebige Permutation von $X_1, X_2, \cdots, X_n, X_{n+1}$. Greifen wir in willkürlicher Weise $n+1$ Permutationen heraus, so, daß das letzte Element Ξ_{n+1} darin alle Werte $X_1, X_2, \cdots, X_n, X_{n+1}$ genau einmal durchläuft. Mit $\left[\sum_{i=1}^{n+1}\right]$ bezeichnen wir eine Summe, die über diese Permutationen erstreckt ist, wobei jedes Glied mit dem Vorzeichen + oder — versehen wird, je nachdem die Permutation gerade oder ungerade ist. Zum Beispiel ist die Summe

$$\left[\sum_{i=1}^{n+1}\right] dF(P, \Xi_1, \Xi_2, \cdots, \Xi_n; \Xi_{n+1}) \tag{22}$$

eindeutig bestimmt, weil das Differential schiefsymmetrisch in bezug auf n erste Elemente $\Xi_1, \Xi_2, \cdots, \Xi_n$ ist. Man kann in (22) die Permutationen so wählen, daß alle gerade sind; dann sind alle Glieder mit dem Vorzeichen $+$ zu versehen.

Die Summe (22) bezeichnen wir mit

$$\mathfrak{d}F(P, X_1, X_2, \cdots, X_n; X_{n+1}),$$

und nennen sie das *Rotational* des Differentials $F(P, X_1, X_2, \cdots, X_n)$. Der Buchstabe $\mathfrak{d}$ soll an den engen Zusammenhang mit dem Differential erinnern und ist zum Unterschied gotisch. Das Rotational enthält, wie das Differential, eine neue Veränderliche X_{n+1}, die wir etwa den „Zuwachs" nennen können.

Es ist beispielsweise für $F(P, X, Y)$:

$$\mathfrak{d}F(P, X, Y; Z) = dF(P, X, Y; Z) + dF(P, Z, X; Y) + dF(P, Y, Z; X).$$

Ist $n = 0$, so gibt es außer der identischen keine Permutationen, und das Rotational stimmt mit dem Differential überein:

$$\mathfrak{d}F(P; X) = dF(P; X).$$

Aus der Definition des Rotationals folgt die Beziehung:

$$\mathfrak{d}F(P, |X_1, X_2, \cdots, X_n|; X_{n+1}) = dF(P, |X_1, X_2, \cdots, X_n; X_{n+1}|)$$

und mittels der Formel (20),

$$(23)\quad \mathfrak{d}F(P, X_1, X_2, \cdots, X_n; X_{n+1}) = \frac{1}{n!} dF(P, |X_1, X_2, \cdots, X_n; X_{n+1}|).$$

Also ist das Rotational *schiefsymmetrisch*, was auch unmittelbar aus der Definition folgt. Die Formel (23) könnte selbst zur Definition des Rotationals dienen. Man kann das Rotational etwa als das schiefsymmetrisierte Differential bezeichnen.

Wir setzen voraus, daß das Differential $F(P, X_1, X_2, \cdots, X_n)$ ein stetiges zweites Differential $d^2F(P, X_1, X_2, \cdots, X_n; X_{n+1}; X_{n+2})$ besitzt. Wir wollen die Iteration

$$\mathfrak{d}\mathfrak{d}F(P, X_1, X_2, \cdots, X_n; X_{n+1}; X_{n+2})$$

berechnen. Es ist am bequemsten, die Formel (23) zu differenzieren,

$$d\mathfrak{d}F(P, X_1, X_2, \cdots, X_n; X_{n+1}; X_{n+2})$$
$$= \frac{1}{n!} d^2F(P, |X_1, X_2, \cdots, X_n; X_{n+1}|; X_{n+2}),$$

und auf beiden Seiten die Summierung $\left[\sum_{i=1}^{n+2}\right]$ zu verwenden,

$$\begin{aligned}(24)\qquad &\mathfrak{d}\mathfrak{d}F(P, X_1, X_2, \cdots, X_n; X_{n+1}; X_{n+2})\\ &= \frac{1}{n!}d^2F(P, |X_1, X_2, \cdots, X_n; X_{n+1}; X_{n+2}|).\end{aligned}$$

Da $d^2F(P, X_1, X_2, \cdots, X_n; X_{n+1}; X_{n+2})$ stetig ist, so folgt aus dem Satz 1 (wo man $X_1, X_2, \cdots, X_n$ als konstant ansehen soll), daß

$$d^2F(P, X_1, X_2, \cdots, X_n; X_{n+1}; X_{n+2})$$

in bezug auf X_{n+1}, X_{n+2} symmetrisch ist. Daher reduzieren sich paarweise die Glieder von $d^2F(P, |X_1, X_2, \cdots, X_n; X_{n+1}; X_{n+2}|)$, die sich nur durch die Reihenfolge von zwei letzten Elementen voneinander unterscheiden. Also folgt aus (24), daß

$$(25)\qquad \mathfrak{d}\mathfrak{d}F(P, X_1, X_2, \cdots, X_n; X_{n+1}; X_{n+2}) = 0.$$

Die Iteration des Rotationals verschwindet also identisch, falls das zweite Differential von $F(P, X_1, X_2, \cdots, X_n)$ stetig ist.

Dem Differential dF entspricht in gewöhnlicher Tensoranalysis im Falle $n = 0$ der Operator grad. Nur treten in allgemeiner Analysis die linearen Formen an die Stelle der Komponentendarstellung auf, der Bemerkung von § 9 gemäß.

Auch dem Rotational $\mathfrak{d}F$ entspricht im Falle $n = 0$ der Operator grad. Im Falle $n = 1$, wo $F(P, X)$ ein Vektorfeld darstellt, bestimmt

$$\mathfrak{d}F(P, X; Y) = dF(P, X; Y) - dF(P, Y; X)$$

die bilineare Kovariante oder den schiefsymmetrischen Tensoroperator, der gewöhnlich mit rot oder curl bezeichnet wird. Das Rotational verallgemeinert diesen Operator auf einen beliebigen Grad des Tensorfeldes und auf einen beliebigen Banachschen Raum.

Die Beziehung (25) umfaßt für $n = 0$ die Formel:

$$\text{curl grad } F = 0.$$

Dieser Fall war schon in (3) enthalten.

11. **Einfache Hyperflächen.** Es sei in einem durch ein kartesisches System $(x_1, x_2, \cdots, x_n)$ gerichteten n-dimensionalen Euklidischen Raume eine gerichtete Einheitskugel $\mathfrak{K}_n$,

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leqq 1,$$

gegeben. Ihre „Fläche" $\mathfrak{R}_{n-1}$ ist durch

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1$$

bestimmt. Der positive Sinn von $\mathfrak{R}_{n-1}$ werde dadurch definiert, daß die $(n-1)$-dimensionale gerichtete Tangentenebene in einem Punkte von $\mathfrak{R}_{n-1}$

mit der äußeren Normale zusammen ein positiv gerichtetes n-dimensionales System bilden soll[15].

Es sei $\xi_1, \xi_2, \cdots, \xi_n$ eine beliebige *gerade* Permutation von $x_1, x_2, \cdots, x_n$. Ist darin $\xi_n = x_i$, so werden wir diese Permutation mit Π_i bezeichnen. Die Koordinaten $\xi_1, \xi_2, \cdots, \xi_n$ bestimmen auch den positiven Sinn des Raumes. Mit $\mathfrak{K}_{n-1}^{(i)}$ bezeichnen wir die $(n-1)$-dimensionalen gerichteten Einheitskugeln

$$\xi_1^2 + \xi_2^2 + \cdots + \xi_{n-1}^2 \leqq 1, \quad \xi_n = x_i = 0 \quad (i = 1, 2, \cdots, n).$$

Dies sind Diametralschnitte der Kugel $\mathfrak{K}_n$. Die „Fläche" $\mathfrak{R}_{n-1}$ läßt sich für jedes i in zwei Teile zerlegen: den positiven, wo $\xi_n = x_i > 0$, und den negativen, wo $\xi_n = x_i < 0$. Man bemerkt leicht, daß bei der orthogonalen Projektion des positiven Teiles auf $\mathfrak{K}_{n-1}^{(i)}$ der positive Sinn beibehalten wird, bei der des negativen dagegen wird der Sinn vertauscht.

Ein homöomorphes Bild der gerichteten n-dimensionalen Kugel $\mathfrak{K}_n$ werden wir ein *n-dimensionales einfaches Hyperflächenstück* nennen und kurz mit $(\mathfrak{F}_n)$ bezeichnen. Das Bild von $\mathfrak{R}_{n-1}$ werden wir den Rand des Hyperflächenstückes nennen. Er ist aus zwei $(\mathfrak{F}_{n-1})$ zusammengesetzt, deren Sinn mit dem von $(\mathfrak{F}_n)$ verknüpft ist, und bildet eine (nicht die allgemeinste) geschlossene $n-1$-dimensionale Hyperfläche.

Für $n = 1$ ist $(\mathfrak{F}_n) = (\mathfrak{F}_1)$ ein einfacher Bogen (vgl. § 4), und sein Rand besteht aus zwei $(\mathfrak{F}_0)$, das heißt aus zwei Punkten (Endpunkten); für $n = 2$ ist $(\mathfrak{F}_n) = (\mathfrak{F}_2)$ ein einfaches Flächenstück, und sein Rand bildet eine geschlossene Kurve, die aus zwei einfachen Bogen zusammengesetzt ist, usw.

Zum Unterschied von den Elementen des betrachteten Banachschen Raumes werden wir die Punkte $(x_1, x_2, \cdots, x_n)$ des Euklidischen Raumes mit griechischen großen Buchstaben $\Gamma, \Delta, \cdots$ bezeichnen. Ein $(\mathfrak{F}_n)$ wird durch eine stetige Funktion $P(\Gamma)$ bestimmt, die jedem Punkte Γ der Kugel $\mathfrak{K}_n$ ein Element P des Banachschen Raumes zuordnet, und für welche die Gleichung $P(\Gamma) = P(\Gamma')$ nur für $\Gamma = \Gamma'$ bestehen kann.

12. **Reguläre Hyperflächen.** Ein $(\mathfrak{F}_n)$ heißt ein *reguläres n-dimensionales Hyperflächenstück* und wird mit $(\mathfrak{L}_n)$ bezeichnet, wenn

1° die definierende Funktion $P(\Gamma)$ ein stetiges Differential $dP(\Gamma; \Delta)$ besitzt,

2° für jeden Punkt Γ die Funktion $dP(\Gamma; \Delta)$ eine n-dimensionale Ebene beschreibt, falls Δ alle Punkte des n-dimensionalen Euklidischen Raumes durchläuft.

Die Bedingung 2° besagt nichts anderes, als, daß für jeden Punkt Γ und n linear unabhängige Punkte $\Delta_1, \Delta_2, \cdots, \Delta_n$ des Euklidischen Raumes

[15] Diese Verabredung ist der üblichen über den positiven Sinn des Kreises entgegengesetzt.

die Elemente $dP(\Gamma; \Delta_i)$ $(i = 1, 2, \cdots, n)$ linear unabhängig sind. Die n-dimensionale Ebene

$$P(\Gamma) + dP(\Gamma; \Delta)$$

heißt die Tangentenebene im Elemente $P(\Gamma)$.

Für $n = 1$ ist $(\mathfrak{L}_n) = (\mathfrak{L}_1)$ ein regulärer Bogen (vgl. § 4).

Ist ausnahmsweise für einen Punkt Γ eine der Bedingungen 1° und 2° nicht erfüllt, so kann man das entsprechende Element des Hyperflächenstückes singulär nennen. Wir schließen jedoch die singulären Elemente aus den Betrachtungen aus, obwohl eine passende Beschränkung der Menge dieser Elemente die Durchführung aller folgenden Untersuchungen gestatten würde.

Ist der Punkt Γ, wie auch der Zuwachs Δ durch kartesische Koordinaten bestimmt,

$$\Gamma = (x_1, x_2, \cdots, x_n),$$
$$\Delta = (h_1, h_2, \cdots, h_n),$$

so ist $(\mathfrak{L}_n)$ durch eine Funktion $P(x_1, x_2, \cdots, x_n)$ definiert. Wir behaupten, daß die Bedingungen der Regularität den folgenden äquivalent sind:

a) die Funktion $P(x_1, x_2, \cdots, x_n)$ besitzt stetige partielle Ableitungen $\frac{\partial P}{\partial x_i}$ $(i = 1, 2, \cdots, n)$,

b) diese Ableitungen sind linear unabhängig.

Die Behauptung a) ist mit 1° äquivalent.

Was b) betrifft, so nehme man n linear unabhängige Punkte $\Delta_i = (h_{i1}, h_{i2}, \cdots, h_{in})$ $(i = 1, 2, \cdots, n)$. Dann ist

$$dP(\Gamma; \Delta_i) = \sum_{j=1}^{n} \frac{\partial P}{\partial x_j} h_j.$$

Da die aus h_{ij} gebildete Determinante von Null verschieden ist, so sind $dP(\Gamma; \Delta_i)$ und $\frac{\partial P}{\partial x_j}$ gleichzeitig linear unabhängig oder linear abhängig, was die Äquivalenz von 2° und b) beweist.

Oft werden wir anstatt $x_1, x_2, \cdots, x_n$ andere Parameter $v_1, v_2, \cdots, v_n$ gebrauchen. Dabei soll die Abbildung ein-eindeutig und stetig differenzierbar, und die Funktionaldeterminante überall positiv sein:

$$\frac{d(x_1, x_2, \cdots, x_n)}{d(v_1, v_2, \cdots, v_n)} > 0. \tag{26}$$

Die Parameter $v_1, v_2, \cdots, v_n$ bestimmen den positiven Sinn von $(\mathfrak{L}_n)$. Man bemerkt leicht, daß man an Stelle von $x_1, x_2, \cdots, x_n$ in den Bedingungen a) und b) der Regularität $v_1, v_2, \cdots, v_n$ einsetzen kann; die Bedingungen

bleiben unverändert. Definiert man die Zahlen $k_1, k_2, \cdots, k_n$ durch die Gleichungen

$$h_j = \sum_{i=1}^{n} \frac{\partial x_j}{\partial v_i} k_i \qquad (j = 1, 2, \cdots, n)$$

mit nicht verschwindender Determinante (26), so bekommt man

$$dP(\Gamma; \Delta) = \sum_{j=1}^{n} \frac{\partial P}{\partial x_j} h_j = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial P}{\partial x_j} \frac{\partial x_j}{\partial v_i} k_i = \sum_{i=1}^{n} \frac{\partial P}{\partial v_i} k_i.$$

Wir wollen noch einige Bemerkungen über den Rand von $(\mathfrak{L}_n)$ machen. Wir projizieren stereographisch die positive Hälfte $(x_n > 0)$ der Kugelfläche $\mathfrak{K}_{n-1}$ auf die Ebene $x_n = 0$ vom Punkte $(0, 0, \cdots, 0, -1)$ (Südpol). Ihr Bild fällt mit $\mathfrak{K}^{(n)}_{n-1}$ zusammen. Die kartesischen Koordinaten $y_1, y_2, \cdots, y_{n-1}$ des Bildpunktes bestimmen den Punkt von $\mathfrak{K}_{n-1}$. Die Abbildung ist stetig differenzierbar, und ihre Funktionalmatrix

$$\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial y_{n-1}} & \frac{\partial x_2}{\partial y_{n-1}} & \cdots & \frac{\partial x_n}{\partial y_{n-1}} \end{pmatrix} \tag{27}$$

hat den Rang $n-1$.

Die Funktion

$$Q(y_1, y_2, \cdots, y_{n-1}) = P(x_1, x_2, \cdots, x_n)$$

bildet einen Teil des Randes von $(\mathfrak{L}_n)$ auf $\mathfrak{K}^{(n)}_{n-1}$ ab. Dabei sind

$$\frac{\partial Q}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial P}{\partial x_j} \frac{\partial x_j}{\partial y_i} \qquad (i = 1, 2, \cdots, n-1)$$

stetig und linear unabhängig (sonst hätte die Matrix (27) nicht den Rang $n-1$). Also ist der betrachtete Teil des Randes ein $(\mathfrak{L}_{n-1})$.

Ebenso beweist man, vom Punkte $(0, 0, \cdots, 0, 1)$ projizierend, daß auch der Teil des Randes, welcher der negativen Hälfte von $\mathfrak{K}_{n-1}$ entspricht, ein $(\mathfrak{L}_{n-1})$ ist. Doch bestimmen $y_1, y_2, \cdots, y_{n-1}$ auf dem Bilde der positiven Halbkugelfläche den positiven Sinn, auf dem der negativen dagegen den negativen.

Also ist der Rand eines $(\mathfrak{L}_n)$ aus zwei $(\mathfrak{L}_{n-1})$ zusammengesetzt. Sie bilden zusammen eine (nicht die allgemeinste) geschlossene reguläre $n-1$-dimensionale Hyperfläche. Die Punkte des Randes von $(\mathfrak{L}_n)$, die dem gemeinsamen Rande der beiden $(\mathfrak{L}_{n-1})$ angehören, sind nur durch unsere

Beweisführung, nicht aber durch die Natur der Sache ausgezeichnet. Man kann $\mathfrak{R}_{n-1}$ anders in Hälften zerlegen, wobei diese Punkte nicht auf dem erwähnten gemeinsamen Rande liegen.

13. **Das invariante Hyperflächenintegral.** Es sei $F(P, X_1, X_2, \cdots, X_n)$ ein in einem Bereiche $\mathfrak{B}$ erklärtes n-faches schiefsymmetrisches stetiges Differential. Es sei weiter $\mathfrak{R}$ ein n-dimensionales reguläres Hyperflächenstück im Bereiche $\mathfrak{B}$, das durch die Funktion

$$P(\Gamma) = P(x_1, x_2, \cdots, x_n)$$

der Definition von $(\mathfrak{L}_n)$ gemäß auf die Kugel $\mathfrak{K}_n$ abgebildet wird.

Die Funktion

$$F[P(\Gamma), P_{x_1}(\Gamma), P_{x_2}(\Gamma), \cdots, P_{x_n}(\Gamma)]$$

ist stetig, und man kann sie über $\mathfrak{K}_n$ integrieren:

$$U = \int\int \cdots \int_{\mathfrak{K}_n} F[P(\Gamma), P_{x_1}(\Gamma), P_{x_2}(\Gamma), \cdots, P_{x_n}(\Gamma)]\, dx_1\, dx_2 \cdots dx_n.$$

Wir drücken $x_1, x_2, \cdots, x_n$ durch neue Parameter $v_1, v_2, \cdots, v_n$ aus, wobei alle Voraussetzungen, und insbesondere (26), des § 12, über den Zusammenhang von $x_1, x_2, \cdots, x_n$ und $v_1, v_2, \cdots, v_n$ erfüllt sein sollen. Dann ist:

$$U = \int\int \cdots \int F[P(\Gamma), P_{x_1}(\Gamma), P_{x_2}(\Gamma), \cdots, P_{x_n}(\Gamma)] \times \frac{d(x_1, x_2, \cdots, x_n)}{d(v_1, v_2, \cdots, v_n)}\, dv_1\, dv_2 \cdots dv_n,$$

wo die Integration über den Veränderlichkeitsbereich von $v_1, v_2, \cdots, v_n$ erstreckt werden soll. Setzt man

$$P(\Gamma) = Q(v_1, v_2, \cdots, v_n),$$

so wird

$$Q_{v_i}(v_1, v_2, \cdots, v_n) = \sum_{j=1}^{n} P_{x_j}(\Gamma) \frac{\partial x_j}{\partial v_i},$$

und da $F(P, X_1, X_2, \cdots, X_n)$ schiefsymmetrisch ist, so folgt aus (21)

$$F(Q, Q_{v_1}, Q_{v_2}, \cdots, Q_{v_n}) = F(P, P_{x_1}, P_{x_2}, \cdots, P_{x_n}) \frac{d(x_1, x_2, \cdots, x_n)}{d(v_1, v_2, \cdots, v_n)}.$$

U nimmt die Gestalt

$$U = \int\int \cdots \int F(Q, Q_{v_1}, Q_{v_2}, \cdots, Q_{v_n})\, dv_1\, dv_2 \cdots dv_n \tag{28}$$

an. Also ist das Integral U invariant gegenüber der ausgeführten Parametertransformation. Wir können es mit

$$\int\int \cdots \int_{\mathfrak{R}} F(P, d_1 P, d_2 P, \cdots, d_n P)$$

bezeichnen, wobei formell

$$d_i P = P_{x_i} dx_i \qquad (i = 1, 2, \cdots, n)$$

zu setzen ist.

Ist $F(P, X_1, X_2, \cdots, X_{n-1})$ $n-1$-fach, so kann man auch das Integral

$$\int\int \cdots \int_{\mathfrak{R}} F(P, d_1 P, d_2 P, \cdots, d_{n-1} P)$$

über den geschlossenen Rand $\mathfrak{R}$ eines $(\mathfrak{L}_n)$ erstrecken. Man hat nur $\mathfrak{R}$ in zwei $(\mathfrak{L}_{n-1})$ zu zerlegen und über diese zu integrieren. Dabei ist zu beachten, daß der Sinn der $(\mathfrak{L}_{n-1})$ mit dem der entsprechenden Hälfte von $\mathfrak{R}_{n-1}$ übereinstimme.

Man kann ein invariantes Integral nicht nur über ein reguläres Hyperflächenstück, sondern auch über analoge Bilder von mehrdimensionalen Polyedern, wie z. B. von einem n-dimensionalen Simplex, definieren.

Führen wir die Parameter $v_1, v_2, \cdots, v_n$ ein, die alle Bedingungen außer (26) erfüllen, und ist die Funktionaldeterminante negativ, so soll man (28) nur mit dem Vorzeichen — versehen. Dabei bestimmen $v_1, v_2, \cdots, v_n$ den negativen Sinn von $\mathfrak{R}$.

14. Der Satz von Stokes.

Satz 6. *Ist $n \geqq 2$, ist $F(P, X_1, X_2, \cdots, X_{n-1})$ ein $n-1$-faches schiefsymmetrisches Differential, das in einem Bereiche $\mathfrak{B}$ ein stetiges Differential $dF(P, X_1, X_2, \cdots, X_{n-1}; X_n)$ besitzt, ist $\mathfrak{R}$ ein n-dimensionales reguläres Hyperflächenstück in $\mathfrak{B}$ und $\mathfrak{R}$ sein Rand, so gilt die Stokessche Formel:*

$$\int\int \cdots \int_{\mathfrak{R}} \mathfrak{d} F(P, d_1 P, d_2 P, \cdots, d_{n-1} P; d_n P)$$
$$= \int\int \cdots \int_{\mathfrak{R}} F(P, d_1 P, d_2 P, \cdots, d_{n-1} P).$$

Zusatz. *Ist $F(P)$ eine Operation, die in einem Bereiche $\mathfrak{B}$ ein stetiges Differential $dF(P; X)$ besitzt, ist $\mathfrak{C}$ ein regulärer Bogen in $\mathfrak{B}$ und A, B sein Anfangs- und Endpunkt, so ist:*

$$\int_{\mathfrak{C}} \mathfrak{d} F(P, dP) = F(B) - F(A).$$

Beweis. Was den Zusatz betrifft, der dem Falle $n = 1$ entspricht, so ist er im wesentlichen im § 6, Formel (12), bewiesen worden (man beachte, daß für $n = 1$ das Rotational mit dem Differential übereinstimmt, § 10).

Wir wollen den Satz beweisen. Wir gebrauchen dabei die Bezeichnungen des § 11. Außerdem soll $\left[\sum_{i=1}^{n}\right]$, wie im § 10, eine Summe bezeichnen, die über n Permutationen $\Pi_i (i = 1, 2, \cdots, n)$ erstreckt ist (die Bedeutung von Π_i wie im § 11).

Ist Π_i für ein bestimmtes i gleich $\xi_1, \xi_2, \cdots, \xi_n$, so bestimmen $\xi_1, \xi_2, \cdots, \xi_{n-1}$ zwei Punkte auf $\mathfrak{R}_{n-1}$, für die $\xi_n > 0$ bzw. $\xi_n < 0$ ist. Die entsprechenden Elemente von $\mathfrak{R}$ bezeichnen wir mit $P^+(\xi_1, \xi_2, \cdots, \xi_{n-1})$ und $P^-(\xi_1, \xi_2, \cdots, \xi_{n-1})$, oder kurz mit P^+ und P^-. Dabei kann der Punkt $(\xi_1, \xi_2, \cdots, \xi_{n-1})$ der Ebene $\xi_n = x_i = 0$ die $n-1$-dimensionale Kugel $\mathfrak{K}^{(i)}_{n-1}$ durchlaufen.

Wir führen die Bezeichnungen

$$U = \int\int \cdots \int_{\mathfrak{R}} F(P, d_1 P, d_2 P, \cdots, d_{n-1} P),$$

$$V = \left[\sum_{i=1}^{n}\right] \int\int \cdots \int_{\mathfrak{K}^{(i)}_{n-1}} F(P^+, P^+_{\xi_1}, P^+_{\xi_2}, \cdots, P^+_{\xi_{n-1}})\, d\xi_1\, d\xi_2 \cdots d\xi_{n-1}$$

$$- \left[\sum_{i=1}^{n}\right] \int\int \cdots \int_{\mathfrak{K}^{(i)}_{n-1}} F(P^-, P^-_{\xi_1}, P^-_{\xi_2}, \cdots, P^-_{\xi_{n-1}})\, d\xi_1\, d\xi_2 \cdots d\xi_{n-1}$$

ein und wollen zuerst beweisen, daß

$$U = V. \tag{29}$$

Um in U Parameter einzuführen, zerlegen wir $\mathfrak{R}_{n-1}$ in zwei Hälften, für die $x_n > 0$ und $x_n < 0$. Auf der positiven Hälfte führen wir, wie im § 12, die Parameter $y_1, y_2, \cdots, y_{n-1}$ ein, welche die kartesischen Koordinaten der stereographischen Projektion auf die Ebene $x_n = 0$ vom Punkte $(0, 0, \cdots, 0, -1)$ darstellen. Ebenso sollen für die negative Hälfte $y_1, y_2, \cdots, y_{n-1}$ die Koordinaten der Projektion vom Punkte $(0, 0, \cdots, 0, 1)$ bezeichnen. Zum Unterschied werden die Punkte der beiden Teile von $\mathfrak{R}$, die denselben $y_1, y_2, \cdots, y_{n-1}$ entsprechen, mit $Q^+(y_1, y_2, \cdots, y_{n-1})$ und $Q^-(y_1, y_2, \cdots, y_{n-1})$, oder kurz mit Q^+ und Q^- bezeichnet. Dann ist

$$\begin{aligned} U = &\int\int \cdots \int_{\mathfrak{K}^{(n)}_{n-1}} F(Q^+, Q^+_{y_1}, Q^+_{y_2}, \cdots, Q^+_{y_{n-1}})\, dy_1\, dy_2 \cdots dy_{n-1} \\ &- \int\int \cdots \int_{\mathfrak{K}^{(n)}_{n-1}} F(Q^-, Q^-_{y_1}, Q^-_{y_2}, \cdots, Q^-_{y_{n-1}})\, dy_1\, dy_2 \cdots dy_{n-1}. \end{aligned} \tag{30}$$

Es sei ε eine kleine positive Zahl $\left(\varepsilon < \frac{1}{\sqrt{n}}\right)$. Wir zerlegen $\mathfrak{R}_{n-1}$ in zwei Teile: $\overline{\mathfrak{R}}_{n-1}$, der aus den Punkten besteht, deren alle Koordinaten die Ungleichung

$$|x_i| \geqq \varepsilon \qquad (i = 1, 2, \cdots, n)$$

erfüllen, und $\overline{\overline{\mathfrak{R}}}_{n-1}$, der aus allen übrigen Punkten besteht. Durch beide stereographischen Projektionen wird auch $\mathfrak{K}^{(n)}_{n-1}$ in (dieselben) zwei Teile zerlegt. Ist m der Inhalt des zweiten Teiles von $\mathfrak{K}^{(n)}_{n-1}$, so sieht man leicht, daß

$$\lim_{\varepsilon \to 0} m = 0. \tag{31}$$

Das Integral U zerfällt in zwei Teile:

$$U = \bar{U} + \bar{\bar{U}}.$$

Der zweite Teil $\bar{\bar{U}}$ läßt sich leicht abschätzen. In der Tat sind in beiden Integralen von (30) $Q^+, Q^-, Q^+_{y_1}, Q^-_{y_1}, \cdots$ stetige Funktionen von $y_1, y_2, \cdots, y_{n-1}$. Da nach § 2 F eine stetige Operation von allen Veränderlichen zusammen ist, so ist F auch eine stetige Funktion von endlich vielen numerischen Veränderlichen $y_1, y_2, \cdots, y_{n-1}$. Also ist F beschränkt:

$$\|F\| \leqq a.$$

Daraus folgt nach (30), daß

$$\|\bar{\bar{U}}\| \leqq 2am,$$

und nach (31), daß

$$\lim_{\varepsilon \to 0} \bar{\bar{U}} = 0. \tag{32}$$

Wir gehen zum Integral $\bar{U}$ über, das über $\bar{\mathfrak{R}}_{n-1}$ erstreckt ist. Nach der Definition zerfällt $\bar{\mathfrak{R}}_{n-1}$ in 2^n Kugelsimplexe, die etwa durch die Vorzeichen von $x_1, x_2, \cdots, x_n$ bestimmt werden können. Wir bezeichnen diese Simplexe mit $\mathfrak{S}_j (j = 1, 2, \cdots, 2^n)$ und führen auf jedem die Veränderlichen $v_1, v_2, \cdots, v_{n-1}$ ein, die für die Simplexe, für welche $x_n > 0$, mit $y_1, y_2, \cdots, y_{n-1}$ übereinstimmen, und für die übrigen sich von $y_1, y_2, \cdots, y_{n-1}$ nur durch eine einzige Transposition unterscheiden. Damit wird erreicht, daß auf jedem Simplex $v_1, v_2, \cdots, v_{n-1}$ den positiven Sinn bestimmen. Nimmt man in (30) in Betracht, daß $F(P, X_1, X_2, \cdots, X_{n-1})$ schiefsymmetrisch ist, so folgt

$$\bar{U} = \sum_{j=1}^{2^n} \int\int \cdots \int_{\mathfrak{S}_j} F(Q, Q_{v_1}, Q_{v_2}, \cdots, Q_{v_{n-1}})\, dv_1\, dv_2 \cdots dv_{n-1}. \tag{33}$$

Man bemerke, daß auf jedem Simplex $x_1, x_2, \cdots, x_n$ stetige Ableitungen in bezug auf $v_1, v_2, \cdots, v_{n-1}$ besitzen, und daß alle möglichen Funktionaldeterminanten $\dfrac{d(\xi_1, \xi_2, \cdots, \xi_{n-1})}{d(v_1, v_2, \cdots, v_{n-1})}$ von Null verschieden sind. Doch ist eine solche Determinante für eine bestimmte Permutation Π_i auf einer Hälfte der Gesamtzahl von Simplexen positiv, auf der anderen dagegen negativ.

Wir bemerken, daß

$$Q_{v_\alpha} = \sum_{\beta=1}^{n} P_{x_\beta} \frac{\partial x_\beta}{\partial v_\alpha} \quad (\alpha = 1, 2, \cdots, n-1).$$

Führen wir diese Werte in (33) ein, und nehmen die Schiefsymmetrie von F in Betracht, so folgt

$$\bar{U} = \sum_{j=1}^{2^n}\Big[\sum_{i=1}^{n}\Big]\int\int\cdots\int_{S_j} F(P, P_{\xi_1}, P_{\xi_2}, \cdots, P_{\xi_{n-1}}) \times \frac{d(\xi_1, \xi_2, \cdots, \xi_{n-1})}{d(v_1, v_2, \cdots, v_{n-1})}\, dv_1\, dv_2 \cdots dv_{n-1}. \tag{34}$$

Wir greifen eine Permutation Π_i heraus und untersuchen das entsprechende Glied von $\bar{U}$. Für die Simplices, für welche $\xi_n = x_i > 0$, ist nach § 11 der positive Sinn durch $\xi_1, \xi_2, \cdots, \xi_{n-1}$ bestimmt, und die Funktionaldeterminante ist positiv. Für die übrigen Simplexe ist sie negativ. Bezeichnet man mit $\bar{\mathfrak{K}}^{(i)}_{n-1}$ die orthogonale Projektion aller Simplexe auf die Ebene $\xi_n = x_i = 0$, so nimmt (34) die Gestalt

$$\begin{aligned}\bar{U} = &\Big[\sum_{i=1}^{n}\Big]\int\int\cdots\int_{\bar{\mathfrak{K}}^{(i)}_{n-1}} F(P, P^{+}_{\xi_1}, P^{+}_{\xi_2}, \cdots, P^{+}_{\xi_{n-1}})\, d\xi_1\, d\xi_2 \cdots d\xi_{n-1} \\ &-\Big[\sum_{i=1}^{n}\Big]\int\int\cdots\int_{\bar{\mathfrak{K}}^{(i)}_{n-1}} F(P, P^{-}_{\xi_1}, P^{-}_{\xi_2}, \cdots, P^{-}_{\xi_{n-1}})\, d\xi_1\, d\xi_2 \cdots d\xi_{n-1}\end{aligned} \tag{35}$$

an.

Bezeichnen wir mit $\bar{\bar{\mathfrak{K}}}^{(i)}_{n-1}$ die Mengen $\mathfrak{K}^{(i)}_{n-1} - \bar{\mathfrak{K}}^{(i)}_{n-1}$. Dann läßt sich das Integral V in zwei Teile zerlegen,

$$V = \bar{V} + \bar{\bar{V}},$$

von denen der erste mit (35) übereinstimmt:

$$\bar{U} = \bar{V}. \tag{36}$$

Ist m_1 der Inhalt von $\bar{\bar{\mathfrak{K}}}^{(i)}_{n-1}$ (alle Mengen $\bar{\bar{\mathfrak{K}}}^{(i)}_{n-1}$ besitzen denselben Inhalt), so ist

$$\lim_{\varepsilon \to 0} m_1 = 0. \tag{37}$$

Man beweist ebenso wie für (30), daß F in allen Integralen von V beschränkt ist:

$$\|F\| \leqq b.$$

Daraus folgt, daß

$$\|\bar{\bar{V}}\| \leqq 2nbm_1,$$

und nach (37), daß

$$\lim_{\varepsilon \to 0} \bar{\bar{V}} = 0. \tag{38}$$

Aus den Beziehungen (32), (36) und (38) folgt die geforderte Gleichung (29).

Jetzt wollen wir noch V umformen. Wir greifen aus V zwei Integrale heraus, die einer bestimmten Permutation Π_i entsprechen. Sind $\xi_1, \xi_2, \cdots, \xi_{n-1}$ die Koordinaten eines Punktes von $\mathfrak{K}^{(i)}_{n-1}$, so gibt es zwei Werte von ξ_n,

die auf $\mathfrak{R}_{n-1}$ den $\xi_1, \xi_2, \cdots, \xi_{n-1}$ entsprechen. Wir bezeichnen diese mit $\xi_n^+(\xi_1, \xi_2, \cdots, \xi_{n-1})$ und $\xi_n^-(\xi_1, \xi_2, \cdots, \xi_{n-1})$, oder kurz mit ξ_n^+ und ξ_n^- (es ist $\xi_n^- = -\xi_n^+$). Die herausgegriffenen Integrale haben den Wert:

$$\begin{aligned}
&\int\int\cdots\int_{\mathfrak{K}_{n-1}^{(i)}} \{F[P(\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n^+), P_{\xi_1}(\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n^+), \cdots \\
&\qquad \cdots, P_{\xi_n}(\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n^+)] \\
&\quad - F[P(\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n^-), P_{\xi_1}(\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n^-), \cdots \\
&\qquad \cdots, P_{\xi_n}(\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n^-)]\}\, d\xi_1 d\xi_2 \cdots d\xi_{n-1} \\
=&\int\int\cdots\int_{\mathfrak{K}_{n-1}^{(i)}} d\xi_1 d\xi_2 \cdots d\xi_{n-1} \int_{\xi_n^-}^{\xi_n^+} dF[P(\xi_1, \xi_2, \cdots, \xi_n), P_{\xi_1}(\xi_1, \xi_2, \cdots, \xi_n), \cdots \\
&\qquad \cdots, P_{\xi_{n-1}}(\xi_1, \xi_2, \cdots, \xi_n); P_{\xi_n}(\xi_1, \xi_2, \cdots, \xi_n)]\, d\xi_n \\
=&\int\int\cdots\int_{\mathfrak{K}_n} dF(P, P_{\xi_1}, P_{\xi_2}, \cdots, P_{\xi_{n-1}}; P_{\xi_n})\, d\xi_1 d\xi_2 \cdots d\xi_n.
\end{aligned}$$

Führt man diese Umformung für alle Glieder von V aus, so folgt:

$$\begin{aligned}
V &= \left[\sum_{i=1}^{n}\right] \int\int\cdots\int_{\mathfrak{K}_n} dF(P, P_{\xi_1}, P_{\xi_2}, \cdots, P_{\xi_{n-1}}; P_{\xi_n})\, d\xi_1 d\xi_2 \cdots d\xi_n \\
&= \int\int\cdots\int_{\mathfrak{K}_n} \mathfrak{d} F(P, P_{x_1}, P_{x_2}, \cdots, P_{x_{n-1}}; P_{x_n})\, dx_1 dx_2 \cdots dx_n \\
&= \int\int\cdots\int_{\mathfrak{R}} \mathfrak{d} F(P, d_1 P, d_2 P, \cdots, d_{n-1} P; d_n P).
\end{aligned}$$

Erinnert man sich an die Bedeutung von U und an (29), so wird der Beweis des Satzes beendet.

15. **Folgerungen aus dem Stokesschen Satze.** Verschwindet $\mathfrak{d} F(P, X_1, X_2, \cdots, X_{n-1}; X_n)$ identisch und ist $\mathfrak{R}$ der Rand eines beliebigen $(\mathfrak{L}_n)$, so ist:

$$\int\int\cdots\int_{\mathfrak{R}} F(P, d_1 P, d_2 P, \cdots, d_{n-1} P) = 0.$$

Zerlegen wir $\mathfrak{R}$ in zwei $(\mathfrak{L}_{n-1})$, $\mathfrak{S}_1$ und $\mathfrak{S}_2$, und wechseln auf $\mathfrak{S}_2$ den Sinn um, so folgt:

$$\begin{aligned}
&\int\int\cdots\int_{\mathfrak{S}_1} F(P, d_1 P, d_2 P, \cdots, d_{n-1} P) \\
&= \int\int\cdots\int_{\mathfrak{S}_2} F(P, d_1 P, d_2 P, \cdots, d_{n-1} P).
\end{aligned} \tag{39}$$

Also ist das Integral eine Funktion des gemeinsamen Randes $\mathfrak{T}$ von $\mathfrak{S}_1$ und $\mathfrak{S}_2$, der eine geschlossene reguläre $(n-2)$-dimensionale Hyperfläche bildet. Wir haben mit Hyperflächenfunktionen im Sinne von Herrn Volterra zu tun.[16] Doch sind diese Funktionen nicht für alle selbst regulären

[16] Vgl. [4], S. 74–86.

geschlossenen Hyperflächen bestimmt, weil man erstens im voraus nicht wissen kann, ob sich durch diese eine geschlossene $(n-1)$-dimensionale Hyperfläche $\mathfrak{R}$ führen läßt (man erinnere sich an die Kurven mit Knoten im dreidimensionalen Raume), und zweitens entsteht dieselbe Frage bei Bildung eines $(\mathfrak{L}_n)$, dessen Rand mit $\mathfrak{R}$ zusammenfällt.

Für $n = 3$ wird $\mathfrak{R}$ eine geschlossene Fläche und $\mathfrak{T}$ eine geschlossene Kurve. Wir haben es mit Linienfunktionen zu tun.

Für $n = 2$ ist $\mathfrak{R}$ eine geschlossene Kurve, und die Integrale (39) sind längs zweier regulären Bogen zu nehmen, die zwei Punkte verbinden. Das Verschwinden von $\mathfrak{d}F(P, X; Y)$ ist jetzt mit der Bedingung (6) der Symmetrie identisch und die Unabhängigkeit des Integrals (39) vom Integrationswege fällt in mancher Hinsicht mit dem ersten Teile des Beweises des Satzes 3 zusammen. Doch sind jetzt die Bogen $\mathfrak{S}_1$ und $\mathfrak{S}_2$ nicht nur durch Differenzierbarkeitsbedingungen, sondern auch durch gegenseitige Lage im Raume bedeutend mehr beschränkt.

Es folgt daraus, daß man auch die Untersuchungen des ersten Kapitels auf den Fall $n > 2$ auszudehnen suchen kann. Man muß zuerst das von Herrn Volterra vorgeschlagene[17] Problem der Linien- und Hyperflächenfunktionen in allgemeinen Räumen untersuchen. Dann kann man eine Linien- oder Hyperflächenfunktion durch ihr Differential zu bestimmen versuchen. Ich hoffe, mich mit diesen Gegenständen an anderem Orte beschäftigen zu können.

Zum Schluß wollen wir noch mittels des Stokesschen Satzes die Identität (25) von neuem begründen. Es seien $P_0, X_1, X_2, \cdots, X_{n+2}$ gegeben. Das Element

$$P_0 + u_1 X_1 + u_2 X_2 + \cdots + u_{n+2} X_{n+2},$$

wo

$$u_1^2 + u_2^2 + \cdots + u_{n+2}^2 \leqq r^2,$$

durchläuft eine $(n+2)$-dimensionale Kugel vom Radius r, die wir mit $\mathfrak{K}$ bezeichnen. Ihr Rand $\mathfrak{R}$ werde durch die Äquatorkugelfläche

$$u_1^2 + u_2^2 + \cdots + u_{n+1}^2 = r^2, \quad u_{n+2} = 0,$$

die wir mit $\mathfrak{T}$ bezeichnen, in zwei Teile $\mathfrak{S}_1$ und $\mathfrak{S}_2$ zerlegt. Dann folgt aus dem Stokesschen Satze, daß

$$\begin{aligned}\int\int\cdots\int_{\mathfrak{T}} & F(P, d_1P, d_2P, \cdots, d_nP) \\ &= \int\int\cdots\int_{\mathfrak{S}_1} \mathfrak{d}F(P, d_1P, d_2P, \cdots, d_nP;\ d_{n+1}P) \\ &= -\int\int\cdots\int_{\mathfrak{S}_2} \mathfrak{d}F(P, d_1P, d_2P, \cdots, d_nP;\ d_{n+1}P),\end{aligned}$$

[17] Siehe [4], S. 86.

woraus

$$\int\int\cdots\int_{\mathfrak{R}} \mathfrak{d}F(P, d_1P, d_2P, \cdots, d_nP;\ d_{n+1}P) = 0.$$

Verwendet man noch einmal den Stokesschen Satz, so folgt:

$$\int\int\cdots\int_{\mathfrak{R}} \mathfrak{d}\mathfrak{d}F(P, d_1P, d_2P, \cdots, d_nP;\ d_{n+1}P;\ d_{n+2}P) = 0$$

oder

$$\int\int\cdots\int_{K_{n+2}} \mathfrak{d}\mathfrak{d}F(P_0 + u_1X_1 + u_2X_2 + \cdots + u_{n+2}X_{n+2},\ X_1,\ X_2, \cdots \cdots, X_n;\ X_{n+1};\ X_{n+2})\, du_1\, du_2 \cdots du_{n+2} = 0.$$

Strebt r gegen Null, so folgt endlich:

$$\mathfrak{d}\mathfrak{d}F(P_0, X_1, X_2, \cdots, X_n;\ X_{n+1};\ X_{n+2}) = 0,$$

was zu beweisen war.

Literaturverzeichnis.

[1] Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae **3** (1922), S. 133—181.

[2] Fréchet, La notion de différentielles dans l'analyse générale. Annales de l'Ecole Normale Supérieure (3) **42** (1925), S. 293—323.

[3] Graves, Riemann Integration and Taylor's Theorem in General Analysis. Transactions of the American Mathematical Society **29** (1927), S. 163—177.

[4] Volterra, Theory of Functionals and of Integral and Integro-Differential Equations. London and Glasgow (1930).

[5] Kerner, Zur Theorie der impliziten Funktionaloperationen. Studia Mathematica **3** (1931), S. 156—173.

[6] Kerner, Sur les variations faibles et fortes d'une fonctionelle. Annali di Matematica (4) **10** (1932), S. 145—164.

[7] Kerner, L'extremum dans l'espace hilbertien. Annali di Matematica (4) **10** (1932), S. 183—202.

[8] Kerner, Gewöhnliche Differentialgleichungen der allgemeinen Analysis. Prace Matematyczno-Fizyczne **40** (1932), S. 47–67.

LINIENOPERATIONEN[1].

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1. **Einleitung.** Herr Volterra hat bekanntlich in die Analysis Funktionen eingeführt, deren Wert von einer geschlossenen Kurve abhängt, die sogenannten Funktionen der Linien[2]. Und insbesondere hat er die additiven Linienfunktionen untersucht, welche Ableitungen besitzen. Die letzten sind reelle Zahlen, die den Doppelrichtungen des Raumes entsprechen und in mancher Hinsicht dem Gradienten analog sind. Derartige Untersuchungen lassen sich unmittelbar auf Räume von endlicher Dimensionenzahl und auf geschlossene Flächen und Hyperflächen ausdehnen. Nun hat Herr Volterra das Problem vorgeschlagen, seine Untersuchungen auf den Raum stetiger Funktionen zu übertragen[3].

In vorliegender Arbeit haben wir einen Versuch gemacht, dieses Problem zu lösen. Doch wollten wir uns nicht auf den Fall des Raumes stetiger Funktionen beschränken. Die Beziehungen sind gar nicht verwickelter im Falle eines allgemeineren *Banachschen* Raumes (d. h. eines linearen, normierten und vollständigen Raumes). Wir haben unseren Versuch für einen solchen Raum getan. Und noch in einer Hinsicht stellten wir uns auf einen allgemeineren Grund. Die Funktionen von Herrn Volterra sollen reelle Zahlen sein. Wir setzen voraus, daß die Funktionswerte selbst Elemente[4] eines (desselben oder eines anderen) Banachschen Raumes sind. Deshalb sprechen wir von *Linienoperationen*, *Flächenoperationen* und *Hyperflächenoperationen*.

Im letzten Paragraph wollen wir jedoch etwas näher den Fall des Raumes der stetigen Funktionen und der reellen Werte von Operationen (die man deshalb Funktionale zu nennen pflegt) entwickeln. Dabei werden die unmittelbaren Verallgemeinerungen der Volterraschen Ableitungen definiert, während im allgemeinen Falle an ihre Stelle die Differentiale treten.

Dem Titel gemäß beschränken wir uns hauptsächlich auf den Fall der Linienoperationen. Was die n-dimensionalen Hyperflächenoperationen betrifft, so haben wir nur die Unterschiede und Besonderheiten am Ende jedes Paragraphen unterstrichen (siehe die Abschnitte „Der allgemeine Fall").

Manche Probleme der vorliegenden Arbeit sind Problemen über die gewöhnlichen Operationen analog, die in der Arbeit über Differentiale

[1] Received August 5, 1932.

[2] Rendiconti della Reale Accademia dei Lincei (4) 3 (1887).

[3] *Theory of Functionals.* London and Glasgow, 1930. S. 86.

[4] Wir werden unten von *Punkten* des Banachschen Raumes sprechen, was bei geometrischer Sprechweise bequemer ist.

vom Verfasser behandelt wurden[5]. Wir benutzen unten manche Ergebnisse jener Arbeit, und insbesondere diese des zweiten Kapitels, die den Satz von Stokes betreffen. Wir haben in §§ 2—4 der Bequemlichkeit halber einen kurzen Auszug aus diesen Ergebnissen gemacht und sie ein wenig unseren Zwecken angepaßt. Doch verzichten wir auf strenge Beweise der etwas veränderten Behauptungen, weil diese Beweise ganz analog den in jener Arbeit vorhandenen verlaufen. Auch die Definition des einfachen Zusammenhanges unterscheidet sich von der dortigen. Das führt uns schneller zum Ziel, und topologische Untersuchungen sind keineswegs ein Gegenstand der vorliegenden Arbeit.

Wir bezeichnen mit kleinen Buchstaben die reellen Zahlen, und mit großen die Elemente (Punkte) eines Banachschen Raumes.

2. **Differentiale.** Eine Operation $F(P, X_1, X_2, \cdots, X_n)$, die den Punkten $P, X_1, X_2, \cdots, X_n$ eines Banachschen Raumes ein Element F desselben oder auch eines anderen Raumes zuordnet, heißt ein *n-faches Differential*, wenn es in bezug auf $X_1, X_2, \cdots, X_n$ n-linear ist. Ein Differential heißt *stetig*, wenn es in bezug auf P stetig ist. Ein stetiges Differential ist in bezug auf alle Veränderlichen zusammen stetig[6]. Ein Differential heißt *symmetrisch* (*schiefsymmetrisch*), wenn es in bezug auf $X_1, X_2, \cdots, X_n$ symmetrisch (schiefsymmetrisch) ist. Ein Differential kann selbst ein Differential in bezug auf P besitzen, das wir mit $dF(P, X_1, X_2, \cdots, X_n; X_{n+1})$ bezeichnen und einfach das Differential des gegebenen Differentials nennen.

Für ein schiefsymmetrisches Differential (der Fall $n = 1$ werde immer als schiefsymmetrisch gerechnet) wird ein Operator, das *Rotational*

$$\mathfrak{d}F(P, X_1, X_2, \cdots, X_n; X_{n+1})$$

definiert, der aus dem Differential $dF(P, X_1, X_2, \cdots, X_n; X_{n+1})$ dadurch entsteht, daß man darin $n+1$ Permutationen von $X_1, X_2, \cdots, X_n, X_{n+1}$ ausübt, in denen der letzte Punkt genau einmal jeden Wert $X_1, X_2, \cdots, X_n, X_{n+1}$ durchläuft, dann das so permutierte Differential mit dem Zeichen $+$ oder $-$ versieht, je nachdem die Permutation gerade oder ungerade ist, und endlich alle erhaltenen Ausdrücke addiert. Es ist beispielsweise für $n = 1$

$$\mathfrak{d}F(P, X; Y) = dF(P, X; Y) - dF(P, Y; X),$$

und für $n = 2$

$$\mathfrak{d}F(P, X, Y; Z) = dF(P, X, Y; Z) + dF(P, Y, Z; X) + dF(P, Z, X; Y).$$

3. **Stückweise-glatte Gebilde.** Es sei Γ_0 ein Punkt eines n-dimensionalen Euklidischen Raumes, der durch ein kartesisches Koordinatensystem $(x_1, x_2, \cdots, x_n)$ gerichtet ist. Es sei weiter $P(\Gamma)$ eine Operation, die

[5] Vorangehende Arbeit dieses Bandes.

[6] Loc. cit. unter [5], S. 548.

jedem Punkte Γ einer Umgebung von Γ_0 einen Punkt P des von uns betrachteten Banachschen Raumes zuordnet.

Wir nennen die Operation $P(\Gamma)$ glatt für $\Gamma = \Gamma_0$, wenn sie in einer Umgebung von Γ_0 ein Differential $dP(\Gamma; \Delta)$ besitzt, das für $\Gamma = \Gamma_0$ stetig ist. Geht man zur Koordinatendarstellung über, so soll die Funktion $P(x_1, x_2, \cdots, x_n)$ in der Umgebung von Γ_0 Ableitungen $\partial P / \partial x_i$ $(i = 1, 2, \cdots, n)$ besitzen, die für Γ_0 stetig sind.

Ein Bild eines Simplex eines n-dimensionalen Raumes mittels einer glatten Operation werden wir ein glattes n-dimensionales Hyperflächenelement nennen. Es kann auch auf einen Punkt zusammenschrumpfen.

Mit $\mathfrak{K}_n$ bezeichnen wir die n-dimensionale Einheitskugel

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leqq 1,$$

und mit $\mathfrak{R}_{n-1}$ ihren Rand

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

Der Sinn von $\mathfrak{R}_{n-1}$ wird dadurch festgelegt, daß er mit der äußeren Normalen der Kugel zusammen den positiven Sinn des Raumes bestimmt[7]. Ein $\mathfrak{K}_n$, wie auch ein $\mathfrak{R}_{n-1}$, kann in eine endliche Anzahl von Teilen zerlegt werden, die eineindeutige Bilder von Simplexen mittels glatter Operationen sind, und die wir auch Simplexe nennen werden; der Rand zweier aneinanderliegenden Simplexe soll dabei zu den beiden gerechnet werden.

Ist eine Operation $P(\Gamma)$ für alle Punkte der Kugel $\mathfrak{K}_n$ stetig, und läßt sich die Kugel $\mathfrak{K}_n$ in eine endliche Anzahl von Simplexen so zerlegen, daß für jeden Simplex $P(\Gamma)$ glatt ist, so heiße das Bild von $\mathfrak{K}_n$ mittels $P(\Gamma)$ ein *stückweise-glattes n-dimensionales Hyperflächenstück* (kurz: st.-gl. n-dimensionales Hpflstück).

Ein ebensolches Bild von $\mathfrak{R}_{n-1}$ heiße eine *stückweise-glatte* $(n-1)$*-dimensionale geschlossene Hyperfläche* (kurz: st.-gl. $(n-1)$-dimensionale gschl. Hpfläche).

Beide Gebilde sind nach der Definition aus einer endlichen Anzahl glatter n- oder $(n-1)$-dimensionaler Hpflächenelemente zusammengesetzt. Unter dem *Rande* eines st.-gl. n-dimensionalen Hpflstückes, das mittels $P(\Gamma)$ auf die Kugel $\mathfrak{K}_n$ abgebildet ist, verstehe man das Bild von $\mathfrak{R}_{n-1}$ mittels derselben Operation $P(\Gamma)$, also eine st.-gl. $(n-1)$-dimensionale gschl. Hpfläche. Diese läßt sich in zwei st.-gl. $(n-1)$-dimensionale Hpflstücke zerlegen, welche auf zwei einfach zusammenhängende Teile von $\mathfrak{R}_{n-1}$ abgebildet sind.

Die Definitionen der Hpflächen umfassen selbstverständlich die Kurven und Flächen. Insbesondere werden wir hauptsächlich mit den st.-gl. gschl.

[7] Loc. cit. unter [5], S. 561–562.

Kurven zu tun haben. Diese bestehen aus einer endlichen Anzahl aufeinanderfolgender glatter Bogen.

4. Invariantes Hyperflächenintegral und Satz von Stokes. Es sei nun $\mathfrak{R}$ ein reguläres[8] n-dimensionales Hpflstück, das mittels einer Funktion $P(x_1, x_2, \cdots, x_n)$ auf $\mathfrak{K}_n$ regulär abgebildet wird. Es sei weiter $F(P, X_1, X_2, \cdots, X_n)$ ein für alle Punkte P von $\mathfrak{R}$ erklärtes, stetiges, schiefsymmetrisches n-faches Differential. Dann ist das Integral

$$\int\int\cdots\int_{\mathfrak{K}_n} F(P, P_{x_1}, P_{x_2}, \cdots, P_{x_n})\, dx_1\, dx_2 \cdots dx_n$$

von der Parameterwahl unabhängig und wird mit

$$\int\int\cdots\int_{\mathfrak{R}} F(P, d_1 P, d_2 P, \cdots, d_n P)$$

bezeichnet. In derselben Weise läßt sich dieses Hyperflächenintegral für ein glattes Hpflächenelement und dann durch Addition für st.-gl. Hpflstücke definieren.

Ist $\mathfrak{R}$ ein st.-gl. n-dimensionales Hpflstück und $\mathfrak{R}$ sein (st.-gl. $(n-1)$-dimensionaler geschlossener) Rand, ist weiter $F(P, X_1, X_2, \cdots, X_{n-1})$ ein für alle Punkte P von $\mathfrak{R}$ erklärtes, schiefsymmetrisches $(n-1)$-faches Differential, das selbst ein stetiges Differential $dF(P, X_1, X_2, \cdots, X_{n-1}; X_n)$ besitzt, so läßt sich, ähnlich wie für reguläre Gebilde, die *Stokessche Formel*

$$\begin{aligned}(1)\qquad &\int\int\cdots\int_{\mathfrak{R}} F(P, d_1 P, d_2 P, \cdots, d_{n-1} P)\\ &= \int\int\cdots\int_{\mathfrak{R}} \mathfrak{d} F(P, d_1 P, d_2 P, \cdots, d_{n-1} P; d_n P)\end{aligned}$$

beweisen. Es empfiehlt sich am besten, diese Formel für ein glattes Hpflächenelement und dann durch Addition für den allgemeinen Fall zu beweisen.

In dem (für unsere Zwecke wichtigsten) Falle $n = 2$ hat (1) eine der beiden Gestalten:

$$(2)\qquad \int_{\mathfrak{R}} F(P, dP) = \int\int_{\mathfrak{R}} \mathfrak{d} F(P, d_1 P; d_2 P),$$

$$(3)\qquad \int_{\mathfrak{R}} F(P, dP) = \int\int_{\mathfrak{R}} [dF(P, d_1 P; d_2 P) - dF(P, d_2 P; d_1 P)],$$

wo $\mathfrak{R}$ ein st.-gl. Flstück und $\mathfrak{R}$ sein Rand (eine st.-gl. gschl. Kurve) ist.

5. Additive Linienoperationen. Wir wollen eine Summe von zwei st.-gl. gschl. Kurven $\mathfrak{C}$ definieren, falls diese *mindestens einen gemeinsamen Bogen enthalten, auf dem die zu beiden Kurven gehörenden positiven Sinne entgegengesetzt sind.* Wir nennen eine *Summe von* $\mathfrak{C}$ *und* $\mathfrak{C}'$, und bezeichnen mit

$$\mathfrak{C} + \mathfrak{C}'$$

[8] Loc. cit. unter [5], S. 562.

eine (st.-gl. gschl.) *Kurve, die durch Zusammensetzung von* $\mathfrak{C}$ *und* $\mathfrak{C}'$ *nach Entfernung des Inneren eines gemeinsamen Bogens entsteht.* Diese Definition ist nicht eindeutig, weil beide Kurven mehrere gemeinsame Bogen enthalten können, und insbesondere, weil sie mit jedem Bogen auch jeden Teilbogen gemeinsam haben.

Wir definieren eine *Linienoperation* als eine *Zuordnung, in welcher jeder st.-gl. gschl. Kurve* $\mathfrak{C}$ *ein Punkt eines,* im allgemeinen anderen, *Banachschen Raumes entspricht.* Wir bezeichnen sie mit

$$F(\mathfrak{C}).$$

Die Kurve $\mathfrak{C}$ soll dabei einem bestimmten Bereiche $\mathfrak{B}$ angehören, der auch mit dem ganzen Raume zusammenfallen kann. Ist der Raum der zugeordneten Punkte eine Zahlenachse, so spricht man auch von einem *Linienfunktional.*

Definition. *Ist für jede Bestimmung von*

$$\mathfrak{C} = \mathfrak{C}' + \mathfrak{C}''$$

immer

$$F(\mathfrak{C}) = F(\mathfrak{C}') + F(\mathfrak{C}''),$$

so heißt die Operation $F(\mathfrak{C})$ *additiv.*

Man beweist leicht, daß

$$F(\mathfrak{C}'') = -F(\mathfrak{C}'),$$

falls $\mathfrak{C}'$ und $\mathfrak{C}''$ sich untereinander nur dem Sinne nach unterscheiden, und daß

$$F(\mathfrak{C}) = 0,$$

falls $\mathfrak{C}$ auf einen Punkt zusammenschrumpft.

Es seien hinfort alle Linienoperationen additiv, wenn auch dies nicht explizit betont wird.

Der allgemeine Fall. Die Definition der Summe läßt sich ohne weiteres auf st.-gl. n-dimensionale gschl. Hpflächen ausdehnen. Nur soll man den gemeinsamen Bogen durch ein gemeinsames Hpflstück ersetzen. Die Definition der *Hyperflächenoperation* und des *Hyperflächenfunktionals,* wie auch der Additivität bietet keine Schwierigkeiten.

6. **Stetige Linienoperationen.** Es sei P_0 ein Punkt des Raumes. Eine glatte Funktion $P(u, v)$, die den Wertepaaren (u, v) einer Umgebung von (u_0, v_0) Punkte des betrachteten Raumes zuordnet, und insbesondere dem Wertepaare (u_0, v_0) den Punkt P_0, heiße ein *zweidimensionales Netz durch* P_0. Ein Netz bestimmt eine glatte Fläche durch P_0, doch kann dieselbe Fläche verschiedenen Netzen entsprechen.

Es seien zwei Wertepaare (u_1, v_1) und (u_2, v_2) in einer Umgebung von (u_0, v_0) gegeben. Die Kurven des Netzes $P(u, v)$, die durch Gleichungen

$u = u_1$, $u = u_2$, $v = v_1$, $v = v_2$ bestimmt sind, bilden ein krummliniges Parallelogramm, das wir *Parallelogramm des Netzes* nennen, und mit $\mathfrak{P}$ bezeichnen. Die Punkte $P(u, v)$, für welche u zwischen u_1 und u_2, und v zwischen v_1 und v_2 enthalten ist, bilden ein glattes Flstück, das wir *Parallelogrammfläche des Netzes* nennen, und mit $\mathfrak{Q}$ bezeichnen. Der positive Sinn von $\mathfrak{Q}$ sei durch (u, v) bestimmt, falls

$$(u_2 - u_1)(v_2 - v_1) > 0, \tag{4}$$

und durch (v, u) im entgegengesetzten Falle. Der Sinn von $\mathfrak{P}$ sei dadurch bestimmt, daß es der Rand von $\mathfrak{Q}$ sein soll. In dem durch (4) bestimmten Falle ist die Reihenfolge der Parallelogrammecken:

$$P(u_1, v_1),\ P(u_1, v_2),\ P(u_2, v_2),\ P(u_2, v_1).$$

Gehört P_0 dem Definitionsbereiche einer Linienoperation $F(\mathfrak{C})$ an, so entspricht jedem Parallelogramm des Netzes durch P_0 ein bestimmter Wert von $F(\mathfrak{C})$.

Definition. *Eine in einer Umgebung von P_0 erklärte, additive Linienoperation $F(\mathfrak{C})$ heißt im Punkte P_0 stetig, wenn für jedes zweidimensionale Netz $P(u, v)$ durch P_0 der Wert $F(\mathfrak{P})$ gegen Null konvergiert, falls (u_1, v_1) und (u_2, v_2) gegen (u_0, v_0) streben.*

Der allgemeine Fall. Das zweidimensionale Netz wird durch ein $(n+1)$-dimensionales, und das Parallelogramm durch den Rand eines $(n+1)$-dimensionalen Parallelepipeds ersetzt.

7. Das Differential einer Linienoperation. Wir führen diesen Hauptbegriff ein durch die

Definition. *Ist $F(\mathfrak{C})$ eine in einer Umgebung von P_0 erklärte, additive Linienoperation, und gibt es eine bilineare Operation $\Phi(X, Y)$ derart, daß für jedes zweidimensionale Netz $P(u, v)$ durch P_0*

$$\frac{F(\mathfrak{P})}{(u_2 - u_1)(v_2 - v_1)} \to \Phi[P_u(u_0, v_0),\ P_v(u_0, v_0)], \tag{5}$$

wenn (u_1, v_1) und (u_2, v_2) gegen (u_0, v_0) streben,

so heißt $\Phi(X, Y)$ das Differential von $F(\mathfrak{C})$ im Punkte P_0.

Es hängt von P_0 ab und wird mit

$$dF(P_0; X, Y)$$

bezeichnet. Wir haben solche Operationen zweifache Differentiale genannt.

Wir lassen den Beweis aus, daß eine Linienoperation in P_0 stetig ist, falls sie in P_0 ein Differential besitzt.

Das *Differential $dF(P_0; X, Y)$ ist schiefsymmetrisch.* In der Tat, führt man das Netz $P'(u, v) = P(v, u)$ ein, und hält man (u_1, v_1) und (u_2, v_2)

fest, so wechselt $\mathfrak{P}$ nur seinen Sinn und $F(\mathfrak{P})$ das Zeichen. Aus (5) folgt, daß

$$\Phi(P_v, P_u) = -\Phi(P_u, P_v).$$

Der allgemeine Fall. Das Differential einer n-dimensionalen Hpflächenoperation ist ein $(n+1)$-faches schiefsymmetrisches Differential.

Beispiel. Ist $\Psi(P, X)$ ein einfaches Differential, das selbst ein stetiges Differential $d\Psi(P, X; Y)$ besitzt, so ist das invariante Integral

$$F(\mathfrak{C}) = \int_{\mathfrak{C}} \Psi(P, dP) \tag{6}$$

längs einer st.-gl. gschl. Kurve $\mathfrak{C}$ eine additive Linienoperation. Aus der Stokesschen Formel (2) folgt für ein Parallelogramm $\mathfrak{P}$ des Netzes durch P_0, daß

$$F(\mathfrak{P}) = \int_{\mathfrak{P}} \Psi(P, dP) = \int\int_{\Omega} \mathfrak{d}\Psi(P, d_1P; d_2P)$$
$$= \int_{v_1}^{v_2} dv \int_{u_1}^{u_2} \mathfrak{d}\Psi(P, P_u; P_v)\, du,$$

woraus nach leichter Überlegung[9] folgt, daß $F(\mathfrak{C})$ ein Differential besitzt, und daß

$$dF(P_0; X, Y) = \mathfrak{d}\Psi(P_0, X; Y). \tag{7}$$

Im Falle eines vielfachen Integrals

$$F(\mathfrak{C}) = \int\int\cdots\int_{\mathfrak{C}} \Psi(P, d_1P, d_2P, \cdots, d_nP)$$

nimmt die Formel (7) die Gestalt an:

$$dF(P_0; X_1, X_2, \cdots, X_n, X_{n+1}) = \mathfrak{d}\Psi(P_0, X_1, X_2, \cdots, X_n; X_{n+1}).$$

8. **Variation einer Linienoperation.** Es sei eine st.-gl. gschl. Kurve $\mathfrak{C}_0$ in eine einparametrige Schar ebensolcher Kurven $\mathfrak{C}$, die vom Parameter α abhängen, eingebettet. Wir machen dabei folgende Voraussetzungen:

1° *für α, die einem bestimmten Werte α_0 hinreichend nahe sind, ist die gschl. Kurve $\mathfrak{C}$ aus endlich vielen Bogen*

$$P^{(i)}(u, \alpha), \qquad 0 \leqq u \leqq 1, \qquad i = 1, 2, \cdots, m$$

zusammengesetzt,

2° *für $\alpha = \alpha_0$ fällt $\mathfrak{C}$ mit $\mathfrak{C}_0$ zusammen,*

3° *für hinreichend kleine Werte von $(\alpha - \alpha_0)$ und für $0 \leqq u \leqq 1$ sind alle Funktionen $P^{(i)}(u, \alpha)$ glatt.*

Damit ist durch jeden Punkt der Kurve $\mathfrak{C}_0$ ein Netz $P^{(i)}(u, \alpha)$ gelegt, dem die Kurve selbst angehört. Den Grenzpunkten zweier Nachbarbogen von $\mathfrak{C}_0$ entsprechen zwei Netze.

[9] Vgl. den expliziten Beweis in § 11.

Satz. *Liegt $\mathfrak{C}_0$ im Definitionsbereiche einer additiven Linienoperation $F(\mathfrak{C})$, welche ein stetiges Differential $dF(P; X, Y)$ besitzt, so existiert die erste Variation von $F(\mathfrak{C})$ längs $\mathfrak{C}_0$ für die oben bestimmte Schar $\mathfrak{C}$; diese Variation ist gleich*

$$\left[\frac{dF(\mathfrak{C})}{d\alpha}\right]_{\alpha=\alpha_0} = \sum_{i=1}^{m}\int_0^1 dF[P^{(i)}(u, \alpha_0);\ P_u^{(i)}(u, \alpha_0),\ P_\alpha^{(i)}(u, \alpha_0)]\,du = \int_{\mathfrak{C}_0} dF(P; dP, P_\alpha). \tag{8}$$

Beweis. Es sei $(\alpha - \alpha_0)$ hinreichend klein, damit $\mathfrak{C}$ im Definitionsbereiche von $F(\mathfrak{C})$ liege, und die Bedingungen 1° und 3° erfüllt werden. Für jeden Wert von $i \leqq m$ und $0 \leqq u_0 \leqq 1$ konvergiere der Ausdruck

$$\frac{F(\mathfrak{P})}{(u_2-u_1)(\alpha_2-\alpha_1)} - dF[P^{(i)}(u_0, \alpha_0);\ P_u^{(i)}(u_0, \alpha_0),\ P_\alpha^{(i)}(u_0, \alpha_0)]$$

mit (u_1-u_0), (u_2-u_0), $(\alpha_1-\alpha_0)$, $(\alpha_2-\alpha_0)$ gegen Null, wo $\mathfrak{P}$ das Parallelogramm des Netzes $P^{(i)}(u, \alpha)$ durch den Punkt $P^{(i)}(u_0, \alpha_0)$ bezeichnet, welches den Wertepaaren (u_1, α_1) und (u_2, α_2) entspricht. Wir behaupten, daß diese Konvergenz in bezug auf alle i und u_0 gleichmäßig ist. Es genügt natürlich, dies für einen bestimmten Wert von i zu beweisen.

Setzen wir im Gegenteil voraus, daß es für ein bestimmtes positives ε und für jedes positive δ Werte von $u_0, u_1, u_2, \alpha_1, \alpha_2$ gibt derart, daß

$$|u_1-u_0|<\delta, \quad |u_2-u_0|<\delta, \quad |\alpha_1-\alpha_0|<\delta, \quad |\alpha_2-\alpha_0|<\delta,$$

und trotzdem

$$\left\| \frac{F(\mathfrak{P})}{(u_2-u_1)(\alpha_2-\alpha_1)} - dF[P^{(i)}(u_0, \alpha_0);\ P_u^{(i)}(u_0, \alpha_0),\ P_\alpha^{(i)}(u_0, \alpha_0)] \right\| > \varepsilon.$$

Es sei δ_n eine gegen Null strebende Folge positiver Zahlen. Für jedes δ_n bilden wir Werte $u_0^{(n)}, u_1, u_2, \alpha_1, \alpha_2$ derart, daß

$$|u_1-u_0^{(n)}|<\delta_n, \quad |u_2-u_0^{(n)}|<\delta_n, \quad |\alpha_1-\alpha_0|<\delta_n, \quad |\alpha_2-\alpha_0|<\delta_n, \tag{9}$$

$$\left\| \frac{F(\mathfrak{P})}{(u_2-u_1)(\alpha_2-\alpha_1)} - dF[P^{(i)}(u_0^{(n)}, \alpha_0);\ P_u^{(i)}(u_0^{(n)}, \alpha_0),\ P_\alpha^{(i)}(u_0^{(n)}, \alpha_0)] \right\| > \varepsilon. \tag{10}$$

Es sei $u_0^{(0)}$ ein Häufungswert von $u_0^{(n)}$, und setzen wir der Einfachheit halber voraus, daß n nur die Teilfolge der natürlichen Zahlen durchläuft, für die

$$\lim_{n\to\infty} u_0^{(n)} = u_0^{(0)}.$$

Jetzt wählen wir eine solche positive Zahl δ_0, daß für

$$|u_1-u_0^{(0)}|<\delta_0, \quad |u_2-u_0^{(0)}|<\delta_0, \quad |\alpha_1-\alpha_0|<\delta_0, \quad |\alpha_2-\alpha_0|<\delta_0 \tag{11}$$

die Ungleichung

$$(12)\quad \left\| \frac{F(\mathfrak{P})}{(u_2-u_1)(\alpha_2-\alpha_1)} - dF[P^{(i)}(u_0^{(0)}, \alpha_0);\ P_u^{(i)}(u_0^{(0)}, \alpha_0),\ P_\alpha^{(i)}(u_0^{(0)}, \alpha_0)] \right\| < \frac{\varepsilon}{2}$$

erfüllt wird, und daß für

$$|u - u_0^{(0)}| < \delta_0$$

die Ungleichung

$$(13)\quad \begin{aligned} &\| dF[P^{(i)}(u, \alpha_0);\ P_u^{(i)}(u, \alpha_0),\ P_\alpha^{(i)}(u, \alpha_0)] \\ &- dF[P^{(i)}(u_0^{(0)}, \alpha_0);\ P_u^{(i)}(u_0^{(0)}, \alpha_0),\ P_\alpha^{(i)}(u_0^{(0)}, \alpha_0)] \| < \frac{\varepsilon}{2} \end{aligned}$$

gilt (was der Stetigkeit von $dF[P(u, \alpha_0);\ P_u(u, \alpha_0),\ P_\alpha(u, \alpha_0)]$ in bezug auf u wegen möglich ist).

Für hinreichend großes n liegt das Intervall

$$(u_0^{(n)} - \delta_n,\ u_0^{(n)} + \delta_n)$$

im Intervalle

$$(u_0^{(0)} - \delta_0,\ u_0^{(0)} + \delta_0),$$

und es ist

$$\delta_n < \delta_0.$$

Aus (9) folgt dann (11), daraus (12). Die Ungleichung (13), wo man u durch $u_0^{(n)}$ ersetzen darf, und (12) widersprechen der Ungleichung (10). Also muß die Konvergenz gleichmäßig sein.

Ist für ein ε die Zahl δ bestimmt, derart, daß für jedes $0 \leqq u_0 \leqq 1$ und jedes i aus

$$(14)\qquad |u - u_0| < \delta, \qquad |\alpha - \alpha_0| < \delta$$

die Ungleichung

$$(15)\quad \left\| \frac{F(\mathfrak{P})}{(u-u_0)(\alpha-\alpha_0)} - dF[P^{(i)}(u_0, \alpha_0);\ P_u^{(i)}(u_0, \alpha_0),\ P_\alpha^{(i)}(u_0, \alpha_0)] \right\| < \varepsilon$$

[$\mathfrak{P}$ durch (u_0, α_0) und (u, α) bestimmt] folgt, so zerlege man das Intervall $0 \leqq u \leqq 1$ in Teilintervalle

$$0 = u^{(0)} < u^{(1)} < \cdots < u^{(n)} = 1,$$

so daß für jedes j

$$u^{(j)} - u^{(j-1)} < \delta,$$

und wähle man einen Wert α, für den

$$(16)\qquad |\alpha - \alpha_0| < \delta.$$

Mit $\mathfrak{P}_j^{(i)}$ bezeichnen wir das Parallelogramm des Netzes $P^{(i)}(u, \alpha)$ durch den Punkt $P^{(i)}(u^{(j-1)}, \alpha_0)$, welches durch die Wertepaare $(u^{(j-1)}, \alpha_0)$ und $(u^{(j)}, \alpha)$ bestimmt ist. Ist $\mathfrak{C}$ die Kurve, die dem Werte α entspricht, so ist

$$(17)\qquad F(\mathfrak{C}) - F(\mathfrak{C}_0) = \sum_{i=1}^{m} \sum_{j=1}^{n} F(\mathfrak{P}_j^{(i)}).$$

Andererseits gelten für jedes $\mathfrak{P}_j^{(i)}$ die Ungleichungen (14), wo man u_0 durch $u^{(j-1)}$ und u durch $u^{(j)}$ ersetzt. Die Ungleichung (15) wird zu

$$\left\| \frac{F(\mathfrak{P}_j^{(i)})}{(u^{(j)}-u^{(j-1)})(\alpha-\alpha_0)} - dF[P^{(i)}(u^{(j-1)}, \alpha_0); P_u^{(i)}(u^{(j-1)}, \alpha_0), P_\alpha^{(i)}(u^{(j-1)}, \alpha_0)] \right\| < \varepsilon.$$

Wir multiplizieren beiderseits mit $(u^{(j)}-u^{(j-1)})$, summieren über j und i, und nehmen (17) in Betracht:

$$\left\| \frac{F(\mathfrak{C})-F(\mathfrak{C}_0)}{\alpha-\alpha_0} - \sum_{i=1}^{m}\sum_{j=1}^{n} dF[P^{(i)}(u^{(j-1)}, \alpha_0); P_u^{(i)}(u^{(j-1)}, \alpha_0), P_\alpha^{(i)}(u^{(j-1)}, \alpha_0)](u^{(j)}-u^{(j-1)}) \right\| < m\varepsilon.$$

Streben alle $(u^{(j)}-u^{(j-1)})$ gegen Null, so folgt daraus

$$\left\| \frac{F(\mathfrak{C})-F(\mathfrak{C}_0)}{\alpha-\alpha_0} - \sum_{i=1}^{m}\int_0^1 dF[P^{(i)}(u, \alpha_0); P_u^{(i)}(u, \alpha_0), P_\alpha^{(i)}(u, \alpha_0)]\,du \right\| < m\varepsilon, \tag{18}$$

weil die zu integrierende Funktion stetig ist[10]. Da m konstant, ε willkürlich ist, und α nur die Ungleichung (16) erfüllen soll, wo δ durch ε bestimmt ist, so folgt aus (18) die gewünschte Beziehung (8).

Der allgemeine Fall. Ist die st.-gl. n-dimensionale gschl. Hpfläche $\mathfrak{C}$ in eine Schar ebensolcher Gebilde eingebettet, die analoge Bedingungen wie oben erfüllen (glatte Hpflächenelemente an der Stelle von Teilbogen), so gilt die Formel

$$\left[\frac{dF(\mathfrak{C})}{d\alpha}\right]_{\alpha=\alpha_0} = \int\int\cdots\int_{\mathfrak{C}_0} dF(P; d_1P, d_2P, \cdots, d_nP, P_\alpha).$$

Der Beweis unterscheidet sich nur durch etwa mehr komplizierte Formulierungen.

9. **Ein Integralausdruck für Linienoperationen.** Ist eine st.-gl. gschl. Kurve Rand eines st.-gl. Flstückes, welches ganz im Definitionsbereiche von $F(\mathfrak{C})$ liegt, so läßt sich $F(\mathfrak{C})$ einfach mittels eines Integrals darstellen. Es gilt nämlich der

Satz. *Liegt ein st.-gl. Flstück $\mathfrak{K}$ im Definitionsbereiche einer additiven Linienoperation $F(\mathfrak{C})$, welche ein stetiges Differential $dF(P; X, Y)$ besitzt, ist $\mathfrak{C}$ der Rand von $\mathfrak{K}$, so gilt die Formel*

$$F(\mathfrak{C}) = \int\int_{\mathfrak{K}} dF(P; d_1P, d_2P). \tag{19}$$

[10] Vgl. Kerner, *Prace matematyczno-fizyczne* **40** (1932), S. 54.

Beweis. Es genügt die Formel (19) im Falle zu beweisen, wo $\mathfrak{R}$ ein glattes Dreieck (Sonderfall des glatten Hpflächenelementes) ist. Der allgemeine Fall folgt durch Addition.

Es sei $\mathfrak{R}$ auf ein ebenes Dreieck $\mathfrak{K}$ mittels einer glatten Operation abgebildet. Mit $\mathfrak{C}$ bezeichne man den Rand von $\mathfrak{K}$, dem der Rand $\mathfrak{C}$ entspricht. Wählen wir einen inneren Punkt A von $\mathfrak{K}$ und bilden wir $\mathfrak{C}$ in bezug auf A homothetisch im Verhältnis α ab. Wächst α von 0 bis 1, so durchläuft das Bild $\mathfrak{C}_\alpha$ von $\mathfrak{C}$ eine einparametrige Schar von Dreiecken, die $\mathfrak{K}$ überdeckt; $\mathfrak{C}_0$ fällt mit dem Punkt A zusammen, und $\mathfrak{C}_1$ mit dem Dreieck $\mathfrak{C}$. Auf $\mathfrak{R}$ entsteht durch Abbildung eine Schar $\mathfrak{C}_\alpha$ von st.-gl. gschl. Kurven, welche für jedes $0 \leqq \alpha \leqq 1$ die Bedingungen des § 8 erfüllen. Für $\alpha = 0$ reduziert sich $\mathfrak{C}_\alpha$ auf einen Punkt (Bild von A) und für $\alpha = 1$ ist $\mathfrak{C}_1 = \mathfrak{C}$. Nach Formel (8) erhalten wir:

$$\frac{dF(\mathfrak{C}_\alpha)}{d\alpha} = \int_{\mathfrak{C}_\alpha} dF(P; dP, P_\alpha).$$

Integriert man von 0 bis 1 und nimmt man in Betracht, daß

$$F(\mathfrak{C}_0) = 0, \quad F(\mathfrak{C}_1) = F(\mathfrak{C}),$$

so folgt

$$F(\mathfrak{C}) = \int_0^1 d\alpha \int_{\mathfrak{C}_\alpha} dF(P; dP, P_\alpha),$$

was mit Formel (19) gleichbedeutend ist.

Man kann diese Formel auch unmittelbar, in ähnlicher Weise wie (8), beweisen. Sie ist eine Verallgemeinerung der Stokesschen Formel (2), die man aus (19) erhalten kann, wenn man als $F(\mathfrak{C})$ die Linienoperation (6) wählt und (7) in Betracht nimmt.

Der allgemeine Fall. Ist $\mathfrak{C}$ n-dimensional und $\mathfrak{R}$ $(n+1)$-dimensional, so verwandelt sich die Formel (19) in

$$F(\mathfrak{C}) = \int\int \cdots \int_{\mathfrak{R}} dF(P; d_1 P, d_2 P, \cdots, d_n P, d_{n+1} P).$$

10. **Das zweite Differential einer Linienoperation.** Besitzt das Differential $dF(P; X, Y)$ einer Linienoperation $F(\mathfrak{C})$ ein Differential, so nennen wir das letzte das *zweite Differential* von $F(\mathfrak{C})$ und bezeichnen es mit

$$d^2F(P; X, Y; Z).$$

Satz. *Besitzt eine additive Linienoperation $F(\mathfrak{C})$ in einer Umgebung von P_0 ein stetiges zweites Differential $d^2F(P; X, Y; Z)$, so ist*

$$\mathfrak{d}\, dF(P_0; X, Y; Z) = 0, \tag{20}$$

oder

$$d^2F(P_0; X, Y; Z) + d^2F(P_0; Y, Z; X) + d^2F(P_0; Z, X; Y) = 0.$$

Beweis. Sind X, Y, Z gegeben, so bilde man eine dreidimensionale Hyperebene

$$P = P_0 + xX + yY + zZ. \tag{21}$$

Ist r genügend klein, so liegt das Bild $\mathfrak{K}$ der Kugel $\mathfrak{K}_3$:

$$x^2 + y^2 + z^2 \leqq r^2,$$

mittels (21) in der erwähnten Umgebung von P_0. Ist $\mathfrak{R}$ der Rand von $\mathfrak{K}$, so folgt aus der Stokesschen Formel (1):

$$\int\int_{\mathfrak{R}} dF(P; d_1 P, d_2 P) = \int\int\int_{\mathfrak{K}} \mathfrak{d}\, dF(P; d_1 P, d_2 P; d_3 P). \tag{22}$$

Es sei $\mathfrak{C}$ das Bild des Kreises

$$x^2 + y^2 = r^2, \qquad z = 0$$

mittels (21). Die geschlossene Kurve (Kreis) $\mathfrak{C}$ zerschneidet $\mathfrak{R}$ in zwei Teile: $\mathfrak{R}_1$ (wo $z > 0$) und $\mathfrak{R}_2$ (wo $z < 0$); $\mathfrak{C}$ bildet den Rand der beiden Teile, doch muß man im Falle von $\mathfrak{R}_2$ den Sinn von $\mathfrak{C}$ umkehren. Nach der Formel (19) schreibt man:

$$F(\mathfrak{C}) = \int\int_{\mathfrak{R}_1} dF(P; d_1 P, d_2 P) = -\int\int_{\mathfrak{R}_2} dF(P; d_1 P, d_2 P),$$

woraus das Verschwinden von (22) folgt. Also ist

$$\int\int\int_{\mathfrak{K}_3} \mathfrak{d}\, dF(P_0 + xX + yY + zZ; X, Y; Z)\, dx\, dy\, dz = 0 \tag{23}$$

für jedes hinreichend kleine r. Wäre (20) nicht erfüllt, so würde (23) für kleine r zu einem Widerspruch führen.

Der allgemeine Fall. Ist $\mathfrak{C}$ n-dimensional, so lautet die Bedingung (20):

$$\mathfrak{d}\, dF(P_0; X_1, X_2, \cdots, X_n, X_{n+1}; X_{n+2}) = 0.$$

11. **Integration eines Differentials.** Wir wenden uns einem, so zu sagen, inversen Problem zu. Es ist ein schiefsymmetrisches Differential $\Phi(P, X, Y)$ gegeben, und es wird gefragt, ob es eine Linienoperation gibt, deren Differential gleich $\Phi(P, X, Y)$ ist. Wir führen dazu den Begriff des einfachen Zusammenhanges ein:

Ein Bereich $\mathfrak{B}$ heiße n-dimensional einfach zusammenhängend, wenn jede st.-gl. n-dimensionale gschl. Hpfläche in $\mathfrak{B}$ ein Rand eines st.-gl. $(n+1)$-dimensionalen, ganz in $\mathfrak{B}$ gelegenen, Hpflstückes ist. Im Falle $n = 1$ und $n = 2$ sprechen wir auch von einfach linien- und flächenzusammenhängenden Bereichen.

Satz. *Besitzt das schiefsymmetrische Differential $\Phi(P, X, Y)$ in einem einfach linien- und flächenzusammenhängenden Bereiche $\mathfrak{B}$ ein stetiges Differential $d\Phi(P, X, Y; Z)$, und ist*

$$\mathfrak{d}\,\Phi(P, X, Y; Z) = 0, \tag{24}$$

so gibt es genau eine additive Linienoperation $F(\mathfrak{C})$, für welche

$$dF(P; X, Y) = \Phi(P, X, Y). \tag{25}$$

Der Satz des § 10 hat uns gelehrt, daß (24) eine notwendige Bedingung der Lösbarkeit von (25) in bezug auf $F(\mathfrak{C})$ ist. Der ausgesprochene Satz behauptet die Hinlänglichkeit der Bedingung (24).

Beweis. Wir beginnen mit der Eindeutigkeit der gesuchten Linienoperation $F(\mathfrak{C})$. Zu diesem Zwecke setzen wir voraus, daß es eine Linienoperation $F(\mathfrak{C})$ gibt, welche (25) erfüllt. Es sei $\mathfrak{C}$ eine st.-gl. gschl. Kurve in $\mathfrak{B}$. Da der Bereich $\mathfrak{B}$ einfach linienzusammenhängend ist, so läßt sich in $\mathfrak{B}$ ein st.-gl. Flstück $\mathfrak{R}$ bilden, das $\mathfrak{C}$ zum Rand besitzt. Dann aber muß nach der Formel (19), wegen (25), der Ausdruck gelten:

$$F(\mathfrak{C}) = \int\int_{\mathfrak{R}} \Phi(P, d_1 P, d_2 P), \tag{26}$$

was die Eindeutigkeit von $F(\mathfrak{C})$, wenn es existiert, beweist.

Jetzt wollen wir die Existenz von $F(\mathfrak{C})$ beweisen. Dazu zeigen wir, daß (26) wirklich eine additive Linienoperation definiert, und daß diese die Beziehung (25) erfüllt.

In der Tat sei $\mathfrak{C}$ eine st.-gl. gschl. Kurve und $\mathfrak{R}$ ein st.-gl. Flstück, das $\mathfrak{C}$ zum Rand hat. Die Formel (26) definiert ein Element F. Ist $\mathfrak{R}_1$ ein anderes st.-gl. Flstück, das $\mathfrak{C}$ zum Rand hat, so bilden $\mathfrak{R}$ und $\mathfrak{R}_1$ zusammen, wenn man auf $\mathfrak{R}_1$ den positiven Sinn umkehrt, eine st.-gl. gschl. Fläche in $\mathfrak{B}$. Da $\mathfrak{B}$ auch einfach flächenzusammenhängend ist, so gibt es ein st.-gl. dreidimensionales Hpflstück $\mathfrak{S}$, das diese Fläche zum Rand hat. Die Stokessche Formel (1) liefert:

$$\int\int_{\mathfrak{R}} \Phi(P, d_1 P, d_2 P) - \int\int_{\mathfrak{R}_1} \Phi(P, d_1 P, d_2 P) = \int\int\int_{\mathfrak{S}} \mathfrak{d}\,\Phi(P, d_1 P, d_2 P; d_3 P),$$

oder nach (24):

$$\int\int_{\mathfrak{R}} \Phi(P, d_1 P, d_2 P) = \int\int_{\mathfrak{R}_1} \Phi(P, d_1 P, d_2 P).$$

Also bestimmt (26) eine Linienoperation, die nur von $\mathfrak{C}$, nicht aber von $\mathfrak{R}$, abhängt. Die Additivität folgt ohne weiteres.

Es bleibt nur für (26) die Formel (25) zu beweisen. Es sei wirklich P_0 ein beliebiger Punkt von $\mathfrak{B}$, $P(u, v)$ ein Netz durch P_0, das noch in $\mathfrak{B}$ liegt, $\mathfrak{P}$ ein durch Wertepaare (u_1, v_1) und (u_2, v_2) bestimmtes Parallelogramm des Netzes, Ω die entsprechende Parallelogrammfläche. Es sei der Bequemlichkeit halber $u_1 < u_2$ und $v_1 < v_2$. Aus (26) folgt:

$$F(\mathfrak{P}) = \int\int_{\Omega} \Phi(P, d_1 P, d_2 P) = \int_{v_1}^{v_2} dv \int_{u_1}^{u_2} \Phi[P(u, v), P_u(u, v), P_v(u, v)]\, du. \tag{27}$$

Es sei ε eine willkürliche positive Zahl. Wählen wir (u_1, v_1) und (u_2, v_2) dem dem Punkte P_0 entsprechenden Wertepaare (u_0, v_0) hinreichend nah, damit die Ungleichung

$$\| \Phi[P(u, v), P_u(u, v), P_v(u, v)] - \Phi[P(u_0, v_0), P_u(u_0, v_0), P_v(u_0, v_0)] \| < \varepsilon \tag{28}$$

für alle $u_1 \leqq u \leqq u_2$, $v_1 \leqq v \leqq v_2$ gelte. Die Möglichkeit dieser Wahl folgt aus der Stetigkeit von $\Phi[P(u, v), P_u(u, v), P_v(u, v)]$. Aus (27) und (28) schließt man, daß

$$\left\| F(\mathfrak{P}) - \int_{v_1}^{v_2} dv \int_{u_1}^{u_2} \Phi[P(u_0, v_0), P_u(u_0, v_0), P_v(u_0, v_0)]\, du \right\| < \varepsilon (u_2 - u_1)(v_2 - v_1),$$

oder

$$\left\| \frac{F(\mathfrak{P})}{(u_2 - u_1)(v_2 - v_1)} - \Phi[P(u_0, v_0), P_u(u_0, v_0), P_v(u_0, v_0)] \right\| < \varepsilon,$$

woraus die behauptete Eigenschaft (25) folgt.

Bemerkung. Ist der Bereich $\mathfrak{B}$ nicht einfach linienzusammenhängend, so wird die Linienoperation $F(\mathfrak{C})$ nur für solche st.-gl. gschl. Kurven erklärt, die einen Rand eines st.-gl. Flstückes in $\mathfrak{B}$ bilden.

Der allgemeine Fall. Der Satz fordert, daß der Bereich $\mathfrak{B}$ n- und $(n+1)$-dimensional einfach zusammenhängend sei. Die Bedingung (24) wird zu

$$\mathfrak{d}\, \Phi(P, X_1, X_2, \cdots, X_n, X_{n+1}; X_{n+2}) = 0.$$

12. **Der Fall eines Funktionalraumes.** Setzen wir voraus, daß der Banachsche Raum, in welchem die Linienoperationen untersucht werden, ein Funktionalraum, nämlich der Raum der stetigen Funktionen $p(t)$ im Intervalle $0 \leqq t \leqq 1$ sei. Dabei ist unter $\|p(t)\|$ das Maximum von $|p(t)|$ zu verstehen.

Es sei $f(\mathfrak{C})$ eine Linienoperation, die jeder st.-gl. gschl. Kurve eines Bereiches des Raumes eine reelle Zahl f zuordnet, also ein Linienfunktional. Dies ist der von Herrn Volterra[3] vorgeschlagene Fall. Wir setzen voraus, daß $f(\mathfrak{C})$ in einem Punkte $P = p(t)$ ein Differential

$$df(P; X, Y) = df[p(t); x(t), y(t)]$$

besitzt. Dies ist ein bilineares schiefsymmetrisches Funktional von $x(t)$ und $y(t)$. Setzen wir weiter voraus, es besitze eine normale Form[11]:

[11] Vgl. Kerner, *Annali di Matematica* (4) **10** (1932), S. 162.

(29) $$\varphi[x(t), y(t)] = \int_0^1 g(t)\, x(t)\, y(t)\, dt + \int_0^1 \int_0^1 h(s, t)\, x(s)\, y(t)\, ds\, dt,$$

wo $g(t)$ und $h(s, t)$ stetig sind. Da $\varphi[x(t), y(t)]$ schiefsymmetrisch ist, so kann man es in der Form schreiben:

$$\varphi[x(t), y(t)] = \frac{1}{2}\{\varphi[x(t), y(t)] - \varphi[y(t), x(t)]\}$$
$$= \int_0^1 \int_0^1 \frac{1}{2}[h(s, t) - h(t, s)]\, x(s)\, y(t)\, ds\, dt,$$

oder kurz

(30) $$\varphi[x(t), y(t)] = \int_0^1 \int_0^1 j(s, t)\, x(s)\, y(t)\, ds\, dt,$$

wo $j(s, t)$ eine schiefsymmetrische stetige Funktion von s und t ist. Man kann auch unmittelbar beweisen, daß (29) die Form (30) haben muß, und daß $j(s, t)$, als stetige Funktion, durch $\varphi[x(t), y(t)]$ eindeutig bestimmt ist.

Wir schreiben

(31) $$j(s, t) = f'[p(t); s, t]$$

und nennen es die *Ableitung von* $f(\mathfrak{C})$ *im Punkte* $p(t)$. Die Formel (30) nimmt die Form

$$df[p(t); x(t), y(t)] = \int_0^1 \int_0^1 f'[p(t); s, t]\, x(s)\, y(t)\, ds\, dt$$

an. Die Ableitung (31) verallgemeinert auf unendlich viele Veränderliche die Volterraschen Ableitungen einer Linienfunktion in verschiedenen Doppelrichtungen, ganz analog wie die Funktionalableitung eines Funktionals nach Herrn Volterra die gewöhnliche Ableitung einer Funktion.

Der allgemeine Fall. Die Ableitung

$$f'[p(t); t_1, t_2, \cdots, t_n, t_{n+1}]$$

ist eine schiefsymmetrische stetige Funktion von $t_1, t_2, \cdots, t_n, t_{n+1}$.

REMARQUES SUR LES FONCTIONS SOUSHARMONIQUES.[1]

PAR EDWARD SZPILRAJN.

Nous considérons dans cette note une classe de fonctions qui présente une extension naturelle de la classe des fonctions sousharmoniques au sens de M. F. Riesz.[2]

Les théorèmes que nous allons démontrer permettent de résoudre un problème posé par M. Mazurkiewicz, concernant des suites de fonctions sousharmoniques.

Notations et remarques préliminaires. 1°. Nous considérons des fonctions réelles de deux variables réelles qui admettent des valeurs finies ou infinies.

Toutes nos considérations peuvent être appliquées dans le cas de n variables réelles.

2°. Z étant un ensemble plan quelconque, $\overline{Z}$ désigne sa fermeture, CZ son complémentaire.

3°. (x, y) et (x', y') étant deux points arbitraires du plan et Z un ensemble plan quelconque, $\varrho(x, y; x', y')$ désigne la distance entre (x, y) et (x', y'), et $\varrho(x, y; Z)$ la distance entre (x, y) et Z, c.-à-d. la borne inférieure des nombres $\varrho(x, y; \xi, \eta)$ où $(\xi, \eta) \in Z$.

4°. $f(x, y)$ étant une fonction quelconque définie dans un ensemble plan Z, on désigne par $\overline{\operatorname{Max}} f(x, y)$ [pour tout point $(x, y) \in \overline{Z}$] la borne supérieure des nombres $\overline{\lim_{n\to\infty}} f(x_n, y_n)$, pour toute suite $\{(x_n, y_n)\}$ de points de Z, tendant vers (x, y).

$\operatorname{Max} f(x, y)$ est une fonction semi-continue supérieurement et on a $\operatorname{Max} f(x, y) \geqq f(x, y)$ pour tout $(x, y) \in Z$.

5°. Le cercle fermé de rayon r et de centre (x_0, y_0), c.-à-d. l'ensemble $\underset{(x,y)}{E}\,[\varrho(x_0, y_0; x, y) \leqq r]$ sera désigné par $Q(x_0, y_0; r)$.

6°. $f(\xi, \eta)$ étant une fonction sommable sur $Q(x, y; r)$, posons:

$$J_r f(x, y) = \frac{1}{\pi r^2} \iint\limits_{Q(x,y;r)} f(\xi, \eta)\, d\xi\, d\eta .$$

La fonction $J_r f(x, y)$ est — comme on le sait — une fonction continue de (x, y).

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[1] Received June 22, 1932.—Les résultats contenus dans cette note ont été présentés à la Société Polonaise de Mathématique (section de Varsovie) à la séance du 18. III. 1932.

[2] Cf. F. Riesz: *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel;* 1ère partie, Acta Mathematica **48** (1926); 2ième partie, Acta Mathematica **54** (1930).

1. M. Littlewood a démontré le théorème suivant:[3]

(*) *Pour qu'une fonction* $u(x, y)$ *soit sousharmonique dans un domaine* D, *il faut et il suffit que*

a) $u(x, y)$ *soit sommable sur tout cercle fermé contenu dans* D,

b) *pour tout* x, y *et* r *tels que* $0 < r < \varrho(x, y; CD)$, *on ait*

$$J_r u(x, y) \geqq u(x, y),$$

c) $u(x, y)$ *soit semi-continue supérieurement.*

M. Littlewood a prouvé de même que la condition b) peut être remplacée par la suivante:

b′) *pour tout* $(x, y) \in D$ *il existe des nombres* r *positifs, arbitrairement petits tels que*

$$J_r u(x, y) \geqq u(x, y).$$

Remarque. M. Saks a établi des relations entre les fonctions sousharmoniques et les propriétés d'une expression introduite par M. Blaschke.[4] Dans le même ordre d'idées, et en se servant du même raisonnement, on peut lier, à l'aide du théorème (*), les fonctions sousharmoniques à une autre expression, introduite par M. Privaloff.[5]

$f(x, y)$ étant une fonction sommable sur tout cercle fermé contenu dans le domaine D, posons d'après M. Privaloff:

$$\overline{\Delta}^* f(x, y) = \overline{\lim_{r \to 0}} \frac{8[J_r f(x, y) - f(x, y)]}{r^2}$$

et

$$\underline{\Delta}^* f(x, y) = \underline{\lim_{r \to 0}} \frac{8[J_r f(x, y) - f(x, y)]}{r^2}.$$

Or, nous allons démontrer que la condition b) du théorème (*) peut être remplacée par chacune des conditions suivantes:

$$\beta') \quad \underline{\Delta}^* u(x, y) \geqq 0 \quad \textit{pour tout} \quad (x, y) \in D;$$

$$\beta'') \quad \overline{\Delta}^* u(x, y) \geqq 0 \quad \textit{pour tout} \quad (x, y) \in D.$$

Il est évident que ces conditions résultent de b).

Admettons donc qu'une fonction $u(x, y)$ satisfasse aux conditions a), β'') et c).

[3] J. E. Littlewood. *On the definition of a subharmonic function*, Journal of the London Mathematical Society **2** (1927), p. 189, et F. Riesz l. c., 2ième partie, p. 344.

[4] S. Saks. *Sur les fonctions convexes et sousharmoniques* (en polonais) Mathesis Polska **6** (1931), p. 57; aussi: *On subharmonic functions*, Acta. Univ. Franc. Jos. Szeged **5** (1932), p. 190.

[5] J. Privalloff. *Sur les fonctions harmoniques*, Recueil de la Société Mathém. de Moscou **32** (1925).

Posons

$$u_n(x, y) = u(x, y) + \frac{x^2}{n}.$$

On a

$$\underline{\Delta}^* \frac{x^2}{n} = \overline{\Delta}^* \frac{x^2}{n} = \frac{2}{n}$$

$\left(\text{car dans ce cas } \underline{\Delta}^* = \overline{\Delta}^* = \Delta, \text{ où } \Delta f \text{ désigne } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$, donc

$$\overline{\Delta}^* u_n(x, y) > 0.$$

Il en résulte que les fonctions $u_n(x, y)$ satisfont aux conditions a), b') et c), et qu'elles sont par conséquent sousharmoniques.

La suite $u_n(x, y)$ étant uniformément convergente, $u(x, y)$ est sousharmonique.[6]

2. DÉFINITION. *Une fonction $u(x, y)$ est dite presque sousharmonique dans un domaine D lorsque*

a) *$u(x, y)$ est sommable sur tout cercle fermé, contenu dans D,*

b°) *pour presque tout point $(x, y) \in D$ et tout r tel que $0 < r < \varrho(x, y; CD)$ on a*

$$J_r u(x, y) \geqq u(x, y).$$

Nous allons démontrer le

THÉORÈME 1. *Pour qu'une fonction $u(x, y)$ soit presque sousharmonique dans D, il faut et il suffit qu'il existe une fonction $v(x, y)$ sousharmonique dans D, presque partout égale à $u(x, y)$.*

Démonstration. La suffisance de cette condition étant évidente, il ne reste qu'à en démontrer la nécessité.

Soient $u(x, y)$ une fonction presque sousharmonique dans D et D^* l'ensemble de tout les points $(x, y) \in D$ tels que pour tout r positif $< \varrho(x, y; CD)$:

$$J_r u(x, y) \geqq u(x, y).$$

On a alors $m(D - D^*) = 0$, et par conséquent $\bar{D}^* \supset D$.

Désignons par $u^*(x, y)$ la fonction $u(x, y)$ considérée seulement sur D^*, et posons pour tout $(x, y) \in D$:

$$v(x, y) = \text{Max } u^*(x, y).$$

Nous voulons démontrer que $v(x, y)$ est une fonction sousharmonique.

On a

(1) $\qquad u(x, y) \leqq v(x, y)$ *presque partout dans D.*

[6] Voir Riesz l. c., 1ère partie, p. 335.

Pour tout point $(x, y) \in D$ il existe une suite $\{(x_n, y_n)\}$ de points de D^*, tendant vers (x, y), tels que

$$\lim_{n\to\infty} u^*(x_n, y_n) = v(x, y). \tag{2}$$

Soit $0<r<\varrho(x, y; CD)$, et par conséquent $r<\varrho(x_n, y_n; CD)$ pour $n>N$. Il résulte de la définition de la fonction $u^*(x, y)$ que

$$u^*(x_n, y_n) \leqq J_r u(x_n, y_n) \text{ pour } n>N. \tag{3}$$

La fonction $J_r u(x, y)$ étant continue, on a d'après (2) et (3):

$$v(x, y) \leqq J_r u(x, y). \tag{4}$$

Dans la suite nous nous servirons de cette inégalité plusieurs fois.

Soit Q un cercle fermé contenu dans D. Pour r suffisamment petit la fonction $J_r u(x, y)$ est définie et continue dans un ensemble contenant Q. Il résulte de (1) et (4) que $v(x, y)$ est sommable sur Q.

D'autre part on a toujours, en vertu de (1),

$$J_r u(x, y) \leqq J_r v(x, y),$$

et ainsi d'après (4):

$$v(x, y) \leqq J_r v(x, y).$$

La fonction $v(x, y)$ satisfait donc aux conditions a), b) et c) et, d'après le théorème (*), elle est sousharmonique.

Nous allons démontrer à présent qu'on a presque partout $u(x, y) = v(x, y)$.

Supposons au contraire que

$$m \underset{(x,y)}{E} [u(x, y) \neq v(x, y)] > 0,$$

d'où, en vertu de (1):

$$m \underset{(x,y)}{E} [v(x, y) > u(x, y) > -\infty] > 0.$$

Il existe par conséquent un nombre $\alpha > 0$ tel que

$$m \underset{(x,y)}{E} [v(x, y) - \alpha > u(x, y) > -\infty] > 0.$$

Posons

$$H = \underset{(x,y)}{E} [v(x, y) - \alpha > u(x, y) > -\infty]. \tag{5}$$

Il résulte du théorème bien connu de M. Lebesgue que l'ensemble H contient un point de densité, soit (ξ_0, η_0).

Il existe alors un nombre $r_1 > 0$ tel que l'inégalité $0<r<r_1$ entraîne:

$$m[H \cdot Q(\xi_0, \eta_0; r)] > \frac{1}{2}\pi r^2. \tag{6}$$

D'autre part, la fonction $v(x, y)$ étant semi-continue supérieurement, il existe un nombre $r_2 > 0$ tel qu'on ait

$$v(\xi, \eta) < v(\xi_0, \eta_0) + \frac{\alpha}{2} \text{ pour } (\xi, \eta) \in Q(\xi_0, \eta_0; r_2).$$

Par conséquent l'inégalité suivante subsiste pour $0 < r < r_2$:

$$J_r v(\xi_0, \eta_0) < v(\xi_0, \eta_0) + \frac{\alpha}{2}. \tag{7}$$

Soit $0 < r < \text{Min } (r_1, r_2)$. Posons

$$K_1 = H \cdot Q(\xi_0, \eta_0; r),$$
$$K_2 = Q(\xi_0, \eta_0; r) - H.$$

On a

$$J_r v(\xi_0, \eta_0) - J_r u(\xi_0, \eta_0) = \frac{1}{\pi r^2} \int\int_{K_1} [v(x, y) - u(x, y)]\, dx\, dy + \frac{1}{\pi r^2} \int\int_{K_2} [v(x, y) - u(x, y)]\, dx\, dy.$$

Les relations (5) et (6) entraînent:

$$\frac{1}{\pi r^2} \int\int_{K_1} [v(x, y) - u(x, y)]\, dx\, dy > \frac{1}{\pi r^2} \cdot \frac{\pi r^2}{2} \cdot \alpha = \frac{\alpha}{2}$$

et l'inégalité (1) nous donne

$$\frac{1}{\pi r^2} \int\int_{K_2} [v(x, y) - u(x, y)]\, dx\, dy \geqq 0,$$

d'où il vient:

$$J_r v(\xi_0, \eta_0) - J_r u(\xi_0, \eta_0) > \frac{\alpha}{2}.$$

Il en résulte, d'après (7):

$$J_r u(\xi_0, \eta_0) < v(\xi_0, \eta_0),$$

ce qui est incompatible avec (4).

Le théorème se trouve ainsi démontré.

3. Nous ferons dans ce numéro quelques remarques concernant la notion de fonction presque sousharmonique.

1°. Nous allons démontrer d'abord que la condition b°) dans la définition de fonction presque sousharmonique *ne peut pas être remplacée* par la condition:

b*) pour presque tout point $(x, y) \in D$ il existe un nombre $R > 0$ tel que l'inégalité $0 < r < R$ entraîne

$$J_r u(x, y) \geqq u(x, y).$$

La fonction

$$u_0(x, y) = -\log \sqrt{x^2 + y^2}$$

remplit en effet les conditions a) et b*), mais elle n'est pas presque sousharmonique.

Cette fonction est d'une part sommable et harmonique dans tout le plan, le point (0,0) étant exclu. — Il en résulte immédiatement que $u_0(x, y)$ satisfait aux conditions a) et b*) (dans *tout* le plan).

Supposons d'autre part que $u_0(x, y)$ soit presque sousharmonique. Il existe, d'après le théorème 1, une fonction sousbarmonique $v(x, y)$ égale presque partout à $u_0(x, y)$. Mais $v(x, y)$ ne peut pas admettre d'infinis positifs, et en même temps elle est semi-continue supérieurement, ce qui est incompatible avec l'allure de $v(x, y)$ dans le voisinage du point (0,0).

2°. Remarquons encore, que *toute fonction presque sousharmonique n'est égale presque partout qu'à une seule fonction sousharmonique.*

A cet effet il suffit de démontrer que deux fonctions sousharmoniques $u_1(x, y)$ et $u_2(x, y)$, égales presque partout, sont identiques. On a pour tout $r > 0$:

$$J_r u_1(x, y) = J_r u_2(x, y)$$

et comme

$$\lim_{r \to 0} J_r u(x, y) = u(x, y)$$

pour toute fonction sousharmonique $u(x, y)$,[7] nous obtenons

$$u_1(x, y) = u_2(x, y).$$

3°. Nous prouvons enfin que *toute fonction $u(x, y)$, continue et presque sousharmonique dans un domaine D, est sousharmonique dans D.*

Soit $(x_0, y_0) \in D$ et $0 < r < \varrho(x_0, y_0; CD)$. La fonction continue $J_r u(x, y)$ est définie dans un ensemble K contenant un voisinage de (x_0, y_0). On a

$$J_r u(x, y) \geqq u(x, y)$$

presque partout dans K et, en vertu de la continuité de ces deux fonctions, aussi au point (x_0, y_0).

4. Une suite $\{f_n(x, y)\}$ de fonctions définies dans un domaine D et sommables dans tout cercle fermé contenu dans D est dite *convergente en moyenne* vers une fonction $f(x, y)$ lorsqu'on a pour tout cercle fermé Q, contenu dans D:

$$\lim_{n \to \infty} \int\int_Q |f_n(x, y) - f(x, y)| \, dx \, dy = 0.$$

On sait qu'il existe dans toute suite $\{f_n(x, y)\}$, convergente en moyenne vers $f(x, y)$, une suite partielle convergente (au sens ordinaire) presque partout vers $f(x, y)$.

[7] Cela résulte aisément des conditions b) et c). Cf. Riesz l. c., 2ième partie, p. 344.

THÉORÈME 2. *Si une suite* $\{u_n(x, y)\}$ *de fonctions presque sousharmoniques dans* D *converge en moyenne vers une fonction* $u(x, y)$, *cette dernière est de même presque sousharmonique.*

Démonstration. Soit D_1 l'ensemble de tous les points $(x, y) \in D$ tels que pour tout entier n et tout nombre positif $r < \varrho(x, y; CD)$ on ait

$$J_r u_n(x, y) \geqq u_n(x, y).$$

Les fonctions $u_n(x, y)$ étant presque sousharmoniques, $D - D_1$ est un ensemble de mesure nulle.

Soit à présent $\{u_{k_n}(x, y)\}$ une suite partielle de $\{u_n(x, y)\}$, convergente presque partout vers $u(x, y)$; désignons par D_2 l'ensemble des points pour lesquels $\{u_{k_n}(x, y)\}$ converge vers $u(x, y)$. On a donc: $m(D - D_2) = 0$. Posons $D^* = D_1 \cdot D_2$; il en résulte que $m(D - D^*) = 0$.

Soit $(x_0, y_0) \in D^*$ et $0 < r < \varrho(x_0, y_0; CD)$.

Nous avons d'après la définition de D^*:

$$J_r u_{k_n}(x_0, y_0) \geqq u_{k_n}(x_0, y_0) \quad \textit{pour} \quad n = 1, 2, \cdots \tag{1}$$

et

$$\lim_{n \to \infty} u_{k_n}(x_0, y_0) = u(x_0, y_0). \tag{2}$$

Mais la suite $\{u_{k_n}(x, y)\}$ étant convergente en moyenne vers $u(x, y)$, on a

$$\lim_{n \to \infty} J_r u_{k_n}(x_0, y_0) = J_r u(x_0, y_0),$$

donc, en vertu de (1) et (2),

$$J_r u(x_0, y_0) \geqq u(x_0, y_0).$$

Notre théorème est ainsi démontré.

Les Théorèmes 1 et 2 nous donnent le corollaire suivant, ne concernant que les fonctions sousharmoniques:

Si une suite $u_n(x, y)$ *de fonctions sousharmoniques converge en moyenne vers une fonction* $u(x, y)$, *cette dernière est égale presque partout à une fonction sousharmonique.*

C'est la réponse à une question posée par M. Mazurkiewicz.

VARSOVIE.

ON A THEOREM OF HAHN-STEINHAUS.*

By S. Saks and J. D. Tamarkin.

1. **Introduction.** In the present note we consider the function-spaces $\mathfrak{M}$, consisting of all measurable functions $f(x)$ essentially bounded on (a, b) with the metric

$$\|f\|_{\mathfrak{M}} = \text{ess. l. u. b.}\,|f(x)|,$$

and $\mathfrak{C}$, consisting of all functions $g(t)$ continuous on $\alpha \leqq t \leqq \beta$ with the metric

$$\|g\|_{\mathfrak{C}} = \max_{\alpha \leqq t \leqq \beta} |g(t)|.$$

According to custom we say that $f(x)$ is essentially bounded on (a, b) if there exists a finite constant M such that $|f(x)| \leqq M$ almost everywhere in (a, b), and define the essential least upper bound ($\equiv$ ess. l. u. b.) of $|f(x)|$ as the greatest lower bound of these constants M. A sequence of functions $\{f_n(x)\}$ is said to be essentially bounded on (a, b) if the inequalities

$$|f_n(x)| \leqq M \qquad (n = 1, 2, \cdots)$$

are satisfied simultaneously except for a set of values of x of measure zero.

In his investigations on the representation of functions by means of singular integrals Hahn[1] has established and used the following result:

Let $\{\omega_n(x)\}$ be an orthonormal set of functions for the interval (a, b), where each $\omega_n(x)$ is essentially bounded. Let $g(x)$ be an arbitrary continuous function and let

$$g_n = \int_a^b g(x)\,\omega_n(x)\,dx \qquad (n = 1, 2, \cdots)$$

be the sequence of its Fourier coefficients with respect to the set $\{\omega_n(x)\}$. If the sequence of functions

$$\sum_{\nu=1}^{n} g_\nu\,\omega_\nu(x) \qquad (n = 1, 2, \cdots)$$

is essentially bounded for each fixed $g(x)$, then the sequence of functions

$$\int_a^b \left| \sum_{\nu=1}^{n} \omega_\nu(x)\,\omega_\nu(t) \right| dt \qquad (n = 1, 2, \cdots)$$

is also essentially bounded.

* Received March 6, 1933.

[1] H. Hahn, *Über die Darstellung gegebener Funktionen durch singuläre Integrale*, II. Denkschriften der K. Akademie in Wien, Math.-naturwiss. Kl., vol. 93 (1916), pp. 657–692, esp. pp. 678–679.

This result of Hahn was stated (without proof) by Steinhaus[2] in the following, more general form:

Let $K(x, t)$ be integrable over the square $a \leqq x,\ t \leqq b$. If the sequence of functions

$$\int_a^b |K_n(x, t)|\, dt \qquad (n = 1, 2, \cdots)$$

is not essentially bounded, then there exists a continuous function $h(x)$ such that the sequence of functions

$$\int_a^b K_n(x, t)\, h(t)\, dt \qquad (n = 1, 2, \cdots)$$

is not essentially bounded.

In the present note we establish a theorem which is more general than that of Hahn-Steinhaus, using a method of proof which is entirely different from that of Hahn. To state this theorem in a concise form we need a suitable notation.

Let $H(x, t)$ be a function of two variables x, t. We say that $H(x, t)$ satisfies condition (A) if (i) for each fixed x in (a, b), which does not belong to a set $E(H)$ of measure zero, the function $H(x, t)$ is of bounded variation in t over the interval $\alpha \leqq t \leqq \beta$, and (ii) the function $H(x, t)$ is normalized according to the condition

$$H(x, t) = \frac{1}{2}[H(x, t+0) + H(x, t-0)], \quad \alpha < t < \beta,\ x \subset CE(H) = D(H).$$

We set

$$V(x; H) \equiv V(x) \equiv \int_\alpha^\beta |d_t H(x, t)|, \qquad x \subset D(H).$$

Our theorem can now be stated as follows:

THEOREM 1. *Let all the functions of the sequence $\{H_n(x, t)\}$ satisfy condition* (A). *If for every continuous function $g(t)$ the sequence of functions*

$$\int_\alpha^\beta g(t)\, d_t H_n(x, t) \qquad (n = 1, 2, \cdots)$$

is essentially bounded on (a, b), then the sequence of functions

$$V(x; H_n) \equiv V_n(x) \equiv \int_\alpha^\beta |d_t H_n(x, t)| \qquad (n = 1, 2, \cdots)$$

is also essentially bounded.

As we shall see, this theorem appears as an immediate consequence of two lemmas of the theory of linear transformations which might be of interest in themselves. These lemmas and the proof of Theorem 1 are

[2] H. Steinhaus, *Sur les développements orthogonaux,* Bulletin International de l'Académie Polonaise, Cl. des sciences math. et nat. (A), 1926, pp. 11–39, esp. p. 13.

dealt with in § 2 of the present note. In § 3 a theorem is proved which is analogous to Theorem 1, but which corresponds to the case where the space $\mathfrak{C}$ is replaced by the space $\mathfrak{R}$ of all the characteristic functions of measurable sets.

2. **Lemmas.** We are concerned here with the linear transformation of $\mathfrak{C}$ to $\mathfrak{M}$ as defined by

$$U(x; g) \equiv U(g) \equiv \int_\alpha^\beta g(t)\, d_t H(x, t), \tag{1}$$

where $H(x, t)$ satisfies condition (A). This transformation is said to be limited if there exists a constant M such that

$$\|U(x; g)\|_{\mathfrak{M}} \leqq M\|g\|_{\mathfrak{C}}, \qquad g \subset \mathfrak{C}.$$

The modulus $\|U\|$ of the transformation $U(g)$ is then defined as the greatest lower bound of the set of all admissible constants M. We can also write

$$\|U\| = \text{l. u. b. } \|U(x; g)\|_{\mathfrak{M}} \quad \text{when} \quad \|g\|_{\mathfrak{C}} = 1.$$

We shall prove two lemmas which state necessary and sufficient conditions that the transformation (1) be limited.

LEMMA 1. *If for every $g(t) \subset \mathfrak{C}$ the function*

$$U(x; g) = \int_\alpha^\beta g(t)\, d_t H(x, t) \subset \mathfrak{M}$$

so that $U(x; g)$ is essentially bounded on (a, b), then the transformation $U(x; g)$ is limited. The converse is also true.

Proof. First we prove that whenever $g^{(n)}(t)$ is a sequence of elements of $\mathfrak{C}$ which converges in $\mathfrak{C}$ to $g^{(0)}(t)$ so that

$$\|g^{(n)} - g^{(0)}\|_{\mathfrak{C}} \to 0 \text{ as } n \to \infty,$$

then

$$U(x; g^{(n)}) \to U(x; g^{(0)}) \tag{2}$$

almost everywhere in (a, b). Indeed, for each $x \subset D(H)$ we have

$$\begin{aligned} |U(x; g^{(n)}) - U(x; g^{(0)})| &= \left| \int_\alpha^\beta [g^{(n)}(t) - g^{(0)}(t)]\, d_t H(x; t) \right| \\ &\leqq \|g^{(n)} - g^{(0)}\|_{\mathfrak{C}}\, V(x) \to 0. \end{aligned}$$

Since meas $D(H) = b - a$, (2) holds almost everywhere in (a, b). Now let m be any integer $\geqq 0$. Let P_m be the set of elements $g(t) \subset \mathfrak{C}$ such that the inequality

$$|U(x; g)| \leqq m$$

is satisfied for all x in (a, b) aside from a set of values of x of measure zero, which might depend on $g(t)$. Under the hypothesis of Lemma 1 we have

$$\mathfrak{C} = \sum_{m=0}^{\infty} P_m.$$

On the other hand, from the above argument it follows that each P_m is closed in $\mathfrak{C}$. A familiar argument of Baire will then show the existence of at least one integer $m = p$ and of a sphere in $\mathfrak{C}$, say

$$(S) \qquad \| g - g_0 \|_{\mathfrak{C}} \leqq r,$$

which is entirely contained in P_p. Hence, for an arbitrary $g(t) \subset S$ and for all x aside from a set $E(g)$ of values of x of measure zero,

$$(3) \qquad | U(x; g) | \leqq p.$$

Thus

$$\| U(x; g) \|_{\mathfrak{M}} \leqq p, \quad g \subset S.$$

Since $U(x; g)$ is a linear transformation it follows that $U(x; g)$ is limited.[3] The converse statement of Lemma 1 is obvious.

LEMMA 2. *A necessary and sufficient condition that the transformation* (1) *be limited is that the function*

$$(4) \qquad V(x; H) = V(x) = \int_{\alpha}^{\beta} | d_t H(x, t) | \subset \mathfrak{M}$$

or else that $V(x)$ *be essentially bounded. If this condition is satisfied, then*

$$(5) \qquad \| U \| = \| V \|_{\mathfrak{M}}.$$

Proof. The sufficiency of condition (4) follows immediately from the inequality

$$\| U(x; g) \| \leqq V(x) \| g \|_{\mathfrak{C}} \leqq \| V \|_{\mathfrak{M}} \| g \|_{\mathfrak{C}}$$

which is satisfied almost everywhere in (a, b). We also note that

$$(6) \qquad \| U \| \leqq \| V \|_{\mathfrak{M}}.$$

Now let $\{g_i\}$ be a denumerable set of elements dense in $\mathfrak{C}$. For example, let $\{g_i\}$ be the set of all polynomials with rational coefficients. Let E_i be the set of values of x such that

$$| U(x; g_i) | > \| U \| \| g_i \|_{\mathfrak{C}}.$$

[3] Lemma 1 might be considered as a supplement to an analogous theorem of Banach, *Théorie des opérations linéaires*, Warsaw, 1932, p. 87, Theorem 9.

We put

$$E = \sum_{i=0}^{\infty} E_i, \qquad E_0 = E(H).$$

Since each set $E_i (i = 0, 1, 2, \cdots)$ is of measure zero, the same is true of the set E. The inequality

$$|U(x; g)| \leqq \|U\| \, \|g\|_{\mathfrak{C}} \tag{7}$$

is satisfied for each $g = g_i (i = 1, 2, \cdots)$ and all $x \subset CE$. Since, however, the set $\{g_i\}$ is dense in $\mathfrak{C}$ it is readily seen that the inequality (7) is satisfied for all $g \subset \mathfrak{C}$, $x \subset CE$. Furthermore, $U(x; g)$, for any fixed $x \subset CE$, may be interpreted as a linear operation on $\mathfrak{C}$ of modulus $V(x)$. Then it follows from (7) that

$$V(x) \leqq \|U\|$$

for all $x \subset CE$, that is, almost everywhere in (a, b). Thus $V(x)$ is essentially bounded and

$$\|V\|_{\mathfrak{M}} \leqq \|U\|.$$

On the other hand we have proved that

$$\|U\| \leqq \|V\|_{\mathfrak{M}}.$$

Hence

$$\|U\| = \|V\|_{\mathfrak{M}},$$

and the proof of Lemma 2 is completed.

Proof of Theorem 1. Under the hypotheses of Theorem 1 each transformation

$$U_n(x; g) \equiv \int_{\alpha}^{\beta} g(t) \, d_t H_n(x, t) \qquad (n = 1, 2, \cdots)$$

is limited and has the modulus

$$\|U_n\| = \|V_n\|_{\mathfrak{M}} = \left\| \int_{\alpha}^{\beta} |d_t H_n(x, t)| \right\|,$$

while, by Lemmas 1 and 2, each function $V_n(x)$ is essentially bounded. Furthermore, at each point g of the space $\mathfrak{C}$ the sequence $\{U_n(x; g)\}$ is bounded in $\mathfrak{M}$. It follows[4] that the sequence of moduli $\{\|U_n\|\}$ is bounded. Hence the sequence of functions $\{V_n(x)\}$ is essentially bounded on (a, b), as is stated in Theorem 1.

3. **A modification of Theorem 1.** In this section we deal with the space $\mathfrak{R}$ consisting of all the characteristic functions of measurable subsets of the interval $I \equiv (0, 1)$, that is, of all the functions $e(t)$ defined by

[4] Banach, loc. cit., p. 80, Theorem 5.

$$e(t) = 1, \text{ or } 0, \text{ according as } t \subset E, \text{ or } t \subset CE,$$

where E is an arbitrary measurable set $\subset I$. If we define the metric of $\mathfrak{R}$ by

$$\|e\|_{\mathfrak{R}} = \int_0^1 |e(t)|\, dt = \text{meas } E,$$
$$\text{distance } (e_1, e_2) = \|e_1 - e_2\|_{\mathfrak{R}},$$

we obtain a metric complete separable space which, however, is not linear.[5] If we replace the space $\mathfrak{C}$ by $\mathfrak{R}$ we are naturally led to the following statement.

THEOREM 2. *Let $K_n(x, t)$ be defined for almost all (x, t) in the square $0 \leqq x,\ t \leqq 1$, and be integrable in t over $(0, 1)$ for almost all values of x. If, for an arbitrary measurable set $E \subset I$, the sequence of functions*

$$\int_E K_n(x, t)\, dt = \int_0^1 K_n(x, t)\, e(t)\, dt \qquad (n = 1, 2, \cdots) \tag{8}$$

is essentially bounded, then the sequence of functions

$$\int_0^1 |K_n(x, t)|\, dt \qquad (n = 1, 2, \cdots) \tag{9}$$

is also essentially bounded.

Proof. Let m be any integer $\geqq 0$. We designate by P_m the set of elements $e(t) \subset \mathfrak{R}$ such that the simultaneous inequalities

$$\left| \int_0^1 K_n(x, t)\, e(t)\, dt \right| \leqq m \qquad (n = 1, 2, \cdots)$$

are satisfied for all x aside from a set of values of x of measure zero, which might depend on $e(t)$. On the one hand it is plain that the set P_m is closed in $\mathfrak{R}$. On the other hand, under the hypotheses of Theorem 2, we have

$$\mathfrak{R} = \sum_{m=0}^{\infty} P_m.$$

A familiar argument of Baire will then show the existence of at least one integer $m = p$ and of a sphere in $\mathfrak{R}$, say

$$\|e - e_0\|_{\mathfrak{R}} \leqq r, \tag{S}$$

which is entirely contained in P_p. In other words, the simultaneous inequalities

$$\left| \int_0^1 K_n(x, t)\, e(t)\, dt \right| \leqq p \qquad (n = 1, 2, \cdots)$$

[5] Throughout this section we use the results of the paper by S. Saks, *On some functionals*. Trans. Amer. Math. Soc., vol. 35 (1933), pp. 549–556. This paper will be referred to by (*F*).

are satisfied for an arbitrary $e(t) \subset S$, and for all x aside from a set $E(e)$ of values of x of measure zero.

Now we can apply the lemma of § 2 of (F). By part (i) of this lemma, for an arbitrary element $e(t)$ of the sphere

$$(S_0) \qquad \|e\|_{\mathfrak{R}} \leqq r$$

there exist two elements $e'(t)$, $e''(t)$ such that

$$e(t) = e'(t) - e''(t), \qquad e'(t) \subset S, \qquad e''(t) \subset S.$$

Thus the simultaneous inequalities

$$\left|\int_0^1 K_n(x,t)\,e(t)\,dt\right| \leqq \left|\int_0^1 K_n(x,t)\,e'(t)\,dt\right| + \left|\int_0^1 K_n(x,t)\,e''(t)\,dt\right| \leqq 2p$$
$$(n = 1, 2, \cdots)$$

are satisfied for an arbitrary $e(t) \subset S_0$ and for all x aside from those values of x belonging to a set $E'(e)$ of measure zero. Again, by part (iii) of the same lemma, an arbitrary element of the space $\mathfrak{R}$ can be represented as the sum of a fixed number of elements $e_i(t) \subset S_0$, namely,

$$e(t) = e_1(t) + e_2(t) + \cdots + e_k(t), \qquad k = \left[\frac{1}{r}\right] + 1.$$

Now we see that the inequalities

$$\left|\int_0^1 K_n(x,t)\,e(t)\,dt\right| \leqq 2kp \qquad (n = 1, 2, \cdots)$$

are satisfied simultaneously for an arbitrary element $e(t) \subset \mathfrak{R}$, and for all values of x, aside from those values belonging to a set $E''(e)$ of measure zero.

Since the space $\mathfrak{R}$ is separable, a simple argument of the type used in the proof of Lemma 2 above will establish the existence of a *fixed* set D such that meas $D = 1$, while the inequalities

$$\left|\int_0^1 K_n(x,t)\,e(t)\,dt\right| \leqq 2kp \qquad (n = 1, 2, \cdots)$$

are satisfied simultaneously regardless of the element $e(t) \subset \mathfrak{R}$ and $x \subset D$. This immediately yields

$$\int_0^1 |K_n(x,t)|\,dt \leqq 4kp \qquad (n = 1, 2, \cdots)$$

for all $x \subset D$, that is, almost everywhere. This shows that the sequence of functions (9) is essentially bounded, and Theorem 2 is therefore proved.

ADDITION TO THE PAPER "ON THE SUMMABILITY OF FOURIER SERIES II".[19] [20]

By Einar Hille and J. D. Tamarkin.

6. (B)-effective transformations. In § 1 we explained the reasons which compelled us to denote by L_∞ the class of all continuous functions rather than the class B of all essentially bounded functions. It appears, however, that results of some interest may be obtained for transformations of Fourier series of functions $\subset B$, provided the limit concept in this space is defined not by means of the metric of B, but rather by means of *asymptotic convergence.*[21] We shall denote asymptotic convergence by the symbol $\xrightarrow{as}$. The following known property is essential for our discussion.

LEMMA 6.1. *If a sequence of functions $\{f_n(x)\} \subset L_1$ converges in L_1, it also converges asymptotically. Conversely, if $\{f_n(x)\}$ converges asymptotically and is bounded in B (that is, the functions $f_n(x)$ are uniformly essentially bounded), then $\{f_n(x)\}$ converges in L_1.*

We now introduce

DEFINITION 6.1. *A transformation $\mathfrak{A}$ is said to be (B)-effective if it satisfies the following conditions*

(E′.1) *The series*

$$\tau_m(x;f) \sim \sum_{n=1}^{\infty} a_{mn} f_n \varphi_n(x) \subset B \qquad m = 1, 2, \cdots, \tag{6.1}$$

whenever $f(x) \subset B$.

(E′.2) *The sequence of the m-th transforms converges asymptotically to $f(x)$,*

$$\tau_m(x;f) \xrightarrow{as} f(x) \text{ when } m \to \infty. \tag{6.2}$$

(E′.3) *For every fixed function $f(x) \subset B$ the sequence of the m-th transforms is bounded in B. In other words, there exists a constant $K(f)$ depending only on $f(x)$ such that*

$$\| \tau_m(x;f) \|_B \leqq K(f). \tag{6.3}$$

Our purpose is now to prove

THEOREM 6.1. *The classes of (B)-effective, (L_1)-effective and (L_∞)-effective transformations are identical.*

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[19] Received March 15, 1933.

[20] The present note constitutes § 6 of our article in the preceding issue of these Annals, pp. 329–348.

[21] Banach [1], pp 3–4. We are indebted to Professor M. Riesz for calling our attention to such a possibility.

Proof. Let $\{a_n\}$ be a factor-sequence which transforms the Fourier series of an arbitrary function $f(x) \subset B$ into a Fourier series $\subset B$. The class of all such factor-sequences is denoted by (B, B). It is well-known (see footnote [8] on p. 336) that the classes (B, B) and $(1, 1) = (\infty, \infty)$ are identical. Hence, whenever a transformation $\mathfrak{A}$ satisfies condition (E. 1) of p. 331 with $p = 1$ or $p = \infty$, it also satisfies condition (E′.1), and vice versa. Now assume that $\mathfrak{A}$ is (L_1)-effective. Let $f(x)$ be an arbitrary function $\subset B$. Since (E.1), and therefore (E′.1), is satisfied, each transform $\tau_m(x; f) \subset B$. By hypothesis

$$(6.4)\quad \|\tau_m(x; f) - f(x)\|_1 \to 0, \quad \|\tau_m(x; f)\|_1 \leqq M_m^{(1)} \|f\|_1 \leqq M_m^{(1)} \|f\|_B.$$

Consequently, by Lemma 6.1,

$$\tau_m(x; f) \xrightarrow{as} f(x) \text{ when } m \to \infty,$$

and so condition (E′.2) is satisfied. To prove that (E′.3) is also satisfied we have to show that the sequence $\{\tau_m(x; f)\}$ is uniformly essentially bounded. Let $g(x)$ be an arbitrary function $\subset L_1$. We can apply relation (2.18) of p. 337 which holds for an arbitrary pair of functions $f(x) \subset B$, $g(x) \subset L_1$ (see footnote [10] on p. 337). Thus, upon putting

$$(6.5)\quad U_m(g) = \frac{1}{b-a} \int_a^b g(x) \tau_m(x; f)\, dx,$$

we have

$$(6.6)\quad U_m(g) = \frac{1}{b-a} \int_a^b f(x) \tau_m(x; g)\, dx,$$

and, by (6.4),

$$(6.7)\quad |U_m(g)| \leqq M_m^{(1)} \|f\|_B \|g\|_1, \qquad m = 1, 2, \cdots.$$

If we consider $U_m(g)$ as a linear operation on L_1, we see that $\{U_m(g)\}$ is a sequence of linear limited operations bounded on L_1. By a theorem of Banach[22] the sequence of their bounds, $\{\|\tau_m(x; f)\|_B\}$, is bounded. Thus condition (E′.3) is satisfied and $\mathfrak{A}$ is (B)-effective.

Conversely, assume that $\mathfrak{A}$ is (B)-effective. Condition (E′.1), hence also condition (E.1) with $p = 1$ and $p = \infty$, is satisfied. Moreover, by (E′.2) and (E′.3), for an arbitrary function $f(x) \subset B$ we have

$$\tau_m(x; f) \xrightarrow{as} f(x), \qquad \|\tau_m(x; f)\|_B \leqq K(f).$$

In view of Lemma 6.1 this implies

$$\|\tau_m(x; f) - f(x)\|_1 \to 0 \quad \text{when} \quad m \to \infty.$$

The set of all essentially bounded functions is dense in L_1, and so the transformation $\mathfrak{A}$ is (L_1)-effective over a dense set of L_1. Consider the

[22] Banach [1], p. 80, Theorem 5.

sequence of the mth transforms $\{\tau_m(x; g)\}$, where $g(x)$ is an arbitrary function of L_1. To apply Theorem 3.1 of p. 338 we have only to show that this sequence is uniformly limited on L_1. We now write

$$(6.8)\quad V_m(f) = \frac{1}{b-a}\int_a^b f(x)\tau_m(x; g)\,dx = \frac{1}{b-a}\int_a^b g(x)\,\tau_m(x; f)\,dx.$$

This is a linear limited operation on B with the bound $\|\tau_m(x; g)\|_1$. For each element $f(x) \subset B$ we have

$$(6.9)\qquad |V_m(f)| \leqq \|\tau_m(x; f)\|_B \|g\|_1 \leqq K(f)\|g\|_1, \qquad m = 1, 2, \cdots,$$

whence the sequence of operations $\{V_m(f)\}$ is bounded on B. By the same theorem of Banach[22] the sequence of bounds $\{\|\tau_m(x; g)\|_1\}$ is bounded. Consequently the sequence $\{\tau_m(x; g)\}$ of linear limited transformations on L_1 is bounded on L_1. Another application of Banach's theorem will show that the sequence $\{\tau_m(x; g)\}$ is uniformly limited on L_1. Since all the conditions of Theorem 3.1 are satisfied, we conclude that $\mathfrak{A}$ is (L_1)-effective, and Theorem 6.1 is completely proved.

Remark 6.1. Let $f(x)$ be an arbitrary function $\subset B$. By referring to the papers mentioned in footnotes [9] and [10] on p. 337 it is readily found that Lemma 2.3 remains valid if the space L_∞ is replaced by B. Hence, for a fixed m, $\tau_m(x; f)$ defines a linear limited transformation on B to B, with the bound $M_m^{(B)} = M_m^{(1)}$. Thus

$$(6.10)\qquad \|\tau_m(x; f)\|_B \leqq M_m^{(1)}\|f\|_B.$$

The quantity

$$M_{\mathfrak{A}}^{(B)} = \overline{\lim_{m\to\infty}} M_m^{(B)} = \overline{\lim_{m\to\infty}} M_m^{(1)} = M_{\mathfrak{A}}^{(1)}$$

may be designated as the (B)-bound of the transformation $\mathfrak{A}$. We conclude that the (B)-bound of a (B)-effective transformation $\mathfrak{A}$ is equal to its (L_1)- and (L_∞)-bounds.

Remark 6.2. Let a sequence $\{f_k(x)\} \subset B$ be uniformly essentially bounded and let

$$f_k(x) \xrightarrow{\text{a.u.}} f(x) \quad \text{when} \quad k \to \infty.$$

If $\{a_n\}$ is an arbitrary sequence $\subset (B, B)$, then

$$T_a(x; f) \sim \sum_{n=1}^{\infty} a_n f_n \varphi_n(x), \qquad f(x) \sim \sum_{n=1}^{\infty} f_n \varphi_n(x),$$

is a linear limited transformation on B to B. We shall prove that it is continuous in the sense that the sequence of functions $\{T_a(x; f_k)\}$ is uniformly essentially bounded, while

(6.11) $$T_a(x; f_k) \xrightarrow{as} T_a(x; f)$$

whenever

(6.12) $$f_k(x) \xrightarrow{as} f(x), \qquad \|f_k(x)\|_B \leqq K.$$

Indeed, since

$$\|T_a(x; f_k)\|_B \leqq M_a^{(1)} \|f_k(x)\|_B \leqq K M_a^{(1)},$$

the sequence of functions $\{T_a(x; f_k)\}$ is essentially uniformly bounded. On the other hand, by Lemma 6.1,

$$\|f_k(x) - f(x)\|_1 \to 0, \qquad \|T_a(x; f_k) - T_a(x; f)\|_1 \to 0,$$

whence

$$T_a(x; f_k) \xrightarrow{as} T_a(x; f) \quad \text{when} \quad k \to \infty.$$

Remark 6.3. We call a transformation $\mathfrak{A}$ which satisfies only conditions (E′.1) and (E′.2), but not (E′.3), (B')-effective. The argument of the first part of the proof of Theorem 6.1 shows that if $\mathfrak{A}$ is (L_1)-effective, it is also (B')-effective. It is an open question whether the class of (B')-effective transformations coincides with the class of (L_1)-effective transformations or is actually larger than this class.

ON A THEOREM OF PALEY AND WIENER.*

By Einar Hille and J. D. Tamarkin.

1. In their recent work on the theory of quasi-analytic functions Paley and Wiener prove the following important theorem which is the basis of their discussion.

Let $\omega(x)$ be a real non-negative function not equivalent to zero defined for $-\infty<x<\infty$, and of integrable square in this range. A necessary and sufficient condition that there exist a real- or complex-valued function $F(x)$ defined in the same range, vanishing for $x<0$, and such that the Fourier transform $G(x)$ of $F(x)$ satisfy the condition $|G(x)|=\omega(x)$ is that

$$\int_{-\infty}^{\infty}\frac{|\log\omega(x)|}{1+x^2}\,dx<\infty.$$[1]

In the present note we prove a slightly more general theorem. This proof uses essentially the same idea as that of Paley and Wiener, but seems to throw some additional light on the situation.

2. We shall deal with two complex variables, $z=x+iy$, and $w=\varrho e^{i\theta}$ which are related by

$$w=\frac{1+iz}{1-iz}, \tag{1}$$

so that the half-plane $\Im(z)>0$ is mapped on the interior of the unit circle $|w|<1$. The variables $\xi=e^{i\varphi}$ and $\zeta=\tan\frac{\varphi}{2}$ will trace the circumference of the unit-circle in the w-plane and the real axis in the z-plane respectively.

Let p be any number such that $1\leqq p<\infty$. The classes of functions $\Phi(\xi)$, $F(\zeta)$ for which the integrals

$$\int_{-\pi}^{\pi}|\Phi(\xi)|^p\,d\varphi,\qquad \int_{-\infty}^{\infty}|F(\zeta)|^p\,d\zeta$$

exist will be designated by L_p, $\mathfrak{L}_p$ respectively. The formula

$$2\int_{-\infty}^{\infty}\frac{|F(\zeta)|^p}{1+\zeta^2}\,d\zeta=\int_{-\pi}^{\pi}|\Phi(\xi)|^p\,d\varphi,\qquad \Phi(\xi)=F\left(\tan\frac{\varphi}{2}\right),$$

* Received March 16, 1933.

[1] R. E. A. C. Paley and N. Wiener, *Notes on the theory and application of Fourier transforms. I. On a theorem of Carleman.* Trans. Amer. Math. Soc., 35 (1933), pp. 348-355. Paley and Wiener write $x>0$ instead of $x<0$. This distinction is due to a different notation for the Fourier transform of a function.

shows that $\Phi(\xi) \subset L_p$ whenever $F(\zeta) \subset \mathfrak{L}_p$. We designate by H_p the class of functions $\gamma(w)$, analytic in $|w|<1$, such that the integral

$$\int_{-\pi}^{\pi} |\gamma(\varrho e^{i\theta})|^p \, d\theta$$

is bounded for $0 \leqq \varrho < 1$.

The integrals

$$I(z; F) = (2\pi i)^{-1} \int_{-\infty}^{\infty} \frac{F(\zeta)}{\zeta - z} \, d\zeta, \tag{2}$$

$$J(w; \Phi) = (2\pi i)^{-1} \int_{-\pi}^{\pi} \frac{\Phi(\xi)}{\xi - w} \, d\xi \tag{3}$$

will define the integrals of Cauchy type associated with the functions $F(\zeta)$, $\Phi(\xi)$, while the integrals

$$Q(z; F) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{F(\zeta) y}{(\zeta - x)^2 + y^2} \, d\zeta = I(z; F) - I(\bar{z}; F), \tag{4}$$

$$\bar{z} = x - iy,$$

$$P(w; \Phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(\xi) \frac{1 - \varrho^2}{1 + \varrho^2 - 2\varrho \cos(\varphi - \theta)} \, d\varphi \tag{5}$$

$$= J(w; \Phi) - J(w^*; \Phi), \qquad w^* = \bar{w}^{-1},$$

will define the Poisson integrals associated with the same functions. It is plain that $I(z; F)$, $J(w; \Phi)$ are analytic in their respective variables whenever $\Im(z) \neq 0$, $|w| \neq 1$, $F(\zeta) \subset \mathfrak{L}_p$, $\Phi(\xi) \subset L_p$. It is well known that, for almost all ζ, ξ,

$$\lim_{z \to \zeta} Q(z; F) = F(\zeta),$$

$$\lim_{w \to \xi} P(w; \Phi) = \Phi(\xi),$$

where z and w tend to their respective limits along arbitrary non-tangential paths. If the variables z, w are related by (1), the variables ζ, ξ by $\xi = \dfrac{1 + i\zeta}{1 - i\zeta} = e^{i\varphi}$, $\zeta = \tan \dfrac{\varphi}{2}$, and the functions $F(\zeta)$, $\Phi(\xi)$ by $F(\zeta) = F\left(\tan \dfrac{\varphi}{2}\right) = \Phi(\xi)$, then a direct computation shows that

$$J(w; \Phi) = I(z; F) - C, \quad C = I(-i; F), \tag{6}$$

$$P(w; \Phi) = Q(z; F). \tag{7}$$

Let $g(z)$ be analytic in the half-plane $\Im(z) > 0$. For almost all ζ let $g(z)$ tend to a finite limit as $z \to \zeta$ along all non-tangential paths. These

limiting values define the limit-function $G(\zeta)$ of $g(z)$. In an analogous fashion we define the limit-function $\Gamma(\xi)$ of the function $\gamma(w)$ analytic in the unit circle $|w|<1$. We shall be particularly concerned with the analytic functions which are represented by their own Cauchy integrals[2]

$$g(z) = I(z; G), \quad \gamma(w) = J(w; \Gamma).$$

It will be assumed that $G(\zeta) \subset \mathfrak{L}_p$, $\Gamma(\xi) \subset L_p$ for some fixed p, $1 \leqq p < \infty$.

3. We now are in a position to state and prove our results.

THEOREM 1. *A necessary and sufficient condition that a function $G(\zeta) \subset \mathfrak{L}_p$ be the limit-function of a function $g(z)$ which is analytic in the half-plane $\Im(z)>0$ and which is represented by its Cauchy integral, is that $G(\zeta)$ satisfy the condition*

$$(8) \qquad I(\bar{z}; G) = (2\pi i)^{-1} \int_{-\infty}^{\infty} \frac{G(\zeta)\, d\zeta}{\zeta - \bar{z}} = 0, \qquad \bar{z} = x - iy, \quad y>0.$$

Proof.[3] If $g(z)$ is represented by its Cauchy integral then

$$g(z) = I(z; G)$$

and, in view of (4),

$$(9) \qquad g(z) = Q(z; G) + I(\bar{z}; G).$$

Since the limit-functions of $g(z)$ and of $Q(z; G)$ are the same ($= G(\zeta)$), the limit-function of $I(\bar{z}; G)$ is zero for almost all ζ. Hence $I(\bar{z}; G)$, being an analytic function of the variable $\bar{z}$, vanishes identically for $y>0$ by the uniqueness theorem of analytic functions. Conversely, if (8) is satisfied, then (9) shows that $g(z) \equiv Q(z; G)$ is the analytic function satisfying the conditions of Theorem 1.

THEOREM 2. *Let $g(z)$ be analytic in the half-plane $\Im(z)>0$ and let its limit-function $G(\zeta) \subset \mathfrak{L}_p$. If $g(z)$ is represented by its Cauchy integral it is also represented by its Poisson integral, and vice versa.*

Proof. We observe that the analogous theorem in the case of a function analytic in the unit circle is well known. If now $g(z)$ is represented by its Cauchy integral, then Theorem 1 and formula (9) show that

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[2] We refer to the following two articles for the various properties of the Cauchy and Poisson integrals, and of the classes H_p, needed in the subsequent discussion: G. Fichtenholz, *Sur l'intégrale de Poisson et quelques questions qui s'y rattachent.* Fund. Math., 13 (1929), pp. 1–33; W. Smirnoff, *Sur les valeurs limites des fonctions regulières à l'intérieur d'un circle.* Jour. Soc. Phys.-Math. de Léningrade, 2 (1928–29), pp. 22–37. Other bibliographical references are found in these articles.

[3] Cf. Fichtenholz, loc. cit. [2], pp. 21–23.

$g(z) = Q(z; G)$ is also represented by its Poisson integral. Assume conversely that

$$g(z) = Q(z; G) = I(z; G) - I(\bar{z}; G). \tag{10}$$

If we pass to the variable w and set $\gamma(w) = g(z)$ we conclude from (7) that $\gamma(w) = P(w; \Gamma)$ is represented by its Poisson integral since $\Gamma(\xi) = G\left(\tan\frac{\varphi}{2}\right) = G(\zeta)$ obviously is the limit-function of $\gamma(w)$. Then $\gamma(w)$ is represented also by its Cauchy integral, and we have

$$\gamma(w) = J(w; \Gamma).$$

On returning to the variable z and using (6) we get

$$g(z) = I(z; G) - C. \tag{11}$$

A comparison of (10) and (11) yields

$$I(\bar{z}; G) = C, \qquad y > 0.$$

Let x be fixed and $y \to \infty$. It is readily seen that $I(\bar{z}; G) \to 0$. This is obvious when $p = 1$, and the general case follows from the inequality

$$\left| \int_{-\infty}^{\infty} \frac{G(\zeta)}{\zeta - \bar{z}} d\zeta \right| \leq \left[\int_{-\infty}^{\infty} |G(\zeta)|^p \, d\zeta \right]^{1/p} \left[\int_{-\infty}^{\infty} \frac{d\zeta}{\{(\zeta - x)^2 + y^2\}^{p'}} \right]^{1/p'},$$

$$p' = \frac{p}{p-1}, \quad p > 1.$$

It results that $I(\bar{z}; G) = 0$, and (10) shows that $g(z)$ is represented by its Cauchy integral.

Remark. From the above argument it follows incidentally that if $G(\zeta) \subset \mathfrak{L}_p$ and $\Gamma(\xi) \subset L_p$ are the limit-functions of the analytic functions $g(z) = \gamma(w)$, where $w = \frac{1+iz}{1-iz}$, then the corresponding Cauchy integrals are transformed into one another,

$$I(z; G) = J(w; \Gamma).$$

4. In the subsequent discussion we shall employ Fourier transforms defined in a slightly more general manner than usual. Let $F(\zeta)$ be measurable over $(-\infty, \infty)$ and integrable over every finite range. Let

$$\mathfrak{F}_N(u) \equiv \mathfrak{F}_N(u; F) \equiv (2\pi)^{-1/2} \int_{-N}^{N} e^{-iu\zeta} F(\zeta) \, d\zeta. \tag{12}$$

We assume that:

(i) For every fixed finite $A > 0$

$$\int_{-A}^{A} |\mathfrak{F}_N(u)| \, du \leq M_A, \tag{13}$$

where the constant M_A depends only on A and on $F(\zeta)$, but not on N.

(ii) There exists a fixed number $\varepsilon_0 \geqq 0$ such that for each $\varepsilon > \varepsilon_0$

$$\int_{-\infty}^{-A} |\mathfrak{F}_N(u)|\, e^{\varepsilon u}\, du \to 0, \quad \int_A^{\infty} |\mathfrak{F}_N(u)|\, e^{-\varepsilon u}\, du \to 0 \tag{14}$$

as $A \to \infty$, the approach being uniform in N.

Under these assumptions it is readily proved that there exists a sequence of values $\{N_k\}$, $N_k \uparrow \infty$, and a function $\mathfrak{F}^0(u) \equiv \mathfrak{F}^0(u; F)$ such that

$$\int_0^u \mathfrak{F}_{N_k}(t)\, dt \to \mathfrak{F}^0(u) \text{ as } N_k \to \infty \tag{15}$$

for every finite range of u. This function $\mathfrak{F}^0(u)$ will be called a Fourier transform of $F(\zeta)$ of order 1. To prove these statements we have only to observe that, for a fixed A, the functions

$$\int_0^u \mathfrak{F}_N(t)\, dt, \qquad -A \leqq u \leqq A, \tag{16}$$

are of uniformly bounded variation. Hence, by a classical theorem of Helly, there exists a subsequence $\mathfrak{F}_i(A; u)$ of this set and a function of bounded variation $\mathfrak{F}(A; u)$ such that

$$\int_0^u \mathfrak{F}_i(A; t)\, dt \to \mathfrak{F}(A; u) \quad \text{as} \quad i \to \infty, \quad |u| \leqq A. \tag{17}$$

We now take a sequence $\{A_j\}$, $A_j \uparrow \infty$, of values of A and apply the "diagonal process" to the double sequence of functions $\{\mathfrak{F}_i(A_j; t)\}$. This process establishes the existence of the sequence $\{N_k\}$ and of the function $\mathfrak{F}^0(u)$ satisfying (15).

From the construction it follows that $\mathfrak{F}^0(u)$ is of bounded variation over every finite range, while

$$\int_\alpha^\beta \psi(t)\, \mathfrak{F}_{N_k}(t)\, dt \to \int_\alpha^\beta \psi(t)\, d\mathfrak{F}^0(t) \quad \text{as} \quad N_k \to \infty$$

holds for an arbitrary continuous function $\psi(t)$. On setting $\psi(t) = e^{-\varepsilon|t|}$ and using property (ii) we see at once that the integral

$$\int_{-\infty}^{\infty} e^{-\varepsilon|t|}\, |d\mathfrak{F}^0(t)| \tag{18}$$

exists (as an absolutely convergent improper Stieltjes integral). On the basis of these facts it is now readily seen that the relation

$$\int_{-\infty}^{\infty} e^{-\varepsilon|t|}\, \psi(t)\, \mathfrak{F}_{N_k}(t)\, dt \to \int_{-\infty}^{\infty} e^{-\varepsilon|t|}\, \psi(t)\, d\mathfrak{F}^0(t), \qquad N_k \to \infty, \tag{19}$$

holds for an arbitrary continuous function $\psi(t)$ bounded in $-\infty < u < \infty$. Finally we note that

$$\mathfrak{F}^0(0) = 0, \qquad \mathfrak{F}^0(u) = \frac{1}{2}[\mathfrak{F}^0(u+0) + \mathfrak{F}^0(u-0)]. \tag{20}$$

The first of these relations follows from (17) while the second may be assumed without modifying (19).

An important special case is obtained if we impose the additional restriction:

(iii) For every fixed A the set of integrals (16) is uniformly (with respect to N) absolutely continuous.

Under this assumption the function $\mathfrak{F}^0(u)$ is itself absolutely continuous. On setting

$$\mathfrak{F}^0(u) = \int_0^u \mathfrak{F}(t)\, dt \tag{21}$$

we call the function $\mathfrak{F}(u) = \mathfrak{F}(u; F)$ the Fourier transform of $F(\zeta)$. According as the conditions (i–ii) or (i–iii) are satisfied we shall say that $F(\zeta)$ has a Fourier transform of order 1 or a Fourier transform respectively.

5. This notion of Fourier transform can be generalized in the direction of the Hahn-Wiener-Bochner theory of generalized trigonometric integrals. We do not intend to enter here into a discussion of such generalizations and their relationship with the theories mentioned. Neither do we investigate here the question of the uniqueness of the Fourier transform as defined above. We prove only the following

LEMMA. *The Fourier transform of order* 1 *of* $F(\zeta)$ *is uniquely determined at its points of continuity whenever* $F(\zeta) \subset \mathfrak{L}_p$, $1 \leqq p$. *Moreover, at all such points we have*

$$\int_0^u \mathfrak{F}_N(t)\, dt \to \mathfrak{F}^0(u) \tag{22}$$

no matter how N *tends to infinity. The function* $\mathfrak{F}^0(u)$ *is uniquely determined everywhere if it is normalized according to* (20).

Proof. We start from the directly verifiable formulas

$$\begin{aligned} \frac{1}{x-\zeta+iy} &= \frac{1}{i}\int_0^\infty e^{i(x-\zeta+iy)t}\, dt, \\ \frac{1}{x-\zeta-iy} &= -\frac{1}{i}\int_{-\infty}^0 e^{i(x-\zeta-iy)t}\, dt, \qquad y>0, \\ \frac{2y}{(\zeta-x)^2+y^2} &= \int_{-\infty}^\infty e^{i(x-\zeta)t} e^{-y|t|}\, dt. \end{aligned} \tag{23}$$

Let $F(\zeta) \subset \mathfrak{L}_p$ and let conditions (i) and (ii) be satisfied. Thus we have

$$2y\int_{-N}^{N} \frac{F(\zeta)\,d\zeta}{(\zeta-x)^2+y^2} = \int_{-\infty}^{\infty} e^{ixt}\,e^{-y|t|}\,dt \int_{-N}^{N} e^{-i\zeta t}\,F(\zeta)\,d\zeta \tag{24}$$

$$= (2\pi)^{1/2}\int_{-\infty}^{\infty} e^{ixt}\,e^{-y|t|}\,\mathfrak{F}_N(t)\,dt.$$

On allowing $N = N_k \to \infty$ we get, in view of (19),

$$\Omega(x, y) = (2/\pi)^{1/2}\,y\int_{-\infty}^{\infty} \frac{F(\zeta)\,d\zeta}{(\zeta-x)^2+y^2} = \int_{-\infty}^{\infty} e^{ixt}\,dV(t), \tag{25}$$

$$V(u) = \int_0^u e^{-y|t|}\,d\mathfrak{F}_0(t), \qquad y > \varepsilon_0. \tag{26}$$

Here $V(u)$ is of bounded variation over the whole infinite range. Hence we can conclude that[4]

$$V(t) = \lim_{A\to\infty} (2\pi)^{-1}\int_{-A}^{A} \Omega(x, y)\,\frac{e^{-ixt}-1}{-ix}\,dx. \tag{27}$$

Thus $\mathfrak{F}^0(u)$ is uniquely determined by $F(\zeta)$ at all the points of continuity, and everywhere if $\mathfrak{F}^0(u)$ is normalized according to (20).

Remark 1. If all three conditions (i–iii) are satisfied then an analogous argument[5] will show that

$$\mathfrak{F}(u) = e^{y|u|}\lim_{A\to\infty} (2\pi)^{-1}\int_{-A}^{A} \Omega(x, y)\,e^{-ixu}\,dx. \tag{28}$$

Finally, if in condition (ii) $\varepsilon_0 = 0$, we can allow $y \to 0$ in (25) with the result

$$F(x) = \lim_{y\to 0} (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{ixt}\,e^{-y|t|}\,\mathfrak{F}(t)\,dt, \tag{29}$$

almost everywhere in $-\infty < x < \infty$. Thus we have obtained the classical Sommerfeld integral.

Remark 2. Conditions (i—iii) are satisfied when $1 \leqq p \leqq 2$. Then $F(\zeta)$ has a Fourier transform in $\mathfrak{L}_{p'}$, $p' = \dfrac{p}{p-1}$, that is

$$\mathfrak{F}(u) = \underset{N\to\infty}{\text{l.i.m.}}\ \mathfrak{F}_N(u),$$

[4] Cf. S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932, pp. 65–68. Bochner assumes that $V(u)$ is increasing, but, as he remarks himself, his argument is valid for any function of bounded variation in $(-\infty, \infty)$.

[5] Cf. Bochner, loc. cit. [4], p. 42.

where l.i.m. stands for the limit in the mean of order p' if $p'<\infty$, and for the uniform limit when $p'=\infty$[6]. In this case we have $\varepsilon_0=0$. But there exist functions $F(\zeta)\subset\mathfrak{L}_p$, $p>2$, possessing no Fourier transform in any $\mathfrak{L}_q$, but such that the Fourier transform in the above more general sense exists, while $\varepsilon_0>0$ in condition (ii)[7].

6. We now return to our main subject matter.

THEOREM 3. *Let $G(\zeta)\subset\mathfrak{L}_p$, $p\geqq 1$, and suppose $G(\zeta)$ possesses the Fourier transform $\mathfrak{F}^0(u)$ of order* 1. *A necessary and sufficient condition that $G(\zeta)$ be a limit-function of a function $g(z)$ which is analytic in the half-plane $\mathfrak{I}(z)>0$ and which is represented by its Cauchy integral, is that $\mathfrak{F}^0(u)$ vanish for $u<0$.*[8]

Proof. As in § 5 it is readily seen that

$$\int_{-\infty}^{\infty}\frac{G(\zeta)\,d\zeta}{\zeta-x+iy}=-i(2\pi)^{1/2}\int_{-\infty}^{0}e^{ixt}\,e^{yt}\,d\mathfrak{F}^0(t),\qquad y>\varepsilon_0. \tag{30}$$

By the Lemma of § 5 the function $\mathfrak{F}^0(t)$ is uniquely determined if it is normalized according to (20). Consequently the vanishing of the integral (8) implies, and is implied by, the vanishing of $\mathfrak{F}^0(t)$ for $t<0$. Now it remains only to apply Theorem 1.

THEOREM 4. *The condition $\mathfrak{F}^0(u;\,G)=0$ for $u<0$ is necessary and sufficient that the Poisson integral $Q(z;\,G)$ represent a function $g(z)$ analytic in the half-plane $\mathfrak{I}(z)>0$.*

Proof. This theorem is a simple consequence of Theorems 1, 2 and 3.

THEOREM 5 (Paley and Wiener). *Let $0\leqq F(\zeta)\subset\mathfrak{L}_p$, $1\leqq p\leqq 2$. The condition*

$$\int_{-\infty}^{\infty}\frac{|\log F(\zeta)|\,d\zeta}{1+\zeta^2}<\infty \tag{31}$$

is necessary and sufficient that there exist a function $G(\zeta)\subset\mathfrak{L}_p$ such that

$$|G(\zeta)|=F(\zeta), \tag{32}$$

while the Fourier transform $\mathfrak{F}(u;\,G)$ of $G(\zeta)$ vanishes for $u<0$.

Proof. To prove the necessity of (31) assume that $G(\zeta)$ is a function satisfying the conditions of Theorem 5. By Theorem 4, the function

[6] E. C. Titchmarsh, *A contribution to the theory of Fourier transforms.* Proc. London Math. Soc., (2) 23 (1924–25), pp. 279–289.

[7] E. Hille and J. D. Tamarkin, *On the summability of Fourier Series.* III. To appear shortly in the Math. Annalen. It should be mentioned, however, that the method employed in § 18 of this paper could be used to construct functions $F(\zeta)\subset\mathfrak{L}_p$, $p>2$, such that $\lim \mathfrak{F}_x(u)$ exists, but the theory developed in the present paper does not apply.

[8] For a special case of this theorem see N. Wiener, *The operational calculus,* Math. Ann., 95 (1926), pp. 557–584 (580).

$g(z) = Q(z; G)$ is analytic in the half-plane $\Im(z) > 0$ and is represented by its Cauchy, as well as by its Poisson, integral. The same will be true in the w-plane for the corresponding function $\gamma(w) = g(z)$. This implies, however, the integrability of $\log|\Gamma(\xi)|$, where $\Gamma(\xi) = G(\zeta)$ is the limit-function of $\gamma(w)$.[9] Hence

$$\int_{-\infty}^{\infty} \frac{|\log F(\zeta)|\, d\zeta}{1+\zeta^2} = \frac{1}{2}\int_{-\pi}^{\pi} |\log|\Gamma(\xi)||\, d\varphi < \infty.$$

Conversely, let (31) be satisfied. Then $\Phi(\xi) = F\left(\tan\frac{\varphi}{2}\right) \subset L_p$, while $\log\Phi(\xi) \subset L_1$. Under these circumstances[9] there exists an analytic function $\gamma(w)$ such that the limit-function $\Gamma(\xi)$ satisfies the relation $|\Gamma(\xi)| = \Phi(\xi)$, and $\gamma(w)$ is represented by its Cauchy, as well as by its Poisson, integral. In view of the preceding theorems, the same will be true of the corresponding function $g(z) = \gamma(w)$, and the Fourier transform of the limit-function $G(\zeta)$ must vanish for the negative values of its argument. Finally (32) holds. This completes the proof of Theorem 5.

Remark. The conditions of Theorem 5 remain necessary when $p > 2$. It is an open question whether they will be also sufficient for such values of p.

[9] Cf. Smirnoff, loc. cit. [2].

ON THE LACUNARY COEFFICIENTS OF POWER SERIES.[1]

By R. E. A. C. Paley†.

1. Let $U(\theta)$ be a real function defined in $(-\pi, \pi)$, and let

$$U(\theta) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

Let $\lambda_1, \lambda_2, \lambda_3, \cdots$ be a series of integers for which $\lambda_{n+1}/\lambda_n \geqq q > 1$. Zygmund[2] has shown that the existence of the integral

$$\int_{-\pi}^{\pi} |U(\theta)| \overset{+}{\log}^{1/2} |U(\theta)| \, d\theta \tag{1}$$

implies the convergence of the series

$$\sum_{n=1}^{\infty} (a_{\lambda_n}^2 + b_{\lambda_n}^2). \tag{2}$$

The condition (1) is the best possible, and no weaker condition on U will imply the convergence of (2).

Now let $V(\theta)$ be the function conjugate to $U(\theta)$. The object of this paper is to prove the following theorem.

THEOREM. *If*

$$\int_{-\pi}^{\pi} |U(\theta)| \, d\theta < \infty, \qquad \int_{-\pi}^{\pi} |V(\theta)| \, d\theta < \infty, \tag{3}$$

then the series (2) *converges.*

2. Let

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{U(\varphi)(1-r^2)\, d\varphi}{1-2r\cos(\theta-\varphi)+r^2},$$

$$v(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{V(\varphi)(1-r^2)\, d\varphi}{1-2r\cos(\theta-\varphi)+r^2},$$

and

$$f(z) = f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta}).$$

Then (3) is equivalent to the condition

$$\int_{-\pi}^{\pi} |f(re^{i\theta})| \, d\theta = O(1). \tag{4}$$

Suppose first that $f(z)$ has no zeros in the circle $|z|<1$, and that $f(z) = g^2(z)$,

[1] Received March 16, 1933.

[2] A. Zygmund, "*On the convergence of lacunary trigonometric series*", Fundamenta Math., 16 (1930), pp. 90–107, Theorem G (106).

$$f(z) = \frac{1}{2}a_0 + c_1 z + c_2 z^2 + \cdots \qquad (c_n = a_n + i b_n),$$
$$g(z) = \quad d_0 + d_1 z + d_2 z^2 + \cdots,$$

where $\sum |d_n|^2 = O(1)$. Then

$$\begin{aligned} a_{\lambda_n}^2 + b_{\lambda_n}^2 &= |a_{\lambda_n} + i b_{\lambda_n}|^2 = |c_{\lambda_n}|^2 \\ &= \left| \sum_{m=0}^{\lambda_n} d_m d_{\lambda_n - m} \right|^2 \\ &\leqq 4 \left(\sum_{m=\left[\frac{1}{2}\lambda_n\right]}^{\lambda_n} |d_m d_{\lambda_n - m}| \right)^2 \\ &\leqq 4 \left(\sum_{m=0}^{\left[\frac{1}{2}\lambda_n\right]+1} |d_m|^2 \right) \left(\sum_{m=\left[\frac{1}{2}\lambda_n\right]}^{\lambda_n} |d_m|^2 \right) \\ &\leqq 4 \left(\sum_{m=0}^{\infty} |d_m|^2 \right) \left(\sum_{m=\left[\frac{1}{2}\lambda_n\right]}^{\lambda_n} |d_m|^2 \right), \end{aligned}$$

by Schwarz' inequality. Thus

$$\begin{aligned} \sum_{n=1}^{\infty} (a_{\lambda_n}^2 + b_{\lambda_n}^2) &\leqq 4 \left(\sum_{m=0}^{\infty} |d_m|^2 \right) \left(\sum_{n=1}^{\infty} \sum_{m=\left[\frac{1}{2}\lambda_n\right]}^{\lambda_n} |d_m|^2 \right) \\ &= 4 \left(\sum_{m=0}^{\infty} |d_m|^2 \right) \left(\sum_{m=0}^{\infty} |d_m|^2 \sum_{m \leqq \lambda_n \leqq 2m+1} 1 \right) \qquad (5) \\ &\leqq 4 \left(\sum_{m=0}^{\infty} |d_m|^2 \right)^2 \left(\frac{\log 2}{\log q} + 1 \right) < \infty. \end{aligned}$$

3. Finally if $f(z)$ has zeros, we can write[3]

$$f(z) = f_1(z) + f_2(z),$$

where f_1 and f_2 both satisfy inequalities of the form (4) and neither of them has a zero inside the unit circle. Thus the coefficients of each of them satisfies an inequality of the form (5), and hence the coefficients of f will also satisfy such an inequality.

[3] In virtue of a theorem of Hardy and Littlewood; see G. H. Hardy and J. E. Littlewood, "*Some new properties of Fourier constants*", Math. Annalen, 97 (1927), pp. 159-209 [207]. I am indebted to Professor Littlewood who suggested this point of the argument to me.

ÜBER ANALYTISCHE DEFORMATIONEN EINES RECHTECKS[1].

Von G. Pólya.

Eine Überlegung von L. Ahlfors, die übrigens in seinen vielversprechenden Untersuchungen über den Picardschen Satz eine wesentliche Rolle spielt[2], führte mich zu einer geometrischen Aussage über analytische Abbildungen, die wegen ihrer Einfachheit und Anschaulichkeit einer Mitteilung wert zu sein scheint (vgl. Satz I); mit ihrer Hilfe und zu ihrer Erläuterung will ich noch einen wohlbekannten funktionentheoretischen Hilfssatz (vgl. Satz II) auf neue Art und in verschärfter Form (vgl. Satz III) beweisen.

I.

Ich spreche im folgenden, der Kürze halber, von der Deformation des Rechtecks, wo ich eigentlich genauer von der Deformation des Rechtecks*randes* sprechen sollte.

Alle Figuren, die wir im folgenden zu betrachten haben, liegen in einer Ebene, deren Punkte auf die übliche Weise mit komplexen Zahlen bezeichnet sind. Unter *Deformation* eines Rechtecks verstehe ich eine eindeutige (aber im allgemeinen nicht ein-eindeutige!) stetige Abbildung des Rechtecksrandes. Die Deformation des Rechtecks ergibt also eine geschlossene stetige Kurve, die mehrfache Punkte haben kann.

Unter einer *analytischen* Deformation des Rechtecks verstehe ich eine Abbildung des Rechtecksrandes, die durch eine im abgeschlossenen Rechteck reguläre analytische Funktion entworfen wird. Heißt eine solche Funktion $f(z)$, so wird der Punkt $w = f(z)$ die aus der analytischen Deformation resultierende Kurve dann durchlaufen, wenn z den Rand des Rechtecks beschreibt.

Zu je zwei gegenüberliegenden Seiten des Rechtecks gehört ein Parallelstreifen; ich verstehe darunter das ins Unendliche sich erstreckende offene Gebiet, das zwischen den beiden Geraden liegt, in welche die betreffenden Rechtecksseiten verlängert werden können. Unter einer *einseitigen Kompression* des Rechtecks verstehe ich eine solche spezielle Deformation des Rechtecks, bei welcher die Bilder zweier gegenüberliegenden Rechtecksseiten ganz innerhalb des entsprechenden Parallelstreifens, die der beiden anderen

[1] Received March 8, 1933.

[2] Comptes Rendus, 194 (1932), pp. 245—247. Vgl. Lemma I.

Rechtecksseiten hingegen ganz außerhalb des zu ihnen gehörigen Parallelstreifens, u. zw. nach *verschiedenen Seiten* davon zu liegen kommen (wie beispielsweise in Fig. 1, wo das Entsprechen der Teile von Urbild und Abbild durch gleiche Ausziehung hervorgehoben ist). Die Seiten des Rechtecks und die entsprechenden Bögen der Bildkurve sind als abgeschlossen zu denken. Es gilt der Satz:

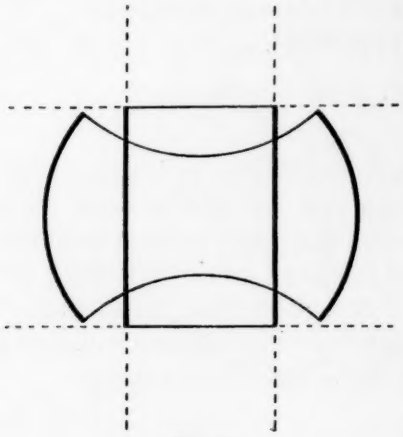

Fig. 1.

I. *Eine analytische Deformation eines Rechtecks ist nie eine einseitige Kompression.*

Die Konformität der analytischen Abbildung läßt diese Aussage als ziemlich „plausibel" erscheinen. Der Beweis ist sehr einfach. Es darf angenommen werden, daß die Seiten des in der z-Ebene gelegenen Rechtecks achsenparallel sind, die Südwestecke des Rechtecks sei mit $a+ib$, die Nordostecke mit $a+\alpha+i(b+\beta)$ bezeichnet, so daß a, b reell, α, β positiv sind. Die die Deformation des Rechtecksrandes hervorbringende, im abgeschlossenen Rechteck reguläre analytische Funktion heiße

$$w = f(z) = f(x+iy) = u(x, y)+iv(x, y);$$

es ist $z = x+iy$ gesetzt, x, y, u, v sind reell. Es genügt offenbar, den Fall zu betrachten, in welchem die Bilder der beiden vertikalen Rechtecksseiten außerhalb des durch sie bestimmten offenen vertikalen Streifens liegen. Ferner darf man annehmen, daß das Bild der rechten Vertikalseite rechts, das der linken links vom Streifen liegt. (Wäre es umgekehrt, so genügte es, die Bildfigur um den Mittelpunkt des Rechtecks um 180° zu drehen: lineare ganze Transformation von $f(z)$!) Die vorausgesetzte Situation der Bilder der beiden Vertikalseiten ergibt die Ungleichung

$$\text{(1)} \qquad \operatorname*{Min}_{b\leqq y, y'\leqq b+\beta} [u(a+\alpha, y)-u(a, y')] \geqq \alpha.$$

Um so mehr gilt

$$\text{(2)} \qquad \alpha \leqq \operatorname*{Min}_{b\leqq y\leqq b+\beta} [u(a+\alpha, y)-u(a, y)].$$

Man beachte, daß

$$\int f(z)\, dz = \int (u+iv)\, (dx+i\, dy),$$

entlang des betrachteten Rechtecksrandes erstreckt, verschwindet. Das Verschwinden des Imaginärteils ergibt

$$\int_b^{b+\beta} [u(a+\alpha, y) - u(a, y)]\, dy = \int_a^{a+\alpha} [v(x, b+\beta) - v(x, b)]\, dx.$$

Folglich ist

$$\beta \cdot \operatorname*{Min}_{b \leqq y \leqq b+\beta} [u(a+\alpha, y) - u(a, y)] \leqq \alpha \cdot \operatorname*{Max}_{a \leqq x \leqq a+\alpha} [v(x, b+\beta) - v(x, b)].$$

Dies mit (2) konfrontiert, ergibt

$$\beta \leqq \operatorname*{Max}_{a \leqq x \leqq a+\alpha} [v(x, b+\beta) - v(x, b)]. \tag{3}$$

Um so mehr gilt

$$\operatorname*{Max}_{a \leqq x,\, x' \leqq a+\alpha} [v(x, b+\beta) - v(x', b)] \geqq \beta. \tag{4}$$

Wir haben (4) aus (1) erschlossen. Das bedeutet aber: Wenn (vgl. (1)) die Bilder der Vertikalseiten aus dem offenen Vertikalstreifen von der Breite α ganz ausgeschlossen sind, so können (dies besagt (4)) die Bilder der Horizontalseiten nicht ganz in den offenen Horizontalstreifen von der Breite β eingeschlossen werden. Damit ist aber der ausgesprochene Satz bewiesen.

II.

Der eben bewiesene Satz I kann zu einem, wie mir scheint, neuen Beweise des folgenden wohlbekannten Satzes[3] verwendet werden.

II. *Die analytische Funktion $f(z)$ sei regulär im Halbstreifen*

$$-1 \leqq x \leqq 1, \quad y \geqq 0 \tag{5}$$

($z = x+iy$; x, y reell). Ferner sei

$$\lim_{y \to +\infty} f(-1+iy) = a, \quad \lim_{y \to +\infty} f(1+iy) = b. \tag{6}$$

Wenn $a \neq b$, so nimmt $f(z)$ im Streifen (5) *beliebig große Werte an.*

Ohne Beeinträchtigung der Allgemeinheit darf angenommen werden, daß $a = -2$, $b = 2$ ist. Es gibt dann, gemäß (6), ein y_1, so daß

$$\Re f(-1+iy) < -1, \qquad \Re f(1+iy) > 1 \quad \text{für } y \geqq y_1 \tag{7}$$

($\Re w$ heißt Realteil von w). Es sei $y_2 > y_1$. Die Vertikalseiten des Rechtecks

$$-1 \leqq x \leqq 1, \qquad y_1 \leqq y \leqq y_2$$

werden durch $f(z)$ auf Kurvenbögen abgebildet, die aus dem diesen Seiten entsprechenden Vertikalstreifen, gemäß (7), ganz hinausgedrängt sind; daher können, gemäß Satz I, die Bilder der Horizontalseiten des Rechtecks, ent-

[3] E. Lindelöf, *Mémoire sur certaines inégalités etc.*, Acta Societatis Scientiarum Fennicae, **25** (1908), No. 7; vgl. S. 28.

worfen sei es durch $f(z)$, sei es durch $f(z)+ic$, wo c reell, nicht ganz in den durch das Rechteck bestimmten Horizontalstreifen hineingedrängt sein: Für passende x_1, x_2, wo $-1 \leqq x_1, x_2 \leqq 1$, muß somit

$$|f(x_2+iy_2)-f(x_1+iy_1)| \geqq y_2-y_1$$

sein. Die rechte Seite wird aber mit y_2 beliebig groß: Dies beweist den Satz II.

Eigentlich haben wir etwas mehr bewiesen, nämlich:

III. *Die analytische Funktion $f(z)$ sei regulär im Halbstreifen* (5), *und sie soll für $x=-1$ nur Werte annehmen, die einer abgeschlossenen Halbebene A, und für $x=+1$ nur solche Werte, die einer abgeschlossenen Halbebene B angehören, wobei A und B keinen gemeinsamen Punkt haben. Dann ist $f(z)$ im Streifen* (5) *unbeschränkt.*

Es kann Satz III, wie Satz II, durch konforme Abbildung verallgemeinert werden, was ich wohl nicht genauer auszuführen brauche.

PRINCETON UNIVERSITY, PRINCETON, N. J.

THÉORIE GÉNÉRALE DES VARIÉTÉS ET DE LEURS THÉORÈMES DE DUALITÉ[1].

Par EDUARD ČECH.

La topologie combinatoire a fait dans les dernières années des progrès très remarquables et elle constitue, tant intrinsèquement qu'au point de vue des applications, un des plus importants chapitres de la topologie. Ceci est vrai tout particulièrement pour la théorie de l'homologie; c'est en effet la branche la plus avancée de la topologie combinatoire[2]. D'autre part, la topologie générale, fondée sur la théorie abstraite des ensembles, constitue désormais un édifice immense occupant une des places centrales dans le vaste champ des mathématiques modernes.

A mon avis, on peut espérer d'arriver à étendre d'une manière inattendue le champ de recherches topologiques si on réussit à approcher mutuellement le plus près possible les méthodes combinatoires et celles de la théorie générale des ensembles. Le premier pas essentiel et général dans cette direction et qui a déjà eu des conséquences très remarquables a été fait par M. Vietoris qui a réussi à fonder la théorie de l'homologie dans chaque espace métrique et compact[3]. Néanmoins il y a encore un profond abîme entre les recherches combinatoires et celles abstraites dans la topologie. Mon opinion est *d'une part* que la topologie combinatoire doit complètement abandonner l'usage des polyèdres, c'est-à-dire des figures dont la nature topologique (au sens d'une description axiomatique complète) nous restera probablement pour longtemps cachée; *d'autre part* qu'il est impossible d'étudier profondément déjà p. ex. la topologie des sous-ensembles de l'espace ordinaire si on s'obstine à négliger systématiquement les notions combinatoires.

Un des plus beaux sujets de la topologie combinatoire est sans doute la théorie des *variétés* (Mannigfaltigkeiten, manifolds). Or toutes les définitions connues[4] d'une variété V supposent ou que V soit homéomorphe à un polyèdre ou du moins que chaque point de V possède un entourage[5] homéomorphe à un

[1] Received February 15, 1933. — J'ai exposé quelques résultats de ce Mémoire dans une communication au Congrès Int. de Zurich (septembre 1932) ainsi que dans un cycle de quatre conférences que j'ai faites à l'Université de Varsovie (novembre 1932).

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[3] Math. Annalen, 97, 1927, pp. 454—472. Cf. déjà L. E. J. Brouwer, Math. Annalen, 72, 1912, pp. 422—425.

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Un des plus beaux sujets de la topologie combinatoire est sans doute la théorie des *variétés* (Mannigfaltigkeiten, manifolds). Or toutes les définitions connues[4] d'une variété V supposent ou que V soit homéomorphe à un polyèdre ou du moins que chaque point de V possède un entourage[5] homéomorphe à un

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logique de la notion de variété[6]. Dans le présent Ouvrage, je donne justement une telle définition comprenant du reste, comme on pourrait le démontrer sans difficulté, toutes les définitions connues comme des cas particuliers.

Je m'appuie dans ce qui suit très essentiellement sur mon Mémoire *Théorie générale de l'homologie dans un espace quelconque*[7] dont la connaissance est indispensable au lecteur; je le cite par l'abréviation *Homologie*. Outre ce quit a été exposé dans l'*Homologie*, je ne suppose de la topologie *combinatoire* que la connaissance des nombres de Betti d'un simplexe. Je suppose d'ailleurs quelques connaissances de la topologie générale, en particulier de la théorie de la dimension (v. n° 2.2). Relativement à l'*Homologie*, quatre remarques sont nécéssaires:

I. Pour un (p, R)-cycle, j'emploie ici la courte notation C^p (p. ex.) au lieu de $\{C^p(\mathfrak{U})\}$ (*Homologie*, II, 20). Donc C^p est l'ensemble de tous les $C^p(\mathfrak{U})$, $\mathfrak{U}$ parcourant tous les réseaux dans R.

II. La famille fondamentale de réseaux (*Homologie*, II, 1) est dans ce qui suit toujours la famille de tous les réseaux ouverts (v. *Homologie*, III, 2).

III. La notion d'un *affinement normal* (*Homologie*, II, 15) dépend essentiellement des deux ensembles A et α (v. l. c.); je parle donc toujours explicitement d'un *affinement normal rel. aux cycles mod α dans A* (l'attribut dans A sera omis si $A = R$; l'attribut mod α sera omis si $\alpha = 0$).

IV. Dans ce qui suit, les coefficients de toutes les chaînes appartiennent à l'ensemble $\mathfrak{R}$ des *nombres rationnels* (v. *Homologie*, I, 1 et II, 3); du reste, toute la théorie vaut sans aucune modification si $\mathfrak{R}$ désigne l'ensemble des entiers réduits mod p, p étant un nombre *premier*[8] (v. *Homologie*, V, 1).

Le présent Ouvrage est divisé en huit Chapitres.

Au Chap. I, j'introduis successivement les axiomes A_1 (n° 1), A_2 (n° 2), A_3 (n° 3) et A_4 (n° 7). Les deux axiomes A_1 et A_2 sont vérifiés en particulier si R est un espace *métrique et compact*. Bien que ce soit le cas le plus important, je n'ai pas voulu introduire explicitement la métrisabilité de R, d'autant mieux que cette hypothèse ne simplifie guère les démonstrations. On pourrait se passer entièrement de l'ensemble S introduit dans l'axiome A_3, en remplaçant A_1 et A_3 par l'axiome unique que $R - S$ soit un espace polyèdre. *Il n'existe donc jusqu'à présent aucune définition purement topo-*

[6] Après avoir terminé cet Ouvrage, j'ai pris connaissance d'un manuscript de M. Lefschetz intitulé *On generalized manifolds* (à paraître dans le Amer. Journal of Math.). L'illustre géomètre americain y étudie, avec une méthode entièrement différente, presque la même notion. Je voudrais insister sur une différence essentielle: pour démontrer le théorème de dualité entre les p-cycles et les $(n-p)$-cycles ($0 \leqq p \leqq n - p \leqq n$), je n'ai besoin d'aucun axiome relatif aux cycles de dimensions $< n - p - 1$.

[7] Fund. Math., 19, 1932, pp. 149—183.

[8] J'espère de revenir ailleurs sur la possibilité d'étendre ma théorie des variétés au cas d'autres domaines $\mathfrak{R}$.

localement bicompact[9]; ce procédé, logiquement équivalent à celui du texte, introduirait peut-être quelques difficultés d'exposition. L'axiome A_4 dit que la *dimension* n (au sens de Menger-Urysohn) de l'espace $R-S$ est finie (et >0). En appliquant le *allgemeiner Zerlegungssatz* de M. Menger[10] je prouve au n° 8 que, un réseau $\mathfrak{Z}$ étant donné, on peut définir une famille complète de réseaux « *commodes* » $\mathfrak{U}$ jouissant de la propriété suivante: chaque $(k, \mathfrak{U})$-simplexe τ^k est situé dans un certain sens sur un $(h, \mathfrak{Z})$-simplexe σ^h[11] de manière qu'on ait toujours $k \leqq n-h$. On voit que les σ^h sont pour notre variété abstraite ce que les faces sont pour une variété polyèdrale (à un σ^h correspondant une $(n-h)$-face du polyèdre).

Au Chap. II, j'introduis successivement les axiomes B (n° 9), D_1 (n° 11), D_2 (n° 12), E (n° 13) et F (n° 16) relatifs à la manière dont se comportent les cycles à n ou à $n-1$ dimensions de l'espace $R-S$. A l'aide de ces axiomes, on démontre (n° 17) l'existence d'un (n, R)-cycle G^n mod S, recouvrant tout l'espace R et appelé le *cycle principal*. En considérant un réseau $\mathfrak{Z}$ et les réseaux commodes $\mathfrak{U}$ relatifs, on attache (n° 19) à chaque $(0, \mathfrak{Z})$-simplexe σ^0 une $(n, \mathfrak{U})$-chaîne $K^n(\sigma^0, \mathfrak{U})$ située dans σ^0 de manière que la somme de tous les K^n soit égale à $G^n(\mathfrak{U})$. Successivement, on attache alors à chaque $(h, \mathfrak{Z})$-simplexe σ^h une $(n-h, \mathfrak{U})$-chaîne $K^{n-h}(\sigma^h, \mathfrak{U})$ située dans σ^h de manière que *les relations d'incidence entre les* σ^h *soient duelles de celles entre les* $K^{n-h}(\sigma^h, \mathfrak{U})$. On voit déjà que le théorème de dualité subsiste entre les $(p, \mathfrak{Z})$-chaînes et entre les $(n-p, \mathfrak{U})$-*chaînes élémentaires,* c'est-à-dire de la forme $\sum c_i K^{n-p}(\sigma_i^p, \mathfrak{U})$. Ce théorème de dualité n'a d'ailleurs encore aucune signification géométrique et on doit encore vaincre trois différentes difficultés:

α. On doit prouver que l'on peut s'arranger de façon que $K^{n-p}(\sigma_i^p, \mathfrak{U}) \neq 0$.

β. On doit prouver que, C^{n-p} étant un $(n-p, R)$-cycle, $C^{n-p}(\mathfrak{U})$ est homologue à une chaîne élémentaire.

γ. On doit prouver que si la chaîne élémentaire $C^{n-p}(\mathfrak{U})$ est homologue à zéro, il existe une $(n-p+1)$-chaîne *élémentaire* $D^{n-p+1}(\mathfrak{U})$ telle que $D^{n-p+1}(\mathfrak{U}) \to C^{n-p}(\mathfrak{U})$[12].

Pour vaincre les difficultés α, β et γ, on a besoin de nouveaux axiomes (le nombre de ces axiomes croit avec p; pour $p=0$ aucun nouvel axiome n'est plus nécessaire).

La difficulté α est résolue au Chap. III (n° 25) à l'aide des axiomes G^k (v. n° 21), où $n-p \leqq k \leqq n-1$. L'axiome G^k dit que, dans $R-S$, les petits cycles à k dimensions sont ~ 0.

[9] Tous les axiomes ultérieurs se rapportent uniquement à l'espace $R-S$.

[10] C'est peut-être la première application de ce théorème très important à mon avis.

[11] τ^k est d'espèce σ^h dans la terminologie du texte (v. n° 8.3).

[12] En réalité, je démontrerai un énoncé un peu plus compliqué que γ.

La difficulté β est résolue au Chap. IV (n° 31) à l'aide des axiomes G^k $(n-p \leqq k \leqq n-1)$ et des nouveaux axiomes H^k (v. n° 27), où $\max(0, n-p-1) \leqq k \leqq n-2$. La démonstration est très compliquée et presque entièrement combinatoire.

La difficulté γ est résolue au Chap. V (n° 44); on n'a plus besoin d'aucun nouvel axiome. La démonstration est très analogue à celle du Chap. IV.

Au Chap. VI, j'introduis (n° 56) l'indice de Kronecker d'un couple de cycles dont les dimensions sont resp. p et $n-p$. Cette théorie est complète à l'exception de la loi commutative que je démontrerai dans un Mémoire qui fera suite au présent, et où j'exposerai la théorie complète des *intersections des cycles* sur une variété abstraite.

Au Chap. VII, je démontre le théorème général de dualité dans une variété abstraite; pour une variété polyèdrale, c'est la formule (7), p. 142 de la *Topology* de M. Lefschetz.

Au Chap. VIII je prouve d'abord que les axiomes $G^k\left(0 \leqq k \leqq \frac{n-1}{2}\right)$ et $H^k\left(0 \leqq k \leqq \frac{n}{2}-1\right)$ sont des conséquences des autres axiomes. Ensuite, je généralise à mes variétés quelques autres théorèmes de dualité (théorèmes de Poincaré et de MM. Alexander, Lefschetz et Pontrjagin).

I.

1. Axiome A_1: *L'espace R est bicompact*[13]. Cela signifie que de chaque famille $\mathfrak{F}$ de sous-ensembles ouverts on peut extraire une famille finie recouvrant R (= réseau dans R).

1.1. *Un sous-ensemble A de R est bicompact si et seulement si A est fermé dans R.*

Démonstration. Supposons en premier lieu que $A = \bar{A}$. Soit $\mathfrak{F}_0$ un recouvrement de A. A chaque $U_0 \in \mathfrak{F}_0$ attachons un ensemble U ouvert dans R et tel que $U_0 = U \cdot A$; en ajoutant $R - A$ à ces ensembles U, on obtient un recouvrement $\mathfrak{F}$ de R. L'espace R étant bicompact, il existe un recouvrement fini $\mathfrak{F}_1$ extrait de $\mathfrak{F}$. Les $U_0 = U \cdot A$, où $U \in \mathfrak{F}_1$, forment un recouvrement fini de A extrait de $\mathfrak{F}_0$. Supposons en second lieu que l'ensemble $A \subset R$ soit bicompact. Alors $A = \bar{A}$ (v. Alexandroff et Urysohn, l. c. sub [13]), p. 47, corollaire 1).

1.2. *L'espace R est normal.* Cela signifie que, U étant un entourage d'un sous-ensemble fermé A de R, il existe un entourage V de A tel que $V \Subset U$.

Pour la démonstration, v. Alexandroff et Urysohn, l. c., p. 26, I.

[13] L. Vietoris, *Stetige Mengen*, Monatshefte f. Math. u. Phys., t. 31, 1921; P. Alexandroff et P. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verhandelingen Akad. Amsterdam, 1929.

1.3. *Chaque réseau (dans R) $\mathfrak{U}$ possède un affinement $\mathfrak{V}$ jouissant de la propriété suivante: A chaque sommet V de $\mathfrak{V}$ on peut attacher un sommet U de $\mathfrak{U}$ de manière que non seulement $V \subset U$ mais aussi $V' \subset U$ pour chaque sommet V' de $\mathfrak{V}$ tel que $VV' \neq 0$.*

Démonstration. Soient U_i $(1 \leqq i \leqq m)$ les sommets de $\mathfrak{U}$. D'après un lemme de M. Menger[14] on peut trouver des ensembles ouverts U_i' $(1 \leqq i \leqq m)$ tels que $U_i' \Subset U_i$, $\sum_1^m U_i' = R$. A chaque point a de R attachons un entourage V si petit que $1°$ $a \in U_i'$ entraîne $V \subset U_i'$; $2°$ $a \in \overline{U}_i'$ entraîne $V \subset U_i$; $3°$ $a \in R - \overline{U}_i'$ entraîne $VU_i' = 0$. D'après l'axiome A_1, il existe des points a_ν en nombre fini tels que les entourages V_ν correspondants constituent un réseau $\mathfrak{V}$. Pour chaque valeur de ν, il existe une valeur de i telle que $a_\nu \in U_i'$, d'où (d'après $1°$) $V_\nu \subset U_i'$. Il suffit de montrer que $V_\mu V_\nu \neq 0$ entraîne $V_\mu \subset U_i$. Or soit $b \in V_\mu V_\nu$; alors $b \in V_\mu U_i'$, d'où $V_\mu U_i' \neq 0$ et donc (d'après $3°$) $a_\mu \in \overline{U}_i'$ et par suite (d'après $2°$) $V_\mu \subset U_i$.

2. Axiome A_2: *Chaque sous-ensemble ouvert de R est un F_σ dans R.*

2.1. *Chaque sous-ensemble de R satisfait à l'axiome A_2*[15].

2.2. Nous allons rappeler les théorèmes de la théorie de la dimension dont nous ferons usage dans ce qui suit. Ces théorèmes sont vrais dans chaque espace satisfaisant aux axiomes A_1 et A_2, et même plus généralement dans chaque espace topologique normal satisfaisant à l'axiome A_2. Pour les démonstrations je renvoie au livre de M. Menger: *Dimensionstheorie* (cité par M) pour le cas particulier d'un espace métrique séparable ainsi qu'à mon Mémoire: *Sur la dimension des espaces parfaitement normaux*[16] (cité par Č) pour le cas général.

2.21[17]. $\dim R = -1$ signifie que $R = 0$. A étant un sous-ensemble fermé de R, $\dim_A R \leqq n$ signifie qu'à chaque entourage U de A on peut attacher un entourage $V \subset U$ de A tel qu'on ait $\dim \Phi \leqq n-1$, où $\Phi = \overline{V} - V$. $\dim R = n$ signifie que: $1°$ $\dim_A R \leqq n$ pour chaque sous-ensemble fermé A de R; $2°$ la relation $\dim_A R \leqq n-1$ n'est pas vraie pour chaque sous-ensemble fermé A de R[18].

2.22[19]. $S \subset R$ entraîne que $\dim S \leqq \dim R$.

[14] *Dimensionstheorie*, pp. 159–160, „Bemerkung". Le lemme est évidemment vrai dans chaque espace normal.

[15] V. Urysohn, *Über die Mächtigkeit zusammenhängender Mengen*, Math. Annalen, t. 94, p. 286, note [41]) au bas de la page.

[16] Bull. intern. de l'Acad. des Sc. de Bohème, 1932.

[17] M, Chap. II (v. aussi p. 116); Č, Chap. II.

[18] Dans cette définition, il est permis de limiter A aux points de R, si l'espace R jouit de la propriété suivante: de chaque famille de sous-ensembles ouverts recouvrant R on peut extraire une famille dénombrable recouvrant R (cette propriété vaut si R est un sous-ensemble d'un espace satisfaisant aux axiomes A_1 et A_3).

[19] M, p. 81; Č, n° 23.

2.23[20]. Soit $R = \sum_{\nu=1}^{\infty} R_\nu$, les R_ν étant fermés dans R. Alors $\dim R = \max.\dim R_\nu$.

2.24[21]. Soit $\dim R \leqq n$; soit $\mathfrak{U}$ un réseau dans R. Il existe un affinement $\mathfrak{V}$ de $\mathfrak{U}$ dont l'ordre (v. cet Ouvrage, n° 7.2) est $\leqq n$.

2.25[22]. Soit $\dim R \leqq n$. Soit $\mathfrak{U}$ un réseau dans R; soient $U_1, U_2, \cdots, U_m$ les sommets de $\mathfrak{U}$. Il existe des sous-ensembles fermés $F_1, F_2, \cdots, F_m$ de R tels que 1° $F_\nu \subset U_\nu$; 2° $\dim F_{\nu_0} \cdot F_{\nu_1} \cdot \cdots \cdot F_{\nu_h} \leqq n-h$ pour $0 \leqq h \leqq n+1$; 3° $\sum_1^m F_\nu = R$[23].

2.26[24]. Soit $R = R_0 \supset R_1 \supset \cdots \supset R_k$, les R_i étant fermés dans R. Soit $\dim R_i = n_i$. Soit $a \in R_k$. Soit U un entourage de a. Il existe un entourage $V \subset U$ de a tel que $\dim R_i(\bar{V} - V) \leqq n_i - 1$ pour $0 \leqq i \leqq k$.

2.3. Si $R = R_0 \supset R_1 \supset \cdots \supset R_n$, les R_k étant des sous-ensembles fermés de R tels que $\dim R_k \leqq n-k$ pour $0 \leqq k \leqq n$, alors chaque réseau $\mathfrak{U}$ dans R possède un affinement $\mathfrak{V}$ jouissant de la propriété suivante: Si chaque sommet d'un $(h, \mathfrak{V})$-simplexe σ^h rencontre R_k $(0 \leqq k \leqq n)$, on a $h \leqq n-k$.

Démonstration. Le théorème étant évident pour $n = 0$ (v. 2.24), supposons le vrai pour la dimension $n-1$ (= l'hypothèse H). Soit donné le réseau $\mathfrak{U}$. A chaque point a de R attachons un entourage $V \subset U \in \mathfrak{U}$ tel que (v. 2.26) $\dim R_k F(V) \leqq n-k-1$ pour $0 \leqq k \leqq n$, en particulier $R_n F(V) = 0$. D'après l'axiome A_1, il existe des points a_ν $(1 \leqq \nu \leqq s)$ en nombre fini tels que $\sum_1^s V_\nu = R$. Posons

$$V_1' = V_1, \qquad V_\nu' = V_\nu - \sum_{i=1}^{\nu-1} \bar{V}_i \quad \text{pour} \quad 2 \leqq \nu \leqq s.$$

On a alors $\sum_1^s \bar{V}_\nu' = R$, $V_\mu' V_\nu' = 0$, d'où $V_\mu' F(V_\nu') = 0$ et enfin $F(V_\nu') \subset \sum_{i=1}^{\nu} F(V_\nu)$ et par suite (v. 2.23) $\dim R_k F(V_\nu') \leqq n-k-1$ pour $0 \leqq k \leqq n$, $1 \leqq \nu \leqq s$, en particulier $R_n \cdot F(V_\nu') = 0$. Posons $R_0' = \sum_1^s F(V_\nu')$, $R_k' = R_k \cdot R_0'$ $(1 \leqq k \leqq n-1)$. Alors $\dim R_k' \leqq n-1-k$ pour $0 \leqq k \leqq n-1$. En vertu de 1.1 et 2.1, l'espace R_0' satisfait aux axiomes A_1 et A_2. Donc,

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[20] M, Chap. III; Č, Chap. III.

[21] M, pp. 158–161; Č, n° 26.

[22] M, pp. 161–174; Č, n° 26.

[23] En réalité on trouve l. c. seulement la démonstration de l'existence d'un réseau fermé $\mathfrak{F}$ qui est un affinement de $\mathfrak{U}$ et tel que $\dim \Phi_{i_0} \Phi_{i_1} \cdots \Phi_{i_h} \leqq n-h$ pour $\Phi_i \in \mathfrak{F}$. Or il suffit de partager les sommets Φ de $\mathfrak{F}$ en m groupes de manière que $\Phi \subset U_\nu$ pour chaque Φ du $\nu^{\text{ème}}$ groupe et de désigner par F_ν la somme de tous les Φ de $\nu^{\text{ème}}$ groupe.

[24] M, p. 169; Č, n° 24.2.

d'après l'hypothèse H, il existe des ensembles v_μ $(1 \leqq \mu \leqq t)$ en nombre fini ouverts dans R_0' et tels que: 1° $\sum_1^t v_\mu = R_0'$; 2° chaque v_μ fait partie d'un sommet U_u de $\mathfrak{U}$; 3° si les indices $\mu_0, \mu_1, \cdots, \mu_h$ différents l'un de l'autre sont tels que $\prod_0^h v_{\mu_i} \neq 0$ et que (pour une certaine valeur de k, $0 \leqq k \leqq n-1$) $R_k' v_{\mu_i} \neq 0$ pour $0 \leqq i \leqq h$, alors $h \leqq n-k-1$. Pour $1 \leqq \mu \leqq t$, soit V_μ'' un sous-ensemble ouvert de R tel que 1° $v_\mu \subset V_\mu'' \subset U_\mu - R_n$; 2° si $v_\mu \cdot R_k' = 0$ $(1 \leqq \mu \leqq t,\ 1 \leqq k \leqq n-1)$, alors $V_\mu'' \cdot R_k = 0$; 3° si $\prod_0^h v_{\mu_i} = 0$ alors $\prod_0^h V_{\mu_i}'' = 0$[25]. Les ensembles V_ν' $(1 \leqq \nu \leqq s)$ et V_μ'' $(1 \leqq \mu \leqq t)$ constituent l'affinement cherché $\mathfrak{V}$ de $\mathfrak{U}$.

3. Axiome A_3. *S est un sous-ensemble fermé de R; $S \neq R$.*

3.1. *Il existe une suite dénombrable $\{F_i\}$ de sous-ensembles bicompacts de $R-S$ telle que* 1° $F_{i+1} \supset F_i$; 2° $R-S = \sum_1^\infty F_i$; 3° *pour chaque sous-ensemble bicompact A de $R-S$ il existe une valeur de i telle que $A \subset F_i$. On peut même supposer $F_i = \bar{G}_i$, G_i étant un sous-ensemble ouvert de $R-S$, $G_i \Subset G_{i+1}$.*

Démonstration. D'après les axiomes A_2 et A_3, il existe une suite $\{\Phi_i\}$ de sous-ensembles bicompacts (v. 1.1) de $R-S$ telle que $R-S = \sum_1^\infty \Phi_i$. S et Φ_1 sont deux sous-ensembles fermés disjoints de l'espace R. D'après 1.2, il existe un ensemble ouvert G_1 tel que $\Phi_1 \subset G_1 \Subset R-S$. Plus généralement, supposons que, pour une certaine valeur de k $(= 2, 3, \cdots)$, on ait déjà construit des ensembles ouverts G_i $(1 \leqq i \leqq k-1)$ tels que $G_{i-1} \Subset G_i$, $\Phi_i \subset G_i \Subset R-S$. Les ensembles $\Phi_k + \bar{G}_{k-1}$ et S étant fermés dans l'espace normal R et disjoints, il existe un ensemble ouvert G_k tel que $\Phi_k + \bar{G}_{k-1} \subset G_k \Subset R-S$. La suite $\{G_i\}$ étant ainsi construite par récurrence, il suffit de poser $F_i = \bar{G}_i$. En effet, si A est un sous-ensemble bicompact de $R-S$, on a $\sum_1^\infty AG_i = A$, d'où, d'après la définition même de la bicompacticité, $A = \sum_1^k AG_i$ pour une certaine valeur de k, et donc $A \subset \sum_1^k G_i = G_k \subset F_k$.

4. Un *réseau gén.* (= généralisé) $\mathfrak{P}$ est une famille $\{P_i\}$ au plus dénombrable de sous-ensembles ouverts et non vides de $R-S$ telle que: 1° $R-S = \sum_1^\infty P_i$; 2° pour chaque sous-ensemble bicompact A de $R-S$ on a $AP_i = 0$ pour presque toutes les valeurs de i. En particulier, un

[25] V. *Homologie*, III, 21. L'hypothèse que l'espace soit complètement normal est ici vérifiée en vertu de 1.3 et 2.3 (v. Urysohn, l. c. sub [15]).

point $a \in R - S$ ne peut appartenir qu'à un nombre fini des P_i. Les P_i sont les *sommets* du réseau gén. $\mathfrak{P}$.

Dans le cas particulier $S = 0$ [ou, plus généralement, si $R - S$ est fermé dans R] l'ensemble $R - S$ est bicompact; en vertu de la condition 2°, les ensembles P_i sont alors en nombre fini et le réseau gén. est simplement un réseau dans $R - S$.

4.1. Soit $\mathfrak{P}$ une famille de sous-ensembles ouverts de $R - S$ (en particulier, $\mathfrak{P}$ peut être un réseau gén.). Soit $\mathfrak{Q}$ un réseau gén. On dit que $\mathfrak{Q}$ constitue un *affinement* de $\mathfrak{P}$, si chaque sommet Q de $\mathfrak{Q}$ est contenu dans un élément (sommet) de $\mathfrak{P}$.

4.2. *Soit $\mathfrak{P}$ une famille comme dans* 4.1. *Il existe un réseau gén. $\mathfrak{Q}$ qui soit un affinement de $\mathfrak{P}$.*

Démonstration. Déterminons $\{F_i\}$ d'après 3.1. Les ensembles F_i étant bicompacts et $\subset R - S$, pour chaque valeur de i il existe un nombre fini d'éléments $P_{i1}, P_{i2}, \cdots, P_{i\lambda_i}$ de $\mathfrak{P}$ recouvrant F_i. Posons $Q_{ik} = P_{ik} - F_{i-1}$ $(1 \leqq k \leqq \lambda_i)$, où $F_0 = 0$. La suite Q_{ik} $(i = 1, 2, 3, \cdots; k = 1, 2, \cdots, \lambda_i)$ dont on supprime les membres vides, constitue le réseau gén. $\mathfrak{Q}$ cherché.

4.3. *Chaque réseau gén. $\mathfrak{P}$ possède un affinement $\mathfrak{Q}$ jouissant de la propriété suivante: Pour chaque $Q \in \mathfrak{Q}$ il existe un $P \in \mathfrak{P}$ tel que $Q \in P$. En particulier $Q \in R - S$ pour chaque sommet Q de $\mathfrak{Q}$.*

Démonstration. Déterminons les P_{ik} $(i = 1, 2, 3, \cdots; k = 1, 2, \cdots, \lambda_i)$ comme dans 4.2. Pour chaque valeur de i les ensembles ouverts $P_{i1}, P_{i2}, \cdots, P_{i\lambda_i}$ recouvrent le sous-ensemble fermé F_i de l'espace normal R. Il existe donc (v. [14]) des ensembles Q_{ik} $(k = 1, 2, \cdots, \lambda_i)$ tels que $Q_{ik} \in P_{ik}$, $\sum_{k=1}^{\lambda_i} Q_{ik} \supset F_i$. Les Q_{ik} $(i = 1, 2, 3, \cdots; k = 1, 2, \cdots, \lambda_i)$ constituent le réseau gén. $\mathfrak{Q}$ cherché.

4.4. *Chaque réseau gén. $\mathfrak{P}$ possède un affinement $\mathfrak{Q}$ jouissant de la propriété suivante: A chaque sommet Q de $\mathfrak{Q}$ on peut attacher un sommet P de $\mathfrak{P}$ de manière que non seulement $Q \subset P$ mais aussi $Q' \subset P$ pour chaque sommet Q' de $\mathfrak{Q}$ tel que $QQ' \neq 0$.*

Démonstration. Déterminons $\{F_i\}$ d'après 3.1. On peut (v. 4.3) déterminer deux suites $\{P_i\}$ et $\{P_i'\}$ d'ensembles ouverts telles que $P_i \in \mathfrak{P}$, $P_i' \in P_i$, $\sum_1^\infty P_i' = \sum_1^\infty P_i = R - S$. A chaque point a de $R - S$ attachons un entourage Q si petit que: 1° $a \in P_i'$ entraîne $Q \subset P_i'$; 2° $a \in \bar{P}_i'$ entraîne $Q \subset P_i$; 3° $a \in R - \bar{P}_i'$ entraîne $QP_i' = 0$[26]; 4° $a \in R - F_i$ entraîne $QF_i = 0$. Les

[26] On ne peut prescrire qu'un nombre fini de telles conditions à l'entourage Q. Or d'après la définition même du réseau gén., la relation $a \in P_i$ (et *a fortiori* $a \in P_i'$ ou $a \in \bar{P}_i'$) ne peut avoir lieu que pour un nombre fini d'indices i de manière que 1° et 2° n'imposent qu'un nombre fini de conditions. Quant à 3°, déterminons un entourage U de a tel que

ensembles F_i étant bicompacts, de la relation $R - S = \sum_1^\infty F_i$ on voit qu'il existe une suite dénombrable $\{a_\nu\}$ de points de $R - S$ telle que les Q_ν correspondants recouvrent $R - S$; de plus, pour une valeur donnée de k, on a $a_\nu \in F_k$ pour un nombre fini d'indices ν, d'où, en vertu de 4°, $Q_\nu F_k = 0$ pour presque toutes les valeurs de ν. Les Q_ν constituent donc (v. 3.1) un réseau gén. $\mathfrak{Q}$. Pour chaque valeur de ν, il existe une valeur de i telle que $a_\nu \in P_i'$, d'où $Q_\nu \subset P_i'$ d'après 1°. Si $Q_\mu Q_\nu \neq 0$, on a $Q_\mu P_i' \neq 0$ et donc (d'après 3°) $a_\mu \in \overline{P}_i'$ et par suite (d'après 2°) $Q_\mu \subset P_i$.

5.1. On dit qu'un réseau $\mathfrak{U}$ est un *affinement mod S* d'un réseau gén. $\mathfrak{P}$, si chaque sommet U de $\mathfrak{U}$ tel que $US = 0$ est un sous-ensemble d'un sommet de $\mathfrak{P}$.

5.2. Une famille Φ de réseaux dans R s'appellera *parfaitement complète* si, $\mathfrak{U}$ étant un réseau et $\mathfrak{P}$ étant un réseau gén., il existe dans Φ un affinement de $\mathfrak{U}$ qui soit un affinement mod S de $\mathfrak{P}$. Une famille parfaitement complète est complète (dans le sens de *Homologie*, II, 30 et relativement à la famille fondamentale Z constituée par tous les réseaux ouverts dans R).

5.3. *Soit $\mathfrak{U}$ un réseau; soit $\mathfrak{P}$ un réseau gén. Il existe un réseau $\mathfrak{V}$ tel que* 1° *$\mathfrak{V}$ est un affinement de $\mathfrak{U}$*; 2° *$\mathfrak{V}$ est un affinement mod S de $\mathfrak{P}$*; 3° *les sommets rencontrant S sont les mêmes dans les deux réseaux $\mathfrak{U}$ et $\mathfrak{V}$*; 4° *$V \in \mathfrak{V}$, $VS = 0$ entraîne $\overline{V}S = 0$.*

Démonstration. Soit G la somme de tous les sommets de $\mathfrak{U}$ rencontrant S. Soient $U_i\,(1 \leqq i \leqq k)$ les autres sommets de $\mathfrak{U}$. Soit $P_\nu\,(1 \leqq \nu \leqq m)$ les sommets de $\mathfrak{P}$ rencontrant $R - G$ (ces sommets sont en nombre fini, car $R - G$ est un sous-ensemble bicompact de $R - S$). L'ensemble $R - G$ étant fermé dans l'espace normal R et recouvert par les ensembles $U_i \cdot P_\nu\,(1 \leqq i \leqq k,\ 1 \leqq \nu \leqq m)$, il existe, d'après le lemme [14], des ensembles ouverts, $V_{i\nu}'$ tels que $V_{i\nu} \in U_i \cdot P_\nu$, $\sum V_{i\nu} \supset R - G$. Le réseau $\mathfrak{V}$ s'obtient de $\mathfrak{U}$ en remplaçant chaque $U_i\,(1 \leqq i \leqq k)$ par les $V_{i\nu}\,(1 \leqq \nu \leqq m)$, dont on omet ceux qui sont vides.

5.4. *Soit $\mathfrak{U}$ un réseau; soit $\mathfrak{P}$ un réseau gén.; soit Φ une famille parfaitement complète de réseaux. La famille de tous les réseaux $\mathfrak{V}$ tels que* 1° *$\mathfrak{V} \in \Phi$*; 2° *$\mathfrak{V}$ est un affinement de $\mathfrak{U}$*; 3° *$\mathfrak{V}$ est un affinement mod S de $\mathfrak{P}$, est parfaitement complète.*

La démonstration est banale.

5.5. *Soit Φ la famille de tous les réseaux $\mathfrak{U}$ tels que:* 1° *$\mathfrak{U}$ est régulier par rapport à S* (v. *Homologie*, III, 22); 2° *$U \in \mathfrak{U}$, $US = 0$ entraîne $\overline{U}S = 0$. La famille Φ est parfaitement complète.*

Cela résulte de 5.3 et de *Homologie*, III, 23.

$U \in R - S$ et posons la nouvelle condition 5°: $Q \subset U$. La condition 3° est alors vérifiée pour tous les indices i tels que $0 = \overline{U} P_i \supset \overline{U} P_i'$. Or $\overline{U}$ étant un sous-ensemble bicompact de $R - S$, l'inégalité $\overline{U} P_i \neq 0$ n'a lieu que pour un nombre fini d'indices i. Enfin, d'après 3.1 on a $a \in F_i$ pour presque toutes les valeurs de i.

6.1. *Déterminons* $\{G_i\}$ *d'après* 3.1. *Soit* $p = 0, 1, 2, \cdots$. *Pour chaque valeur de* i *soit* L_i *un système linéaire de* (p, R)*-cycles mod* $(R - G_i)$ *tel que* $C^p \in L_i$ *pour chaque* (p, R)*-cycle* C^p *mod* $(R - G_i)$ *jouissant de la propriété que, pour chaque réseau* $\mathfrak{U}$ *d'une famille complète donnée il existe un* $C_i^p \in L_i$ *tel que* $C^p(\mathfrak{U}) \sim C_i^p(\mathfrak{U})$ *mod* $(R - G_i)$. *Si* $i < k$, *supposons qu'à chaque* $C_k^p \in L_k$ *on puisse attacher un* $C_i^p \in L_i$ *de manière que* $C_i^p \sim C_k^p$ *mod* $(R - G_i)$. *Il existe un* (p, R)*-cycle* C^p *mod* S *tel que pour chaque valeur de* i *il existe un* $C_i^p \in L_i$ *jouissant de la propriété* $C^p \sim C_i^p$ *mod* $(R - G_i)$.

Démonstration. Soit Φ la famille complète du n° 5.5. Soit $\mathfrak{U} \in \Phi$. Soit K la somme de tous les sommets de $\mathfrak{U}$ disjoints à S. D'après la définition de la famille Φ, on a $K \in R - S$, de manière qu'il existe (v. 3.1) un indice i_0 tel que $K \subset G_i$ pour $i \geqq i_0$. Alors si un sommet U de $\mathfrak{U}$ rencontre $R - G_i$ $(i \geqq i_0)$, il rencontre aussi S; le réseau $\mathfrak{U}$ étant (v. 5.5) régulier par rapport à S, on a plus généralement: si le noyau d'un $\mathfrak{U}$-simplexe rencontre $R - G_i$ $(i \geqq i_0)$, il rencontre aussi S. Donc les frontières, cycles, homologies mod S et mod $(R - G_i)$ $(i \geqq i_0)$ coïncident dans le réseau $\mathfrak{U}$.

Pour $i \geqq i_0$, soit $\mathcal{A}_i^p(\mathfrak{U})$ la famille de tous les $(p, \mathfrak{U})$-cycles $C^p(\mathfrak{U})$ mod S (ou, ce qui est la même chose, mod $(R - G_i)$) tels qu'il existe un $C_i^p \in L_i$ tel que $C^p(\mathfrak{U}) \sim C_i^p(\mathfrak{U})$ mod $(R - G_i)$ (et donc mod S). On voit sans peine que $\mathcal{A}_i^p(\mathfrak{U}) \neq 0$ et que $\mathcal{A}_k^p(\mathfrak{U}) \subset \mathcal{A}_i^p(\mathfrak{U})$ pour $k > i \geqq i_0$. Donc

$$\prod_{i \in N} \mathcal{A}_i^p(\mathfrak{U}) = \Pi(N) \neq 0$$

pour chaque ensemble fini N d'indices $i \geqq i_0$, car $\Pi(N) = \mathcal{A}_k^p(\mathfrak{U})$ pour $k = \max. N$. $\mathcal{A}_i^p(\mathfrak{U})$ est évidemment un système linéaire contenu dans le module fini de toutes les $(p, \mathfrak{U})$-chaînes. Par suite (v. *Homologie*, I, 15) la partie commune $\mathcal{A}^p(\mathfrak{U})$ de tous les $\mathcal{A}_i^p(\mathfrak{U})$ $(i \geqq i_0)$ est non vide; c'est donc un système linéaire.

Or, soit $\mathfrak{U}, \mathfrak{V} \in \Phi$ et supposons que $\mathfrak{V}$ soit un affinement de $\mathfrak{U}$, $\pi = Pr.(\mathfrak{V}, \mathfrak{U})$. On voit sans peine que $\pi \mathcal{A}^p(\mathfrak{V}) \subset \mathcal{A}^p(\mathfrak{U})$. Donc (v. *Homologie*, II, 21) il existe un (p, R)-cycle C^p mod S tel que $C^p(\mathfrak{U}) \in \mathcal{A}^p(\mathfrak{U})$ pour chaque réseau $\mathfrak{U} \in \Phi$. Soit donnée une valeur de i; on doit prouver que $C^p \sim C_i^p$ mod $(R - G_i)$, $C_i^p \in L_i$. On peut même démontrer que $C^p \in L_i$. Il suffit de prouver que pour chaque $\mathfrak{U} \in \Phi$ il existe un $C_i^p \in L_i$ tel que $C^p(\mathfrak{U}) \sim C_i^p(\mathfrak{U})$ mod $(R - G_i)$. Or soit $\mathfrak{U} \in \Phi$; soit $k \geqq i_0$. On a $C^p(\mathfrak{U}) \in \mathcal{A}^p(\mathfrak{U}) \subset \mathcal{A}_k^p(\mathfrak{U})$, d'où $C^p(\mathfrak{U}) \sim C_k^p(\mathfrak{U})$ mod S avec $C_k^p \in L_k$. Or on peut attacher à C_k^p un $C_i^p \in L_i$ tel que $C_k^p \sim C_i^p$ mod $(R - G_i)$, d'où $C^p(\mathfrak{U}) \sim C_i^p(\mathfrak{U})$ mod $(R - G_i)$.

6.2. *Déterminons* G_i *d'après* 3.1. *Soit* $p = 0, 1, 2, \cdots$. *Soit* C^p *un* (p, R)*-cycle mod* S *tel que* $C^p \sim 0$ *mod* $(R - G_i)$ *pour chaque* i. *Alors* $C^p \sim 0$ *mod* S.

Démonstration. La famille Φ du n° 5.5 étant complète, il suffit de voir que $C^p(\mathfrak{U}) \sim 0$ mod S pour chaque réseau $\mathfrak{U} \in \Phi$. Or nous venons de voir

que, $\mathfrak{U}$ étant donné, il existe une valeur de i telle que $C^p(\mathfrak{U}) \sim 0 \bmod (R - G_i)$ (ce qui a été supposé pour chaque i) entraîne $C^p(\mathfrak{U}) \sim 0 \bmod S$.

6.3. *Soient R_1, R_2 deux espaces; soit $S_1 \subset R_1$, $S_2 \subset R_2$. Supposons que R_1 et S_1, ainsi que R_2 et S_2, vérifient les axiomes A_1, A_2, A_3. Supposons de plus que $R_1 - S_1$ soit homéomorphe à $R_2 - S_2$. Alors, pour $p = 0, 1, 2, \cdots$, le $p^{\text{ème}}$ groupe de Betti de R_1 mod S_1 est isomorphe au $p^{\text{ème}}$ groupe de Betti de R_2 mod S_2.*

Démonstration. Soit φ l'homéomorphie donnée entre $R_1 - S_1$ et $R_2 - S_2$. Dans l'espace $R_1 - S_1$, définissons la suite $\{G_i\}$ comme dans 3.1. La suite $\{G_i'\}$, où $G_i' = \varphi G_i$, jouit de propriétés analogues par rapport à $R_2 - S_2$. Soit C^p un (p, R_1)-cycle mod S_1. Pour $i = 1, 2, \cdots$, soit C_i^p un $(p, \bar{G}_i)$-cycle mod $(\bar{G}_i - G_i)$ tel qu'on ait $C^p \sim C_i^p \bmod (R_1 - G_i)$. On voit sans peine que de tels cycles existent. Posons ${}^*C_i^p = \varphi C_i^p$; c'est un $(p, \bar{G}_i')$-cycle mod $(\bar{G}_i' - G_i')$ que l'on peut considérer aussi comme un (p, R_2)-cycle mod $(R_2 - G_i')$. Evidemment on a pour $i < k$: ${}^*C_i^p = {}^*C_k^p \bmod (R_2 - G_i')$. D'après 6.1 il existe un (p, R_2)-cycle mod S_2 : ${}^*C^p$ tel que ${}^*C^p \sim {}^*C_i^p \bmod (R_2 - G_i')$ pour chaque i. Posons ${}^*C^p = \psi C^p$. De 6.2 il résulte aisément que ψ définit une isomorphie entre les deux groupes de Betti considérés.

7. Axiome A_4 : *La dimension* (v. 2.21) *de $R - S$ est égale à n* ($= 1, 2, 3, \cdots$).

7.1. Soit $\mathfrak{U}$ un réseau dans R. Un $\mathfrak{U}$-simplexe (en particulier un sommet de $\mathfrak{U}$) sera dit *intérieur* si aucun de ses sommets ne rencontre S, *extérieur* dans le cas contraire.

7.2. L'*ordre* d'un réseau $\mathfrak{U}$ dans R est la plus grande dimension d'un $\mathfrak{U}$-simplexe. L'*ordre mod S* de $\mathfrak{U}$ est la plus grande dimension m d'un $\mathfrak{U}$-simplexe intérieur; si chaque sommet de $\mathfrak{U}$ rencontre S, on pose $m = -1$.

7.3. *Soit $\mathfrak{U}$ un réseau; soit $\mathfrak{P}$ un réseau gén. Il existe un réseau $\mathfrak{V}$ tel que 1° $\mathfrak{V}$ est un affinement de $\mathfrak{U}$; 2° $\mathfrak{V}$ est un affinement mod S de $\mathfrak{P}$; 3° les sommets rencontrant S sont les mêmes dans les deux réseaux $\mathfrak{U}$ et $\mathfrak{V}$; 4° l'ordre mod S de $\mathfrak{V}$ est $\leqq n$.*

Démonstration. Soit G la somme des sommets extérieures de $\mathfrak{U}$. Soient U_i $(1 \leqq i \leqq k)$ les sommets intérieurs de $\mathfrak{U}$. Les ensembles $(R - G)\, U_i$ constituent un réseau $\mathfrak{U}'$ dans $R - G$. Soient $P_1, P_2, \cdots, P_m$ les sommets de $\mathfrak{P}$ rencontrant $R - G$; les ensembles $P_\nu - G$ $(1 \leqq \nu \leqq m)$ constituent un réseau $\mathfrak{W}'$ dans $R - G$. Puisque $R - G$ est un espace bicompact de dimension $\leqq n$, il existe un affinement $\mathfrak{V}'$ simultané de $\mathfrak{U}'$ et de $\mathfrak{W}'$ dont l'ordre est $\leqq n$. Soient V_j' $(1 \leqq j \leqq h)$ les sommets de $\mathfrak{V}'$. Déterminons (v. *Homologie*, III, 21) des sous-ensembles ouverts V_j de $R - S$ de manière que $(R - G)\, V_j = V_j'$ et que $\Pi\, V_{j_\nu}' = 0$ entraîne $\Pi\, V_{j_\nu} = 0$. $\mathfrak{V}'$ étant un affinement de $\mathfrak{U}'$ et de $\mathfrak{W}'$ on peut s'arranger de manière que chaque V_j fasse partie d'un U_i et d'un sommet de $\mathfrak{P}$. En ajoutant aux V_j les sommets extérieurs de $\mathfrak{U}$, on obtient le réseau cherché $\mathfrak{V}$.

7.4. *Soit Φ_1 la famille de tous les réseaux $\mathfrak{U}$ de la famille Φ du n° 5.5 dont l'ordre mod S est $\leqq n$. La famille Φ_1 est parfaitement complète.*

D'après 5.5 et 7.3.

8. Soit $\mathfrak{Z}$ un réseau de la famille Φ_1 du n° 7.4 possédant des sommets intérieurs. Soit G la somme de tous les sommets extérieurs de $\mathfrak{Z}$. Pour $h = 0, 1, 2, \cdots$ soient $\sigma_i^h (1 \leqq i \leqq \alpha_h)$ tous les $(h, \mathfrak{Z})$-simplexes intérieurs rangés dans un ordre déterminé. (On a $\alpha_h = 0$ pour $h > n$.) En particulier $\sigma_i^0 (1 \leqq i \leqq \alpha_0)$ sont les sommets intérieurs de $\mathfrak{Z}$ de manière que les $\sigma_i^0 - G$ constituent un réseau dans l'espace bicompact $R - G$. Puisque $\dim(R-S) \leqq n$, d'après 2.25 il existe des sous-ensembles bicompacts T_i $(1 \leqq i \leqq \alpha_0)$ de $R - S$ tels que $T_i \subset \sigma_i^0$, $\sum_1^{\alpha_0} T_i \supset R - G$[27] et enfin $\dim R_k \leqq n - k$ pour $0 \leqq k \leqq n+1$, où R_k désigne l'ensemble de tous les points de R appartenant à $k+1$ au moins parmi les ensembles T_i. (On a en particulier $R_0 = \sum_1^{\alpha_0} T_i$, $R_{n+1} = 0$.) Désignons par $\mathfrak{T}$ le réseau fermé dans R_0 aux sommets T_i. Pour $0 \leqq h \leqq n$, $1 \leqq i \leqq \alpha_h$, $\sigma_i^h = (\sigma_{i_0}^0, \sigma_{i_1}^0, \cdots, \sigma_{i_h}^0)$ soit $T(\sigma_i^h) = \prod_0^h T_{i_\nu}$. En particulier $T(\sigma_i^0) = T_i$. Les $T(\sigma_i^h)$ sont d'ailleurs les noyaux des $\mathfrak{T}$-simplexes (si on exclut les $T(\sigma_i^h) = 0$).

8.1. $\mathfrak{Z}$ et $\mathfrak{T}$ étant donnés, un réseau $\mathfrak{U}$ dans R s'appellera *commode par rapport à* $\mathfrak{Z} + \mathfrak{T}$, ou *commode* tous court[28], s'il jouit des propriétés suivantes: 1° $\mathfrak{U}$ appartient à la famille Φ du n° 5.5; 2° aucun sommet de $\mathfrak{U}$ ne rencontre simultanément R_0 et S; 3° l'ordre mod S de $\mathfrak{U}$ est $\leqq n$; 4° pour $1 \leqq k \leqq n$, si chaque sommet d'un $(h, \mathfrak{U})$-simplexe rencontre R_k, on a $h \leqq n - k$; 5° pour $U \in \mathfrak{U}$, $0 \leqq h \leqq n$, $1 \leqq i \leqq \alpha_h$ la relation $UT(\sigma_i^h) = 0$ entraîne $\overline{U}T(\sigma_i^h) = 0$; 6° si le sommet U de $\mathfrak{U}$ recontre $T_{i_0}, T_{i_1}, \cdots, T_{i_h}$, alors les $\sigma_{i_0}^0, \sigma_{i_1}^0, \cdots, \sigma_{i_h}^0$ sont les sommets d'un $(h, \mathfrak{Z})$-simplexe σ_i^h et on a $UT(\sigma_i^h) \neq 0$.

8.2. *Les réseaux commodes forment une famille parfaitement complète.* La démonstration fera l'objet des n°s 8.21—8.22.

8.21. Soit R_0^* un ensemble bicompact tel que $R_0 \subset R_0^* \subset R - S$. D'après 2.1, l'espace R_0^* satisfait à l'axiome A_2.

LEMME L_p $(0 \leqq p \leqq n)$: Soit $\mathfrak{V}$ un réseau dans R_0^*. Il existe un affinement $\mathfrak{V}_p$ de $\mathfrak{V}$ jouissant des propriétés P_1, P_2, P_3 et P_4^q $(p \leqq q \leqq n)$ ci-après. *Propriété* P_1: L'ordre (7.2) du réseau $\mathfrak{V}_p$ est $\leqq n$. *Propriété* P_2: Pour $1 \leqq k \leqq n$, chaque $\mathfrak{V}_p$-simplexe dont tous les sommets rencontrent R_k a la

[27] On pourrait même supposer que $\sum T_i = R - G$; mais ce ne serait pas commode plus tard (v. n° 40).

[28] Cette définition est provisoire. Plus tard (dans 18.1) nous allons un peu restreindre cette notion.

dimension $\leqq n-k$. *Propriété* P_3 : $V_0, V_1, \cdots, V_k$ étant des sommets de $\mathfrak{V}_p$, la relation $\prod_0^k V_\nu = 0$ entraîne $\prod_0^k \bar{V}_\nu = 0$; en outre, pour $0 \leqq h \leqq n$, $1 \leqq i \leqq \alpha_h$, la relation $\prod_0^k V_\nu T(\sigma_i^h) = 0$ entraîne $\prod_0^k \bar{V}_\nu T(\sigma_i^h) = 0$. Nous exprimerons cette propriété comme il suit : Les incidences dans $\bar{\mathfrak{V}}_p + \mathfrak{T}$ sont les mêmes que dans $\mathfrak{V}_p + \mathfrak{T}$. [Une *incidence* dans $\mathfrak{V}_p + \mathfrak{T}$, p. ex., est un groupe de sommets de $\mathfrak{V}_p$ et de $\mathfrak{T}$ dont le produit n'est pas vide.] *Propriété* P_4^q : Soit V_p un sommet de $\mathfrak{V}_p$ tel que 1° pour $q+1 \leqq Q \leqq n$, $1 \leqq i \leqq \alpha_Q$, on a $V_p T(\sigma_i^Q) = 0$; 2° il existe une valeur i_0, $1 \leqq i_0 \leqq \alpha_q$ telle que $V_p T(\sigma_{i_0}^q) \neq 0$; alors, pour $1 \leqq i \leqq \alpha_0$, ou bien $V_p T_i = 0$, ou bien σ_i^0 est un sommet du simplexe $\sigma_{i_0}^q$. La démonstration du lemme L_p sera donnée dans 8.211 et 8.212.

8.211. Démontrons d'abord le lemme L_n. Pour chaque couple i, j tel que $1 \leqq i \leqq \alpha_0$, $1 \leqq j \leqq n$ et tel que σ_i^0 ne soit pas un sommet de σ_j^n, on a $T_i T(\sigma_j^n) = 0$ (car autrement on aurait $R_{n+1} \neq 0$). Il existe un affinement $\mathfrak{U}$ de $\mathfrak{V}$ tel qu'aucun sommet de $\mathfrak{U}$ ne rencontre simultanément T_i et $T(\sigma_j^n)$, pour aucun de nos couples i, j[29]. Puisque $R_0^* \supset R_1 \supset \cdots \supset R_n$, dim $R_0^* \leqq n$, dim $R_k \leqq n-k$, $R_k = \bar{R}_k$, d'après 2.3 il existe un affinement $\mathfrak{V}_n'$ du réseau $\mathfrak{U}$ jouissant des propriétés P_1 et P_2. Le réseau $\mathfrak{U}$, et donc évidemment aussi son affinement $\mathfrak{V}_n'$, jouit de la propriété P_4^n. Il ne reste qu'à satisfaire à P_3. Or les propriétés P_1, P_2 et P_4^n du réseau $\mathfrak{V}_n'$ restent évidemment intactes si l'on rapetisse[30] les sommets de $\mathfrak{V}_n'$. On peut donc choisir le réseau $\mathfrak{V}_n'$ de manière que, si $\mathfrak{V}_n''$ en est un rapetissement quelconque, les incidences dans $\mathfrak{V}' + \mathfrak{T}$ (qui sont en nombre fini) soient les mêmes que dans $\mathfrak{V}_n'' + \mathfrak{T}$. Le réseau $\mathfrak{V}_n'$ étant ainsi choisi, on voit sans peine que chaque rapetissement fort $\mathfrak{V}_n$ de $\mathfrak{V}_n'$ jouit de la propriété P_3.

8.212. Le lemme L_n étant démontré, supposons pour une certaine valeur de p $(0 \leqq p \leqq n-1)$ la validité du lemme L_{p+1}. Il s'agit d'en déduire le lemme L_p. Soit donc $\mathfrak{V}$ un réseau donné dans R_0^*. Soit $\mathfrak{V}'$ un rapetissement fort de $\mathfrak{V}$. D'après L_{p+1}, déterminons un affinement $\mathfrak{V}_{p+1}$ de $\mathfrak{V}'$ jouissant des propriétés P_1, P_2, P_3 et P_4^q $(p+1 \leqq q \leqq n)$.

Soit $V_1, V_2, \cdots, V_r$ la suite de tous les sommets de $\mathfrak{V}_{p+1}$, $M_1 = \bar{V}_1$, $M_2 = \bar{V}_2, \cdots, M_r = \bar{V}_r$ celle de tous les sommets de $\bar{\mathfrak{V}}_{p+1}$ et $M_{r+1}, M_{r+2}, \cdots, M_{r+s}$

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[29] En effet, les ensembles disjoints T_i et $T(\sigma_j^n)$ étant fermés dans l'espace normal R_0^*, il existe deux sous-ensembles ouverts U_{ij}' et U_{ij}'' de R_0 tels que $U_{ij}' \supset T_i$, $U_{ij}'' \supset T(\sigma_j^n)$, $U_{ij}' \cdot U_{ij}'' = 0$. Les ensembles U_{ij}', U_{ij}'' et $R_0 - (T_i + T(\sigma_j^n))$ constituent un réseau $\mathfrak{U}_{ij}$ dans R_0. Il suffit de prendre comme $\mathfrak{U}$ un affinement simultané de $\mathfrak{V}$ et de tous les réseaux $\mathfrak{U}_{ij}$.

[30] *Rapetisser* les sommets $U_1, U_2, \cdots, U_m$ d'un réseau $\mathfrak{U}$ signifie que l'on passe de $\mathfrak{U}$ à un réseau $\mathfrak{V}$ aux sommets $V_1, V_2, \cdots, V_m$ tels que $V_i \subset U_i$. Le rapetissement $\mathfrak{V}$ de $\mathfrak{U}$ est dit *fort* si $V_i \in U_i$. *Chaque réseau* $\mathfrak{U}$ *possède un rapetissement fort* (v. le lemme [14]).

celle de tous les noyaux $T(\sigma_i^h)$ des $\mathfrak{T}$-simplexes. D'après *Homologie*, III, 21 il existe des sous-ensembles ouverts $U_1', U_2', \cdots, U_{r+s}'$ de R_0^* tels que 1° $U_\nu' \supset M_\nu$ $(1 \leqq \nu \leqq r+s)$; 2° pour chaque combinaison $(\nu_1, \nu_2, \cdots, \nu_h)$ d'indices $1, 2, \cdots, r+s$ $(1 \leqq h \leqq r+s)$ la relation $\prod_{i=1}^{h} M_{\nu_i} = 0$ entraîne $\prod_{i=1}^{h} U_{\nu_i}' = 0$. Pour chaque ν $(1 \leqq \nu \leqq r)$ il existe un sommet V_ν' de $\mathfrak{V}$ tel que $M_\nu \subset V_\nu'$. Posons $U_\nu = U_\nu' V_\nu'$ $(1 \leqq \nu \leqq r)$. Les ensembles $U_1, U_2, \cdots, U_r$ constituent un réseau $\mathfrak{U}$ dans R_0^* tel que 1° $V_\nu \in U_\nu$ pour $1 \leqq \nu \leqq r$; 2° les incidences dans $\mathfrak{U}+\mathfrak{T}$ sont les mêmes que dans ($\bar{\mathfrak{V}}_{p+1}+\mathfrak{T}$ et par suite, $\mathfrak{V}_{p+1}$ jouissant de la propriété P_3, les mêmes que dans) $\mathfrak{V}_{p+1}+\mathfrak{T}$; 3° $\mathfrak{U}$ est un affinement de $\mathfrak{V}$.

Pour chaque couple composé d'un sommet V_ν de $\mathfrak{V}_p$ et du noyau $T(\sigma_i^h)$ d'un $\mathfrak{T}$-simplexe tel que $p+1 \leqq h \leqq n$, $V_\nu T(\sigma_i^h) \neq 0$, choisissons un point $b_{\nu hi} \in V_\nu T(\sigma_i^h)$. Appelons *points distingués* ces points $b_{\nu hi}$ (qui sont en nombre fini).

Désignons par M la somme de tous les sommets de $\bar{\mathfrak{V}}_{p+1}$ n'ayant aucun point commun avec $R_{p+1} = \sum_{i=1}^{\alpha_{p+1}} T(\sigma_i^{p+1})$. Considérons tous les couples T_i, $T(\sigma_j^p)$ $(1 \leqq i \leqq \alpha_0,\ 1 \leqq j \leqq \alpha_p)$ tels que σ_i^0 ne soit pas un sommet de σ_j^p. Évidemment les ensembles MT_i et $MT(\sigma_j^p)$ sont fermés et disjoints. Il existe donc[31] un réseau $\mathfrak{W}_1$ dans R_0^* dont aucun sommet ne rencontre simultanément les deux ensembles MT_i et $MT(\sigma_j^p)$ (pour aucun de nos couples T_i, $T(\sigma_j^p)$).

L'espace R_0^* étant bicompact et le réseau $\mathfrak{V}_{p+1}$ un rapetissement fort de $\mathfrak{U}$, il existe évidemment un réseau $\mathfrak{W}_2$ dans R_0^* tel que si $W \in \mathfrak{W}_2$, $WV_\nu \neq 0$, on a $W \subset U_\nu$.

D'après 1.3, on construit un réseau $\mathfrak{W}_3$ dans R_0^* tel que, si W, $W' \in W_3$, $WW' \neq 0$ et si W contient le point distingué $b_{\nu hi}$, on a $W+W' \subset V_\nu$.

En vertu du lemme L_{p+1}, il existe un affinement simultané $\mathfrak{W}$ des réseaux $\mathfrak{V}$, $\mathfrak{W}_1$, $\mathfrak{W}_2$ et $\mathfrak{W}_3$ jouissant des propriétés P_1, P_2, P_3 et P_4^q $(p+1 \leqq q \leqq n)$. Le réseau $\mathfrak{W}$ possède évidemment les propriétés de chacun des trois réseaux $\mathfrak{W}_1$, $\mathfrak{W}_2$ et $\mathfrak{W}_3$.

Partageons les sommets V_ν $(1 \leqq \nu \leqq p+1)$ de $\mathfrak{V}_{p+1}$ en deux espèces, les sommets de première espèce étant ceux qui rencontrent un des ensembles $T(\sigma_i^{p+1})$ $(1 \leqq i \leqq \alpha_{p+1})$.

Partageons aussi les sommets de $\mathfrak{W}$ en deux espèces de manière que $W \in \mathfrak{W}$ est de première espèce si, et seulement si, $WV_\nu \neq 0$ pour un sommet V_ν de première espèce de $\mathfrak{V}_{p+1}$. $\mathfrak{W}$ étant un affinement de $\mathfrak{W}_2$, on a alors $W \subset U_\nu$.

[31] Cf. la note [29].

W étant un sommet de première espèce de $\mathfrak{W}$, distiguons deux cas : Si W ne contient aucun point distingué, choisissons, parmi les sommets V_ν de première espèce de $\mathfrak{V}_{p+1}$ tels que $WV_\nu \neq 0$, *un* sommet V_{ν_0} et attachons W à V_{ν_0}. Si W contient un point distingué, attachons W à *chaque* sommet V_ν de première espèce de $\mathfrak{V}_{p+1}$ tel que $WV_\nu \neq 0$. Pour chaque sommet V_ν de première espèce de $\mathfrak{V}_{p+1}$, soit V'_ν la somme de tous les $W \in \mathfrak{W}$ attachés à V_ν. On a alors $V'_\nu \subset U_\nu$.

Les sommets du réseau $\mathfrak{V}_p$ sont : 1° les ensembles V'_ν que nous venons d'attacher à chaque sommet V_ν de première espèce de $\mathfrak{V}_{p+1}$ (ce sont les sommets de première espèce de $\mathfrak{V}_p$) ; 2° les sommets de seconde espèce de $\mathfrak{W}$ (ce sont les sommets de seconde espèce de $\mathfrak{V}_p$). Les réseaux $\mathfrak{U}$ et $\mathfrak{W}$ étant des affinements de $\mathfrak{V}$, $\mathfrak{V}_p$ est aussi un affinement de $\mathfrak{V}$. Il s'agit de voir si le réseau $\mathfrak{V}_p$ possède les propriétés P_1, P_2, P_3 et P_4^q $(p \leqq q \leqq n)$.

Commençons par les propriétés P_1 et P_2. A cet effet, choisissons un point $a \in R_0^*$. On doit prouver que les propriétés P_1 et P_2 sont vérifiées pour tous les $\mathfrak{V}_p$-simplexes dont le noyau contient a. Deux cas sont à distinguer. *En premier lieu,* supposons qu'aucun des sommets de $\mathfrak{W}$ contenant a ne contient aucun point distingué. Chaque sommet de $\mathfrak{U}_p$ était la somme d'un certain groupe de sommets de $\mathfrak{W}$; dans notre cas, chaque sommet de $\mathfrak{W}$ contenant a appartient à un seul de ces groupes. Le réseau $\mathfrak{W}$ jouissant des propriétés P_1 et P_2, on voit que cela a lieu aussi pour le réseau $\mathfrak{V}_p$ relativement aux $\mathfrak{V}_p$-simplexes dont le noyau contient a. *En second lieu,* supposons que, parmi les sommets de $\mathfrak{W}$ contenant a, il y en ait au moins un, soit W_0, contenant un point distingué, soit $b_{\nu_0 h_0 i_0}$.

Le réseau $\mathfrak{W}$ étant un affinement de $\mathfrak{W}_3$, il en résulte que pour chaque sommet W de $\mathfrak{W}$ contenant a on a l'inclusion $W \subset V_{\nu_0}$. Puisque $p+1 \leqq h_0 \leqq n$, $b_{\nu_0 h_0 i_0} \in V_{\nu_0} T(\sigma_{i_0}^{h_0}) \neq 0$, V_{ν_0} est un sommet de première espèce de $\mathfrak{V}_{p+1}$, donc $W \subset V_{\nu_0}$ en est un de $\mathfrak{W}$. Il en résulte que tous les sommets de $\mathfrak{V}_p$ contenant a sont de première espèce ; soient $V'_{\nu_1}, V'_{\nu_2}, \cdots, V'_{\nu_k}$ ces sommets. On a alors $V'_{\nu_i} \subset U_{\nu_i}$ $(1 \leqq i \leqq k)$. Puisque $V_\nu \subset U_\nu$, et puisque les incidences dans $\mathfrak{W}_{p+1} + \mathfrak{T}$ sont les mêmes que dans $\mathfrak{U} + \mathfrak{T}$, le réseau $\mathfrak{U}$ jouit des propriétés P_1 et P_2 (car $\mathfrak{V}_{p+1}$ en jouit par supposition). De $V'_{\nu_i} \subset U_{\nu_i}$ on conclut que le réseau $\mathfrak{V}_p$ jouit des propriétés P_1 et P_2 relativement aux $\mathfrak{V}_p$-simplexes dont le noyau contient a.

Chaque sommet de $\mathfrak{U}_p$ étant la somme d'un groupe de sommets de $\mathfrak{W}$ et le réseau $\mathfrak{W}$ jouissant de la propriété P_3, on voit sans peine que le réseau $\mathfrak{U}_p$ en jouit également.

Ensuite montrons que pour $p+1 \leqq q \leqq n$ le réseau $\mathfrak{V}_p$ jouit de la propriété P_4^q. Soit donc V^* un sommet de $\mathfrak{V}_p$ tel que 1° pour $q+1 \leqq Q \leqq n$, $1 \leqq i \leqq \alpha_Q$, on a $V^* T(\sigma_i^Q) \neq 0$; 2° il existe une valeur i_0, $1 \leqq i_0 \leqq \alpha_q$

telle que $V^* T(\sigma_{i_0}^q) \neq 0$. Soit $i\,(1 \leqq i \leqq \alpha_0)$ un indice tel que σ_i^0 n'est pas un sommet de $\sigma_{i_0}^q$. On doit montrer que $V^* T_i = 0$. Si le sommet V^* de $\mathfrak{V}_p$ est de seconde espèce, cela est clair, car V^* est alors un sommet du réseau $\mathfrak{W}$ qui jouit de la propriété P_4^q. Supposons donc que $V^* = V_\nu'$ soit un sommet de première espèce de $\mathfrak{V}_p$. On a alors $V_\nu' \subset U_\nu$ de manière qu'il suffit de montrer que $U_\nu T_i = 0$. L'inclusion $V_\nu' \subset U_\nu$ montre d'ailleurs que $U_\nu T(\sigma_{i_0}^q) \neq 0$. Or les incidences dans $\mathfrak{U} + \mathfrak{T}$ étant les mêmes que dans $\mathfrak{V}_{p+1} + \mathfrak{T}$, on voit que $V_\nu \cdot T(\sigma_{i_0}^q) \neq 0$, et qu'il suffit de montrer que $V_\nu T_i = 0$. Supposons par impossible que $V_\nu T_i \neq 0$. Le réseau $\mathfrak{V}_{p+1}$ jouissant de la propriété P_4^q, il en résulte qu'il existe des indices Q, j tels que $V_\nu T(\sigma_j^Q) \neq 0$. Il existe alors un point distingué $b_{\nu jQ} \in V_\nu T(\sigma_j^Q)$. Soit W un sommet de $\mathfrak{W}$ tel que $b_{\nu jQ} \in W$. Le sommet W contenant le point distingué $b_{\nu jQ}$, on a l'inclusion $W \subset V_\nu'$ d'après la définition même de V_ν'. Donc $b_{\nu jQ} \in V_\nu' \cdot T(\sigma_j^Q) = V^* T(\sigma_j^Q) \neq 0$, ce qui est une contradiction.

Il ne reste qu'à prouver que le réseau $\mathfrak{V}_p$ jouit de la propriété P_4^p. Soit donc V^* un sommet de $\mathfrak{V}_p$ tel que 1° pour $p+1 \leqq Q \leqq n$, $1 \leqq i \leqq \alpha_Q$, on a $V^* T(\sigma_i^Q) = 0$; 2° il existe une valeur i_0, $1 \leqq i_0 \leqq \alpha_p$ telle que $V^* T(\sigma_{i_0}^p) \neq 0$. Soit $i\,(1 \leqq i \leqq \alpha_0)$ un indice tel que σ_i^0 n'est pas un sommet de $\sigma_{i_0}^p$. On doit montrer que $V^* T_i = 0$. Remarquons d'abord que V^* est un sommet de seconde espèce de $\mathfrak{V}_p$. En effet, dans le cas contraire on aurait $V^* = V_\nu'$. V_ν étant un sommet de première espèce de $\mathfrak{V}_{p+1}$, il existerait un indice j $(1 \leqq j \leqq \alpha_{p+1})$ tel que $V_\nu T(\sigma_j^{p+1}) \neq 0$, d'où $b_{\nu,j,p+1} \in V_\nu T(\sigma_j^{p+1})$. W étant un sommet de $\mathfrak{W}$ contenant $b_{\nu,j,p+1}$, on aurait $W \subset V_\nu'$ (par la définition même de V_ν') d'où $b_{\nu,j,p+1} \in V_\nu' T(\sigma_j^{p+1}) = V^* T(\sigma_j^{p+1}) \neq 0$, ce qui est une contradiction. Donc V^* est un sommet de seconde espèce de $\mathfrak{V}_p$; cela signifie que $V^* = W$ est un sommet de $\mathfrak{W}$ ne rencontrant aucun sommet de première espèce de $\mathfrak{V}_{p+1}$, d'où $V^* \subset M$ en vertu de la définition des sommets de première espèce de $\mathfrak{V}_{p+1}$ et de la définition de M. Le réseau $\mathfrak{W}$ ètant un affinement de $\mathfrak{W}_1$, le sommet V^* de $\mathfrak{W}$ ne peut rencontrer simultanément les deux ensembles (MT_i et $MT(\sigma_{i_0}^p)$ et par suite, en vertu de l'inclusion $V^* \subset M$, les deux ensembles) T_i et $T(\sigma_{i_0}^p) \neq 0$. Or $V^* T(\sigma_{i_0}^p) \neq 0$, donc $V^* T_i = 0$.

8.22. Nous venons de démontrer le lemme L_0. Or soit $\mathfrak{V}$ un réseau donné dans R et soit $\mathfrak{P}$ un réseau gén. donné. Les ensembles R_0 et S étant fermés et disjoints, il existe un affinement $\mathfrak{V}_1$ de $\mathfrak{V}$ tel qu'aucun sommet de $\mathfrak{V}_1$ ne rencontre simultanément R_0 et S. Soit (v. 5.4) $\mathfrak{V}_2$ un réseau de la famille Φ du n° 5.5 qui soit un affinement de $\mathfrak{V}_1$ et un affinement mod S de $\mathfrak{P}$. Soi M la somme de tous les sommets intérieurs (v. 7.1) de $\mathfrak{V}_2$. Soit N la somme de tous les autres sommets de $\mathfrak{V}_2$. Soit $R_0^* = R - N$, de manière que R_0^* est un ensemble bicompact tel que $R_0 \subset R_0^* \subset M \subset R - S$.

V_2 parcourant les sommets du réseau $\mathfrak{V}_2$, les ensembles $R_0^* V_2$ (dont on supprime ceux qui sont vides) sont les sommets d'un réseau $\mathfrak{V}^*$ dans R_0^*. D'après le lemme L_0, il existe dans l'espace R_0^* un affinement $\mathfrak{U}^*$ du réseau $\mathfrak{V}^*$ jouissant des propriétés P_1, P_2, P_3 et P_4^q $(0 \leqq q \leqq n)$. $U_1^*, U_2^*, \cdots, U_m^*$ étant les sommets du réseau $\mathfrak{U}^*$, déterminons [v. *Homologie*, III, 21] des ensembles $U_1, U_2, \cdots, U_m$ ouverts dans $M \supset R_0^*$ et tels que $1°$ $U_\nu^* \subset U_\nu$ $(1 \leqq \nu \leqq m)$; $2°$ $R_0^* U_i = U_i^*$; $3°$ $R_0^* \bar{U}_i = \bar{U}_i^*$; $4°$ $\nu_1, \nu_2, \cdots, \nu_h$ étant une combinaison des indices $1, 2, \cdots, m$, la relation $\prod_1^h U_{\nu_i}^* = 0$ entraîne $\prod_1^h U_{\nu_i} = 0$; chaque U_ν $(1 \leqq \nu \leqq m)$ est un sous-ensemble d'un sommet intérieur de $\mathfrak{V}_2$. En ajoutant aux $U_1, U_2, \cdots, U_m$ les sommets extérieurs de $\mathfrak{V}_2$, on obtient un réseau $\mathfrak{U}$ dans R qui est évidemment un affinement du réseau $\mathfrak{V}$ et un affinement mod S de $\mathfrak{P}$. On voit sans peine que le réseau $\mathfrak{U}$ est commode.

8.3. Soit U un sommet d'un réseau commode $\mathfrak{U}$. Si $UR_0 = 0$, disons que U est d'espèce σ^{-1} (où le symbole σ^{-1}, appelé aussi $(-1, \mathfrak{Z})$-simplexe intérieur, n'a aucune signification). Si $UR_0 \neq 0$, d'après la propriété $6°$ d'un réseau commode (v. 8.1), il existe des indices h, i $(0 \leqq h \leqq n,\ 1 \leqq i \leqq \alpha_h)$ tels que $UT(\sigma_i^h) \neq 0$, tandis que $UT(\sigma_j^0) = 0$ pour toutes les valeurs de j $(1 \leqq j \leqq \alpha_0)$ telles que σ_j^0 n'est pas un sommet de σ_i^h [et donc, plus généralement, $UT(\sigma_j^k) \neq 0$ $(0 \leqq k \leqq n,\ 1 \leqq j \leqq \alpha_k)$ si, et seulement si, le simplexe σ_i^k est une k-face du simplexe σ_i^h, ce qui exige en particulier $k \leqq h$]. Nous disons alors que U est d'espèce σ_i^h. Chaque sommet U du rèseau commode $\mathfrak{U}$ possède donc une espèce bien déterminée; cette espèce est un $(h, \mathfrak{Z})$-simplexe intérieur; le nombre h $(-1 \leqq h \leqq n)$ est appelé *rang* du sommet U.

Soit maintenant $\tau^k = (U_0, U_1, \cdots, U_k)$ un $(k, \mathfrak{U})$-simplexe. Supposons que le sommet U_ν $(0 \leqq \nu \leqq k)$ soit d'espèce $\sigma_{i_\nu}^{h_\nu}$. Soit σ_i^h la face commune de dimension maxima de tous les $k+1$ $\mathfrak{Z}$-simplexes $\sigma_{i_\nu}^{h_\nu}$. [Si $h_\nu = -1$ pour une valeur de ν $(0 \leqq \nu \leqq k)$, ou bien si les $k+1$ $\mathfrak{Z}$-simplexes n'ont aucun sommet commun à tous, on a $\sigma_i^h = \sigma^{-1}$.] Nous disons alors que τ^k est d'espèce σ_i^h; le nombre h $(-1 \leqq h \leqq n)$ est le *rang* de τ^k.

8.31. *Si le $(k, \mathfrak{U})$-simplexe τ^k est extérieur* (v. 7.1), *en particulier si $k \geqq n+1$, alors τ^k est d'espèce σ^{-1}.*

C'est une conséquence immédiate des propriétés $2°$ et $3°$ d'un réseau commode (v. 8.1).

8.32. *Si le noyau d'un $(k, \mathfrak{U})$-simplexe τ^k rencontre R_h $(0 \leqq h \leqq n)$, le rang de τ^k est $\geqq h$. En effet, il existe un indice i $(1 \leqq i \leqq \alpha_h)$ tel que $\mathfrak{J}(\tau^k)\ T(\sigma_i^h) \neq 0$, d'où $UT(\sigma_i^h) \neq 0$ pour chaque sommet U de τ^k.*

8.33. *Soit τ^k un $(k, \mathfrak{U})$-simplexe de rang $h \geqq 0$. Alors $h \leqq n - k$.* C'est une conséquence immédiate des propriétés $3°$ et $4°$ d'un réseau commode.

8.34. *Soit $\tau^{k'}$ une k'-face d'un $(k, \mathfrak{U})$-simplexe τ^k, avec τ^k et $\tau^{k'}$ resp. d'espèce σ_i^h, $\sigma_{i'}^{h'}$. Alors le $\mathfrak{Z}$-simplexe σ_i^h est une h-face du $\mathfrak{Z}$-simplexe $\sigma_{i'}^{h'}$; en particulier on a $h \leqq h'$, et $h = h'$ entraîne $\sigma_{i'}^{h'} = \sigma_i^h$.* [Le symbole σ^{-1} est ici considéré comme une (-1)-face de chaque $\mathfrak{Z}$-simplexe intérieur.] C'est une conséquence aisée des définitions.

8.35. *Soient $\mathfrak{U}$, $\mathfrak{U}_1$ deux réseaux commodes, $\mathfrak{U}_1$ étant un affinement de $\mathfrak{U}$; soit $\pi = Pr.(\mathfrak{U}_1, \mathfrak{U})$. Soit τ^k un $(k, \mathfrak{U}_1)$-simplexe d'espèce σ_i^{n-k}. Alors ou bien $\pi\tau^k = 0$, ou bien $\pi\tau^k$ est un $(k, \mathfrak{U})$-simplexe d'espèce σ_i^{n-k}.* C'est une conséquence facile de la définition de l'espèce et de 8.33.

II.

9. Axiome B. *Chaque point $a \in R - S$ possède un entourage U tel que, Γ^n étant un (n, R)-cycle (absolu) dans $\overline{U}$, on a l'homologie $\Gamma^n \sim 0$ mod S*[32].

9.1. *Chaque entourage suffisamment petit de chaque point $a \in R - S$ possède la propriété suivante: Chaque (n, R)-cycle Γ^n dans $\overline{U}$ est ~ 0 dans $\overline{U}$. Il existe même* (v. 7.4) *un réseau $\mathfrak{V}$ dans R tel que, si Γ^n est un (n, R)-cycle dans $\overline{U}$, on a $\Gamma^n(\mathfrak{U}) = 0$ pour chaque affinement $\mathfrak{U}$ de $\mathfrak{V}$ qui soit d'ordre $\leqq n$ mod S.*

Démonstration. Déterminons un entourage $U \Subset R - S$ du point a d'après l'axiome B. Soit $\mathfrak{V}_1$ un réseau tel qu'aucun sommet de $\mathfrak{V}_1$ ne rencontre simultanément $\overline{U}$ et S. Déterminons un affinement $\mathfrak{V}$ de $\mathfrak{V}_1$ jouissant de la propriété du n° 1.3. Alors, si V', V'' sont deux sommets de $\mathfrak{V}$ tels que $V'\overline{U} \neq 0$, $SU'' \neq 0$, on a $V'V'' = 0$ et la même propriété appartient à chaque affinement de $\mathfrak{V}$. Soit $\mathfrak{U}$ un affinement de $\mathfrak{V}$ d'ordre $\leqq n$ mod S. Puisque $\Gamma^n \sim 0$ mod S, il existe une $(n+1, \mathfrak{U})$-chaîne $\Delta^{n+1}(\mathfrak{U})$ telle que

$$(*) \qquad F\Delta^{n+1}(\mathfrak{U}) - \Gamma^n(\mathfrak{U}) \subset S.$$

Supposons, par impossible, que $\Gamma^n(\mathfrak{U}) \neq 0$. Alors il existe un $(n, \mathfrak{U})$-simplexe σ^n qui se trouve dans la chaîne $\Gamma^n(\mathfrak{U})$ [avec un coefficient $\neq 0$]. Puisque $\Gamma^n \subset \overline{U}$, chaque sommet de σ^n rencontre $\overline{U}$ et ne peut donc rencontrer S. Par suite $(*)$ montre que la chaîne $\Delta^{n+1}(\mathfrak{U})$ contient un $(n+1, \mathfrak{U})$-simplexe σ^{n+1} dont σ^n est une n-face. Puisque $\mathfrak{U}$ èst un affinement de $\mathfrak{V}$ et puisque les sommets de σ^n rencontrent $\overline{U}$, aucun sommet de σ^{n+1} ne peut rencontrer S. Or ceci est une contradiction, car le réseau $\mathfrak{U}$ est d'ordre $\leqq n$ mod S.

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[32] En vertu de *Homologie*, II, 28, cet axiome peut être énoncé comme il suit: *Chaque point $a \in R - S$ possède un entourage U tel que, $\mathfrak{U}$ étant un réseau et $\Gamma^n(\mathfrak{U})$ étant un $(n, \mathfrak{U})$-cycle essentiel dans $\overline{U}$, on a l'homologie $\Gamma^n(\mathfrak{U}) \sim 0$ dans $\overline{U}$.* De la même manière on peut modifier aussi l'énoncé de presque tous les axiomes suivants. Il est important que le lecteur se rappelle constamment la possibilité de cette modification.

9.2. *Il existe un réseau gén.* $\mathfrak{P}_1$ *jouissant de la propriété suivante: Lorsque* $P_1 \in \mathfrak{P}_1$, *on a* $P_1 \in R-S$ *et si* Γ^n *est un* (n, R)*-cycle dans* $\overline{P}_1$, *on a* $\Gamma^n \sim 0$ *dans* $\overline{P}_1$; *on peut même attacher à chaque sommet* P_1 *de* $\mathfrak{P}_1$ *un réseau* $\mathfrak{V}(P_1)$ *tel que, si* Γ^n *est un* (n, R)*-cycle dans* $\overline{P}_1$, *on a* $\Gamma^n(\mathfrak{U}) = 0$ *pour chaque affinement* $\mathfrak{U}$ *de* $\mathfrak{V}(P_1)$ *d'ordre* $\leqq n \bmod S$. C'est une conséquence de 4.2 et de 9.1.

9.3. Déterminons un réseau gén. $\mathfrak{P}_1$ d'après 9.2, puis un affinement $\mathfrak{P}_2$ de $\mathfrak{P}_1$ jouissant de la propriété suivante: A chaque sommet P_2 de $\mathfrak{P}_2$ on peut attacher un sommet P_1 de $\mathfrak{P}_1$ de manière que, si P_2' est un sommet de $\mathfrak{P}_2$ tel que $\overline{P}_2 \overline{P}_2' \neq 0$, on ait $\overline{P}_2' \subset P_1$. [Le réseau gén. $\mathfrak{P}_2$ existe en vertu de 4.3 et 4.4.] Soit $\mathfrak{P}_3$ un affinement de $\mathfrak{P}_2$ jouissant par rapport à $\mathfrak{P}_2$ de la même propriété que $\mathfrak{P}_2$ par rapport à $\mathfrak{P}_1$. On suppose dans tout ce Chapitre que le choix des réseaux gén. $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$ ait été fait d'une manière bien determinée.

10. Soit $i = 1, 2,$ ou 3. Soit $A \neq 0$ un sous-ensemble bicompact de $R-S$. Soit Γ^{n-1} un $(n-1, R)$-cycle dans A. Supposons qu'il existe un sommet P_i du réseau gén. $\mathfrak{P}_i$ tel qu'on ait l'homologie $\Gamma^{n-1} \sim 0$ dans $\overline{P}_i$ (d'où $A \subset \overline{P}_i$). Nous dirons alors que Γ^{n-1} est un $(n-1, R)$*-cycle du type* t_i *dans* A.

10.1 *Soit* Γ^{n-1} *un* $(n-1, R)$*-cycle du type* t_i $(i = 1, 2, 3)$ *dans* A. *Soit* P_i *un sommet correspondant du réseau gén.* $\mathfrak{P}_i$. *Il existe un* (n, R)*-cycle* C^n *mod* A *dans* $\overline{P}_i$ *jouissant des propriétés suivantes:* 1° $FC^n \sim \Gamma^{n-1}$ *dans* A; 2° *il existe un réseau* $\mathfrak{V}(P_i)$ [*ne dépendant que de* P_i] *tel que si le réseau* $\mathfrak{U}_2$ *est un affinement de* $\mathfrak{U}_1$ *et si* $\mathfrak{U}_1$ *est un affinement de* $\mathfrak{V}(P_i)$, *si enfin* $\mathfrak{U}_1$ *et* $\mathfrak{U}_2$ *sont d'ordre* $\leqq n \bmod S$, *on a* $\pi C^n(\mathfrak{U}_2) = C^n(\mathfrak{U}_1)$ *mod* A, *où* $\pi = Pr.(\mathfrak{U}_2, \mathfrak{U}_1)$. *Le cycle relatif* C^n *est bien déterminé à une homologie mod* A *dans* $\overline{P}_i$ *près; pour chaque affinement* $\mathfrak{U}$ *de* $\mathfrak{V}(P_i)$ *d'ordre* $\leqq n \bmod S$ *on peut même affirmer que* $C^n(\mathfrak{U})$ *est bien déterminé mod* A. *Le cycle relatif* C^n *reste inaltéré (dans le sens qui vient d'être précisé) si on remplace* Γ^{n-1} *par un* $(n-1, R)$*-cycle* $\Gamma_1^{n-1} \sim \Gamma^{n-1}$ *dans* A.

Démonstration. Déterminons un réseau $\mathfrak{V}(P_i)$ jouissant de la propriété du n° 9.2. Cette propriété restant intacte si on remplace $\mathfrak{V}(P_i)$ par son affinement, on peut supposer qu'aucun sommet de $\mathfrak{V}(P_i)$ ne rencontre simultanément les deux ensembles $\overline{P}_i$ et S (qui sont fermés et disjoints). Désignons par Φ la famille de tous les affinements de $\mathfrak{V}(P_i)$ qui soient d'ordre $\leqq n \bmod S$; c'est (v. 7.3) une famille complète de réseaux.

A chaque réseau $\mathfrak{U} \in \Phi$ attachons un affinement $\mathfrak{U}_1 \in \Phi$ normal relativement aux cycles dans $\overline{P}_i$, et choisissons une projection $\pi_1 = Pr.(\mathfrak{U}_1, \mathfrak{U})$, puis un $(n-1, \mathfrak{U}_1)$-cycle $\Gamma_1^{n-1}(\mathfrak{U}_1) \sim \Gamma^{n-1}(\mathfrak{U}_1)$ dans A, enfin une $(n, \mathfrak{U}_1)$-chaîne dans $\overline{P}_i$: $C_1^n(\mathfrak{U}_1)$ telle que $FC_1^n(\mathfrak{U}_1) = \Gamma_1^{n-1}(\mathfrak{U}_1)$ et posons $C^n(\mathfrak{U}) = \pi_1 C_1^n(\mathfrak{U}_1)$. Démontrons le lemme L suivant: La $(n, \mathfrak{U})$-chaîne $C^n(\mathfrak{U})$ reste inaltérée mod A si l'on fait un autre choix $\mathfrak{U}_2$, π_2, $\Gamma_2^{n-1}(\mathfrak{U}_2)$, $C_2^n(\mathfrak{U}_2)$ de $\mathfrak{U}_1$, π_1,

$\Gamma_1^{n-1}(\mathfrak{U}_1)$, $C_1^n(\mathfrak{U}_1)$. On voit sans peine qu'il résulte du lemme L que le cycle relatif C^n, s'il existe, est bien déterminé dans le sens énoncé dans le théorème.

Démontrons d'abord le lemme L dans deux cas particuliers: I. Soit $\mathfrak{U}_2 = \mathfrak{U}_1$, $\pi_2 = \pi_1$. On a alors $\Gamma_1^{n-1}(\mathfrak{U}_1) \sim \Gamma_2^{n-1}(\mathfrak{U}_1)$ dans A; il existe donc une $(n, \mathfrak{U}_1)$-chaîne dans A: $D^n(\mathfrak{U}_1)$ telle que

$$FD^n(\mathfrak{U}_1) = \Gamma_2^{n-1}(\mathfrak{U}_1) - \Gamma_1^{n-1}(\mathfrak{U}_1) = FC_2^n(\mathfrak{U}_1) - FC_1^n(\mathfrak{U}_1).$$

Alors $\Delta^n(\mathfrak{U}_1) = D^n(\mathfrak{U}_1) + C_1^n(\mathfrak{U}_1) - C_2^n(\mathfrak{U}_1)$ est un $(n, \mathfrak{U}_1)$-cycle dans $\overline{P}_i$. $\mathfrak{U}_1$ étant un affinement de $\mathfrak{U}$ normal relativement aux $(n, \mathfrak{U})$-cycles dans $\overline{P}_1$, il en résulte que $\pi_1\Delta^n(\mathfrak{U}_1) = \pi_1 D^n(\mathfrak{U}_1) + \pi_1 C_1^n(\mathfrak{U}_1) - \pi_1 C_2^n(\mathfrak{U}_1)$ est un $(n, \mathfrak{U})$-cycle *essentiel* dans $\overline{P}i$. Or le réseau $\mathfrak{U}$ est un affinement d'ordre $\leqq n \bmod S$ du réseau $\mathfrak{B}(P_i)$, d'où il résulte (v. 9.2) que $\pi_1\Delta^n(\mathfrak{U}_1) = 0$. Or ceci donne $\pi_1 C_1^n(\mathfrak{U}_1) = \pi_1 C_2^n(\mathfrak{U}_1) \bmod A$, car $D^n(\mathfrak{U}_1) \subset A$ entraîne $\pi_1 D^n(\mathfrak{U}_1) \subset A$. II. Soit $\mathfrak{U}_2 = \mathfrak{U}_1$, $\Gamma_2^{n-1}(\mathfrak{U}_2) = \Gamma_1^{n-1}(\mathfrak{U}_1)$, $C_2^n(\mathfrak{U}_2) = C_1^n(\mathfrak{U}_1)$. On a alors $\pi_2 C_1^n(\mathfrak{U}_1) - \pi_1 C_1^n(\mathfrak{U}_1) \sim 0 \bmod A$ dans $\overline{P}_i$. Or le réseau $\mathfrak{U}_1$ est d'ordre $\leqq n \bmod S$ et aucun sommet de $\mathfrak{U}_1$ ne rencontre simultanément $\overline{P}_i$ et S; on en déduit sans peine que l'homologie précédente entraîne $\pi_2 C_1^n(\mathfrak{U}_1) = \pi_1 C_1^n(\mathfrak{U}_1) \bmod A$.

Ceci étant, passons à la démonstration du lemme L dans le cas général. Soit $\mathfrak{U}_3$ un affinement simultané des réseaux $\mathfrak{U}$, $\mathfrak{U}_1$, $\mathfrak{U}_2$ normal relativement aux cycles dans $\overline{P}_i$; soit $\pi_3 = Pr.(\mathfrak{U}_3, \mathfrak{U}_1)$, $\pi_3' = Pr.(\mathfrak{U}_3, \mathfrak{U}_2)$. Soit $C_3^n(\mathfrak{U}_3)$ une $(n, \mathfrak{U}_3)$-chaîne dans $\overline{P}_i$ telle que $FC_3^n(\mathfrak{U}_3) = \Gamma^{n-1}(\mathfrak{U}_3)$. D'après I on a $\pi_1 C_1^n(\mathfrak{U}_1) = \pi_1\pi_3 C_3^n(\mathfrak{U}_3) \bmod A$, $\pi_2 C_2^n(\mathfrak{U}_2) = \pi_2\pi_3' C_3^n(\mathfrak{U}_3) \bmod A$; d'après II on a $\pi_1\pi_3 C_3^n(\mathfrak{U}_3) = \pi_2\pi_3' C_3^n(\mathfrak{U}_3) \bmod A$. Donc $\pi_1 C_1^n(\mathfrak{U}_1) = \pi_2 C_2^n(\mathfrak{U}_2) \bmod A$.

Le lemme L étant démontré, il reste à voir que les $(n, \mathfrak{U})$-chaînes $C^n(\mathfrak{U})$ construites plus haut pour chaque $\mathfrak{U} \in \Phi$, jouissent des propriétés suivantes: 1° $C^n(\mathfrak{U}) \subset \overline{P}_1$; 2° $FC^n(\mathfrak{U}) \sim \Gamma^{n-1}(\mathfrak{U})$ dans $\overline{P}_i$; 3° si $\mathfrak{U}_1, \mathfrak{U}_2 \in \Phi$, si $\mathfrak{U}_2$ est un affinement de $\mathfrak{U}_1$ et si $\pi = Pr.(\mathfrak{U}_2, \mathfrak{U}_1)$, on a $\pi C^n(\mathfrak{U}_2) = C^n(\mathfrak{U}_1) \bmod A$. Les propriétés 1° et 2° étant évidentes, démontrons 3°. A cet effet, soit $\mathfrak{U}_3$ un affinement simultané des réseaux $\mathfrak{U}_1$, $\mathfrak{U}_2$ normal relativement aux cycles dans $\overline{P}_1$ et soit $\pi_3 = Pr.(\mathfrak{U}_3, \mathfrak{U}_2)$. D'après le lemme L, on a $C^n(\mathfrak{U}_2) = \pi_3 C^n(\mathfrak{U}_3) \bmod A$, $C^n(\mathfrak{U}_1) = \pi\pi_3 C^n(\mathfrak{U}_3) \bmod A$, d'où $\pi C^n(\mathfrak{U}_2) = C^n(\mathfrak{U}_1) \bmod A$.

10.2. Étant donné un $(n-1, R)$-cycle Γ^{n-1} du type t_i $(i = 1, 2, 3)$ dans A, il peut exister *plusieurs* sommets P_i de $\mathfrak{P}_i$ tels que $\Gamma^{n-1} \sim 0$ dans $\overline{P}_i$. Or, *si $i = 2$ ou $i = 3$, le cycle relatif C^n du n° 10.1 est indépendant du choix de P_i dans le sens suivant: Il existe un réseau $\mathfrak{W}$ tel que dans chaque affinement $\mathfrak{U}$ de $\mathfrak{W}$ qui soit d'ordre $\leqq n \bmod S$ le cycle relatif $C^n(\mathfrak{U})$ est bien déterminé à mod A près.*

Démonstration. Soient P_{i1}, P_{i2} deux choix de P_i et soient C_1^n, C_2^n les deux cycles relatifs correspondants. On a $\overline{P}_{i1}\,\overline{P}_{i2} \supset A \neq 0$. Il existe donc (v. 9.3) un sommet P_{i-1} du réseau gén. $\mathfrak{P}_{i-1}$ tel que $P_{i1} + P_{i2} \subset P_{i-1}$. Il en résulte que Γ^{n-1} est un $(n-1, R)$-cycle du type t_{i-1} dans A, P_{i-1} étant le sommet correspondant de $\mathfrak{P}_{i-1}$. Soit C_0^n le cycle relatif correspondant. Soit $\mathfrak{W}$ un affinement simultané des trois réseaux $\mathfrak{V}(P_{i1})$, $\mathfrak{V}(P_{i2})$, $\mathfrak{V}(P_{i-1})$. Chacun des deux cycles relatifs C_1^n, C_2^n jouit évidemment des propriétés du cycle relatif C_0^n. Ces propriétés déterminant C_0^n univoquement, on a pour chaque affinement $\mathfrak{U}$ d'ordre $\leqq n \bmod S$ du réseau $\mathfrak{W}$ les relations $C_1^n(\mathfrak{U}) = C_2^n(\mathfrak{U}) = C_0^n(\mathfrak{U}) \bmod A$.

10.3. *Soit Γ^{n-1} un $(n-1, R)$-cycle du type t_i $(i = 2$ ou $i = 3)$ dans A. Il existe un ensemble bicompact H tel que* 1° $A \subset H \subset R - S$; 2° $\Gamma^{n-1} \sim 0$ *dans* H; 3° $H \subset \overline{P}_i$, *où* $P_i \in \mathfrak{P}_i$; 4° *si* K *est un ensemble bicompact tel que* $A \subset K \subset \overline{P}_i'$, $P_i' \in \mathfrak{P}_i$ *et* $\Gamma^{n-1} \sim 0$ *dans* K, *alors* $H \subset K$.

La démonstration fera l'objet des n^{os} 10.31—10.33. L'ensemble H est évidemment bien déterminé (v. aussi 10.4); nous l'appellerons le *porteur de l'homologie* $\Gamma^{n-1} \sim 0$.

10.31. Lemme. Soit Γ^{n-1} un $(n-1, R)$-cycle du type t_i $(i = 1, 2, 3)$ dans A. Soit $A \subset B_1 B_2$, B_1 et B_2 étant des ensembles bicompacts tels que $B_1 + B_2 \subset \overline{P}_i$, $P_i \in \mathfrak{P}_i$. Soit $\Gamma^{n-1} \sim 0$ et dans B_1 et dans B_2. C^n étant le cycle relatif du n° 10.1, C^n est situé dans $B_1 B_2$.

Démonstration. Soit Φ la famille de tous les affinements $\mathfrak{U}$ de $\mathfrak{V}(P_i)$ d'ordre $\leqq n \bmod S$. On voit sans peine que $C^n(\mathfrak{U}) \subset B_1$ et $C^n(\mathfrak{U}) \subset B_2$ pour chaque $\mathfrak{U} \in \Phi$. Il s'agit d'en déduire que $C^n(\mathfrak{U}) \subset B_1 B_2$. Soit donc $\mathfrak{U}$ un réseau donné de la famille Φ. Soit $\mathfrak{V}$ un affinement de $\mathfrak{U}$ régulier (v. *Homologie*, III, 22 et 23) par rapport à $B_1 B_2$. Déduisons de $\mathfrak{V}$ un nouveau réseau $\mathfrak{W}$ de la manière suivante: Un sommet V de $\mathfrak{V}$ tel que $VB_1 B_2 \neq 0$ est un sommet de $\mathfrak{W}$; au lieu d'un sommet $V \in \mathfrak{V}$ tel que $VB_1 B_2 = 0$ le réseau $\mathfrak{W}$ possède deux sommets $V - B_1$ et $V - B_2$. Évidemment le réseau $\mathfrak{W}$ est un affinement de $\mathfrak{U}$ régulier par rapport à $B_1 B_2$; en outre $\mathfrak{W}$ possède la propriété suivante: Si $W \in \mathfrak{W}$, $WB_1 \neq 0$, $WB_2 \neq 0$, on a aussi $WB_1 B_2 \neq 0$. Soit $\mathfrak{U}_1 \in \Phi$ un affinement de $\mathfrak{W}$ qui soit en même temps un affinement de $\mathfrak{U}$ normal relativement aux cycles dans $\overline{P}_i$. Soit $\pi_1 = Pr.(\mathfrak{U}_1, \mathfrak{W})$, $\pi_2 = Pr.(\mathfrak{W}, \mathfrak{U})$. D'après 10.1, on a $C^n(\mathfrak{U}) = \pi_2 \pi_1 C^n(\mathfrak{U}_1) \bmod A$. Puisque $C^n(\mathfrak{U}_1) \subset B_1$ et $C^n(\mathfrak{U}_1) \subset B_2$, chaque sommet de chaque simplexe de la $\mathfrak{U}_1$-chaîne $C^n(\mathfrak{U}_1)$ rencontre B_1 et B_2. Il en résulte que chaque sommet de chaque simplexe de la $\mathfrak{W}$-chaîne $\pi_1 C^n(\mathfrak{U}_1)$ rencontre B_1 et B_2, et par suite aussi $B_1 B_2$ d'après la propriété du réseau $\mathfrak{W}$. Le réseau $\mathfrak{W}$ étant régulier par rapport à $B_1 B_2$, il en résulte que $\pi_1 C^n(\mathfrak{U}_1) \subset B_1 B_2$, d'où $\pi_2 \pi_1 C^n(\mathfrak{U}_1) \subset B_1 B_2$ et par suite aussi $C^n(\mathfrak{U}) \subset B_1 B_2$.

10.32. Lemme. Soit Γ^{n-1} un $(n-1, R)$-cycle du type t_i $(i = 1, 2, 3)$

dans A. Soit $\Gamma^{n-1} \sim 0$ dans $\bar{P}_i$, $P_i \in \mathfrak{P}_i$. Soit $\alpha > 0$ un nombre ordinal donné. A chaque nombre ordinal $\xi < \alpha$ soit attaché un ensemble bicompact B_ξ de manière que: 1° $B_0 = \bar{P}_i$; 2° Pour $\xi < \eta < \alpha$ on a $B_\xi \supset B_\eta$; 3° $A \subset B_\xi$ pour chaque $\xi < \alpha$; 4° $\Gamma^{n-1} \sim 0$ dans B_ξ pour chaque $\xi < \alpha$. Posons $B_\alpha = \prod_{\xi<\alpha} B_\xi$. Alors $\Gamma^{n-1} \sim 0$ dans B_α.

Démonstration. Soit C^n le cycle relatif du n° 10.1. Pour chaque $\xi < \alpha$ on a évidemment $C^n \subset B_\xi$. Soit Φ la famille de tous les affinements $\mathfrak{U}$ de $\mathfrak{B}(P_i)$ d'ordre $\leqq n \bmod S$. Soit $\mathfrak{U} \in \Phi$. Il s'agit de montrer que $C^n(\mathfrak{U}) \subset B_\alpha$. Soit $\mathfrak{U}_1$ un rapetissement fort (v.[30]) de $\mathfrak{U}$. Évidemment $\mathfrak{U}_1 \in \Phi$. Soient U_ν $(1 \leqq \nu \leqq m)$ les sommets de $\mathfrak{U}$ et $U'_\nu \in U_\nu$ les sommets correspondants de $\mathfrak{U}_1$. En posant $\pi U'_\nu = U_\nu$, on a $\pi = Pr.(\mathfrak{U}_1, \mathfrak{U})$. D'après 10.1 on a $C^n(\mathfrak{U}) = \pi C^n(\mathfrak{U}_1) \bmod A$. Soit $(U_{\nu_0}, U_{\nu_1}, \cdots, U_{\nu_n})$ un simplexe de la chaîne $C^n(\mathfrak{U})$, $\mathfrak{J} = \prod_{k=0}^{n} U_{\nu_k}$ son noyau. On doit montrer que $\mathfrak{J} B_\alpha \neq 0$. Ceci est clair si $\mathfrak{J} A \neq 0$, car $A \subset B_\alpha$. Supposons donc que $\mathfrak{J} A = 0$. Puisque $C^n(\mathfrak{U}) = \pi C^n(\mathfrak{U}_1) \bmod A$, la chaîne $C^n(\mathfrak{U}_1)$ contient le simplexe $\sigma^n = (U'_{\nu_0}, U'_{\nu_1}, \cdots, U'_{\nu_n})$. Pour chaque $\xi < \alpha$ le noyau de σ^n rencontre B_ξ, car $C^n(\mathfrak{U}_1) \subset B_\xi$. En posant $K = \prod_{k=0}^{n} U'_{\nu_k}$, K est un sous-ensemble bicompact de $\mathfrak{J}$ tel que $KB_\xi \neq 0$ pour chaque $\xi < \alpha$. $\{KB_\xi\}$ étant une suite monotone d'ensembles bicompacts et non vides, on a $\prod_{\xi<\alpha} KB_\xi = KB_\alpha \neq 0$, d'où $\mathfrak{J} B_\alpha \neq 0$.

10.33. Passons à la démonstration de l'énoncé du n° 10.3. Choisissons $P_i \in \mathfrak{P}_i$ de manière que $A \subset \bar{P}_i$, $\Gamma^{n-1} \sim 0$ dans $\bar{P}_i$. Choisissons un nombre transfini β dont la puissance soit supérieure à celle de $R - S$ et définissons par récurrence une suite transfinie $\{B_\xi\}$ $(\xi < \beta)$ telle que: 1° les B_ξ sont des sous-ensembles bicompacts de $R - S$; 2° $B_\xi \supset B_\eta$ pour chaque $\xi < \eta < \beta$; 3° $B_0 = \bar{P}_i$; 4° $B_\xi \supset A$ pour chaque $\xi < \beta$; 5° $\Gamma^{n-1} \sim 0$ dans B_ξ pour chaque $\xi < \beta$. Soit $\eta > 0$ et $< \beta$ un nombre ordinal donné et supposons qu'on ait déjà défini les B_ξ pour tous les $\xi < \eta$. Deux cas sont à distinguer: *Premièrement* soit $\eta = \xi + 1$ un nombre de première espèce. Dans ce cas choisissons l'ensemble bicompact B_η de manière que $A \subset B_\eta \subset B_\xi$ et que $\Gamma^{n-1} \sim 0$ dans B_η; c'est vérifié en posant $B_\eta = B_\xi$; or *si cela est possible* choisissons $B_\eta \neq B_\xi$. *En second lieu* soit η un nombre de seconde espèce. Dans ce cas posons $B_\eta = \prod_{\xi<\eta} B_\xi$ ce qui est loisible en vertu du lemme du n° 10.32. La construction de la suite transfinie $\{B_\xi\}$ $(\xi < \beta)$ est ainsi achevée. De la propriété 2° résulte l'existence d'un nombre ordinal $\alpha < \beta$ tel que $B_\alpha = B_{\alpha+1}$. Posons $H = B_\alpha$. Les propriétés 1°, 2°, 3° du n° 10.3 sont évidentes. Soit donc K un ensemble bicompact tel que $A \subset K \subset \bar{P}'_i$, $P'_i \in \mathfrak{P}_i$ et $\Gamma^{n-1} \sim 0$ dans K; il s'agit de montrer que $H \subset K$. Puisque

$\bar{P}_i \bar{P}'_i \supset A \neq 0$, il existe (v. 9.3) un sommet P_{i-1} de $\mathfrak{P}_{i-1}$ tel que $\bar{P}_i + \bar{P}'_i \subset P_{i-1}$, d'où $H+K \subset P_{i-1}$. Puisque $\Gamma^{n-1} \sim 0$ et dans H et dans K, on a $\Gamma^{n-1} \sim 0$ dans HK en vertu du lemme du n° 10.31. Puisque $H = B_\alpha = B_{\alpha+1}$, on ne peut avoir $HK \neq H$ par la définition même de $B_{\alpha+1}$. Donc $HK = H$, c'est-à-dire $H \subset K$.

10.41. Soit Γ^{n-1} un $(n-1, R)$-cycle du type t_3 dans A. On peut alors considérer Γ^{n-1} aussi comme un $(n-1, R)$-cycle du type t_2 dans A. Evidemment ceci est sans influence sur le porteur de l'homologie $\Gamma^{n-1} \sim 0$.

10.42. *Pour $\nu = 1, 2$ soit Γ_ν^{n-1} un $(n-1, R)$-cycle du type t_2 dans A_ν et soit $\Gamma_\nu^{n-1} \sim 0$ dans $\bar{P}_2$, $P_2 \in \mathfrak{P}_2$; soit H_ν le porteur de l'homologie $\Gamma_\nu^{n-1} \sim 0$. Soit $\Gamma_1^{n-1} \sim \Gamma_2^{n-1}$ dans $A_1 + A_2$. Alors $H_1 - (A_1 + A_2) = H_2 - (A_1 + A_2)$.*

Démonstration. Il existe un $(n-1, R)$-cycle Γ^{n-1} dans $A_1 + A_2$ tel que $\Gamma^{n-1} \sim \Gamma_\nu^{n-1}$ dans $A_1 + A_2$ $(\nu = 1, 2)$, $\Gamma^{n-1} \sim 0$ dans $\bar{P}_2$. [Il suffit de poser p. ex. $\Gamma^{n-1} = \Gamma_1^{n-1}$.] Évidemment Γ^{n-1} est un $(n-1, R)$-cycle du type t_2 dans $A_1 + A_2$; soit H le porteur de l'homologie $\Gamma^{n-1} \sim 0$. Il suffit de montrer que $H_1 + A_2 = H_2 + A_1 = H$. Or $H + H_1 + H_2 \subset \bar{P}_2$. On a $H \supset A_1 + A_2$, $\Gamma^{n-1} \sim 0$ dans H, $\Gamma^{n-1} \sim \Gamma_1^{n-1}$ dans $A_1 + A_2$, donc $\Gamma_1^{n-1} \sim 0$ dans H, d'où $H \supset H_1$ (par la définition de H_1), et par suite $H \supset H_1 + A_2$. D'autre part on a $\Gamma_1^{n-1} \sim 0$ dans H_1, donc aussi dans $H_1 + A_2 \supset A_1 + A_2$, et $\Gamma_1^{n-1} \sim \Gamma^{n-1}$ dans $A_1 + A_2$, donc $\Gamma^{n-1} \sim 0$ dans $H_1 + A_2$, d'où $H_1 + A_2 \supset H$ (par la définition de H). Il est ainsi prouvé que $H_1 + A_2 = H$, et on prouve pareillement que $H_2 + A_1 = H$.

10.43. *Pour $\nu = 1, 2$ soit Γ_ν^{n-1} un $(n-1, R)$-cycle du type t_3 dans A_ν. Soit H_ν le porteur de l'homologie $\Gamma_\nu^{n-1} \sim 0$. Soit $A_1 + A_2 \subset \bar{P}_3$, $P_3 \in \mathfrak{P}_3$. Soit $\Gamma_1^{n-1} \sim \Gamma_2^{n-1}$ dans $A_1 + A_2$. Alors $H_1 - (A_1 + A_2) = H_2 - (A_1 + A_2)$.*

Démonstration. Soit $\Gamma_\nu^{n-1} \sim 0$ dans $\bar{P}_{3\nu}$, $P_{3\nu} \in \mathfrak{P}_3$ $(\nu = 1, 2)$. Alors $\bar{P}_3 \bar{P}_{3\nu} \supset A_\nu \neq 0$. Il existe donc (v. 9.3) un sommet P_2 de $\mathfrak{P}_2$ tel que $\bar{P}_3 + \bar{P}_{31} + \bar{P}_{32} \subset \bar{P}_2$. Il suffit donc d'appliquer 10.41 et 10.42.

10.44. *Soit Γ^{n-1} un $(n-1, R)$-cycle qui peut être considéré comme un cycle dans A_1 ou comme un cycle dans A_2, dans les deux cas du type t_2. Soit H_ν $(\nu = 1, 2)$ le porteur de l'homologie $\Gamma^{n-1} \sim 0$, en considérant Γ^{n-1} comme un cycle dans A_ν. Alors $H_1 - (A_1 + A_2) = H_2 - (A_1 + A_2)$.*

Démonstration. D'après la définition d'un $(n-1, R)$-cycle du type t_2, il existe un sommet $P_2 \in \mathfrak{P}_2$ tel que $\Gamma^{n-1} \sim 0$ dans $\bar{P}_2$, d'où $A_1 + A_2 \subset \bar{P}_2$. Il suffit donc appliquer 10.42 en y posant $\Gamma_1^{n-1} = \Gamma_2^{n-1} = \Gamma^{n-1}$.

10.5. *Pour $\nu = 1, 2$ soit Γ_ν^{n-1} un $(n-1, R)$-cycle du type t_3 dans A_ν. Soit H_ν le porteur de l'homologie $\Gamma_\nu^{n-1} \sim 0$. Soit $H_1 H_2 \neq 0$. Soit $c_\nu \in \mathfrak{R}$. Alors $c_1 \Gamma_1^{n-1} + c_2 \Gamma_2^{n-1}$ est un $(n-1, R)$-cycle du type t_2 dans $A_1 + A_2$ et le porteur H de l'homologie $c_1 \Gamma_1^{n-1} + c_2 \Gamma_2^{n-1} \sim 0$ est contenu dans $H_1 + H_2$.*

Démonstration. Soit $\Gamma_\nu^{n-1} \sim 0$ dans $\bar{P}_{3\nu}$, $P_{3\nu} \in \mathfrak{P}_3$ $(\nu = 1, 2)$. Alors (v. 10.3) on a $\bar{P}_{3\nu} \supset H_\nu$, donc $\bar{P}_{31} \bar{P}_{32} \supset H_1 H_2 \neq 0$. Il existe donc (v. 9.3) un

sommet P_2 de $\mathfrak{P}_2$ tel que $P_{31}+P_{32}\subset P_2$. Il en résulte que $A_1+A_2\subset\overline{P}_2$, $c_1\Gamma_1^{n-1}+c_2\Gamma_2^{n-1}\sim 0$ dans $\overline{P}_2$, de manière que $c_1\Gamma_1^{n-1}+c_2\Gamma_2^{n-1}$ est un $(n-1, R)$-cycle du type t_2 dans A_1+A_2. Puisque $c_1\Gamma_1^{n-1}+c_2\Gamma_2^{n-1}\sim 0$ dans $H_1+H_2\subset\overline{P}_{31}+\overline{P}_{32}\subset\overline{P}_2$, on a $H\subset H_1+H_2$ d'après 10.3.

11. Axiome D_1: *Soit Γ^{n-1} un $(n-1, R)$-cycle du type t_2 (ou t_3, v. 10.41) dans A. Soit H le porteur de l'homologie $\Gamma^{n-1}\sim 0$. L'ensemble $H-A$ est ouvert dans R (donc dans $R-S$).*

L'ensemble $H-A$ sera appelé l'*intérieur* du cycle Γ^{n-1}. La manière dont il dépend de A est éclaircie dans les n^{os} 10.42 et 10.43.

11.1. *Soit Γ^{n-1} un $(n-1, R)$-cycle du type t_2 dans A. Soit H le porteur de l'homologie $\Gamma^{n-1}\sim 0$. Soit a un point de l'intérieur de Γ^{n-1}. Il existe un entourage W de a jouissant de la propriété suivante: A chaque entourage $V\subset W$ de a on peut attacher un $(n-1, R)$-cycle Δ^{n-1} du type t_2 dans $F(V)=\overline{V}-V$ tel que: 1° le porteur de l'homologie $\Delta^{n-1}\sim 0$ est $\overline{V}$ (de manière que le point a est à l'intérieur de Δ^{n-1}); 2° on a l'homologie $\Gamma^{n-1}\sim\Delta^{n-1}$ dans $H-V$. V étant donné, le cycle Δ^{n-1} est bien déterminé à une homologie dans $F(V)$ près.*

Démonstration. Soit W un entourage de a si petit que $W\in H-A$. W existe en vertu de l'axiome D_1. Soit V un entourage de a tel que $V\subset W$. Soit C^n le (n, R)-cycle relatif du n° 10.1. C'est donc un (n, R)-cycle mod A dans H. Soit N la famille complète de réseaux dans H qui s'obtient de la famille N définie dans *Homologie*, IV, 2, en y remplaçant R, R_1, R_2, $R_3=R_1R_2$, α, $\alpha_1=R_1\alpha$, $\alpha_2=R_2\alpha$, $\alpha_3=R_3\alpha$ respectivement par H, $\overline{V}$, $H-V$, $F(V)=\overline{V}(H-V)$ (car $V\in H$), A, 0 (car $\overline{V}A=0$), A, 0. Soit Ψ la famille de tous les réseaux $\mathfrak{U}$ dans R qui s'obtiennent d'un réseau $\mathfrak{V}\in N$ de la manière suivante: Soient $V_1, V_2, \cdots, V_m$ les sommets de $\mathfrak{V}$; d'après *Homologie*, III, 21 on peut déterminer des ensembles U_i $(1\leqq i\leqq m)$ ouverts dans R tels que $V_i=HU_i$ et que $\prod_{\nu=0}^{k} V_{i_\nu}=0$ entraîne $\prod_{0}^{k} U_i=0$; soient $U_{m+1}, U_{m+2}, \cdots, U_{m+s}$ les sommets d'un réseau arbitraire dans $R-H$: les sommets de $\mathfrak{U}$ sont $U_1, U_2, \cdots, U_{m+s}$. Ψ est évidemment une famille complète de réseaux dans R et les réseaux de la famille N sont les intersections de ceux de la famille Ψ avec l'espace H. Pour chaque $\mathfrak{U}\in\Psi$, on peut poser (*Homologie*, IV, 6, où l'on remplace n par $n-1$) $C^n(\mathfrak{U})=K_1^n(\mathfrak{U})-K_2^n(\mathfrak{U})$, les $(n, \mathfrak{U})$-chaînes $K_1^n(\mathfrak{U})$ et $K_2^n(\mathfrak{U})$ étant telles que $K_1^n(\mathfrak{U})\subset\overline{V}$, $K_2^n(\mathfrak{U})\subset H-V$, et on peut définir un $(n-1, \mathfrak{U})$-cycle absolu dans $F(V)$: $\Delta^{n-1}(\mathfrak{U})$ tel que $K_1^n(\mathfrak{U})\to\Delta^{n-1}(\mathfrak{U})$, $K_2^n(\mathfrak{U})\to\Delta^{n-1}(\mathfrak{U})$ mod A. Les $\Delta^{n-1}(\mathfrak{U})$ définissent (v. *Homologie*, IV, 13) un $(n-1, R)$-cycle Δ^{n-1} dans $F(V)$. Puisque $K_1^n(\mathfrak{U})\subset\overline{V}$, on a $\Delta^{n-1}\sim 0$ dans $\overline{V}\subset H\subset\overline{P}_2$, $P_2\in\mathfrak{P}_2$. Δ^{n-1} est donc un $(n-1, R)$-cycle du type t_2 dans $F(V)$ et, H_0 étant le porteur de l'homologie $\Delta^{n-1}\sim 0$, on a $H_0\subset\overline{V}$. D'après 10.3, on a pour

chaque $\mathfrak{U} \in \Psi$: $\Gamma^{n-1}(\mathfrak{U}) \sim FC^n(\mathfrak{U}) = FK_1^n(\mathfrak{U}) - FK_2^n(\mathfrak{U})$ dans A. Or $FK_1^n(\mathfrak{U}) = \Delta^{n-1}(\mathfrak{U})$ et $K_2^n(\mathfrak{U}) \subset H - V$, d'où $FK_2^n(\mathfrak{U}) \subset H - V$; enfin $A \subset H - V$. Donc $\Gamma^{n-1}(\mathfrak{U}) \sim \Delta^{n-1}(\mathfrak{U})$ dans $H - V$ pour chaque $\mathfrak{U} \in \Psi$, d'où $\Gamma^{n-1} \sim \Delta^{n-1}$ dans $H - V$. Si Δ_1^{n-1} est un autre $(n-1, R)$-cycle dans $F(V)$ tel que $\Delta_1^{n-1} \sim 0$ dans $\overline{V}$ et $\Gamma^{n-1} \sim \Delta_1^{n-1}$ dans $H - V$, on a $\Delta^{n-1} - \Delta_1^{n-1} \sim 0$ et dans $\overline{V}$ et dans $H - V$. Puisque $\overline{V} + (H - V) = H \subset \overline{P}_2$, $P_2 \in \mathfrak{P}_2$, $\Delta^{n-1} - \Delta_1^{n-1}$ est un $(n-1, R)$-cycle du type t_2 dans $F(V)$ et, H_1 étant le porteur de l'homologie $\Delta^{n-1} - \Delta_1^{n-1} \sim 0$, on a $H_1 \subset \overline{V}$ et $H_1 \subset H - V$, d'où $H_1 \subset F(V)$ et par suite $\Delta^{n-1} \sim \Delta_1^{n-1}$ dans $F(V)$. Le cycle Δ^{n-1} est donc bien déterminé à une homologie dans $F(V)$ près. On doit encore démontrer que $H_0 = \overline{V}$. (Nous savons déjà que $H_0 \subset \overline{V}$.) D'après la définition de H_0, il existe pour chaque $\mathfrak{U} \in \Psi$ une $(n, \mathfrak{U})$-chaîne $L^n(\mathfrak{U}) \subset H_0$ telle que $L^n(\mathfrak{U}) \to \Delta^{n-1}(\mathfrak{U})$. Or $K_1^n(\mathfrak{U}) \to \Delta^{n-1}(\mathfrak{U})$ et

$$FC^n(\mathfrak{U}) = FK_1^n(\mathfrak{U}) - FK_2^n(\mathfrak{U}) \sim \Gamma^{n-1}(\mathfrak{U})$$

dans A, de manière qu'il existe une $(n, \mathfrak{U})$-chaîne $M^n(\mathfrak{U})$ telle que $K_1^n(\mathfrak{U}) - K_2^n(\mathfrak{U}) \to \Gamma^{n-1}(\mathfrak{U}) + FM^n(\mathfrak{U})$. Il en résulte que $L^n(\mathfrak{U}) - K_2^n(\mathfrak{U}) - FM^n(\mathfrak{U}) \to \Gamma^{n-1}(\mathfrak{U})$ pour chaque $\mathfrak{U} \in \Psi$. Or, $L^n(\mathfrak{U}) \subset H_0$, $K_2^n(\mathfrak{U}) \subset H - V$, $FM^n(\mathfrak{U}) \subset A \subset H - V$, de manière que $\Gamma^{n-1} \sim 0$ dans $H_0 + (H - V) \subset H$. H étant le porteur de l'homologie $\Gamma^{n-1} \sim 0$, on a donc $H_0 + (H - V) = H$, d'où $V = H - (H - V) \subset H_0$ et par suite, H_0 étant fermé, $\overline{V} \subset H_0$. Comme nous savons que $H_0 \subset \overline{V}$, nous arrivons bien à la conclusion $\overline{V} = H_0$.

12. Axiome D_2: *Chaque point $a \in R - S$ est intérieur à un $(n-1, R)$-cycle du type t_2.*

12.1. *Chaque point $a \in R - S$ possède un entourage W_0 jouissant de la propriété suivante: Si $V \subset W_0$ est un entourage de a, il existe un $(n-1, R)$-cycle Δ^{n-1} du type t_3 dans $F(V)$ tel que le porteur de l'homologie $\Delta^{n-1} \sim 0$ soit $\overline{V}$.* (Le point a est par suite à l'intérieur de Δ^{n-1}).

Démonstration. Soit P_3 un sommet de réseau gén. $\mathfrak{P}_3$ contenant le point a. En vertu de l'axiome D_2, il existe un ensemble bicompact $A \subset R - S - (a)$ et un $(n-1, R)$-cycle Γ^{n-1} du type t_2 dans A tel que le point a appartienne à l'intérieur de Γ^{n-1}. Déterminons W d'après 11.1 et posons $W_0 = WP_3$. Pour chaque $V \subset W_0 \subset W$, on a, d'après 11.1, un $(n-1, R)$-cycle Δ^{n-1} du type t_2 dans $F(V)$, tel que le porteur de l'homologie $\Delta^{n-1} \sim 0$ soit $\overline{V}$. Puisque $\Delta^{n-1} \sim 0$ dans $\overline{V} \subset \overline{P}_3$, Δ^{n-1} est un $(n-1, R)$-cycle du type t_3 dans $F(V)$.

12.2. *La dimension de l'espace $R - S$ en chaque point a est égale à n.*

Démonstration. D'après l'axiome A_4 (n° 7) on a $\dim_a(R - S) \leqq n$. Supposons par impossible qu'on ait $\dim_a(R - S) < n$. Le point a possède alors des entourages V arbitrairement petits tels que $\dim F(V) \leqq n - 2$. La famille Ψ de tous les réseaux $\mathfrak{U}$ tels que les dimensions des $\mathfrak{U}$-simplexes

dont le noyau rencontre $F(V)$ sont $\leqq n-2$ est alors une famille complète. D'autre part, si V est suffisamment petit, il existe d'après 12.1 un $(n-1, R)$-cycle Δ^{n-1} dans $F(V)$ tel que $\Delta^{n-1} \dotplus 0$ dans $F(V)$. Or pour tous les réseaux $\mathfrak{U} \in \Psi$ on a évidemment $\Delta^{n-1}(\mathfrak{U}) = 0$ ce qui est une contradiction.

13. Axiome E: *A chaque point $a \in R - S$ et à chaque entourage Q de a on peut attacher un entourage $Q_1 \subset Q$ jouissant de la propriété suivante: Pour chaque entourage Q_2 de a tel que $Q_2 \subset Q_1$ il existe un entourage Q_3 de a tel que: 1° $Q_3 \subset Q_2$; 2° si les $(n-1, R)$-cycles Γ_1^{n-1}, Γ_2^{n-1} sont situés dans $\overline{Q_1} - Q_2$ et sont homologues à zéro dans $\overline{Q_1}$, il existe deux nombres r_1, $r_2 \in \mathfrak{R}$, dont un au moins $\neq 0$, tels que $r_1 \Gamma_1^{n-1} + r_2 \Gamma_2^{n-1} \sim 0$ dans $\overline{Q} - Q_3$.*

13.1. De l'axiome E on déduit d'après 4.2: *$\mathfrak{Q}$ étant un réseau gén. donné, il existe un affinement $\mathfrak{Q}_1$ de $\mathfrak{Q}$ et une projection $\pi = Pr.(\mathfrak{Q}_1, \mathfrak{Q})$ qui jouissent de la propriété suivante: $\mathfrak{Q}_2$ étant un affinement quelconque de $\mathfrak{Q}_1$, on peut déterminer un affinement $\mathfrak{Q}_3$ de $\mathfrak{Q}_2$ ainsi qu'une projection $\pi' = Pr.(\mathfrak{Q}_3, \mathfrak{Q}_2)$ de manière que l'énoncé suivant soit vrai: Pour chaque sommet Q_3 de $\mathfrak{Q}_3$ et pour chaque sommet Q_1 de $\mathfrak{Q}_1$ tel que $Q_1 \supset \pi' Q_3$ on peut attacher à chaque couple de $(n-1, R)$-cycles Γ_1^{n-1}, Γ_2^{n-1} situés dans $\overline{Q_1} - \pi' Q_3$ et homologues à zéro dans $\overline{Q_1}$ un couple de nombres r_1, $r_2 \in \mathfrak{R}$ dont un au moins $\neq 0$, tels que $r_1 \Gamma_1^{n-1} + r_2 \Gamma_2^{n-1} \sim 0$ dans $\overline{\pi Q_1} - Q_3$.*

14.1. *Soit $a \in R - S$; soit U un entourage de a. Il existe un entourage $V \subset U$ de a jouissant de la propriété suivante: A chaque couple C_1^n, C_2^n de (n, R)-cycles mod $(R-U)$ on peut attacher deux nombres r_1, $r_2 \in \mathfrak{R}$, dont un au moins $\neq 0$, tels que $r_1 C_1^n + r_2 C_2^n \sim 0$ mod $(R-V)$.*

Démonstration. Soit $Q \Subset U$ un entourage de a si petit que $Q \subset P_2 \in \mathfrak{P}_2$ (v. 9.3). Q étant donné, déterminons un entourage $Q_1 \subset Q$ d'après l'axiome E. Choisissons l'entourage $Q_2 \subset Q_1$ de a arbitrairement et déterminons l'entourage $Q_3 \subset Q_2$ de a d'après l'axiome E.

Soit Ψ la famille complète de réseaux dans R qui s'obtient de la famille N définie dans *Homologie*, IV, 2 en y remplaçant R, R_1, R_2, $R_3 = R_1 R_2$, α, $\alpha_1 = R_1 \alpha$, $\alpha_2 = R_2 \alpha$, $\alpha_3 = R_3 \alpha$ respectivement par R, $\overline{Q_1}$, $R - Q_1$, $F(Q_1)$, $R-U$, 0, $R-U$, 0. Soit $\nu = 1, 2$. D'après l. c. n° 6 (où l'on remplace C^{n+1} par C_ν^n) il existe pour chaque $\mathfrak{U} \in \Psi$ deux $(n, \mathfrak{U})$-chaînes $K_\nu^n(\mathfrak{U}) \subset \overline{Q_1}$, $L_\nu^n(\mathfrak{U}) \subset R - Q_1$ telles que $C_\nu^n(\mathfrak{U}) = K_\nu^n(\mathfrak{U}) - L_\nu^n(\mathfrak{U})$ et un $(n-1, \mathfrak{U})$-cycle $\Gamma_\nu^{n-1}(\mathfrak{U}) \subset F(Q_1)$ tel que $K_\nu^n(\mathfrak{U}) \to \Gamma_\nu^{n-1}(\mathfrak{U})$. D'après l. c. n° 13 $\Gamma_\nu^{n-1}(\mathfrak{U})$ définissent un $(n-1, R)$-cycle Γ_ν^{n-1} dans $F(Q_1)$. Comme $K_\nu^n(\mathfrak{U}) \subset \overline{Q_1}$, on a $\Gamma_\nu^{n-1} \sim 0$ dans $\overline{Q_1}$. Puisque les cycles Γ_ν^{n-1} sont situés dans $F(Q_1) \subset \overline{Q_1} - Q_2$, il résulte de l'axiome E l'existence de deux nombres $r_1, r_2 \in \mathfrak{R}$, dont un au moins $\neq 0$, tels que $r_1 \Gamma_1^{n-1} + r_2 \Gamma_2^{n-1} \sim 0$ dans $\overline{Q} - Q_3$.

Soit $\mathfrak{U} \in \Psi$ un réseau donné. On doit montrer que $r_1 C_1^n(\mathfrak{U}) + r_2 C_2^n(\mathfrak{U}) \sim 0$ mod $(R-V)$. Soit $\mathfrak{V}$ un affinement de $\mathfrak{U}$ tel que $\Delta^n(\mathfrak{V}) = 0$ pour chaque $(n, \mathfrak{V})$-cycle essentiel dans $\overline{P_2}$. [$\mathfrak{V}$ existe en vertu de 9.2.] Soit $\mathfrak{U}' \in \Psi$ un

affinement de $\mathfrak{V}$ normal relativement aux cycles dans $\bar{P}_2$. Soit $\pi = Pr.(\mathfrak{V}, \mathfrak{U})$, $\pi' = Pr.(\mathfrak{U}', \mathfrak{V})$. D'après l'homologie $r_1 \Gamma_1^{n-1} + r_2 \Gamma_2^{n-1} \sim 0$ dans $\bar{Q} - Q_3$, il existe une $(n, \mathfrak{U}')$-chaîne $D^n(\mathfrak{U}')$ dans $\bar{Q} - Q_3$ telle que $D^n(\mathfrak{U}') \to r_1 \Gamma_1^{n-1}(\mathfrak{U}') + r_2 \Gamma_2^{n-1}(\mathfrak{U}')$. Comme $\Gamma_\nu^{n-1}(\mathfrak{U}') = FK_\nu^n(\mathfrak{U}')$ $(\nu = 1, 2)$, on a $D^n(\mathfrak{U}') - r_1 K_1^n(\mathfrak{U}') - r_2 K_2^n(\mathfrak{U}') \to 0$. Le premier membre de cette relation est un $(n, \mathfrak{U}')$-cycle dans $(\bar{Q} - Q_3) + \bar{Q}_1 \subset \bar{Q} \subset \bar{P}_2$. Donc $\pi'[D^n(\mathfrak{U}') - r_1 K_1^n(\mathfrak{U}') - r_2 K_2^n(\mathfrak{U}')]$ est un $(n, \mathfrak{V})$-cycle essentiel dans P_2 et il est par suite égal à zéro. Donc on a aussi

$$\pi\pi'[r_1 K_1^n(\mathfrak{U}') + r_2 K_2^n(\mathfrak{U}')] = \pi\pi' D^n(\mathfrak{U}') \subset \bar{Q} - Q_3 \subset R - Q_3.$$

D'autre part, on a aussi $K_\nu^n(\mathfrak{U}') - C_\nu^n(\mathfrak{U}') = L_\nu^n(\mathfrak{U}') \subset R - Q_1 \subset R - Q_3$. Donc

$$\pi\pi'[r_1 C_1^n(\mathfrak{U}') + r_2 C_2^n(\mathfrak{U}')] = 0 \mod (R - Q_3).$$

Or $\pi\pi' C_\nu^n(\mathfrak{U}') \sim C_\nu^n(\mathfrak{U}) \mod (R - U) \subset R - Q_3$, car C_ν^n est un (n, R)-cycle $\mod (R - U)$. Donc

$$r_1 C_1^n(\mathfrak{U}) + r_2 C_2^n(\mathfrak{U}) \sim 0 \mod (R - Q_3)$$

pour chaque $\mathfrak{U} \in \Psi$. Il suffit donc de poser $V = Q_3$.

14.2. De 14.1 on déduit d'après 4.2 : $\mathfrak{P}$ *étant un réseau gén., il existe un affinement* $\mathfrak{Q}$ *de* $\mathfrak{P}$ *et une projection* $\pi = Pr.(\mathfrak{Q}, \mathfrak{P})$ *jouissant de la propriété suivante : Si* $Q \in \mathfrak{Q}$ *et si* C_1^n, C_2^n *sont deux* (n, R)*-cycles mod* $(R - \pi Q)$, *il existe deux nombres* $r_1, r_2 \in \mathfrak{R}$, *dont un au moins* $\neq 0$, *tels que* $r_1 C_1^n + r_2 C_2^n \sim 0$ *mod* $(R - Q)$.

15. Soit $a \in R - S$. Soit $\Theta_i(a)$ $(i = 2, 3)$ la famille de tous les $(n-1, R)$-cycles du type t_i dans A, A étant assujetti à la condition de ne pas contenir a. *Orienter l'espace* $R - S$ *au point* a signifie que l'on attache à chaque $\Gamma^{n-1} \in \Theta_2(a)$ un nombre $\omega(a, \Gamma^{n-1}) \in \mathfrak{R}$ tel que 1° $\omega(a, \Gamma^{n-1}) = 0$ si et seulement si le porteur de l'homologie $\Gamma^{n-1} \sim 0$ ne contient pas a; 2° pour $\Gamma^{n-1} \in \Theta_2(a)$, $r \in \mathfrak{R}$ on a $\omega(a, r\Gamma^{n-1}) = r\omega(a, \Gamma^{n-1})$; 3° si Γ_1^{n-1}, Γ_2^{n-1}, $\Gamma_1^{n-1} + \Gamma_2^{n-1} \in \Theta_2(a)$, on a $\omega(a, \Gamma_1^{n-1} + \Gamma_2^{n-1}) = \omega(a, \Gamma_1^{n-1}) + \omega(a, \Gamma_2^{n-1})$.

Remarque 1. Si $\Gamma^{n-1} \in \Theta_2(a)$ en le considérant comme un $(n-1, R)$-cycle du type t_2 dans A_1 ou A_2, le nombre $\omega(a, \Gamma^{n-1})$ doit être le même dans les deux cas. Ceci est d'accord avec la condition 1° (v. 10.44).

Remarque 2. Si $\Gamma^{n-1} \in \Theta_i(a)$ $(i = 2, 3)$, $r \in \mathfrak{R}$, évidemment $r\Gamma^{n-1} \in \Theta_i(a)$.

Remarque 3. Si Γ_1^{n-1}, $\Gamma_2^{n-1} \in \Theta_3(a)$ et si $\omega(a, \Gamma_1^{n-1}) \neq 0$, $\omega(a, \Gamma_2^{n-1}) \neq 0$, $r_1, r_2 \in \mathfrak{R}$, on a $r_1 \Gamma_1^{n-1} + r_2 \Gamma_2^{n-1} \in \Theta_2(a)$. *Démonstration.* Soit $(\nu = 1, 2)$ H_ν le porteur de l'homologie $\Gamma_\nu^{n-1} \sim 0$. D'après 1° $a \in H_1 H_2$. Il suffit donc d'appliquer 10.5.

15.1 *Le point* $a \in R - S$ *étant donné, il est possible d'orienter* $R - S$ *en* a. *Si* ω_1, ω_2 *sont deux orientations de* $R - S$ *en* a, *il existe un nombre* $s \in \mathfrak{R}$, $s \neq 0$, *tel que* $\omega_2(a, \Gamma_{n-1}) = s\omega_1(a, \Gamma^{n-1})$ *pour chaque* $\Gamma_{n-1} \in \Theta_2(a)$. *Inver-*

sement, si ω_1 est une orientation donnée de $R-S$ en a et si $s\in\Re$, $s\neq 0$, on obtient une orientation ω_2 de $R-S$ en a en posant $\omega_2(a, \Gamma^{n-1}) = s\omega_1(a, \Gamma^{n-1})$ pour chaque $\Gamma^{n-1}\in\Theta_2(a)$.

Démonstration. D'après l'axiome D_2 (v. 12) il existe un $(n-1, R)$-cycle Γ_0^{n-1} du type t_2 dans A_0 dont l'intérieur H_0-A_0 contient le point a. Choisissons l'entourage W de a correspondant au cycle Γ_0^{n-1} d'après 11.1. Soit P_3 un sommet du réseau gén. $\mathfrak{P}_3$ (v. 9.3) contenant a. Soit $\mathcal{A}$ la famille de tous les entourages V de a tels que $V\subset WP_3$. A chaque $V\in\mathcal{A}$ correspond d'après 11.1 un $(n-1, R)$-cycle $\Delta^{n-1}(V)$ du type t_2 dans $F(V)$ tel que : 1° le porteur de l'homologie $\Delta^{n-1}(V)\sim 0$ est $\overline{V}$; 2° on a l'homologie $\Gamma^{n-1}\sim\Delta^{n-1}(V)$ dans H_0-V. D'ailleurs, comme $\overline{V}\subset\overline{P}_3$, les cycles $\Delta^{n-1}(V)$ sont du type t_3. Si $V_1, V_2\in\mathcal{A}$; $V_2\subset V_1$, on a $\Delta^{n-1}(V_2)-\Delta^{n-1}(V_1)\sim 0$ dans $\overline{V}_1+\overline{V}_2=\overline{V}_1\subset\overline{P}_3$, de manière que $\Delta^{n-1}(V_2)-\Delta^{n-1}(V_1)$ est un cycle du type t_2 (et aussi du type t_3) et le porteur H_{12} de l'homologie $\Delta^{n-1}(V_2)-\Delta^{n-1}(V_1)\sim 0$ satisfait à la condition $H_{12}\subset\overline{V}_1$. D'autre part, on a $\Gamma^{n-1}\sim\Delta^{n-1}(V_1)$ dans $H_0-V_1\subset H_0-V_2$ et $\Gamma^{n-1}\sim\Delta^{n-1}(V_2)$ dans H_0-V_2, d'où $\Delta^{n-1}(V_2)-\Delta^{n-1}(V_1)\sim 0$ dans $H_0-V_2\subset H_0\subset\overline{P}_2$, où $P_2\in\mathfrak{P}_2$ (car Γ_0^{n-1} est un cycle du type t_2), et par suite $H_{12}\subset H_0-V_2$. On a donc la relation $H_{12}\subset\overline{V}_1-V_2$ de manière que le point a n'est pas à l'intérieur de $\Delta^{n-1}(V_2)-\Delta^{n-1}(V_1)$. Donc on a, dans chaque orientation ω de l'espace $R-S$ en a, la relation $\omega[a, \Delta^{n-1}(V_2)-\Delta^{n-1}(V_1)]=0$. Il en résulte que le nombre $s=\omega[a, \Delta^{n-1}(V)]\in\Re$ est indépendant du choix de $V\in\mathcal{A}$. Comme le point a est situé à l'intérieur V du cycle $\Delta^{n-1}(V)$, on a $s\neq 0$.

Soit maintenant $\Gamma^{n-1}\in\Theta_2(a)$. Γ^{n-1} est un $(n-1, R)$-cycle du type t_2 dans A; soit H le porteur de l'homologie $\Gamma^{n-1}\sim 0$. On a $a\in R-A$. Si $a\in R-H$, on a $\omega(a, \Gamma^{n-1})=0$. Si $a\in H$, on a $a\in H-A$. L'ensemble $H-A$ étant ouvert (v. 11), il existe un $V\in\mathcal{A}$ tel que $V\subset H-A$. Si V est suffisamment petit, d'après 11.1 il existe un $(n-1, R)$-cycle $^*\Delta^{n-1}(V)$ du type t_3 dans $F(V)$ tel que $\Gamma^{n-1}\sim{}^*\Delta^{n-1}(V)$ dans $H-V$ d'où $\omega[a, \Gamma^{n-1}-{}^*\Delta^{n-1}(V)]=0$ et par suite $\omega(a, \Gamma^{n-1})=\omega[a, {}^*\Delta^{n-1}(V)]$. Le porteur de l'homologie $^*\Delta^{n-1}(V)$ est $\overline{V}$. Puisque $\Delta^{n-1}(V)$, $^*\Delta^{n-1}(V)$ sont deux cycles dans $F(V)$ homologues à zéro dans $\overline{V}$, il résulte de l'axiome E (v. 13) que, si V est suffisamment petit, il existe un entourage Q_3 de a ainsi que deux nombres $r_1, r_2\in\Re$ dont au moins un est $\neq 0$ tels que $r_1\Delta^{n-1}(V)+r_2{}^*\Delta^{n-1}(V)\sim 0$ dans $H-Q_3$. Donc $\omega[a, r_1\Delta^{n-1}(V)+r_2{}^*\Delta^{n-1}(V)]=0$, d'où $r_1 s+r_2\omega[a, {}^*\Delta^{n-1}(V)]=0$. Si l'on avait $r_2=0$, on aurait $r_1\neq 0$, $r_1 s=0$, d'où la contradiction $s=0$. Donc $r_2\neq 0$ et $\omega(a, \Gamma^{n-1})=\omega[a, {}^*\Delta^{n-1}(V)]=-\frac{r_1 s}{r_2}$.

On voit que l'orientation ω, *si elle existe*, est complètement déterminée par la connaissance du nombre s. Il s'agit encore de voir qu'on arrive

ainsi effectivement à une orientation. En premier lieu, on doit prouver que le nombre $\omega(a, \Gamma^{n-1})$ est déterminé sans aucune ambiguité. Or si l'entourage V est fixe, le rapport $r_1 : r_2$ dans la relation $r_1 \Delta^{n-1}(V) + r_2 {}^*\Delta^{n-1}(V) \sim 0$ dans $H - Q_3$, est bien déterminé, car autrement on arriverait à la contradiction $\Delta^{n-1}(V) \sim 0$ dans $H - Q_3$ (c'est une contradiction, car $H - Q_3$ ne contient pas le porteur $\overline{V}$ de l'homologie $\Delta^{n-1}(V) \sim 0$). D'autre part, si V_1, $V_2 \in \mathcal{A}$; $V_1 \supset V_2$, nous avons vu que $\Delta^{n-1}(V_2) - \Delta^{n-1}(V_1) \sim 0$ dans $\overline{V}_1 - V_2$; et si V_1 est suffisamment petit, on a tout pareillement ${}^*\Delta^{n-1}(V_2) - {}^*\Delta^{n-1}(V_1) \sim 0$ dans $\overline{V}_1 - V_2$, de manière que la relation $r_1 \Delta^{n-1}(V_2) + r_2 {}^*\Delta^{n-1}(V_2) \sim 0$ dans $H - Q_3$ ($Q_3 \subset V_2$) entraîne $r_1 \Delta^{n-1}(V_1) + r_2 {}^*\Delta^{n-1}(V_1) \sim 0$ dans $H - Q_3$. Enfin on vérifie sans peine que le nombre $\omega(a, \Gamma^{n-1})$ satisfait aux conditions 1°, 2°, 3° du n° 15.

16. Axiome F: *L'espace $R - S$ est orientable, c'est-à-dire il existe une orientation ω de $R - S$ simultanée (en tous ses points) jouissant de la propriété suivante: Si $a \in R - S$, $\Gamma^{n-1} \in \Theta_2(a)$, il existe un entourage U de a tel que $x \in U$ entraîne $\Gamma^{n-1} \in \Theta_2(x)$ et $\omega(x, \Gamma^{n-1}) = \omega(a, \Gamma^{n-1})$.*

Dans tout ce qui suit, on suppose que l'espace $R - S$ soit orienté, c'est-à-dire que l'on en ait choisi une orientation ω bien déterminé (jouissant de la propriété qui vient d'être énoncée).

16.1. *Dans la théorie* mod 2 *l'axiome F est superflu et l'orientation de $R - S$ est possible d'une seule manière.*

Démonstration. Soit $\Gamma^{n-1} \in \Theta_2(a)$ un $(n-1, R)$-cycle du type t_2 dans A; soit H le porteur de l'homologie $\Gamma^{n-1} \sim 0$. Il n'y a que deux possibilités: 1° $a \in R - S - H$; alors $\omega(x, \Gamma^{n-1}) = \omega(a, \Gamma^{n-1}) = 0$ pour tous les $x \in R - S - H$, ce qui est un entourage de a; 2° $a \in H - A$, alors $\omega(x, \Gamma^{n-1}) = \omega(a, \Gamma^{n-1}) = 1$ pour tous les $x \in H - A$ ce qui est (v. 11) un entourage de a.

17. Nous allons construire un certain (n, R)-cycle mod S dans R, dit *cycle principal* que nous désignerons par G^n.

17.1. D'après l'axiome D_2 (v. 12 et 12.1) on peut attacher à chaque point b de $R - S$ un $(n-1, R)$-cycle Γ^{n-1} du type t_3 dans A, de manière que $b \in H - A$, où H est le porteur de l'homologie $\Gamma^{n-1} \sim 0$. Les $H - A$ étant des sous-ensembles ouverts de $R - S$, d'après 4.2 et 4.3 il existe une suite $\{M_i\}$ d'ensembles ouverts constituant un réseau gén. $\mathfrak{M}$ et une suite $\{b_i\}$ de points de $R - S$ telle que, si on désigne par Γ_i^{n-1}, A_i, H_i les Γ^{n-1}, A, H correspondant au point b_i, on ait $M_i \Subset H_i - A_i$ pour chaque i. D'après 10.1 à chaque Γ_i^{n-1} correspond un (n, R)-cycle C_i^n mod A_i dans H_i. Déterminons un affinement $\mathfrak{N}$ de $\mathfrak{M}$ jouissant de la propriété du n° 4.4.

17.11. Ceci étant donné à chaque point $a \in R - S$ attachons un entourage $P \Subset R - S$ si petit qu'il satisfasse aux cinq conditions suivantes: 1° $a \in N \in \mathfrak{N}$ entraîne $P \subset N$; 2° $a \in H_i - A_i$ entraîne que pour chaque $x \in P$ on a $x \in H_i - A_i$

et $\omega(x, \Gamma_i^{n-1}) = \omega(a, \Gamma_i^{n-1})$; 3° $a \in M_i$ entraîne $P \subset M_i$; 4° $a \in R - \bar{M}_i$ entraîne $P \subset R - \bar{M}_i$[33]; 5° pour chaque couple d'indices i, j $(i \neq j)$ tel que $a \in M_i M_j$ posons $\omega_{ai} = \omega(a, \Gamma_i^{n-1})$, $\omega_{aj} = \omega(a, \Gamma_j^{n-1})$; comme $\omega_{ai} \neq 0$, $\omega_{aj} \neq 0$ [car $a \in M_i M_j \subset (H_i - A_i)(H_j - A_j)$] et comme $\Gamma_i^{n-1}, \Gamma_j^{n-1} \in \Theta_2(a)$, on a $\omega_{aj} \Gamma_i^{n-1} - \omega_{ai} \Gamma_j^{n-1} = \Gamma_{aij}^{n-1} \in \Theta_2(a)$ (v. 15, remarque 3); donc $\omega(a, \Gamma_{aij}^{n-1}) = \omega_{aj}\, \omega(a, \Gamma_i^{n-1}) - \omega_{ai}\, \omega(a, \Gamma_j^{n-1}) = 0$ de manière que le porteur H_{aij} de l'homologie $\Gamma_{aij}^{n-1} \sim 0$ ne contient pas le point a; choisissons alors P de manière que $\bar{P} H_{aij} \neq 0$.

17.12. D'après 4.2 on peut déterminer une suite $\{a_\nu\}$ de points de $R - S$ telle que les ensembles $P = P_\nu$ correspondant aux points $a = a_\nu$ constituent un réseau gén. $\mathfrak{P}$[34]. Pour chaque couple ν, i tel que $a_\nu \in M_i$ posons $\omega_{\nu i} = \omega(a_\nu, \Gamma_i^{n-1}) \in \mathfrak{R}$ de manière qu'alors (d'après 2° et d'après l'inclusion $M_i \subset H_i - A_i$) $0 \neq \omega_{\nu i} = \omega(x, \Gamma_i^{n-1})$ pour chaque $x \in P_\nu$. Pour ν, i, j tels que $i \neq j$, $a_\nu \in M_i M_j$ soit $H_{\nu ij}$ le porteur de l'homologie $\omega_{\nu j} \Gamma_i^{n-1} - \omega_{\nu i} \Gamma_j^{n-1} \sim 0$ de manière qu'alors (en vertu de 5°) $\bar{P}_\nu H_{\nu ij} = 0$.

17.13. Soit Φ_2 la famille de tous les réseaux $\mathfrak{U}$ de la famille Φ_1 du n° 7.4 qui soient des affinements mod S de $\mathfrak{P}$. Φ_2 est (v. 5.4 et 7.4) une famille parfaitement complète de réseaux dans R. En remarquant qu'un sous-ensemble bicompact de $R - S$ ne peut rencontrer qu'un nombre fini de fermetures de sommets d'un réseau gén., on démontre sans peine (cf. la démonstration de 7.3) que la famille Ψ est aussi parfaitement complète si Ψ désigne la famille de tous les réseaux $\mathfrak{U} \in \Phi_2$ satisfaisant aux deux propriétés suivantes: 1° P_2^* étant un sommet quelconque du réseau gén. $\mathfrak{P}_2$ (v. 9.3), ou bien aucun sommet intérieur de $\mathfrak{U}$ ne rencontre $\bar{P}_2^*$, ou bien $\mathfrak{U}$ est un affinement du réseau $\mathfrak{B}(P_2^*)$ du théorème 10.1; 2° pour $i = 1, 2, 3, \cdots$ aucun sommet intérieur de $\mathfrak{U}$ ne rencontre simultanément les deux ensembles $\bar{M}_i$ et A_i[35].

17.14. Ceci étant donné soient $\mathfrak{U} \in \Psi$ et σ_k^n $(1 \leqq k \leqq m)$ tous les $(n, \mathfrak{U})$-simplexes $\neq 0$ mod S. Le réseau $\mathfrak{U}$ étant régulier par rapport à S (v. 5.5 et 7.4) un sommet au moins de chaque σ_k^n est un sommet intérieur de $\mathfrak{U}$. $\mathfrak{F}_k$ étant le noyau de σ_k^n, il existe donc (par définition même de la famille $\Phi_2 \supset \Psi$) un indice ν tel que $\mathfrak{F}_k \subset P_\nu$. $\mathfrak{M}$ étant un réseau gén., il existe un indice i tel que $a_\nu \in M_i$, d'où $P_\nu \subset M_i$ en vertu de la condition 3° pour les P_ν. Le nombre $\omega_{\nu i} \in \mathfrak{R}$ étant $\neq 0$, il existe un nombre $r_k \in \mathfrak{R}$ tel que le coefficient du simplexe σ_k^n dans la $(n, \mathfrak{U})$-chaîne $C_i^n(\mathfrak{U})$ soit $r_k\, \omega_{\nu i}$[36].

[33] Comme au n° 4.4, on voit que toutes ces conditions sont simultanément réalisables.

[34] En réalité, du théorème 4.2 résulte seulement l'existence d'une suite $\{Q_\nu\}$ constituant un réseau telle que chaque Q_ν fasse partie d'un $P = P_\nu$ attaché au point $a = a_\nu$. Or on voit sans peine que l'on peut s'arranger de manière que $a_\nu \in Q_\nu$ pour chaque ν, d'où résulte immédiatement que les cinq conditions 1°—5° restent intactes en y remplaçant P par Q.

[35] Ces ensembles sont disjoints, car $M_i \in H_i - A_i$.

[36] Soit P_2^* un sommet du réseau gén. $\mathfrak{P}_2$ tel que $H_i \subset \bar{P}_2^*$. Remarquons que $C_i^n(\mathfrak{U}) \subset H_i \subset \bar{P}_2^*$. Donc si aucun sommet intérieur de $\mathfrak{U}$ ne rencontre $\bar{P}_2^*$, la chaîne $C_i^n(\mathfrak{U})$ ne contient

17.15. Il s'agit maintenant de montrer que le nombre $r_k \in \mathfrak{R}$ est bien déterminé, c'est-à-dire qu'il ne dépend pas du choix des indices ν et i. *En premier lieu,* l'indice ν étant donné, remplaçons l'indice i par j et désignons par r'_k la valeur correspondante de r_k. On a $\mathfrak{J}_k \subset P_\nu \subset M_i M_j$, en particulier $a_\nu \in M_i M_j$, d'où $\overline{P}_\nu H_{\nu ij} = 0$. Comme $\mathfrak{J}_k \subset P_\nu$, on a $\mathfrak{J}_k H_{\nu ij} = 0$. Comme $\mathfrak{J}_k \subset M_i M_j \subset (H_i - A_i)(H_j - A_j)$, on a $\mathfrak{J}_k (A_i + A_j) = 0$. Soit C^n le (n, R)-cycle mod $(A_i + A_j)$ dans $H_{\nu ij}$ correspondant au $(n-1, R)$-cycle $\omega_{\nu j} \Gamma_i^{n-1} - \omega_{\nu i} \Gamma_j^{n-1}$ du type t_2 dans $(A_i + A_j)$. Comme $P_\nu \subset M_i M_j \subset H_i H_j$, on a $H_i H_j \neq 0$. Γ_i^{n-1} étant un $(n-1, R)$-cycle du type t_3, il existe un sommet P_{3i}^* du réseau gén. $\mathfrak{P}_3$ tel que $H_i \subset \overline{P}_{3i}^*$; pareillement il existe un sommet P_{3j}^* de $\mathfrak{P}_3$ tel que $H_j \subset \overline{P}_{3j}^*$. Comme $H_i H_j \neq 0$, on a $\overline{P}_{3i}^* \overline{P}_{3j}^* \neq 0$. Il existe donc (v. 9.3) un sommet P_2^* du réseau gén. $\mathfrak{P}_2$ tel que $\overline{P}_{3i}^* + \overline{P}_{3j}^* \subset P_2^*$. Puisque $H_i \subset \overline{P}_{3i}^*$, $H_j \subset \overline{P}_{3j}^*$, $H_{\nu ij} \subset H_i + H_j$, on a $P_2^* \supset H_i + H_j + H_{\nu ij}$. Donc les chaînes $C_i^n(\mathfrak{U})$, $C_j^n(\mathfrak{U})$, $C^n(\mathfrak{U})$ sont dans $\overline{P}_2^*$. Si aucun sommet intérieur de $\mathfrak{U}$ ne rencontre $\overline{P}_2^*$, les chaînes $C_i^n(\mathfrak{U})$ et $C_j^n(\mathfrak{U})$ ne peuvent contenir aucun sommet intérieur et par suite $r_k \omega_{\nu i} = 0$, $r'_k \omega_{\nu j} = 0$, d'où $r_k = r'_k = 0$. Supposons donc qu'il existe un sommet intérieur de $\mathfrak{U}$ rencontrant $\overline{P}_2^*$; en vertu de la propriété 1° de la famille Ψ, le réseau $\mathfrak{U}$ est un affinement de $\mathfrak{B}(P_2^*)$. D'après 10.1, on en déduit sans peine que $C^n(\mathfrak{U}) = \omega_{\nu j} C_i^n(\mathfrak{U}) - \omega_{\nu i} C_j^n(\mathfrak{U})$ mod $(A_i + A_j)$. Comme $\mathfrak{J}_k H_{\nu ij} = 0$, $C^n(\mathfrak{U}) \subset H_{\nu ij}$, la chaîne $C^n(\mathfrak{U})$ ne peut pas contenir le simplexe σ_k^n, dont le noyau est $\mathfrak{J}_k$. Comme $\mathfrak{J}_k (A_i + A_j) = 0$, la chaîne $\omega_{\nu j} C_i^n(\mathfrak{U}) - \omega_{\nu i} C_j^n(\mathfrak{U})$ ne contient non plus le simplexe σ_k^n. Or cette chaîne contient le simplexe σ_k^n avec le coefficient $\omega_{\nu j} \cdot r_k \omega_{\nu i} - \omega_{\nu i} \cdot r'_k \omega_{\nu j}$. Comme $\omega_{\nu i} \neq 0$, $\omega_{\nu j} \neq 0$, on a donc $r_k = r'_k$. *En second lieu,* remplaçons l'indice ν par μ. Quant à l'indice i, nous pouvons le supposer inaltéré. En effet, $\mathfrak{R}$ étant un réseau gén., il existe deux sommets N_ν et N_μ de $\mathfrak{R}$ tels que $a_\nu \in N_\nu$, $a_\mu \in N_\mu$. D'après la condition 1° pour les P_ν, on a alors $P_\nu \subset N_\nu$, $P_\mu \subset N_\mu$. Or $\mathfrak{J}_k \subset P_\nu P_\mu$, d'où $N_\nu N_\mu \neq 0$. D'après la définition du réseau gén. $\mathfrak{R}$, il existe donc un sommet M_i de $\mathfrak{M}$ tel que $N_\nu + N_\mu \subset M_i$, d'où $P_\nu \subset M_i$, $P_\mu \subset M_i$, de manière que cette valeur de i est admissible dans les deux cas. Soient r_k, r'_k les deux valeurs de r_k. D'après la définition même de ces nombres, on a $r_k \omega_{\nu i} = r'_k \omega_{\mu i}$. Or choisissons un point $x \in \mathfrak{J}_k \subset P_\nu P_\mu$. En vertu de la condition 2° pour les P_ν, on a $\omega_{\nu i} = \omega(x, \Gamma_i^{n-1})$, $\omega_{\mu i} = \omega(x, \Gamma_i^{n-1})$, d'où $\omega_{\nu i} = \omega_{\mu i} \neq 0$ et par suite $r_k = r'_k$.

17.16. *Les nombres $r_k \in \mathfrak{R}$ étant bien déterminés posons* $G^n(\mathfrak{U}) = \sum_{k=1}^{m} r_k \sigma_k^n$.

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aucun $(n, \mathfrak{U})$-simplexe intérieur, d'où $r_k \omega_{\nu i} = 0$. Dans le cas contraire, d'après la propriété 1° de la famille Ψ, $\mathfrak{U}$ est un affinement de $\mathfrak{B}(P_2^*)$. D'après 10.1, la chaîne $C_i^n(\mathfrak{U})$ est donc bien déterminée à mod A_i près. Comme $\mathfrak{J}_k \subset P_\nu \subset M_i \subset H_i - A_i$, on a $\mathfrak{J}_k A_i = 0$, de sorte que le coefficient $r_k \omega_{\nu i}$ du simplexe σ_k^n dans la chaîne $C_i^n(\mathfrak{U})$ est bien déterminé.

Remarque. Observons que la $(n, \mathfrak{U})$-chaîne $G^n(\mathfrak{U})$ ainsi définie ne contient que des $(n, \mathfrak{U})$-simplexes $\not\equiv 0 \bmod S$.

17.2. *Pour chaque* $\mathfrak{U} \in \Psi$, $G^n(\mathfrak{U})$ *est un* $(n, \mathfrak{U})$*-cycle mod* S.

Démonstration. Soit σ^{n-1} un $(n-1, \mathfrak{U})$-simplexe $\not\equiv 0 \bmod S$. Il faut montrer que la chaîne $FG^n(\mathfrak{U})$ ne contient pas le simplexe σ^{n-1}. Pour $1 \leqq k \leqq m$, soit $\eta_k \in \mathfrak{R}$ le coefficient de σ^{n-1} dans la chaîne $F\sigma_k^n$. On doit montrer que $\sum_{k=1}^{m} \eta_k r_k = 0$, ou bien que $\sum_{k \in Z} \eta_k r_k = 0$ où Z est l'ensemble de tous les indices k $(1 \leqq k \leqq m)$ tels que $\eta_k \not\equiv 0$. Puisque $\sigma^{n-1} \not\equiv 0 \bmod S$ et puisque le réseau $\mathfrak{U}$ est régulier par rapport à S, σ^{n-1} possède un sommet U tel que $US \not\equiv 0$. Comme $\mathfrak{U} \in \Phi$, il existe un indice ν tel que $U \subset P_\nu$, d'où $K \subset P_\nu$, K étant le noyau de σ^{n-1}. D'après la condition 3° pour les P_ν (v. 17.11), il existe un indice i tel que $P_\nu \subset M_i$. Si $k \in Z$, on a évidemment $\mathfrak{Z}_k \subset K \subset P_\nu \subset M_i$ de manière que le coefficient de σ_k^n dans la chaîne $C_i^n(\mathfrak{U})$ est $r_k \omega_{\nu i}$. Donc le coefficient de σ^{n-1} dans la chaîne $FC_i^n(\mathfrak{U})$ est $\sum_{k \in Z} \eta_k r_k \omega_{\nu i}$. Or $\sigma^{n-1} \not\equiv 0 \bmod S$ et $C_i^n(\mathfrak{U}) \to 0 \bmod S$. Donc $\sum_{k \in Z} \eta_k r_k \omega_{\nu i} = 0$ et par suite $\sum_{k \in Z} \eta_k r_k = 0$, car $\omega_{\nu i} \not\equiv 0$.

17.3. *Soit* $\mathfrak{U}, \mathfrak{U}_1 \in \Psi$; *soit* $\mathfrak{U}_1$ *un affinement de* $\mathfrak{U}$; *soit* $\pi = Pr.(\mathfrak{U}_1, \mathfrak{U})$. *Alors* $\pi G^n(\mathfrak{U}_1) = G^n(\mathfrak{U})$ *mod* S.

Les $G^n(\mathfrak{U})$, $\mathfrak{U} \in \Psi$ définissent donc un (n, R)-cycle mod S. C'est le cycle principal G^n annoncé dans 17.

Démonstration. Soit σ^n un $(n, \mathfrak{U})$-simplexe donné $\not\equiv 0 \bmod S$. Soient τ_h^n $(1 \leqq h \leqq s)$ tous les $(n, \mathfrak{U}_1)$-simplexes tels que $\pi \tau_h^n = \pm \sigma^n$. Nous pouvons d'ailleurs choisir les orientations des τ_h^n de manière que $\pi \tau_h^n = \sigma^n$ $(1 \leqq h \leqq s)$. Soit $\mathfrak{Z}$ le noyau de σ^n; soient K_h $(1 \leqq h \leqq s)$ les noyaux de τ_h^n. On a évidemment $K_h \subset \mathfrak{Z}$ pour $1 \leqq h \leqq s$. Comme nous le savons, il existe des indices ν, i tels que $\mathfrak{Z} \subset P_\nu \subset M_i$. Soit r le coefficient de σ^n dans $G^n(\mathfrak{U})$; soient r_h' $(1 \leqq h \leqq s)$ les coefficients des τ_h^n dans $G^n(\mathfrak{U}_1)$. Il faut montrer que $r = \sum_{h=1}^{s} r_h'$. Soit P_2^* un sommet du réseau gén. $\mathfrak{P}_2$ tel que $H_i \subset \bar{P}_2^*$. Si aucun sommet intérieur de $\mathfrak{U}$ ne rencontre $\bar{P}_2^*$, on a, comme nous le savons, $r = 0$; or dans ce cas $\mathfrak{Z}\bar{P}_2^* = 0$, d'où $K_h \bar{P}_2^* = 0$ $(1 \leqq h \leqq s)$; on en déduit sans peine que $r_h' = 0$ $(1 \leqq h \leqq s)$, donc $r = \sum_{h=1}^{s} r_h'$. Dans le cas contraire, nous savons que $\mathfrak{U}$, par suite aussi $\mathfrak{U}'$, est un affinement du réseau $\mathfrak{B}(P_2^*)$ du n° 10.1, d'où $C_i^n(\mathfrak{U}) = \pi C_i^n(\mathfrak{U}')$. Or le coefficient de σ^n dans $C_i^n(\mathfrak{U})$ est $r \omega_{\nu i}$ et les coefficients de τ_h^n $(1 \leqq h \leqq s)$ dans $C_i^n(\mathfrak{U}')$ sont $r_h' \omega_{\nu i}$. Par suite $r \omega_{\nu i} = \sum_{h=1}^{s} r_h' \omega_{\nu i}$, et donc $r = \sum_{h=1}^{s} r_h'$, car $\omega_{\nu i} \not\equiv 0$.

17.4. La construction (v. 17.1–17.16) du cycle principal G^n possède un certain dégré d'arbitraire. Or on peut démontrer que (l'orientation de $R-S$ étant donnée), le cycle G^n est bien déterminé dans le sens suivant : *Il existe une famille complète Φ de réseaux dans R telle que la $(n, \mathfrak{U})$-chaîne $G^n(\mathfrak{U})$ est déterminée sans aucune ambiguité pour chaque $\mathfrak{U} \in \Phi$.* (Cf. la remarque à la fin du n° 17.16.) Nous laissons la démonstration de cet énoncé, dont nous ne ferons aucun usage, au soin du lecteur.

17.5. *Soit A un sous-ensemble bicompact de R; soit $\overline{R-S}-A \neq 0$. Le cycle principal G^n n'est pas situé dans A.*

Démonstration. Il existe évidemment un point $a \in (R-S)-A$. Choisissons les indices ν, i de manière que $a \in P_\nu \subset M_i \Subset H_i - A_i$. Soit V un entourage de a si petit que 1° $\overline{V} \subset (R-S)-A$; 2° $\overline{V} \subset P_\nu$. Il suffit de démontrer qu'il existe un réseau $\mathfrak{U} \in \Psi$ tel que $G^n(\mathfrak{U})$ contienne un n-simplexe σ^n dont le noyau $\mathfrak{J}$ ne rencontre pas A. Comme H_i est le porteur de l'homologie $\Gamma_i^{n-1} \sim 0$ et comme $H_i - V_i \subset H_i$, $H_i - V = \overline{H_i - V} \neq H_i$, on a $C_i^n(\mathfrak{U}) \subset H_i$ mais il existe des réseaux $\mathfrak{U} \in \Psi$ arbitrairement petits tels que qu'on n'a pas $C_i^n(\mathfrak{U}) \subset H_i - V$. Choisissons un tel réseau $\mathfrak{U}$ assujetti à la condition qu'aucun sommet de $\mathfrak{U}$ ne rencontre simultanément $\overline{V}$ et A, ou bien $\overline{V}$ et $R-P_\nu$, ou enfin $\overline{V}$ et S. Comme la chaîne $C_i^n(\mathfrak{U})$ est dans H_i, mais non dans $H_i - V$, elle contient un $(n, \mathfrak{U})$-simplexe σ^n dont le noyau $\mathfrak{J}$ rencontre $\overline{V}$. On a alors $\mathfrak{J}A = 0$, $\mathfrak{J} \subset P_\nu \subset M_i$, $\mathfrak{J}S = 0$. Comme $\mathfrak{J} \subset P_\nu \subset M_i$, la chaîne $G^n(\mathfrak{U})$, comme $C_i^n(\mathfrak{U})$, doit contenir le simplexe σ^n. Puisque $\mathfrak{J}A = 0$, on ne peut avoir $G^n(\mathfrak{U}) \subset A$.

Remarque. Puisque les réseaux de la famille Ψ sont d'ordre $\leqq n \bmod S$ (v. 7.4), nous avons démontré plus généralement que le *cycle principal G^n n'est pas homologue à zéro mod A* $(A = \bar{A}, \overline{R-S}-A \neq 0)$.

18. Faisons de nouveau les conventions des n^{os} 8, 8.1 et 8.3. Choisissons (arbitrairement) l'orientation de chaque simplexe σ_i^h $(0 \leqq h \leqq n, 1 \leqq i \leqq \alpha_h)$. Puisque les faces d'un $(h+1, \mathfrak{Z})$-simplexe $(0 \leqq h \leqq n-1)$ intérieur sont des $(h, \mathfrak{Z})$-simplexes intérieurs, il existe des nombres $\eta_{ij}^h \in \mathfrak{R}$ $(0 \leqq h \leqq n-1,\ 1 \leqq i \leqq \alpha_{h+1},\ 1 \leqq j \leqq \alpha_h)$ tels que

$$\sigma_i^{h+1} \to \sum_{j=1}^{\alpha_h} \eta_{ij}^h \sigma_j^h \qquad (0 \leqq h \leqq n-1,\ 1 \leqq i \leqq \alpha_{h+1}).$$

Les nombres η ne prennent d'ailleurs que les valeurs 0, 1, -1.

Comme $F\sigma_i^{h+1} \to 0$, on a pour $1 \leqq h \leqq n-1$, $1 \leqq i \leqq \alpha_{h+1}$, $1 \leqq k \leqq \alpha_{h-1}$:

$$\sum_{j=1}^{\alpha_h} \eta_{ij}^h \eta_{jk}^{h-1} = 0.$$

Remarque. τ^k étant un simplexe d'espèce σ^{-1} d'un réseau commode $\mathfrak{U}$, d'après 8.32 le noyau de τ^k ne rencontre pas R_0 et par suite $\tau^k = 0 \bmod \overline{R-R_0}$.

18.1. Soit Ψ_0 la famille de tous les réseaux commodes (dans le sens du n° 8.1) qui appartiennent à la famille Ψ du n° 17.13. On voit sans peine que Ψ_0 est une famille parfaitement complète de réseaux. *Dorénavant seuls les réseaux de la famille Ψ_0 seront appelés commodes.*

19.1. *Soit $\mathfrak{U}$ un réseau commode. A chaque simplexe intérieur σ_i^h $(0 \leqq h \leqq n,$ $1 \leqq i \leqq \alpha_h)$ du réseau $\mathfrak{Z}$ on peut attacher une $(n-h, \mathfrak{U})$-chaîne $K^{n-h}(\sigma_i^h, \mathfrak{U})$ de la manière suivante: 1° Chaque simplexe de chaque chaîne $K^{n-h}(\sigma_i^h, \mathfrak{U})$ est $\neq 0 \bmod \overline{R-R_0}$; 2° chaque simplexe de la chaîne $K^{n-h}(\sigma_i^h, \mathfrak{U})$ est d'espèce σ_i^h; 3° pour $1 \leqq h \leqq n$, $1 \leqq j \leqq \alpha_{h-1}$ on a*

$$K^{n-h+1}(\sigma_j^{h-1}, \mathfrak{U}) \to \sum_{i=1}^{\alpha_h} \eta_{ij}^{h-1} K^{n-h}(\sigma_i^h, \mathfrak{U}) \bmod \overline{R-R_0};$$

4° on a (v. 17)

$$G^n(\mathfrak{U}) = \sum_{i=1}^{\alpha_0} K^n(\sigma_i^0, \mathfrak{U}) \bmod \overline{R-R_0}.$$

Les chaînes $K^{n-h}(\sigma_i^h, \mathfrak{U})$ sont déterminées sans aucune ambiguité par les propriétés 1°, 2°, 3°, 4°. On a d'ailleurs pour $0 \leqq h \leqq n$, $1 \leqq i \leqq \alpha_h$: 5° $K^{n-h}(-\sigma_i^h, \mathfrak{U}) = -K^{n-h}(\sigma_i^h, \mathfrak{U})$.

Nous appellerons les $K^{n-h}(\sigma_i^h, \mathfrak{U})$ les $\mathfrak{U}$-*chaînes fondamentales.* Chaque $(n-h, \mathfrak{U})$-chaîne $(0 \leqq h \leqq n)$ de la forme $\sum_{i=1}^{\alpha_h} c_i K^{n-h}(\sigma_i^h, \mathfrak{U})$, $c_i \in \mathfrak{R}$ sera appelée une $(n-h, \mathfrak{U})$-*chaîne élémentaire.*

Démonstration. Commençons par les $K^n(\sigma_i^0, \mathfrak{U})$. D'après 8.33 chaque simplexe de la chaîne $G^n(\mathfrak{U})$ est de rang 0 ou -1. D'après la remarque du n° 18, les simplexes de rang -1 sont $= 0 \bmod \overline{R-R_0}$. D'après 19 1°, 2°, 4° on doit donc entendre par $K^n(\sigma_i^0, \mathfrak{U})$ $(1 \leqq i \leqq \alpha_0)$ la partie de la chaîne $G^n(\mathfrak{U})$ dont les simplexes sont $\neq 0 \bmod \overline{R-R_0}$ et d'espèce σ_i^0. Les conditions 1°, 2°, 5° $(h=0)$ et 4° du n° 19 sont immédiatement vérifiées.

Supposons donc que pour une certaine valeur de p $(1 \leqq p \leqq n)$ on ait déjà défini les chaînes $K^{n-h}(\sigma_i^h, \mathfrak{U})$ $(0 \leqq h \leqq p-1,$ $1 \leqq i \leqq \alpha_h)$ de manière que les conditions 1°, 2°, 3°, 4° du n° 19 soient vérifiées pour $0 \leqq h \leqq p-1$, et que l'on ait reconnu que les chaînes $K^{n-h}(\sigma_i^h, \mathfrak{U})$ $(0 \leqq h \leqq n-1)$ sont univoquement déterminées par ces conditions. Il s'agit de définir les chaînes $K^{n-p}(\sigma_i^p, \mathfrak{U})$ $(1 \leqq i \leqq \alpha_p)$ de manière à satisfaire aux conditions 1°, 2°, 3° du n° 19 pour $h = p$ et de prouver que cela n'est possible que d'une seule manière[37].

Ce dernier fait est évident; en effet, la condition 3° donne pour $h = p$

$$\sum_{i=1}^{\alpha_p} \eta_{ij}^{p-1} K^{n-p}(\sigma_i^p, \mathfrak{U}) = F K^{n-p+1}(\sigma_j^{p-1}, \mathfrak{U}) \bmod \overline{R-R_0}.$$

[37] On doit aussi observer la validité de 5° pour $h = p$, en la supposant pour $h = p-1$.

Pour définir $K^{n-p}(\sigma_i^p, \mathfrak{U})$ pour une valeur donnée de i $(1 \leqq i \leqq \alpha_p)$, on doit donc choisir l'indice j $(1 \leqq j \leqq \alpha_{p-1})$ de manière que $\eta_{ij}^{p-1} \neq 0$ (ce qui est toujours possible) et définir comme $K^{n-p}(\sigma_i^p, \mathfrak{U})$ la partie de la chaîne $\eta_{ij}^{p-1} FK^{n-p+1}(\sigma_j^{p-1}, \mathfrak{U})$ dont les simplexes sont $\neq 0 \bmod \overline{R-R_0}$ et d'espèce σ_i^p. Les conditions 1°, 2° et 5° $(h = p)$ sont immédiatement vérifiées. Il s'agit de prouver que 1° la définition de $K^{n-p}(\sigma_i^p, \mathfrak{U})$ est indépendante du choix de l'indice j; 2° les conditions 3° $(h = p)$ sont satisfaites.

A cet effet, considérons une chaîne $FK^{n-p+1}(\sigma_j^{p-1}, \mathfrak{U})$ $(1 \leqq j \leqq \alpha_{p-1})$. D'après 8.33, un simplexe de cette chaîne est de rang $\leqq p$; d'après 8.34 et 2°, ce rang est $\geqq p-1$ et si le rang égale $p-1$, l'espèce du simplexe est σ_j^{p-1}, tandis que, si le rang du simplexe égale p, son espèce est σ_i^p, l'indice i étant tel que $\eta_{ij}^{p-1} \neq 0$. Soit donc $H_j^{n-p}(\mathfrak{U})$ la partie de la chaîne $FK^{n-p+1}(\sigma_j^{p-1}, \mathfrak{U})$ dont les simplexes sont $\neq 0 \bmod \overline{R-R_0}$ et d'espèce σ_j^{p-1} et soit $\eta_{ij}^{p-1} H_{ij}^{n-p}(\mathfrak{U})$ la partie de la même chaîne dont les simplexes sont $\neq 0 \bmod \overline{R-R_0}$ et d'espèce σ_i^p, de manière que

$$(*) \qquad FK^{n-p+1}(\sigma_j^{p-1}, \mathfrak{U}) = H_j^{n-p}(\mathfrak{U}) + \sum_{i=1}^{\alpha_p} \eta_{ij}^{p-1} H_{ij}^{n-p}(\mathfrak{U}) \bmod \overline{R-R_0}.$$

Distinguons deux cas. *En premier lieu,* soit $p = 1$. On a d'après 4° $\sum_{j=1}^{\alpha_0} FK^n(\sigma_j^0, \mathfrak{U}) = 0 \bmod \overline{R-R_0}$, car $FG^n(\mathfrak{U}) = 0 \bmod S \subset \overline{R-R_0}$. Donc

$$\sum_{j=1}^{\alpha_0} H_j^{n-1}(\mathfrak{U}) + \sum_{i=1}^{\alpha_1} \sum_{j=1}^{\alpha_0} \eta_{ij}^0 H_{ij}^{n-1}(\mathfrak{U}) = 0 \bmod \overline{R-R_0},$$

d'où

$$H_j^{n-1}(\mathfrak{U}) = 0 \qquad (1 \leqq j \leqq \alpha_0)$$

et

$$(**) \qquad \sum_{j=1}^{\alpha_0} \eta_{ij}^0 H_{ij}^{n-1}(\mathfrak{U}) = 0 \qquad (1 \leqq i \leqq \alpha_1).$$

On voit bien que la définition de $K^{n-1}(\sigma_i^1, \mathfrak{U})$ est indépendante du choix de l'indice j; en effet, pour une valeur donnée de i $(1 \leqq i \leqq \alpha_1)$, il y a justement deux valeurs $j = j_1$ et $j = j_2$ de l'indice j tels que $\eta_{ij}^0 \neq 0$ et on a $\eta_{ij_1}^0 + \eta_{ij_2}^0 = 0$ de manière que $(**)$ donne

$$H_{ij_1}^{n-1}(\mathfrak{U}) = H_{ij_2}^{n-1}(\mathfrak{U}),$$

de manière qu'on définit sans ambiguité

$$K^{n-1}(\sigma_i^1, \mathfrak{U}) = H_{ij_1}^{n-1}(\mathfrak{U}) = H_{ij_2}^{n-1}(\mathfrak{U}).$$

La relation $(*)$ prend la forme

$$FK^n(\sigma_j^0, \mathfrak{U}) = \sum_{i=1}^{\alpha_1} \eta_{ij}^0 K^{n-1}(\sigma_i^1, \mathfrak{U}) \qquad (1 \leqq j \leqq \alpha_0);$$

or c'est précisément la relation 3° $(h = p = 1)$ qui était à démontrer.

En second lieu, soit $p \geqq 2$. Pour $1 \leqq k \leqq \alpha_{p-2}$ on a

$$K^{n-p+2}(\sigma_k^{p-2}, \mathfrak{U}) \to \sum_{j=1}^{\alpha_{p-1}} \eta_{jk}^{p-2} K^{n-p+1}(\sigma_j^{p-1}, \mathfrak{U}) \mod \overline{R - R_0}$$

et par suite

$$\sum_{j=1}^{\alpha_{p-1}} \eta_{jk}^{p-2} F K^{n-p+1}(\sigma_j^{p-1}, \mathfrak{U}) = 0 \mod \overline{R - R_0}$$

c'est-à-dire

$$\sum_{j=1}^{\alpha_{p-1}} \eta_{jk}^{p-2} H_j^{n-p}(\mathfrak{U}) + \sum_{i=1}^{\alpha_p} \sum_{j=1}^{\alpha_{p-1}} \eta_{ij}^{p-1} \eta_{jk}^{p-2} H_{ij}^{n-p}(\mathfrak{U}) = 0$$

d'où

$$H_j^{n-p}(\mathfrak{U}) = 0 \qquad (1 \leqq j \leqq \alpha_{p-1})$$

et

$$(\dagger) \qquad \sum_{j=1}^{\alpha_{p-1}} \eta_{ij}^{p-1} \eta_{jk}^{p-2} H_{ij}^{n-p}(\mathfrak{U}) = 0 \qquad (1 \leqq j \leqq \alpha_{p-1},\ 1 \leqq i \leqq \alpha_p).$$

L'indice i $(1 \leqq i \leqq \alpha_p)$ étant donné, et j_1, j_2 $(1 \leqq j_1, j_2 \leqq \alpha_{p-1})$ étant tels que $\eta_{ij_1}^{p-1} \neq 0 \neq \eta_{ij_2}^{p-1}$, on doit démontrer que $H_{ij_1}^{n-p}(\mathfrak{U}) = H_{ij_2}^{n-p}(\mathfrak{U})$. On voit sans peine qu'il suffit de déduire ceci sous la supposition qu'il existe un indice k $(1 \leqq k \leqq \alpha_{p-2})$ tel que $\eta_{j_1 k}^{p-2} \neq 0 \neq \eta_{j_2 k}^{p-2}$. Or sous cette supposition, pour $j \neq j_1$ ou j_2, on a $\eta_{ij}^{p-1} \eta_{jk}^{p-2} = 0$ et en outre, comme on le voit sans peine, $\eta_{ij_1}^{p-1} \eta_{j_1 k}^{p-2} + \eta_{ij_2}^{p-1} \eta_{j_2 k}^{p-2} = 0$, de manière que l'égalité à démontrer s'obtient immédiatement de (†). On définit donc sans aucune ambiguité

$$K^{n-p}(\sigma_i^p, \mathfrak{U}) = H_{ij}^{n-p}(\mathfrak{U})$$

l'indice j n'étant assujetti qu'à la condition $\eta_{ij}^{p-1} \neq 0$. La relation (*) prend alors la forme 3° $(h = p)$ qui était à démontrer.

19.2. *Soient* $\mathfrak{U}$, $\mathfrak{U}_1$ *deux réseaux commodes,* $\mathfrak{U}_1$ *étant un affinement de* $\mathfrak{U}$; *soit* $\pi = Pr.(\mathfrak{U}_1, \mathfrak{U})$. *Alors*

$$(*) \qquad \pi K^{n-p}(\sigma_i^p, \mathfrak{U}_1) = K^{n-p}(\sigma_i^p, \mathfrak{U}) \mod \overline{R - R_0} \qquad (1 \leqq i \leqq \alpha_p).$$

Démonstration. Soit d'abord $p = 0$. D'après 17.3 et 18.1 on a $\pi G^n(\mathfrak{U}_1) = G^n(\mathfrak{U}) \mod S \subset \overline{R - R_0}$. D'autre part, d'après 19.1, 4° $G^n(\mathfrak{U}) = \sum_{i=1}^{\alpha_0} K^n(\sigma_i^0, \mathfrak{U})$ mod $\overline{R - R_0}$ et pareillement pour $\mathfrak{U}_1$. Donc

$$\sum_{i=1}^{\alpha_0} [\pi K^n(\sigma_i^0, \mathfrak{U}_1) - K^n(\sigma_i^0, \mathfrak{U})] = 0 \mod \overline{R - R_0}.$$

Or d'après 19.1, 2° et 8.35 chaque simplexe de la chaîne

$$(**) \qquad \pi K^n(\sigma_i^0, \mathfrak{U}) - K^n(\sigma_i^0, \mathfrak{U}) \qquad (1 \leqq i \leqq \alpha_0)$$

est d'espèce σ_i^0. Donc les chaînes (**) sont $= 0 \bmod \overline{R - R_0}$.

En second lieu, supposons que pour une certaine valeur de p $(1 \leqq p \leqq n)$ on ait déjà démontré que

$$\pi K^{n-p+1}(\sigma_j^{p-1}, \mathfrak{U}_1) = K^{n-p+1}(\sigma_j^{p-1}, \mathfrak{U}) \mod \overline{R - R_0} \qquad (1 \leqq j \leqq \alpha_{p-1}).$$

Cette relation reste vraie si l'on forme les frontières des deux termes. Cela donne d'après 19.1, 3°

$$\sum_{i=1}^{\alpha_p} \eta_{ij}^{p-1} [\pi K^{n-p}(\sigma_i^p, \mathfrak{U}_1) - K^{n-p}(\sigma_i^p, \mathfrak{U})] = 0 \mod \overline{R - R_0}.$$

D'après 19.1, 2° et 8.35 chaque simplexe de l'$i^{\text{ème}}$ terme de cette somme est d'espèce σ_i^p; d'autre part, l'indice i étant donné, on peut choisir j de manière que $\eta_{ij}^{p-1} \neq 0$. Il en résulte (*).

19.3. *Soit $\mathfrak{U}$ un réseau commode. Pour $0 \leqq h \leqq n$, $1 \leqq i \leqq \alpha_h$ la chaîne $K^{n-h}(\sigma_i^h, \mathfrak{U})$ est située dans $T(\sigma_i^h)$.*

Démonstration. Soit $\mathfrak{V}$ un affinement de $\mathfrak{U}$ régulier par rapport à $T(\sigma_i^h)$. Soit $\mathfrak{U}_1$ un réseau commode qui soit un affinement de $\mathfrak{V}$. Soit $\pi = Pr.(\mathfrak{V}, \mathfrak{U})$, $\pi_1 = Pr.(\mathfrak{U}_1, \mathfrak{V})$. D'après 19.2 on a $\pi\pi_1 K^{n-h}(\sigma_i^h, \mathfrak{U}_1) = K^{n-h}(\sigma_i^h, \mathfrak{U}) \bmod \overline{R - R_0}$. Soit τ^{n-h} un simplexe de la chaîne $K^{n-h}(\sigma_i^h, \mathfrak{U}_1)$. D'après 19.1, 2° le simplexe τ^{n-h} est d'espèce σ_i^h. Chaque sommet de τ^{n-h} rencontre donc (v. 8.3) $T(\sigma_i^h)$. Donc chaque sommet de $\pi_1 \tau^{n-h}$ rencontre $T(\sigma_i^h)$. Le réseau $\mathfrak{V}$ étant régulier par rapport à $T(\sigma_i^h)$, il en résulte que $\pi_1 \tau^{n-h} = 0$ ou bien le noyau de $\pi_1 \tau^{n-h}$ rencontre $T(\sigma_i^h)$. Donc si $\pi\pi_1 \tau^{n-h} \neq 0$, le noyau de ce simplexe rencontre $T(\sigma_i^h)$. Par suite la chaîne $\pi\pi_1 K^{n-h}(\sigma_i^h, \mathfrak{U}_1)$ est située dans $T(\sigma_i^h)$. Or $\pi\pi_1 K^{n-h}(\sigma_i^h, \mathfrak{U}_1)$ est $\bmod \overline{R - R_0}$ égale à la chaîne $K^{n-h}(\sigma_i^h, \mathfrak{U})$ dont tous les simplexes sont (v. 19.1, 1°) $\neq 0 \bmod \overline{R - R_0}$. Par suite $K^{n-h}(\sigma_i^h, \mathfrak{U}) \subset T(\sigma_i^h)$.

20.1. En partant du réseau gén. ne contenant que le seul sommet $R - S$, on construit d'après 14.2 un réseau gén. $\mathfrak{P}$ jouissant de la propriété suivante: Si $P \in \mathfrak{P}$ à chaque couple C_1^n, C_2^n de (n, R)-cycles $\bmod S$ on peut attacher un couple de nombres $r_1, r_2 \in \mathfrak{R}$ dont un au moins $\neq 0$, de manière que $r_1 C_1^n + r_2 C_2^n \sim 0 \bmod (R - P)$. On peut supposer que pour chaque $P \in \mathfrak{P}$ on ait $P \neq R - S$, $P \in R - S$.

20.2. Supposons que chaque sommet intérieur σ_i^0 $(1 \leqq i \leqq \alpha_0)$ du réseau $\mathfrak{Z}$ du n° 8 fasse partie d'un sommet P_i du réseau gén. $\mathfrak{P}$ du n° 20.1. Désignons par Ψ_1 la famille de tous les réseaux commodes $\mathfrak{U}$ jouissant des deux propriétés suivantes: 1° Pour $1 \leqq i \leqq \alpha_0$, $G^n(\mathfrak{U}) \not\sim 0 \bmod (R - P_i)$;

2° si un sommet U de $\mathfrak{U}$ rencontre T_i $(1 \leqq i \leqq \alpha_0)$, on a $U \subset P_i$; 3° $U\overline{P_i} \neq 0$ pour $1 \leqq i \leqq \alpha_0$ et pour chaque sommet extérieur $\mathfrak{U}$ de $\mathfrak{U}$. Puisque $G^n \not\sim 0 \bmod (R - P_i)$ (v. 17.5, *remarque*) et $T_i \subset \sigma_i^0 \subset P_i$, la famille Ψ_1 est évidemment parfaitement complète.

20.3. Ceci étant, on a pour chaque $\mathfrak{U} \in \Psi_1$ le théorème suivant: *Soit $C^n(\mathfrak{U})$ un $(n, \mathfrak{U})$-cycle essentiel mod S. Il existe une $(n, \mathfrak{U})$-chaîne élémentaire $D^n(\mathfrak{U})$ telle que $C^n(\mathfrak{U}) = D^n(\mathfrak{U}) \bmod \overline{R - R_0}$.*

Démonstration. G^n étant le cycle principal, d'après 20.1 il existe pour chaque i $(1 \leqq i \leqq \alpha_0)$ deux nombres $r_1, r_2 \in \mathfrak{R}$, dont un au moins $\neq 0$, tels que $r_1 C^n(\mathfrak{U}) + r_2 G^n(\mathfrak{U}) \sim 0 \bmod (R - P)$. Or $G^n(\mathfrak{U}) \not\sim 0 \bmod (R - P_i)$ d'après la propriété 1° de la famille Ψ_1. Donc $r_1 \neq 0$, c'est-à-dire qu'il existe un nombre $s_i \in \mathfrak{R}$ $(1 \leqq i \leqq \alpha_0)$ tel que $C^n(\mathfrak{U}) \sim s_i G^n(\mathfrak{U}) \bmod (R - P_i)$. Donc il existe une $(n+1, \mathfrak{U})$-chaîne $E^{n+1}(\mathfrak{U})$ telle que

$$FE^{n+1}(\mathfrak{U}) = C^n(\mathfrak{U}) - s_i G^n(\mathfrak{U}) \bmod (R - P_i).$$

Le réseau $\mathfrak{U}$ étant (v. 8.1, 3°) d'ordre $\leqq n \bmod S$, chaque simplexe de la chaîne $E^{n+1}(\mathfrak{U})$ possède un sommet extérieur, de manière que (v. la propriété 3° de la famille Ψ_1) $E^{n+1}(\mathfrak{U}) \subset R - P_i$, d'où $FE^{n+1}(\mathfrak{U}) \subset R - P_i$ et par suite $C^n(\mathfrak{U}) = s_i G^n(\mathfrak{U}) \bmod (R - P_i)$. Or si un $(n, \mathfrak{U})$-simplexe τ^n est d'espece σ_i^0, chacun des ses sommets rencontre P_i d'où $\tau^n \neq 0 \bmod (R - P_i)$ en vertu de la propriété 2° de la famille Ψ_1. Donc la partie de la chaîne $C^n(\mathfrak{U})$ formée des $(n, \mathfrak{U})$-simplexes d'espèce σ_i^0 est égale à celle de la chaîne $s_i G^n(\mathfrak{U})$, c'est-à-dire à $s_i K^n(\sigma_i^0, \mathfrak{U}) \bmod \overline{R - R_0}$. Or nous avons remarqué, en définissant les chaînes $K^n(\sigma_i^0, \mathfrak{U})$ (v. 19.1), qu'un $(n, \mathfrak{U})$-simplexe qui n'est pas d'espèce σ_i^0 pour aucun i $(1 \leqq i \leqq \alpha_0)$, est $= 0 \bmod \overline{R - R_0}$. Donc $C^n(\mathfrak{U}) = \sum_{i=1}^{\alpha_0} s_i K^n(\sigma_i^0, \mathfrak{U}) \bmod \overline{R - R_0}$.

III.

21. Axiome G^k $(0 \leqq k \leqq n-1)$: *Pour chaque entourage Q d'un point $a \in R - S$ il existe un entourage $P \subset Q$ de a jouissant de la propriété suivante: Si Γ^k est un (k, R)-cycle (absolu) dans $\overline{P}$*[38]*, on a $\Gamma^k \sim 0$ dans $\overline{Q}$.*

21.1. D'après 4.2 on déduit de l'axiome G^k: *Pour chaque réseau gén. $\mathfrak{Q}$ il existe un affinement $\mathfrak{P}$ et une projection $\pi = Pr.(\mathfrak{P}, \mathfrak{Q})$ de manière que: Si Γ^k est un (k, R)-cycle dans $\overline{P}$, $P \in \mathfrak{P}$, on a $\Gamma^k \sim 0$ dans $\overline{\pi P}$.*

22. Dans tout ce Chapitre, on se donnera une valeur fixe de p $(0 \leqq p \leqq n)$. On suppose la validité de tous les axiomes des Chap. I et II, ainsi que celle des axiomes G^k (v. 21) pour $n - p \leqq k \leqq n - 1$.

[38] Pour $k = 0$ il faut supposer encore que $I(\Gamma^k) = 0$, $I(\Gamma^k)$ étant la somme des coefficients de $\Gamma^k(\mathfrak{U})$ (somme indépendante du choix du réseau $\mathfrak{U}$).

23. *Soit* Ξ *la famille de tous les réseaux* $\mathfrak{Z}$ *de la famille* Φ_1 *du n° 7.4 possédant des sommets intérieurs et tels que* 1° $\mathfrak{Z}$ *est un affinement mod S du réseau gén.* $\mathfrak{P}$ *du n° 20.1,* 2° *le noyau de chaque* $\mathfrak{Z}$*-simplexe intérieur* σ *contient un point n'appartenant à la fermeture d'aucun sommet de* $\mathfrak{Z}$ *qui ne soit pas un sommet de* σ. *La famille* Ξ *est parfaitement complète.*

Démonstration. Soit $\mathfrak{U}$ un réseau donné; soit $\mathfrak{Q}$ un réseau gén. donné. Il existe évidemment un réseau $\mathfrak{Z}_{-1}$ tel que 1° $\mathfrak{Z}_{-1} \in \Phi_1$; 2° $\mathfrak{Z}_{-1}$ possède des sommets intérieurs; 3° $\mathfrak{Z}_{-1}$ est un affinement de $\mathfrak{U}$; 4° $\mathfrak{Z}_{-1}$ est un affinement mod S de $\mathfrak{P}$ et de $\mathfrak{Q}$. Soit m l'ordre (v. 7.2) de $\mathfrak{Z}_{-1}$. Supposons généralement que, pour une certaine valeur de h $(0 \leqq h \leqq m)$, on ait déjà défini le réseau $\mathfrak{Z}_{h-1}$. Dans le noyau de chaque $\mathfrak{Z}_{h-1}$-simplexe intérieur τ_ν^h choisissons un point $a_{h\nu}$ de manière que $a_{h\nu} \neq a_{g\mu}$ si $0 \leqq g < h$ ou bien $g = h$, $\mu \neq \nu$[39]. Modifions chaque sommet Z_{h-1} de $\mathfrak{Z}_{h-1}$ en le remplaçant par $Z_h = Z_{h-1} - \Sigma(a_{h\nu})$, où ν parcourt toutes les valeurs telles que Z_h ne soit pas un sommet de τ_ν^h. Soit $\mathfrak{Z}_h$ le réseau dont les sommets sont les ensembles Z_h ainsi associés aux sommets Z_{h-1} de $\mathfrak{Z}_{h-1}$. En procédant de cette manière on finit par construire un réseau $\mathfrak{Z}_m$. Soit $\mathfrak{Z}$ un rapetissement fort[30] de $\mathfrak{Z}_m$. On voit sans peine qu'on peut choisir $\mathfrak{Z}$ de manière que ce soit un affinement de $\mathfrak{U}$ et un affinement mod S de $\mathfrak{Q}$ possédant les propriétés voulues.

24. Soit $\mathfrak{z}$ un affinement de $\mathfrak{Z}$, les deux réseaux $\mathfrak{z}$ et $\mathfrak{Z}$ appartenant à la famille Ξ du n° 23. Pour le réseau $\mathfrak{Z}$ gardons la notation

$$\alpha_h,\ \sigma_i^h,\ T_i,\ T(\sigma_i^h),\ \mathfrak{T},\ K^{n-h}(\sigma_i^h, \mathfrak{U})$$

(où $\mathfrak{U}$ parcourt les réseaux commodes par rapport à $\mathfrak{Z}+\mathfrak{T}$). Pour le réseau $\mathfrak{z}$ on prendra la notation analogue

$$\beta_h,\ \tau_\nu^h,\ t_\nu,\ t(\tau_\nu^h),\ \mathfrak{t},\ k^{n-h}(\tau_\nu^h, \mathfrak{U}),$$

$\mathfrak{U}$ parcourant cette fois les réseaux commodes par rapport à $\mathfrak{z}+\mathfrak{t}$. Supposons le réseau fermé $\mathfrak{t}$ attaché à $\mathfrak{z}$ choisi selon 8 d'une manière quelconque; quant à $\mathfrak{T}$, nous le choisirons bientôt d'une manière convenable.

Choisissons une projection $\pi = Pr.(\mathfrak{z}, \mathfrak{Z})$. Un sommet intérieur τ_ν^0 de $\mathfrak{z}$ sera appelé *complètement intérieur,* si sa projection $\pi\tau_\nu^0$ est un sommet intérieur de $\mathfrak{Z}$. Un $(h, \mathfrak{z})$-simplexe intérieur τ_ν^h $(0 \leqq h \leqq n)$ sera appelé *complètement intérieur,* si chacun de ses sommets est complètement intérieur. On peut numéroter les τ_ν^h de manière que, pour chaque h $(0 \leqq h \leqq n)$, les simplexes complètement intérieurs précèdent les autres; soient τ_ν^h $(1 \leqq \nu \leqq \gamma_h)$ tous les $(h, \mathfrak{z})$-simplexes $(0 \leqq h \leqq n)$ complètement intérieurs (donc $0 \leqq \gamma_h \leqq \beta_h$ pour chaque h).

[39] D'après 12.2, ceci est évidemment possible.

Pour $0 \leqq h \leqq n$, $1 \leqq \nu \leqq \gamma_h$ on a $\pi \tau_\nu^h = 0$ (seulement pour $h \geqq 1$) ou bien il existe un indice $i = \varphi_h(\nu)$ bien déterminé ($1 \leqq i \leqq \alpha_h$) tel que $\pi \tau_\nu^h = \pm \sigma_i^h$. Pour $h = 0$ on a toujours le signe plus; et la même chose vaut aussi pour $1 \leqq h \leqq n$ en orientant convenablement les τ_ν^h ce que nous voulons supposer.

Ceci étant, définissons le réseau fermé $\mathfrak{T}$ attaché à $\mathfrak{Z}$ en posant, pour $1 \leqq i \leqq \alpha_0$,

$$T_i = \sum_\nu t_\nu, \tag{1}$$

où ν parcourt toutes les valeurs ($1 \leqq \nu \leqq \gamma_0$) telles que $\varphi_0(\nu) = i$. On voit sans peine que le réseau fermé $\mathfrak{T}$ possède bien par rapport à $\mathfrak{Z}$ les propriétés du n° 8.

R_k ($0 \leqq k \leqq n$) ayant la signification usuelle (v. 8) relativement à $\mathfrak{T}$, soient R_k' ($0 \leqq k \leqq n$) les ensembles analogues relatives à t. Évidemment $R_k' \supset R_k$ pour $0 \leqq k \leqq n$.

On vérifie sans peine que pour $0 \leqq h \leqq n$, $1 \leqq i \leqq \alpha_h$ on a

$$T(\sigma_i^h) = \sum_\nu t(\tau_\nu^h), \tag{2}$$

où ν parcourt toutes les valeurs ($1 \leqq \nu \leqq \gamma_h$) telles que $\varphi_h(\nu) = i$.

On voit aussi sans peine qu'un réseau $\mathfrak{U}$ commode par rapport à $\mathfrak{z} + \mathfrak{t}$ (dans le sens de la définition du n° 8.1) est aussi commode par rapport à $\mathfrak{Z} + \mathfrak{T}$.

$K^{n-h}(\sigma_i^h, \mathfrak{U})$ étant toujours les chaînes fondamentales (v. 19.1) relatives à $\mathfrak{Z} + \mathfrak{T}$, soient $k^{n-h}(\tau_\nu^h, \mathfrak{U})$ celles relatives à $\mathfrak{z} + \mathfrak{t}$. On vérifie sans peine que pour $0 \leqq h \leqq n$, $1 \leqq i \leqq \alpha_h$ on a dans chaque réseau $\mathfrak{U}$ commode par rapport à $\mathfrak{z} + \mathfrak{t}$

$$K^{n-h}(\sigma_i^h, \mathfrak{U}) = \sum_\nu k^{n-h}(\tau_\nu^h, \mathfrak{U}) \mod \overline{R - R_0}, \tag{3}$$

ν parcourant toutes les valeurs ($1 \leqq \nu \leqq \gamma_h$) telles que $\varphi_h(\nu) = i$.

Ψ_0 et Ψ_1 étant les deux familles de réseaux définies resp. dans 18.1 et 20.2 relativement à $\mathfrak{Z} + \mathfrak{T}$, soient ψ_0 et ψ_1 les familles correspondantes relatives à $\mathfrak{z} + \mathfrak{t}$. On voit sans peine que $\psi_0 \subset \Psi_0$, $\psi_1 \subset \Psi_1$. $\Psi_0(\psi_0)$ est d'ailleurs la famille de tous les réseaux commodes (v. 18.1) relativement à $\mathfrak{Z} + \mathfrak{T}(\mathfrak{z} + \mathfrak{t})$. Les familles ψ_0 et ψ_1 sont, comme nous savons, complètes.

25. *Supposons que le réseau $\mathfrak{Z}$ appartienne à la famille Ξ du n° 23. On peut déterminer le réseau fermé $\mathfrak{T}$ correspondant à $\mathfrak{Z}$ (v. 8) de manière que, si le réseau commode $\mathfrak{U}$ est suffisamment fin, on ait $K^{n-h}(\sigma_i^h, \mathfrak{U}) \neq 0$ pour $0 \leqq h \leqq \min(n, p+1)$, $1 \leqq i \leqq \alpha_h$.*

La démonstration fera l'objet des n°s 25.1—25.2.

25.1. Lemme. Soit $W \neq 0$ un ensemble ouvert $\subset R - S$. Soit $0 \leqq q \leqq p+1$. Il existe un réseau $\mathfrak{B}_q$ et un réseau gén. $\mathfrak{M}_q$ jouissant de la propriété sui-

vante: Supposons que le réseau $\mathfrak{Z}$ du n° 8 soit un affinement de $\mathfrak{B}_q$ et un affinement mod S de $\mathfrak{M}_q$. Choisissons arbitrairement le réseau fermé $\mathfrak{T}$ (n° 8) correspondant à $\mathfrak{Z}$. Alors il existe une valeur de $i\,(1 \leqq i \leqq \alpha_q)$ telle que 1° chaque sommet de σ_i^q est un sous-ensemble de W; 2° dans chaque réseau commode $\mathfrak{U}$ suffisamment fin on a $K^{n-q}(\sigma_i^q, \mathfrak{U}) \neq 0$[40].

Démonstration. Soit d'abord $q \geqq 1$. Choisissons un point $a \in W$. Soit W_1 un entourage de a si petit que $W_1 \Subset W$. Soit $\mathfrak{P}^0$ un réseau gén. tel que: 1° si $a \in \bar{P}^0$, $P^0 \in \mathfrak{P}^0$, on a $P^0 \subset W_1$; 2° $\mathfrak{P}^0$ jouit de la propriété du n° 4.3; 3° si $P^0 \in \mathfrak{P}^0$, on a $\Gamma^n \sim 0$ dans $\bar{P}^0$ pour chaque (n, R)-cycle Γ^n dans $\bar{P}^0$. $\mathfrak{P}^0$ existe d'après 4.2 et 9.1. Soit $\mathfrak{Q}^1$ un affinement de $\mathfrak{P}^0$ jouissant des propriétés des n^{os} 4.3 et 4.4. Déterminons un affinement $\mathfrak{P}^1$ de $\mathfrak{Q}^1$ ainsi qu'une projection $\pi_1 = Pr.(\mathfrak{P}^1, \mathfrak{Q}^1)$ d'après 21.1, en y posant $k = n-1$. Supposons généralement que, pour une certaine valeur de h $(1 \leqq h \leqq q-2)$, on ait déjà déterminé les réseaux gén. $\mathfrak{P}^h$ et $\mathfrak{Q}^h$. Soit $\mathfrak{Q}^{h+1}$ un affinement de $\mathfrak{P}^h$ jouissant de la propriété du n° 4.4. Déterminons un affinement $\mathfrak{P}^{h+1}$ de $\mathfrak{Q}^{h+1}$ ainsi qu'une projection $\pi_{h+1} = Pr.(\mathfrak{P}^{h+1}, \mathfrak{Q}^{h+1})$ d'après 21.1, en y posant $k = n-h-1$. En procédant de cette manière, on construit de proche en proche les réseaux gén. $\mathfrak{P}^h$ $(0 \leqq h \leqq q-1)$ et $\mathfrak{Q}^h$ $(1 \leqq h \leqq q-1)$. Soit encore $\mathfrak{Q}^q$ un affinement de $\mathfrak{P}^{q-1}$ jouissant de la propriété du n° 4.4. Posons $\mathfrak{M}_q = \mathfrak{Q}^q$.

Soit $\mathfrak{B}_q$ un réseau tel que: 1° si $V \in \mathfrak{B}_q$ et que $\bar{V}\bar{W}_1 \neq 0$, on a $V \subset W$; 2° si $a \in \bar{V}$, $V \in \mathfrak{B}_q$, on a $V \subset W_1$.

Supposons que le réseau $\mathfrak{Z}$ soit un affinement de $\mathfrak{B}_q$ et un affinement mod S de $\mathfrak{M}_q = \mathfrak{Q}^q$. Puisque l'affinement $\mathfrak{Q}^q$ de $\mathfrak{P}^{q-1}$ jouit de la propriété du n° 4.4, on peut attacher à chaque $(q-1, \mathfrak{Z})$-simplexe intérieur σ_i^{q-1} $(1 \leqq i \leqq \alpha_{q-1})$ un sommet $P^{q-1}(\sigma_i^{q-1})$ de $\mathfrak{P}^{q-1}$ tel que chaque sommet de σ_i^{q-1} en soit un sous-ensemble. Posons $Q^{q-1}(\sigma_i^{q-1}) = \pi_{q-1} P^{q-1}(\sigma_i^{q-1})$. Supposons généralement que, pour une certaine valeur de h $(0 \leqq h \leqq q-2)$, on ait déjà attaché à chaque σ_i^{h+1} $(1 \leqq i \leqq \alpha_{h+1})$ des sommets $P^{h+1}(\sigma_i^{h+1})$, $Q^{h+1}(\sigma_i^{h+1})$ resp. de $\mathfrak{P}^{h+1}$ et de $\mathfrak{Q}^{h+1}$ de manière que chaque sommet de σ_i^{h+1} soit un sous-ensemble de $P^{h+1}(\sigma_i^{h+1}) \subset Q^{h+1}(\sigma_i^{h+1})$. Supposons donnée une valeur de j $(1 \leqq j \leqq \alpha_h)$. Deux cas sont à distinguer. *En premier lieu,* supposons que σ_j^h ne soit une h-face d'aucun $(h+1; \mathfrak{Z})$-simplexe intérieur; dans ce cas, choisissons un sommet $P^h(\sigma_j^h)$ de $\mathfrak{P}^h$ de manière que chaque sommet de σ_j^h en soit un sous-ensemble; ce qui est possible, puisque le réseau gén. $\mathfrak{P}^{q-1}$ est un affinement de $\mathfrak{P}^h$. *En second lieu,* supposons qu'il

[40] Pour $q = n+1$, la thèse du lemme s'énonce comme il suit: Il existe une valeur de $i\,(1 \leqq i \leqq \alpha_n)$ telle que 1° chaque sommet de σ_i^n est un sous-ensemble de W; 2° dans chaque réseau commode $\mathfrak{U}$ suffisamment fin, on a $I[K^0(\sigma_i^0, \mathfrak{U})] \neq 0$, $(1 \leqq i \leqq \alpha_0)$. (V.[38] pour la signification de I.) Du reste, nous n'aurons dans cet Ouvrage aucune occasion d'appliquer le cas $q = n+1$ du lemme.

existe des valeurs de i telles que $\eta_{ij}^h \neq 0$; dans ce cas, choisissons un sommet $P^h(\sigma_j^h)$ de $\mathfrak{P}^h$ de manière que l'on ait $P^h(\sigma_j^h) \supset Q^{h+1}(\sigma_i^{h+1})$ pour chaque valeur de i $(1 \leqq i \leqq \alpha_{h+1})$ telle que $\eta_{ij}^h \neq 0$; ce qui est possible, car $\mathfrak{Q}^{h+1}$ est un affinement de $\mathfrak{P}^h$ jouissant de la propriété du n° 4.4; on voit que, ici encore, chaque sommet de σ_j^h est un sous-ensemble de $P^h(\sigma_j^h)$. Si $h \geqq 1$, posons $Q^h(\sigma_j^h) = \pi_h P^h(\sigma_j^h)$. En procédant de cette manière, on construit de proche en proche les ensembles $P^h(\sigma_i^h)$ $(0 \leqq h \leqq q-1,\ 1 \leqq i \leqq \alpha_h)$ et $Q^h(\sigma_i^h)$ $(1 \leqq h \leqq q-1,\ 1 \leqq i \leqq \alpha_h)$.

Soit W_2 un entourage de a si petit que 1° pour chaque sommet Q^1 de $\mathfrak{Q}^1$ la relation $W_2 \overline{Q^1} \neq 0$ entraîne $a \in \overline{Q^1}$; 2° $\overline{W_2} \sigma_i^0 = 0$ pour chaque valeur de i $(1 \leqq i \leqq \alpha_0)$ telle que $\overline{\sigma_i^0} - W_1 \neq 0$; 3° $\overline{W_2}\,\overline{R - R_0} = 0$. W existe, car: 1° en vertu de la définition même d'un réseau gén., on a $\overline{W_1} Q^1 = 0$ pour tous les sommets Q^1 de $\mathfrak{Q}^1$ à un nombre fini d'exceptions près; 2° si l'ensemble σ_i^0 n'est pas contenu dans $\overline{W_1}$, on déduit, $\mathfrak{Z}$ étant un affinement de $\mathfrak{B}_q$, que le point a n'est pas situé dans $\overline{\sigma_i^0}$; 3° $a \in R - \overline{R - R_0}$, comme nous allons voir.

On a $\overline{W_1}\,\overline{R - R_0} = 0$ (et donc $a \in R - \overline{R - R_0}$). Dans le cas contraire, il existerait un point $b \in \overline{W_1}\,\overline{R - R_0}$. Or, d'après 8, on a $R - R_0 = R - \sum_{i=1}^{\alpha_0} T_i \subset \sum Z$ où Z parcourt tous les sommets extérieurs de $\mathfrak{Z}$. Il existerait donc un sommet Z de $\mathfrak{Z}$ tel que $b \in \overline{Z}$, $ZS \neq 0$. Le réseau $\mathfrak{Z}$ étant un affinement de $\mathfrak{B}_q$, il existerait un sommet V de $\mathfrak{B}_q$ tel que $Z \subset V$, d'où $b \in \overline{V}$, donc $\overline{V}\,\overline{W_1} \neq 0$ et par suite $V \subset W$, ce qui est impossible, car $Z \subset V$, $ZS \neq 0$, $W \subset R - S$.

On peut donc déterminer un réseau commode $\mathfrak{U}_0$ tel qu'aucun sommet de $\mathfrak{U}_0$ ne rencontre simultanément $\overline{W_1}$ et $\overline{R - R_0}$. $\mathfrak{U}_0$ soit en outre tel que la chaîne $G^n(\mathfrak{U})$ ne soit pas située dans $R - W_2$; c'est réalisable, car (v. 17.5) le cycle principal G^n n'est pas situé dans $R - W_2$. Enfin, $\mathfrak{U}_0$ soit si fin qu'aucun sommet de $\mathfrak{U}_0$ ne rencontre simultanément S et un des ensembles $\overline{P^0(\sigma_i^0)}$ $(1 \leqq i \leqq \alpha_0)$; c'est possible, car $\mathfrak{P}^0$ jouit de la propriété du n° 4.3. Supposons généralement que pour une certaine valeur de h $(0 \leqq h \leqq q-1)$, on ait déjà construit le réseau $\mathfrak{U}_h$. Soit alors $\mathfrak{U}_{h+1}$ un réseau commode tel que $\mathfrak{U}_{h+1}$ soit un affinement de $\mathfrak{U}_h$ normal par rapport aux cycles dans $\overline{P^h(\sigma_i^h)}$ pour chaque valeur de i $(1 \leqq i \leqq \alpha_h)$. On définit de cette manière de proche en proche les réseaux commodes $\mathfrak{U}_h$ $(0 \leqq h \leqq q)$. Soit encore $\mathfrak{U}_{q+1}$ un réseau commode qui soit un affinement de $\mathfrak{U}_q$. Pour $0 \leqq h \leqq q$, choisissons une projection $\pi_h' = Pr.(\mathfrak{U}_{h+1}, \mathfrak{U}_h)$.

Pour $0 \leqq h \leqq n$, soit N_h l'ensemble de tous les indices i $(1 \leqq i \leqq \alpha_h)$ tels que le simplexe σ_i^h possède un sommet qui soit un sous-ensemble de $\overline{W_1}$. Evidemment $j \in N_h$, $\eta_{ij}^h \neq 0$ $(0 \leqq h \leqq n-1)$ entraîne $i \in N_{h+1}$.

Pour $0 \leqq k \leqq q$, $0 \leqq h \leqq n$, $1 \leqq i \leqq \alpha_h$ on a d'après 19.2

$$\pi'_k K^{n-h}(\sigma_i^h, \mathfrak{U}_{k+1}) = K^{n-h}(\sigma_i^h, \mathfrak{U}_k) \bmod \overline{R - R_0}.$$

Or je dis que, si $i \in N_h$, on a plus précisément

$$(*) \qquad \pi'_k K^{n-h}(\sigma_i^h, \mathfrak{U}_{k+1}) = K^{n-h}(\sigma_i^h, \mathfrak{U}_k).$$

D'après 19.1, 1° il suffit de prouver que la chaîne $\pi'_k K^{n-h}(\sigma_i^h, \mathfrak{U}_{k+1})$ ne contient aucun simplexe situé dans $\overline{R - R_0}$. Supposons au contraire que τ soit un tel $\mathfrak{U}_k$-simplexe. Alors chaque sommet de τ rencontrera $\overline{R - R_0}$. D'autre part, d'après 19.3, chaque sommet de τ rencontre $T(\sigma_i^h)$ et par suite chaque sommet Z de σ_i^h. Or, puisque $i \in N_h$, on peut choisir Z de manière que $Z \subset \overline{W}_1$. Donc chaque sommet de τ rencontrera simultanément $\overline{W}_1$ et $\overline{R - R_0}$, ce qui est impossible, car $\mathfrak{U}_k$ est un affinement de $\mathfrak{U}_0$.

Par le même raisonnement on déduit de 19.1, 3° que pour $0 \leqq k \leqq q+1$, $1 \leqq h \leqq n$, $j \in N_{h-1}$ on a

$$(**) \qquad K^{n-h+1}(\sigma_j^{h-1}, \mathfrak{U}_k) \to \sum_{i=1}^{\alpha_h} \eta_{ij}^{h-1} K^{n-h}(\sigma_i^h, \mathfrak{U}_k).$$

Ceci étant, supposons que notre lemme ne soit pas vrai. Si $i \in N_h$ $(0 \leqq h \leqq n)$, le simplexe σ_i^h possède un sommet Z_0 tel que $Z_0 \subset \overline{W}_1$. Z étant un sommet arbitraire de σ_i^h, on a $ZZ_0 \neq 0$ et par suite $Z\overline{W}_1 \neq 0$, d'où $Z \subset W$, car le réseau $\mathfrak{Z}$ est un affinement de $\mathfrak{B}_q$. Donc l'hypothèse que le lemme ne soit pas vrai entraîne, en excluant d'abord le cas $q = n+1$, que $K^{n-q}(\sigma_i^q, \mathfrak{U}_{q+1}) = 0$ pour un choix convenable de $\mathfrak{U}_{q+1}$, si $i \in N_q$. Donc, d'après (*), on a $K^{n-q}(\sigma_i^q, \mathfrak{U}_q) = 0$ pour chaque $i \in N_q$. Donc on déduit de (**) que $K^{n-q+1}(\sigma_i^{q-1}, \mathfrak{U}_q)$ est, pour chaque $i \in N_{q-1}$, un $(n-q+1, \mathfrak{U}_q)$-cycle, situé dans $\overline{P^{q-1}(\sigma_i^{q-1})}$ d'après 19.3. Donc $K^{n-q+1}(\sigma_i^{q-1}, \mathfrak{U}_{q-1})$ est, pour chaque $i \in N_{q-1}$, un $(n-q+1, \mathfrak{U}_q)$-cycle essentiel dans $\overline{P^{q-1}(\sigma_i^{q-1})}$ de manière qu'il existe pour $i \in N_{q-1}$ une $(n-q+2, \mathfrak{U}_{q-1})$-chaîne $H^{n-q+2}(\sigma_i^{q-1}, \mathfrak{U}_{q-1})$ dans $Q^{q-1}(\sigma_i^{q-1})$ telle que

$$H^{n-q+2}(\sigma_i^{q-1}, \mathfrak{U}_{q-1}) \to K^{n-q+1}(\sigma_i^{q-1}, \mathfrak{U}_{q-1}).$$

On arrive au même résultat si $q = n+1$. En effet, de l'hypothèse que le lemme ne soit pas vrai il résulte dans ce cas que $I[K^0(\sigma_i^n, \mathfrak{U}_{n+1})] = 0$ pour chaque $i \in N_n$. Puisque $K^0(\sigma_i^n, \mathfrak{U}_{n+1}) \subset \overline{P^n(\sigma_i^n)}$, il existe donc une $(1, \mathfrak{U}_n)$-chaîne $H^1(\sigma_i^n, \mathfrak{U}_{n-1})$ dans $Q^n(\sigma_i^n)$ telle que

$$H^1(\sigma_i^n, \mathfrak{U}_n) \to K^0(\sigma_i^n, \mathfrak{U}_n).$$

Quelleque soit la valeur de q, il résulte de ce que nous venons de prouver, en tenant compte de (**), que pour chaque $j \in N_{q-2}$

existe des valeurs de i telles que $\eta_{ij}^h \neq 0$; dans ce cas, choisissons un sommet $P^h(\sigma_j^h)$ de $\mathfrak{P}^h$ de manière que l'on ait $P^h(\sigma_j^h) \supset Q^{h+1}(\sigma_i^{h+1})$ pour chaque valeur de i $(1 \leqq i \leqq \alpha_{h+1})$ telle que $\eta_{ij}^h \neq 0$; ce qui est possible, car $\mathfrak{Q}^{h+1}$ est un affinement de $\mathfrak{P}^h$ jouissant de la propriété du n° 4.4; on voit que, ici encore, chaque sommet de σ_j^h est un sous-ensemble de $P^h(\sigma_j^h)$. Si $h \geqq 1$, posons $Q^h(\sigma_j^h) = \pi_h P^h(\sigma_j^h)$. En procédant de cette manière, on construit de proche en proche les ensembles $P^h(\sigma_i^h)$ $(0 \leqq h \leqq q-1,\ 1 \leqq i \leqq \alpha_h)$ et $Q^h(\sigma_i^h)$ $(1 \leqq h \leqq q-1,\ 1 \leqq i \leqq \alpha_h)$.

Soit W_2 un entourage de a si petit que 1° pour chaque sommet Q^1 de $\mathfrak{Q}^1$ la relation $W_2 \overline{Q^1} \neq 0$ entraîne $a \in \overline{Q^1}$; 2° $\overline{W}_2 \sigma_i^0 = 0$ pour chaque valeur de i $(1 \leqq i \leqq \alpha_0)$ telle que $\overline{\sigma_i^0} - W_1 \neq 0$; 3° $\overline{W}_2 \overline{R - R_0} = 0$. W existe, car: 1° en vertu de la définition même d'un réseau gén., on a $\overline{W}_1 Q^1 = 0$ pour tous les sommets Q^1 de $\mathfrak{Q}^1$ à un nombre fini d'exceptions près; 2° si l'ensemble σ_i^0 n'est pas contenu dans $\overline{W}_1$, on déduit, $\mathfrak{Z}$ étant un affinement de $\mathfrak{B}_q$, que le point a n'est pas situé dans $\overline{\sigma_i^0}$; 3° $a \in R - \overline{R - R_0}$, comme nous allons voir.

On a $\overline{W}_1 \overline{R - R_0} = 0$ (et donc $a \in R - \overline{R - R_0}$). Dans le cas contraire, il existerait un point $b \in \overline{W}_1 \overline{R - R_0}$. Or, d'après 8, on a $R - R_0 = R - \sum_{i=1}^{\alpha_0} T_i \subset \sum Z$ où Z parcourt tous les sommets extérieurs de $\mathfrak{Z}$. Il existerait donc un sommet Z de $\mathfrak{Z}$ tel que $b \in \overline{Z}$, $ZS \neq 0$. Le réseau $\mathfrak{Z}$ étant un affinement de $\mathfrak{B}_q$, il existerait un sommet V de $\mathfrak{B}_q$ tel que $Z \subset V$, d'où $b \in \overline{V}$, donc $\overline{V}\overline{W}_1 \neq 0$ et par suite $V \subset W$, ce qui est impossible, car $Z \subset V$, $ZS \neq 0$, $W \subset R - S$.

On peut donc déterminer un réseau commode $\mathfrak{U}_0$ tel qu'aucun sommet de $\mathfrak{U}_0$ ne rencontre simultanément $\overline{W}_1$ et $\overline{R - R_0}$. $\mathfrak{U}_0$ soit en outre tel que la chaîne $G^n(\mathfrak{U})$ ne soit pas située dans $R - W_2$; c'est réalisable, car (v. 17.5) le cycle principal G^n n'est pas situé dans $R - W_2$. Enfin, $\mathfrak{U}_0$ soit si fin qu'aucun sommet de $\mathfrak{U}_0$ ne rencontre simultanément S et un des ensembles $\overline{P^0(\sigma_i^0)}$ $(1 \leqq i \leqq \alpha_0)$; c'est possible, car $\mathfrak{P}^0$ jouit de la propriété du n° 4.3. Supposons généralement que pour une certaine valeur de h $(0 \leqq h \leqq q-1)$, on ait déjà construit le réseau $\mathfrak{U}_h$. Soit alors $\mathfrak{U}_{h+1}$ un réseau commode tel que $\mathfrak{U}_{h+1}$ soit un affinement de $\mathfrak{U}_h$ normal par rapport aux cycles dans $\overline{P^h(\sigma_i^h)}$ pour chaque valeur de i $(1 \leqq i \leqq \alpha_h)$. On définit de cette manière de proche en proche les réseaux commodes $\mathfrak{U}_h$ $(0 \leqq h \leqq q)$. Soit encore $\mathfrak{U}_{q+1}$ un réseau commode qui soit un affinement de $\mathfrak{U}_q$. Pour $0 \leqq h \leqq q$, choisissons une projection $\pi'_h = Pr.(\mathfrak{U}_{h+1}, \mathfrak{U}_h)$.

Pour $0 \leqq h \leqq n$, soit N_h l'ensemble de tous les indices i $(1 \leqq i \leqq \alpha_h)$ tels que le simplexe σ_i^h possède un sommet qui soit un sous-ensemble de $\overline{W}_1$. Evidemment $j \in N_h$, $\eta_{ij}^h \neq 0$ $(0 \leqq h \leqq n-1)$ entraîne $i \in N_{h+1}$.

Pour $0 \leqq k \leqq q$, $0 \leqq h \leqq n$, $1 \leqq i \leqq \alpha_h$ on a d'après 19.2

$$\pi'_k K^{n-h}(\sigma_i^h, \mathfrak{U}_{k+1}) = K^{n-h}(\sigma_i^h, \mathfrak{U}_k) \bmod \overline{R - R_0}.$$

Or je dis que, si $i \in N_h$, on a plus précisément

$$(*) \qquad \pi'_k K^{n-h}(\sigma_i^h, \mathfrak{U}_{k+1}) = K^{n-h}(\sigma_i^h, \mathfrak{U}_k).$$

D'après 19.1, 1° il suffit de prouver que la chaîne $\pi'_k K^{n-h}(\sigma_i^h, \mathfrak{U}_{k+1})$ ne contient aucun simplexe situé dans $\overline{R - R_0}$. Supposons au contraire que τ soit un tel $\mathfrak{U}_k$-simplexe. Alors chaque sommet de τ rencontrera $\overline{R - R_0}$. D'autre part, d'après 19.3, chaque sommet de τ rencontre $T(\sigma_i^h)$ et par suite chaque sommet Z de σ_i^h. Or, puisque $i \in N_h$, on peut choisir Z de manière que $Z \subset \overline{W}_1$. Donc chaque sommet de τ rencontrera simultanément $\overline{W}_1$ et $\overline{R - R_0}$, ce qui est impossible, car $\mathfrak{U}_k$ est un affinement de $\mathfrak{U}_0$.

Par le même raisonnement on déduit de 19.1, 3° que pour $0 \leqq k \leqq q+1$, $1 \leqq h \leqq n$, $j \in N_{h-1}$ on a

$$(**) \qquad K^{n-h+1}(\sigma_j^{h-1}, \mathfrak{U}_k) \to \sum_{i=1}^{\alpha_h} \eta_{ij}^{h-1} K^{n-h}(\sigma_i^h, \mathfrak{U}_k).$$

Ceci étant, supposons que notre lemme ne soit pas vrai. Si $i \in N_h$ $(0 \leqq h \leqq n)$, le simplexe σ_i^h possède un sommet Z_0 tel que $Z_0 \subset \overline{W}_1$. Z étant un sommet arbitraire de σ_i^h, on a $ZZ_0 \neq 0$ et par suite $Z\overline{W}_1 \neq 0$, d'où $Z \subset W$, car le réseau $\mathfrak{Z}$ est un affinement de $\mathfrak{B}_q$. Donc l'hypothèse que le lemme ne soit pas vrai entraîne, en excluant d'abord le cas $q = n+1$, que $K^{n-q}(\sigma_i^q, \mathfrak{U}_{q+1}) = 0$ pour un choix convenable de $\mathfrak{U}_{q+1}$, si $i \in N_q$. Donc, d'après (*), on a $K^{n-q}(\sigma_i^q, \mathfrak{U}_q) = 0$ pour chaque $i \in N_q$. Donc on déduit de (**) que $K^{n-q+1}(\sigma_i^{q-1}, \mathfrak{U}_q)$ est, pour chaque $i \in N_{q-1}$, un $(n-q+1, \mathfrak{U}_q)$-cycle, situé dans $\overline{P^{q-1}(\sigma_i^{q-1})}$ d'après 19.3. Donc $K^{n-q+1}(\sigma_i^{q-1}, \mathfrak{U}_{q-1})$ est, pour chaque $i \in N_{q-1}$, un $(n-q+1, \mathfrak{U}_q)$-cycle essentiel dans $\overline{P^{q-1}(\sigma_i^{q-1})}$ de manière qu'il existe pour $i \in N_{q-1}$ une $(n-q+2, \mathfrak{U}_{q-1})$-chaîne $H^{n-q+2}(\sigma_i^{q-1}, \mathfrak{U}_{q-1})$ dans $Q^{q-1}(\sigma_i^{q-1})$ telle que

$$H^{n-q+2}(\sigma_i^{q-1}, \mathfrak{U}_{q-1}) \to K^{n-q+1}(\sigma_i^{q-1}, \mathfrak{U}_{q-1}).$$

On arrive au même résultat si $q = n+1$. En effet, de l'hypothèse que le lemme ne soit pas vrai il résulte dans ce cas que $I[K^0(\sigma_i^n, \mathfrak{U}_{n+1})] = 0$ pour chaque $i \in N_n$. Puisque $K^0(\sigma_i^n, \mathfrak{U}_{n+1}) \subset \overline{P^n(\sigma_i^n)}$, il existe donc une $(1, \mathfrak{U}_n)$-chaîne $H^1(\sigma_i^n, \mathfrak{U}_{n-1})$ dans $Q^n(\sigma_i^n)$ telle que

$$H^1(\sigma_i^n, \mathfrak{U}_n) \to K^0(\sigma_i^n, \mathfrak{U}_n).$$

Quelleque soit la valeur de q, il résulte de ce que nous venons de prouver, en tenant compte de (**), que pour chaque $j \in N_{q-2}$

$$\Gamma^{n-q+2}(\sigma_j^{q-2}, \mathfrak{U}_{q-1}) = K^{n-q+2}(\sigma_j^{q-2}, \mathfrak{U}_{q-1}) - \sum_{i \in N_{q-1}} \eta_{ij}^{q-2} H^{n-q+2}(\sigma_i^{q-1}, \mathfrak{U}_{q-1})$$

est un $(n-q+2, \mathfrak{U}_{q-1})$-cycle dans

$$T(\sigma_j^{q-2}) + \sum_{\eta_{ij}^{q-2} \neq 0} \overline{Q^{q-1}(\sigma_i^{q-1})} \subset \overline{P^{q-2}(\sigma_j^{q-2})}.$$

Supposons généralement que, pour une certaine valeur de h $(1 \leqq h \leqq q-2)$, on ait déjà attaché à chaque $i \in N_{h+1}$ une $(n-h, \mathfrak{U}_{h+1})$-chaîne $H^{n-h}(\sigma_i^{h+1}, \mathfrak{U}_{h+1})$ dans $\overline{Q^{h+1}(\sigma_i^{h+1})}$ de manière que pour chaque $j \in N_h$

$$\Gamma^{n-h}(\sigma_j^h, \mathfrak{U}_{h+1}) = K^{n-h}(\sigma_j^h, \mathfrak{U}_{h+1}) - \sum_{i \in N_{h+1}} \eta_{ij}^h H^{n-h}(\sigma_i^{h+1}, \mathfrak{U}_{h+1})$$

soit un $(n-h, \mathfrak{U}_{h+1})$-cycle dans $\overline{P^h(\sigma_j^h)}$. Posons

$$H^{n-h}(\sigma_i^{h+1}, \mathfrak{U}_h) = \pi_h H^{n-h}(\sigma_i^{h+1}, \mathfrak{U}_{h+1})$$

etc., ce qui est d'accord avec (*). Alors $\Gamma^{n-h}(\sigma_j^h, \mathfrak{U}_h)$ est un $(n-h, \mathfrak{U}_h)$-cycle essentiel dans $\overline{P^h(\sigma_j^h)}$. Il en résulte qu'il existe pour $j \in N_h$ une $(n-h+1, \mathfrak{U}_h)$-chaîne $H^{n-h+1}(\sigma_j^h, \mathfrak{U}_h)$ dans $\overline{Q^h(\sigma_j^h)}$ telle que

$$H^{n-h+1}(\sigma_j^h, \mathfrak{U}_h) \to \Gamma^{n-h}(\sigma_j^h, \mathfrak{U}_h)$$

d'où pour $j \in N_{h-1}$

$$\begin{aligned}\Gamma^{n-h+1}(\sigma_j^{h-1}, \mathfrak{U}_h) &= K^{n-h+1}(\sigma_j^{h-1}, \mathfrak{U}_h) - \sum_{i \in N_h} \eta_{ij}^{h-1} H^{n-h+1}(\sigma_i^h, \mathfrak{U}_h) \\ &\to \sum_{i \in N_h} \eta_{ij}^{h-1} [K^{n-h}(\sigma_i^h, \mathfrak{U}_h) - \Gamma^{n-h}(\sigma_i^h, \mathfrak{U}_h)] \\ &= \sum_{\substack{i \in N_h \\ k \in N_{h+1}}} \eta_{ki}^h \eta_{ij}^{h-1} H^{n-h}(\sigma_k^{h+1}, \mathfrak{U}_h) = 0\end{aligned}$$

de manière que $\Gamma^{n-h+1}(\sigma_j^{h-1}, \mathfrak{U}_h)$ est, pour $j \in N_{h-1}$, un $(n-h+1, \mathfrak{U}_h)$-cycle dans

$$T(\sigma_j^{h-1}) + \sum_{i \in N_h} \overline{Q^h(\sigma_i^h)} \subset P^{h-1}(\sigma_j^{h-1}).$$

En procédant de cette manière on arrive finalement à attacher à chaque $i \in N_1$ une $(n, \mathfrak{U}_1)$-chaîne $H^n(\sigma_i^1, \mathfrak{U}_1)$ dans $\overline{Q^1(\sigma_i^1)}$ de manière que pour chaque $j \in N_0$

$$\Gamma^n(\sigma_j^0, \mathfrak{U}_1) = K^n(\sigma_j^0, \mathfrak{U}_1) - \sum_{i \in N_1} \eta_{ij}^0 H^n(\sigma_i^1, \mathfrak{U}_1)$$

soit un $(n, \mathfrak{U}_1)$-cycle dans $\overline{P^0(\sigma_i^0)}$ de manière que $\Gamma^n(\sigma_j^0, \mathfrak{U}_0)$ est un $(n, \mathfrak{U}_0)$-cycle essentiel dans $\overline{P^0(\sigma_i^0)} \in \mathfrak{P}^0$. D'après la définition du réseau gén. $\mathfrak{P}^0$, il en résulte que $\Gamma^n(\sigma_j^0, \mathfrak{U}_0) \sim 0$ dans $\overline{P^0(\sigma_j^0)}$ pour chaque $j \in N_0$. Or le réseau $\mathfrak{U}_0$ est d'ordre $\leqq n \bmod S$ et aucun sommet de $\mathfrak{U}_0$ ne peut rencontrer simultanément S et $\overline{P^0(\sigma_j^0)}$. Donc

$$K^n(\sigma_j^0, \mathfrak{U}_0) = \sum_{i \in N_1} \eta_{ij}^0 H^n(\sigma_i^1, \mathfrak{U}_0)$$

pour chaque $i \in N_0$. Donc

$$\sum_{j \in N_0} K^n(\sigma_j^0, \mathfrak{U}_0) = \sum_{i \in N_1} \varepsilon_i H^n(\sigma_i^1, \mathfrak{U}_0)$$

pour $j \in N_0$, où

$$\varepsilon_i = \sum_{j \in N_0} \eta_{ij}^0 \quad \text{pour} \quad i \in N_1.$$

Or pour $i \in N_1$ on a $\sigma_i^1 \to \pm(\sigma_s^0 - \sigma_t^0)$, où un au moins des deux ensembles σ_s^0 et σ_t^0 est contenu dans $\overline{W}_1$. Si $\sigma_s^0 + \sigma_t^0 \subset \overline{W}_1$, on a $s, t \in N_0$, d'où $\varepsilon_i = 0$. Donc si l'indice $i \in N_1$ est tel que $\varepsilon_i \neq 0$, on peut supposer que $\sigma_s^0 - \overline{W}_1 \neq 0$. Or $\sigma_s^0 \subset Q^1(\sigma_i^1)$, donc $Q^1(\sigma_i^1) - \overline{W}_1 \neq 0$. Il en résulte que le point a n'est pas situé dans $\overline{Q^1(\sigma_i^1)}$; en effet, dans le cas contraire on aurait $a \in \overline{Q^1(\sigma_i^1)} \subset \overline{P^0(\sigma_s^0)}$, d'où la contradiction $Q^1(\sigma_i^1) \subset P^0(\sigma_s^0) \subset W_1$ d'après la définition de $\mathfrak{P}^0$. Puisque le point a n'est pas situé dans $\overline{Q^1(\sigma_i^1)}$, on a $\overline{Q^1(\sigma_i^1)}\, W_2 = 0$ d'après la définition de W_2. Donc $i \in N_1$, $\varepsilon_i \neq 0$ entraîne que $\overline{Q^1(\sigma_i^1)} \subset R - W_2$ et par suite que la chaîne $H^n(\sigma_i^1, \mathfrak{U}_0)$ est située dans $R - W_2$. Donc pour chaque $i \in N_0$ la chaîne $K^n(\sigma_i^0, \mathfrak{U}_0)$ est $\subset R - W_2$. Or ce fait subsiste pour toutes les autres valeurs de i $(1 \leqq i \leqq \alpha_0)$; en effet, si l'on n'a pas $i \in N_0$, on a $\overline{\sigma_i^0}\, W_2 = 0$ d'après la définition de W_2 et, d'autre part, la chaîne $K^n(\sigma_i^0, \mathfrak{U}_0)$ est (v. 19.3) située dans $T_i \subset \overline{\sigma_i^0}$. Donc la chaîne

$$\sum_{i=1}^{\alpha_0} K^n(\sigma_i^0, \mathfrak{U}_0)$$

est située dans $R - W_2$. En outre on a $\overline{R - R_0} \subset R - W_2$ d'après la définition de W_2. Donc (v. 19.1, 4°) la chaîne $G^n(\mathfrak{U}_0)$ est située dans $R - W_2$, ce qui est une contradiction.

Le cas $q = 0$, qui était exclu jusqu'à présent, se traite bien facilement. On peut choisir arbitrairement le réseau gén. $\mathfrak{M}_0$. Choisissons de nouveau le point $a \in W$ et son entourage $W_1 \in W$. Supposons le réseau $\mathfrak{V}_0$ choisi de telle manière que 1° $V \in \mathfrak{V}_0$, $\overline{V}\,\overline{W}_1 \neq 0$ entraîne $V \subset W$; 2° $a \in \overline{V}$, $V \in \mathfrak{V}_0$ entraîne $V \subset W_1$. Soit W_2 un entourage de a si petit que 1° $\overline{\sigma_i^0} - W_1 \neq 0$ $(1 \leqq i \leqq \alpha_0)$ entraîne $\overline{W}_2\, \overline{\sigma_i^0} = 0$; 2° $\overline{W}_2\, \overline{R - R_0} = 0$. Comme plus haut on voit que ces conditions sont réalisables. Choisissons le réseau commode $\mathfrak{U}_0$ de manière que la chaîne $G^n(\mathfrak{U}_0)$ ne soit pas située dans $R - W_2$. Si le cas $q = 0$ du lemme n'était pas vrai, on aurait [v. (*)] $K^n(\sigma_i^0, \mathfrak{U}_0) = 0$ pour chaque i tel que $\sigma_i^0 \subset W_1$. Comme plus haut, on voit que pour les autres valeurs de i $K^n(\sigma_i^0, \mathfrak{U}_0)$ est située dans $R - W_2$. Puisque $\overline{R - R_0} \subset R - W_2$, on aurait de nouveau la contradiction $G^n(\mathfrak{U}_0) \subset R - W_2$.

25.2. Passons à la démonstration du théorème du n° 25. D'après la définition de la famille Ξ on peut déterminer des points $a_{hi} \in \mathfrak{I}(\sigma_i^h)$

($0 \leqq h \leqq p+1$, $1 \leqq i \leqq \alpha_h$; $\mathfrak{F}$ signifie le noyau) de manière que l'inclusion $a_{hi} \in \overline{Z}$, $Z \in \mathfrak{Z}$ entraîne que Z soit un sommet de σ_i^h. Désignons par G la somme de tous les sommets extérieurs de $\mathfrak{Z}$. On a alors $a_{hi} \in R - \overline{G}$. Les points a_{hi} étant manifestement distincts les uns des autres, on peut déterminer des ensembles ouverts W_{hi} tels que 1° $W_{hi} \subset \mathfrak{F}(\sigma_i^h)$; 2° $W_{hi} W_{kj} \neq 0$ entraîne $h = k$, $i = j$; 3° $W_{hi} \subset R - \overline{G}$. Pour $0 \leqq h \leqq p+1$, $1 \leqq i \leqq \alpha_h$ attachons à l'ensemble ouvert W_{hi} le réseau $\mathfrak{B}_{hi}$ et le réseau gén. $\mathfrak{M}_{hi}$ d'après 25.1 en y posant $q = h$. Choisissons le réseau $\mathfrak{z}$ de manière qu'il soit un affinement de $\mathfrak{Z}$, ainsi que de chaque $\mathfrak{B}_{hi}$ et que chaque sommet intérieur de $\mathfrak{z}$ fasse partie d'un sommet de chaque $\mathfrak{M}_{hi}$. Associons à $\mathfrak{z}$ un réseau fermé t selon 8. D'après 25.2 il existe des $(h, \mathfrak{z})$-simplexes intérieurs $\varrho_i^h = (z_{hi0}, z_{hi1}, \cdots, z_{hih})$ $(0 \leqq h \leqq p+1, 1 \leqq i \leqq \alpha_h)$ tels que 1° $z_{hi\lambda} \subset W_{hi}$ $(0 \leqq \lambda \leqq h)$; 2° il existe une famille parfaitement complète χ de réseaux commodes par rapport à $\mathfrak{z}+\mathfrak{t}$ et tels que $k^{n-h}(\varrho_i^h, \mathfrak{U}) \neq 0$ pour $0 \leqq h \leqq p+1$, $1 \leqq i \leqq \alpha_h$, $\mathfrak{U} \in \chi$. Posons $\sigma_i^h = (Z_{hi0}, Z_{hi1}, \cdots, Z_{hih})$. Les ensembles $z_{hi\lambda}$ sont évidemment distincts les uns des autres et on a $z_{hi\lambda} \subset W_{hi} \subset \mathfrak{F}(\sigma_i^h) \subset Z_{hi\lambda}$. On peut donc choisir $\pi = Pr.(\mathfrak{z}, \mathfrak{Z})$ de manière que

$$\pi z_{hi\lambda} = Z_{hi\lambda} \quad \text{pour} \quad 0 \leqq h \leqq p+1, \quad 1 \leqq i \leqq \alpha_h, \quad 0 \leqq \lambda \leqq h.$$

Déterminons le réseau fermé $\mathfrak{T}$ attaché à $\mathfrak{Z}$ d'après 24. D'après 24, (3) on a pour chaque $\mathfrak{U} \in \chi$ et pour $0 \leqq h \leqq p+1$, $1 \leqq i \leqq \alpha_h$

$$(*) \qquad K^{n-h}(\sigma_i^h, \mathfrak{U}) = \sum_{\nu} k^{n-h}(\tau_\nu^h, \mathfrak{U}) \mod \overline{R - R_0},$$

ν parcourant tous les indices $(1 \leqq \nu \leqq \gamma_h)$ tels que $\pi \tau_\nu^h = \sigma_i^h$. En particulier la somme à droite dans (*) contient le terme $k^{n-h}(\varrho_i^h, \mathfrak{U})$, puisque évidemment $\pi \varrho_i^h = \sigma_i^h$. La somme à droite dans (*) ne peut être égale à zéro mod $\overline{R - R_0}$ que si chaque terme est $= 0 \mod \overline{R - R_0}$, car chaque simplexe de la chaîne $k^{n-h}(\tau_\nu^h, \mathfrak{U})$ est d'espèce τ_ν^h relativement à $\mathfrak{z}+\mathfrak{t}$ de manière que deux termes différents de notre somme n'ont aucun simplexe en commun. Donc l'égalité $K^{n-h}(\sigma_i^h, \mathfrak{U}) = 0$ entraînerait en particulier que $k^{n-h}(\varrho_i^h, \mathfrak{U}) = 0 \mod \overline{R - R_0}$. Or nous savons que $k^{n-h}(\varrho_\nu^h, \mathfrak{U}) \neq 0$. Il suffit donc de montrer que, si le réseau $\mathfrak{U} \in \chi$ est suffisamment fin, un simplexe de la chaîne $k^{n-h}(\varrho_i^h, \mathfrak{U})$ ne peut pas être situé dans $\overline{R - R_0}$. Or on a, d'après 8, $R_0 \supset R - G$, d'où $\overline{R - R_0} \subset \overline{G}$. D'autre part, on a $t(\varrho_i^h) \subset z_{hi0} \subset W_{hi} \subset R - \overline{G}$. Donc $t(\varrho_i^h)$ et $\overline{R - R_0}$ sont deux ensembles fermés disjoints, de manière que, si le réseau $\mathfrak{U}$ est suffisamment fin, un $\mathfrak{U}$-simplexe ne peut être situé simultanément dans $t(\varrho_i^h)$ et dans $\overline{R - R_0}$. Or chaque simplexe de la chaîne $k^{n-h}(\varrho_i^h, \mathfrak{U})$ est situé dans $t(\varrho_i^h)$ d'après 19.3.

26. *Soit* $0 \leqq p \leqq n$. *Soit* $\mathfrak{Z} \in \Xi$ (v. 23). *Déterminons le réseau fermé* $\mathfrak{T}$ *d'après 25. Soit A un sous-ensemble bicompact de R. Il existe un réseau commode* $\mathfrak{U}_0$ *jouissant de la propriété suivante : Si les nombres* $r_i \in \mathfrak{R}$ *sont tels que* $\sum_{i=1}^{\alpha_p} r_i K^{n-p}(\sigma_i^p, \mathfrak{U}_0)$ *est un* $(n-p, \mathfrak{U}_0)$*-cycle mod A* $\overline{R - R_0}$ *homologue à zéro dans A, alors* $\sum_{i=1}^{\alpha_p} r_i K^{n-p}(\sigma_i^p, \mathfrak{U})$ *est un* $(n-p, \mathfrak{U})$*-cycle mod A* $\overline{R - R_0}$ *homologue à zéro dans A pour chaque réseau commode* $\mathfrak{U}$ *qui est un affinement de* $\mathfrak{U}_0$.

Démonstration. Soient $\mathfrak{U}$ et $\mathfrak{U}'$ deux réseaux commodes dont le second un affinement du premier. D'après 19.2 et 19.1, 1° la relation $K^{n-p}(\sigma_i^h, \mathfrak{U}') \subset A$ entraîne $K^{n-p}(\sigma_i^h, \mathfrak{U}) \subset A$. On en déduit sans peine qu'il existe un réseau commode $\mathfrak{U}_1$ tel que $K^{n-p}(\sigma_i^h, \mathfrak{U}_1) \subset A$ entraîne $K^{n-p}(\sigma_i^h, \mathfrak{U}) \subset A$ pour chaque réseau commode $\mathfrak{U}$. Si l'on numérote convenablement les σ_i^p, on a un nombre α_h' $(0 \leqq \alpha_h' \leqq \alpha_h)$ tel que $K^{n-h}(\sigma_i^h, \mathfrak{U}_1) \subset A$ si et seulement si $1 \leqq i \leqq \alpha_h'$. D'après 25 on peut supposer que $K^{n-h}(\sigma_i^h, \mathfrak{U}) \not\equiv 0 \bmod A\, \overline{R - R_0}$ pour $0 \leqq h \leqq \min(n, p+1)$, $1 \leqq i \leqq \alpha_h'$ dans chaque affinement commode $\mathfrak{U}$ de $\mathfrak{U}_1$. Il en résulte (v. 19.1, 3°) que si

$$(*) \qquad \sum_{i=1}^{\alpha_p'} r_i K^{n-p}(\sigma_i^p, \mathfrak{U}) \to 0 \mod A\, \overline{R - R_0}$$

pour $\mathfrak{U} = \mathfrak{U}_1$, la même relation a lieu pour chaque affinement commode $\mathfrak{U}$ de $\mathfrak{U}_1$. Les chaînes (*) forment un module de rang fini $(\leqq \alpha_p')$. Désignons par $M_p(\mathfrak{U})$ le sous-module du module défini par la condition

$$\sum_{i=1}^{\alpha_p'} r_i K^{n-p}(\sigma_i^p, \mathfrak{U}) \sim 0 \mod A\, \overline{R - R_0} \text{ dans } A$$

et soit $\varrho_p(\mathfrak{U})$ le rang de $M_p(\mathfrak{U})$. Il existe un affinement commode $\mathfrak{U}_0$ de $\mathfrak{U}_1$ tel que la valeur de $\varrho_p(\mathfrak{U})$ pour $\mathfrak{U} = \mathfrak{U}_0$ soit un minimum. On voit sans peine que le réseau commode $\mathfrak{U}_0$ jouit de la propriété voulue.

IV.

27. Axiome H^k $(0 \leqq k \leqq n-2)$: *Pour chaque entourage P d'un point* $a \in R - S$ *il existe un entourage* $P_1 \subset P$ *de a jouissant de la propriété suivante : A chaque entourage* $P_2 \subset P_1$ *de a on peut attacher un entourage* $P_3 \subset P_2$ *de a de manière que : Si* Γ^k *est un (k, R)-cycle dans* $\overline{P_1 - P_2}$ *et que l'on ait* $\Gamma^k \sim 0$ *dans* $\overline{P_1}$, *on a aussi* $\Gamma^k \sim 0$ *dans* $\overline{P - P_3}$.

27.1. D'après 4.2 on déduit de l'axiome H^k : *Pour chaque réseau gén.* $\mathfrak{P}$ *il existe un affinement* $\mathfrak{P}_1$ *et une projection* $\pi' = Pr.(\mathfrak{P}_1, \mathfrak{P})$ *jouissant de la propriété suivante : A chaque affinement* $\mathfrak{P}_2$ *de* $\mathfrak{P}_1$ *on peut attacher un affinement* $\mathfrak{P}_3$ *de* $\mathfrak{P}_2$ *ainsi qu'une projection* $\pi'' = Pr.(\mathfrak{P}_3, \mathfrak{P}_2)$ *de manière*

que: Si $P_3 \in \mathfrak{P}_3$, $\pi'' P_3 \subset P_1 \in \mathfrak{P}_1$, *si* Γ^k *est un* (k, R)*-cycle dans* $\bar{P}_1 - \pi'' P_3$, *si* $\Gamma^k \sim 0$ *dans* $\bar{P}_1$, *alors* $\Gamma^k \sim 0$ *dans* $\pi' P_1 - P_3$.

28. Dans tout ce Chapitre, supposons donnée une valeur fixe de p $(1 \leqq p \leqq n)$. Outre les axiomes des Chap. I et II, on suppose la validité des axiomes G^k pour $n-p \leqq k \leqq n-1$ (v. 21) et celle des axiomes H^k pour max. $(0, n-p-1) \leqq k \leqq n-2$.

29. Soit $\mathfrak{P}_1^p$ un réseau gén. donné.

Déterminons un affinement $\mathfrak{Q}^{p-1}$ de $\mathfrak{P}_1^p$ jouissant de la propriété du n° 4.4. Ensuite déterminons un affinement $\mathfrak{P}^{p-1}$ de $\mathfrak{Q}^{p-1}$ ainsi qu'une projection $\pi_{p-1} = Pr.(\mathfrak{P}^{p-1}, \mathfrak{Q}^{p-1})$ d'après 21.1, en y posant $k = n-1$. Enfin déterminons un affinement $\mathfrak{P}_1^{p-1}$ de $\mathfrak{P}^{p-1}$ ainsi qu'une projection $\pi'_{p-1} = Pr.(\mathfrak{P}_1^{p-1}, \mathfrak{P}^{p-1})$ d'après 27.1, en y posant $k = n-2$[41]. Supposons généralement que, pour une certaine valeur de h $(0 \leqq h \leqq p-2)$, on ait déjà défini les réseaux gén. $\mathfrak{Q}^{h+1}$, $\mathfrak{P}^{h+1}$, $\mathfrak{P}_1^{h+1}$. Déterminons un affinement $\mathfrak{Q}^h$ de $\mathfrak{P}_1^{h+1}$ jouissant de la propriété du n° 4.4. Ensuite déterminons un affinement $\mathfrak{P}^h$ de $\mathfrak{Q}^h$ ainsi qu'une projection $\pi_h = Pr.(\mathfrak{P}^h, \mathfrak{Q}^h)$ d'après 21.1 en y posant $k = n-p+h$. Enfin déterminons un affinement $\mathfrak{P}_1^{h+1}$ de $\mathfrak{P}^{h+1}$ ainsi qu'une projection $\pi'_h = Pr.(\mathfrak{P}_1^h, \mathfrak{P}^h)$ d'après 27.1, en y posant $k = n-p+h-1$[42]. En procédant de cette manière, on construit de proche en proche les réseaux gén. $\mathfrak{Q}^h$, $\mathfrak{P}^h$, $\mathfrak{P}_1^h$ pour $0 \leqq h \leqq p-1$.

Déterminons un affinement $\mathfrak{P}_2^0$ de $\mathfrak{P}_1^0$ jouissant de la propriété du n° 4.3. Ensuite d'après 27.1, où l'on remplace $\mathfrak{P}$, $\mathfrak{P}_1$, $\mathfrak{P}_2$, π', k resp. par $\mathfrak{P}^0$, $\mathfrak{P}_1^0$, $\mathfrak{P}_2^0$, π'_0, $n-p-1$ déterminons un affinement $\mathfrak{P}_3^0$ de $\mathfrak{P}_2^0$ ainsi qu'une projection $\pi''_0 = Pr.(\mathfrak{P}_3^0, \mathfrak{P}_2^0)$[43]. Enfin déterminons un affinement $\mathfrak{P}_4^0$ de $\mathfrak{P}_3^0$ jouissant de la propriété du n° 4.3. Supposons généralement que, pour une certaine valeur de h $(0 \leqq h \leqq p-2)$ on ait déjà défini les réseaux gén. $\mathfrak{P}_2^h$, $\mathfrak{P}_3^h$, $\mathfrak{P}_4^h$. Déterminons un affinement $\mathfrak{P}_2^{h+1}$ de $\mathfrak{P}_4^h$ jouissant de la propriété du n° 4.4. Ensuite d'après 27.1, où l'on remplace $\mathfrak{P}$, $\mathfrak{P}_1$, $\mathfrak{P}_2$, π', k resp. par $\mathfrak{P}^{h+1}$, $\mathfrak{P}_1^{h+1}$, $\mathfrak{P}_2^{h+1}$, π'_{h+1}, $n-p+h$, déterminons un affinement $\mathfrak{P}_3^{h+1}$ de $\mathfrak{P}_2^{h+1}$ ainsi qu'une projection $\pi''_{h+1} = Pr.(\mathfrak{P}_3^{h+1}, \mathfrak{P}_2^{h+1})$. Enfin déterminons un affinement $\mathfrak{P}_4^{h+1}$ de $\mathfrak{P}_3^{h+1}$ jouissant de la propriété du n° 4.3. En procédant de cette manière, on construit de proche en proche les réseaux gén. $\mathfrak{P}_2^h$, $\mathfrak{P}_3^h$, $\mathfrak{P}_4^h$ $(0 \leqq h \leqq p-1)$.

Déterminons encore un affinement $\mathfrak{P}_2^p$ de $\mathfrak{P}_4^{p-1}$ jouissant de la propriété du n° 4.4. Ensuite déterminons un affinement $\mathfrak{P}_3^p$ de $\mathfrak{P}_2^p$ ainsi qu'une projection $\pi''_p = Pr.(\mathfrak{P}_3^p, \mathfrak{P}_2^p)$ jouissant de la propriété du n° 14.2; on peut

[41] Si $n = 1$ et par suite $p = 1$, on détermine l'affinement $\mathfrak{P}_1^0$ de $\mathfrak{P}^0$ et la projection π'_0 de manière que $\pi'_0 P_1^0 \neq P_1^0$ pour chaque $P_1^0 \in \mathfrak{P}_1^0$.

[42] Si $p = n$ et $h = 0$, on détermine l'affinement $\mathfrak{P}_1^0$ de $\mathfrak{P}^0$ et la projection π'_0 de manière que $\pi'_0 P_1^0 \neq P_1^0$ pour chaque $P_1^0 \in \mathfrak{P}_1^0$.

[43] Si $p = n$, on pose $\mathfrak{P}_3^0 = \mathfrak{P}_2^0$ et π''_0 est l'identité.

supposer que l'affinement $\mathfrak{P}_3^p$ de $\mathfrak{P}_2^p$ jouisse de la propriété du n° 4.3. Enfin déterminons un affinement $\mathfrak{P}_4^p$ de $\mathfrak{P}_3^p$ jouissant de la propriété du n° 4.4 ainsi qu'un affinement $\mathfrak{P}_5^p$ de $\mathfrak{P}_4^p$ jouissant de la même propriété.

30. Reprenons les notations du n° 8 en supposant que chaque sommet intérieur du réseau $\mathfrak{Z}$ fasse partie d'un sommet du réseau gén. $\mathfrak{P}_5^p$ (et donc aussi d'un sommet du réseau gén. $\mathfrak{P}_1^p$ donné a priori).

Pour abréger, disons que σ_j^0 $(1 \leqq j \leqq \alpha_0)$ est *contigu* à σ_i^h $(0 \leqq h \leqq p,$ $1 \leqq i \leqq \alpha_h)$, si σ_j^0 rencontre un sommet de σ_i^h (en particulier si σ_j^0 est lui-même un sommet de σ_i^h).

Comme le réseau gén. $\mathfrak{P}_4^p$ $(\mathfrak{P}_5^p)$ jouit par rapport à $\mathfrak{P}_3^p$ $(\mathfrak{P}_4^p)$ de la propriété du n° 4.4, on peut attacher à chaque σ_i^p $(1 \leqq i \leqq \alpha_p)$ un sommet $P_3^p(\sigma_i^p)$ de $\mathfrak{P}_3^p$ de manière que chaque sommet intérieur de $\mathfrak{Z}$ contigu à σ_i^p fasse partie de $P_3^p(\sigma_i^p)$. Posons $P_2^p(\sigma_i^p) = \pi'' P_3^p(\sigma_i^p)$ $(1 \leqq i \leqq \alpha_p)$. Supposons généralement que pour une certaine valeur de h $(0 \leqq h \leqq p-1)$, on ait déjà attaché à chaque σ_i^{h+1} $(1 \leqq i \leqq \alpha_{h+1})$ un sommet $P_2^{h+1}(\sigma_i^{h+1})$ du réseau gén. $\mathfrak{P}_2^{h+1}$ tel que chaque sommet intérieur de $\mathfrak{Z}$ contigu à σ_i^{h+1} en soit un sous-ensemble. Soit donnée une valeur de i $(1 \leqq i \leqq \alpha_h)$. Distinguons deux cas. Premièrement, si le $\mathfrak{Z}$-simplexe σ_i^h n'est pas une face d'aucun $(h+1, \mathfrak{Z})$-simplexe intérieur, choisissons un sommet $P_4^h(\sigma_i^h)$ de $\mathfrak{P}_4^h$ de manière que chaque sommet intérieur de $\mathfrak{Z}$ contigu à σ_i^h en soit un sous-ensemble; ceci est possible, car le réseau gén. $\mathfrak{P}_3^{p+1}$ est un affinement de $\mathfrak{P}_4^h$. En second lieu supposons qu'il existe des valeurs de j $(1 \leqq j \leqq \alpha_{h+1})$ telles que $\eta_{ji}^h \neq 0$. Dans ce cas choisissons le sommet $P_4^h(\sigma_i^h)$ de $\mathfrak{P}_4^h$ de manière qu'on ait $P_4^h(\sigma_i^h) \supset P_2^{h+1}(\sigma_j^{h+1})$ pour toutes les valeurs de j telles que $\eta_{ji}^h \neq 0$; ceci est possible, car le réseau gén. $\mathfrak{P}_2^{h+1}$ jouit par rapport à $\mathfrak{P}_4^h$ de la propriété du n° 4.4. Dans les deux cas le sommet $P_4^h(\sigma_i^h)$ est donc construit de manière que chaque sommet intérieur de $\mathfrak{Z}$ contigu à σ_i^h en soit un sous-ensemble. Comme le réseau gén. $\mathfrak{P}_4^h$ jouit par rapport à $\mathfrak{P}_3^h$ de la propriété du n° 4.3, il existe un sommet $P_3^h(\sigma_i^h)$ $(1 \leqq i \leqq \alpha_h)$ tel que $P_3^h(\sigma_i^h) \supseteq P_4^h(\sigma_i^h)$. Posons encore $P_2^h(\sigma_i^h) = \pi_h'' P_3^h(\sigma_i^h)$. En procédant de cette manière, on attache de proche en proche à chaque $(h, \mathfrak{Z})$-simplexe intérieur σ_i^h $(0 \leqq h \leqq p-1, 1 \leqq i \leqq \alpha_h)$ des sommets $P_4^h(\sigma_i^h)$, $P_3^h(\sigma_i^h)$, $P_2^h(\sigma_i^h)$ resp. de $\mathfrak{P}_4^h$, $\mathfrak{P}_3^h$, $\mathfrak{P}_2^h$.

Le réseau gén. $\mathfrak{P}_2^0$ jouissant par rapport à $\mathfrak{P}_1^0$ de la propriété du n° 4.3, il existe (pour $1 \leqq i \leqq \alpha_0$) un sommet $P_1^0(\sigma_i^0)$ de $\mathfrak{P}_1^0$ tel que $P_1^0(\sigma_i^0) \supseteq P_2^0(\sigma_i^0)$. Posons $P^0(\sigma_i^0) = \pi_0' P_1^0(\sigma_i^0)$, $Q^0(\sigma_i^0) = \pi_0 P^0(\sigma_i^0)$. Supposons généralement que, pour une certaine valeur de h $(0 \leqq h \leqq p-2)$, on ait déjà attaché à chaque σ_i^h $(1 \leqq i \leqq \alpha_h)$ un sommet $Q^h(\sigma_i^h)$ de $\mathfrak{Q}^h$ tel que chaque sommet intérieur de $\mathfrak{Z}$ contigu à σ_i^h en soit un sous-ensemble. Soit donnée une valeur de i $(1 \leqq i \leqq \alpha_{h+1})$. Comme le réseau gén. $\mathfrak{Q}^h$ jouit par rapport

à $\mathfrak{P}_1^{h+1}$ de la propriété du n° 4.4, il existe un sommet $P_1^{h+1}(\sigma_i^{h+1})$ du réseau gén. $\mathfrak{P}_1^{h+1}$ tel que $P_1^{h+1}(\sigma_i^{h+1}) \supset Q_j^h(\sigma_j^h)$ pour chaque valeur de j $(1 \leqq j \leqq \alpha_h)$ telle que $\eta_{ij}^h \neq 0$. Posons encore

$$P^{h+1}(\sigma_i^{h+1}) = \pi'_{h+1} P_1^{h+1}(\sigma_i^{h+1}), \qquad Q^{h+1}(\sigma_i^{h+1}) = \pi_{h+1} P^{h+1}(\sigma_i^{h+1})$$
$$(1 \leqq i \leqq \alpha_{h+1}).$$

En procédant de cette manière, on attache de proche en proche à chaque $(h, \mathfrak{Z})$-simplexe intérieur σ_i^h $(0 \leqq h \leqq p-1,\ 1 \leqq i \leqq \alpha_h)$ des sommets $P_1^h(\sigma_i^h)$, $P^h(\sigma_i^h)$, $Q^h(\sigma_i^h)$ resp. de $\mathfrak{P}_1^h$, $\mathfrak{P}^h$, $\mathfrak{Q}^h$.

Comme le réseau gén. $\mathfrak{Q}^{p-1}$ jouit par rapport à $\mathfrak{P}_1^p$ de la propriété du n° 4.4, on peut enfin attacher à chaque σ_i^p $(1 \leqq i \leqq \alpha_p)$ un sommet $P_1^p(\sigma_i^p)$ de $\mathfrak{P}_1^p$ de manière que $P_1^p(\sigma_i^p) \supset Q^{p-1}(\sigma_j^{p-1})$ pour chaque valeur de j $(1 \leqq j \leqq \alpha_{p-1})$ telle que $\eta_{ij}^{p-1} \neq 0$.

31. Gardons les conventions des n^{os} 8, 29 et 30. *Soit A un sous-ensemble bicompact de R; soit B un entourage de A. On peut choisir le réseau gén. $\mathfrak{P}_1^p$ (n° 29) de manière que ($\mathfrak{Z}$ satisfaisant à la condition énoncée au commencement du n° 30) on ait la propriété suivante: Soit $\mathfrak{U}$ un réseau commode (v. 18.1). Soit $C^{n-p}(\mathfrak{U})$ un $(n-p, \mathfrak{U})$-cycle mod SA dans A essentiel. Il existe une $(n-p, \mathfrak{U})$-chaîne élémentaire $D^{n-p}(\mathfrak{U})$ dans $\overline{B}$ telle que $C^{n-p}(\mathfrak{U}) \sim D^{n-p}(\mathfrak{U})$ mod $\overline{B}$ $\overline{R-R_0}$ dans $\overline{B}$.*

En fait il suffit de choisir $\mathfrak{P}_1^p$ de manière que chacun de ses sommets P' qui rencontre un sommet P'' de $\mathfrak{P}_1^p$ tel que $\overline{P''}\,A \neq 0$, fasse partie de B. Un tel choix de $\mathfrak{P}_1^p$ est évidemment possible (v. 4.3 et 4.4). Tous les réseaux introduits dans 29 étant des affinements de $\mathfrak{P}_1^p$, ils possèdent aussi la même propriété.

La démonstration fera l'objet des n^{os} 32—42.5.

32. En tenant compte de 19.2, on voit sans peine qu'il suffit de donner la démonstration en supposant que le réseau commode $\mathfrak{U}$ soit suffisamment fin. Or (v. 29) l'affinement $\mathfrak{P}_4^h$ du réseau gén. $\mathfrak{P}_3^h$ $(0 \leqq h \leqq p-1)$ ainsi que l'affinement $\mathfrak{P}_2^0(\mathfrak{P}_3^p)$ de $\mathfrak{P}_1^0(\mathfrak{P}_2^p)$ jouit de la propriété du n° 4.3. On peut donc supposer que le réseau $\mathfrak{U}$ soit tel que: 1° pour $0 \leqq h \leqq p-1$, $1 \leqq i \leqq \alpha_h$ chaque sommet de $\mathfrak{U}$ qui rencontre $\overline{P_4^h(\sigma_i^h)}$ fasse partie de $P_3^h(\sigma_i^h)$; 2° pour $1 \leqq i \leqq \alpha_0$ (α_p) chaque sommet de $\mathfrak{U}$ qui rencontre $\overline{P_2^0(\sigma_i^0)}\,[P_3^p(\sigma_i^p)]$ fasse partie de $P_1^0(\sigma_i^0)\,[P_2^p(\sigma_i^p)]$. De plus on peut supposer qu'aucun sommet de $\mathfrak{U}$ ne rencontre simultanément S et un des ensembles $\overline{P_1^0(\sigma_i^0)}$ $(1 \leqq i \leqq \alpha_0)$. Ensuite on peut supposer que, si U', U'' sont deux sommets de $\mathfrak{U}$ tels que $U' R_0 \neq 0$, $U'' S \neq 0$, on ait $U' U'' = 0$. Puis, on peut supposer que si un sommet de $\mathfrak{U}$ rencontre simultanément A et un des ensembles $\overline{P_1^0(\sigma_i^0)}$ $(1 \leqq i \leqq \alpha_0)$, on ait $A\ \overline{P_1^0(\sigma_i^0)} \neq 0$. En outre on peut (v. 17.5) supposer que $G^n(\mathfrak{U}) \not\sim 0 \bmod R - P_3^p(\sigma_i^p)$ $(1 \leqq i \leqq \alpha_p)$. Encore,

comme (v. 30) chaque sommet intérieur de $\mathfrak{Z}$ contigu (v. 30) de σ_i^h $(0 \leqq h \leqq p-1,\ 1 \leqq i \leqq \alpha_h)$ est un sous-ensemble de l'ensemble ouvert $P_4^h(\sigma_i^h)$, on peut supposer que, pour $0 \leqq h \leqq p-1$, $0 \leqq k \leqq p-1$, $1 \leqq j \leqq \alpha_k$, les indices i, j étant tels que les $\mathfrak{Z}$-simplexes σ_i^h et σ_j^k soient des faces d'un certain $(l, \mathfrak{Z})$-simplexe intérieur $(l \geq h,\ l \geq k)$, chaque sommet de $\mathfrak{U}$ qui rencontre $T(\sigma_j^k)$ fasse partie de $P_4^h(\sigma_i^h)$. En effet $T(\sigma_j^k)$ est un sous-ensemble bicompact de chaque sommet de σ_j^k; or chaque sommet de σ_j^k étant contigu à σ_i^h, il en résulte que $T(\sigma_j^k) \subset P_4^h(\sigma_i^h)$. Pareillement on voit qu'on peut supposer que, si σ_r^0 est un sommet de σ_i^p, chaque sommet de $\mathfrak{U}$ qui rencontre T_r est un sous-ensemble de $P_8^p(\sigma_i^p)$.

Il est évident que ces propriétés de $\mathfrak{U}$ appartiennent aussi à chaque affinement de $\mathfrak{U}$.

33. Posons $\mathfrak{U}_0 = \mathfrak{U}$. Soit $\mathfrak{V}$ un affinement de $\mathfrak{U}$ régulier par rapport à $\overline{B}\ \overline{R-R_0}$. Soit $\mathfrak{U}_1$ un réseau commode tel que 1° $\mathfrak{U}_1$ soit un affinement de $\mathfrak{V}$; 2° $\mathfrak{U}_1$ est un affinement de $\mathfrak{U}$ normal par rapport aux cycles mod $R-P_2^p(\sigma_i^p)$, pour $1 \leqq i \leqq \alpha_p$. Soit $\pi_{10} = Pr.(\mathfrak{V}, \mathfrak{U})$, $\pi_{20} = Pr.(\mathfrak{U}_1, \mathfrak{V})$, $\pi_0^* = \pi_{10}\,\pi_{20} = Pr.(\mathfrak{U}_1, \mathfrak{U})$. Supposons généralement que, pour une certaine valeur de h $(0 \leqq h \leqq p-1)$, on ait déjà défini le réseau $\mathfrak{U}_{2p-2h-1}$. Alors, soit $\mathfrak{U}_{2p-2h}$ un réseau commode qui soit un affinement de $\mathfrak{U}_{2p-2h-1}$ normal par rapport aux cycles dans $\overline{P^h(\sigma_i^h)}$ pour chaque i $(1 \leqq i \leqq \alpha_h)$ et soit $\pi_{2p-2h-1}^* = Pr.(\mathfrak{U}_{2p-2h}, \mathfrak{U}_{2p-2h-1})$; ensuite, soit $\mathfrak{U}_{2p-2h+1}$ un réseau commode qui soit un affinement de $\mathfrak{U}_{2p-2h}$ normal par rapport aux cycles dans $\overline{P_1^h(\sigma_i^h)} - P_2^h(\sigma_i^h)$ pour chaque $i (1 \leqq i \leqq \alpha_h)$ et soit $\pi_{2p-2h}^* = Pr.(\mathfrak{U}_{2p-2h+1}, \mathfrak{U}_{2p-2h})$. En procédant de cette manière on construit de proche en proche les réseaux commodes $\mathfrak{U}_h$ pour $1 \leqq h \leqq 2p+1$.

33.1. Le cycle $C^{n-p}(\mathfrak{U})$ (v. 31) étant essentiel, il existe un $(n-p, \mathfrak{U}_{2p+1})$-cycle $C^{n-p}(\mathfrak{U}_{2p+1})$ mod S dans A tel que $C^{n-p}(\mathfrak{U}) \sim \pi^*\,\pi_1^*\,\pi_2^* \cdots \pi_{2p}^*\,C^{n-p}(\mathfrak{U}_{2p+1})$ mod S dans A. Il est évident qu'il suffit de démontrer le théorème du n° 31 en supposant que $C^{n-p}(\mathfrak{U}) = \pi^*\,\pi_1^*\,\pi_2^* \cdots \pi_{2p}^*\,C^{n-p}(\mathfrak{U}_{2p+1})$. Définissons de proche en proche les chaînes $C^{n-p}(\mathfrak{U}_h)$ $(1 \leqq h \leqq 2p)$ en posant $\pi_h^*\,C^{n-p}(\mathfrak{U}_{h+1}) = C^{n-p}(\mathfrak{U}_h)$ $(1 \leqq h \leqq 2p)$; on a alors $\pi^*\,C^{n-p}(\mathfrak{U}_1) = C^{n-p}(\mathfrak{U})$. Généralement, introduisons (pour le n° 34) la convention suivante : Si l'on a défini une certaine $\mathfrak{U}_{h+1}$-chaîne $(0 \leqq h \leqq 2p)$ $E^k(\mathfrak{U}_{h+1})$, on pose $E^k(\mathfrak{U}_h) = \pi_h^*\,E^k(\mathfrak{U}_{h+1})$.

34. Lemme. On peut attacher à chaque $(h, \mathfrak{Z})$-simplexe intérieur σ_i^h $(0 \leqq h \leqq p-1)$ une $(n-p+h+1, \mathfrak{U}_1)$-chaîne $L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_1)$ dans $\overline{B}$ de manière que : 1° pour $1 \leqq i \leqq \alpha_0$, la chaîne

$$FL^{n-p+1}(\sigma_i^0, \mathfrak{U}_1) - C^{n-p}(\mathfrak{U}_1)$$

ne contienne pas des simplexes situés dans $\overline{P_4^0(\sigma_i^0)}$; 2° pour $0 \leqq h \leqq p-2$, $1 \leqq i \leqq \alpha_{h+1}$, la chaîne

$$FL^{n-p+h+2}(\sigma_i^{h+1}, \mathfrak{U}_1) - \sum_{j=1}^{\alpha_h} \eta_{ij}^h L^{n-p+h+1}(\sigma_j^h, \mathfrak{U}_1)$$

ne contienne pas des simplexes situés dans $\overline{P_4^{h+1}(\sigma_i^{h+1})}$; 3° pour $1 \leqq i \leqq \alpha_p$, la chaîne

$$\sum_{j=1}^{\alpha_{p-1}} \eta_{ij}^{p-1} L^n(\sigma_j^{p-1}, \mathfrak{U}_1)$$

soit un (n, R)-cycle mod $R - P_2^p(\sigma_i^p)$.

La démonstration fait l'objet des n^{os} 34.1—34.3.

Remarque. Si on change l'orientation d'un simplexe σ_i^h, on doit évidemment changer le signe de $L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_1)$.

34.1. Pour $1 \leqq i \leqq \alpha_0$, soit $C^{n-p}(\sigma_i^0, \mathfrak{U}_{2p+1})$ la partie de la chaîne $C^{n-p}(\mathfrak{U}_{2p+1})$ dont les simplexes sont situés dans $\overline{P_1^0(\sigma_i^0)}$. Excluons pour le moment le cas où $p = n$. Soit

$$\Gamma^{n-p-1}(\sigma_i^0, \mathfrak{U}_{2p+1}) = FC^{n-p}(\sigma_i^0, \mathfrak{U}_{2p+1}).$$

Or $C^{n-p}(\mathfrak{U}_{2p+1})$ est un cycle mod S; puisque (v. 32) aucun sommet de $\mathfrak{U}$ ne rencontre simultanément S et $\overline{P_1^0(\sigma_i^0)}$, on voit qu'aucun simplexe de $FC^{n-p}(\mathfrak{U}_{2p+1})$ n'est situé dans $\overline{P_1^0(\sigma_i^0)}$. Donc $\Gamma^{n-p-1}(\sigma_i^0, \mathfrak{U}_{2p+1})$ est la partie de la chaîne

(*) $$FC^{n-p}(\sigma_i^0, \mathfrak{U}_{2p+1}) - FC^{n-p}(\mathfrak{U}_{2p+1})$$

dont les simplexes sont situés dans $\overline{P_1^0(\sigma_i^0)}$. Or le noyau de chaque simplexe de la chaîne (*) rencontre $R - P_1^0(\sigma_i^0)$ et par suite (v. 32) ce noyau ne peut rencontrer $\overline{P_2^0(\sigma_i^0)}$; donc[44]

$$\Gamma^{n-p-1}(\sigma_i^0, \mathfrak{U}_k) \subset \overline{P_1^0(\sigma_i^0)} - P_2^0(\sigma_i^0) \qquad (0 \leqq k \leqq 2p+1).$$

Or l'affinement $\mathfrak{U}_{2p+1}$ de $\mathfrak{U}_{2p}$ était (v. 33) normal par rapport aux cycles dans $\overline{P_1^0(\sigma_i^0)} - P_2^0(\sigma_i^0)$; donc $\Gamma^{n-p-1}(\sigma_i^0, \mathfrak{U}_{2p})$ est un $(n-p-1, \mathfrak{U}_{2p})$-cycle *essentiel* dans $\overline{P_1^0(\sigma_i^0)} - P_2^0(\sigma_i^0)$. D'après la définition du réseau gén. $\mathfrak{P}_3^0$ (n° 29) et celle de son sommet $P_3^0(\sigma_i^0)$ (n° 30) il en résulte l'existence d'une $(n-p, \mathfrak{U}_{2p})$-chaîne $D_i^{n-p}(\mathfrak{U}_{2p})$ dans $\overline{P^0(\sigma_i^0)} - P_3^0(\sigma_i^0)$ telle que $C^{n-p}(\sigma_i^0, \mathfrak{U}_{2p}) - D_i^{n-p}(\mathfrak{U}_{2p})$ est un $(n-p, \mathfrak{U}_{2p})$-cycle, situé naturellement dans $\overline{P^0(\sigma_i^0)}$. L'affinement $\mathfrak{U}_{2p}$ de $\mathfrak{U}_{2p-1}$ étant (n° 33) normal par rapport aux cycles dans $\overline{P^0(\sigma_i^0)}$, on voit que $C^{n-p}(\sigma_i^0, \mathfrak{U}_{2p-1}) - D^{n-p}(\sigma_i^0, \mathfrak{U}_{2p-1})$ est un $(n-p, \mathfrak{U}_{2p-1})$-cycle essentiel dans $\overline{P^0(\sigma_i^0)}$. D'après la définition du réseau

[44] Il faut tenir compte de ce que $\Gamma^{n-p-1}(\sigma_i^0, \mathfrak{U}_k) = \pi_k^* \Gamma^{n-p-1}(\sigma_i^0, \mathfrak{U}_{k+1})$ et pareillement dans ce qui suit.

gén. $\mathfrak{P}^0$ (n° 29) et celle de son sommet $P^0(\sigma_i^0)$ (n° 30) il en résulte l'existence d'une $(n-p+1, \mathfrak{U}_{2p-1})$-chaîne $L^{n-p+1}(\sigma_i^0, \mathfrak{U}_{2p-1})$ dans $\overline{Q^0(\sigma_i^0)}$ telle que

$$(*) \qquad L^{n-p+1}(\sigma_i^0, \mathfrak{U}_{2p-1}) \to C^{n-p}(\sigma_i^0, \mathfrak{U}_{2p-1}) - D_i^{n-p}(\mathfrak{U}_{2p-1}).$$

On peut arriver au même résultat dans le cas où $p = n$: D'après la définition du réseau gén. $\mathfrak{P}_1^0$, on a $P^0(\sigma_i^0) - P_1^0(\sigma_i^0) \neq 0$ et par suite $\overline{P^0(\sigma_i^0)} - P_3^0(\sigma_i^0) \neq 0$. Soit donc a_i un point de $\overline{P^0(\sigma_i^0)} - P_3^0(\sigma_i^0)$ et soit U_i un sommet de $\mathfrak{U}_{2p}$ tel que $a_i \in U_i$. Déterminons $r_i \in \mathfrak{R}$ de manière que $I[C^0(\sigma_i^0, \mathfrak{U}_{2p}) - D_i^0(\mathfrak{U}_{2p})] = 0$ (v. [38]), où $D_i^0(\mathfrak{U}_{2p}) = r_i U_i$ est une $(0, \mathfrak{U}_{2p})$-chaîne dans $\overline{P^0(\sigma_i^0)} - P_3^0(\sigma_i^0)$. D'après la définition de $\mathfrak{P}^0$ et de son sommet $P^0(\sigma_i^0)$ il existe une $(1, \mathfrak{U}_{2p-1})$-chaîne $L^1(\sigma_i^0, \mathfrak{U}_{2p-1})$ dans $\overline{Q^0(\sigma_i^0)}$ telle qu'on ait la relation (*) avec $p = n$. Dorénavant, nous pouvons traiter simultanément toutes les valeurs de p $(1 \leqq p \leqq n)$.

Chaque sommet de $\mathfrak{U}$ qui rencontre $\overline{P_4^0(\sigma_i^0)}$ étant (v. 32) contenu dans $P_3^0(\sigma_i^0)$, il résulte de l'inclusion $D_i^{n-p}(\mathfrak{U}) \subset \overline{P^0(\sigma_i^0)} - P_3^0(\sigma_i^0)$ qu'aucun simplexe de $D_i^{n-p}(\mathfrak{U})$ n'est situé dans $\overline{P_4^0(\sigma_i^0)}$; pareillement on voit qu'aucun simplexe de $C^{n-p}(\mathfrak{U}) - C^{n-p}(\sigma_i^0, \mathfrak{U})$ n'est situé dans $\overline{P_4^0(\sigma_i^0)}$. Par suite aucun simplexe de la chaîne

$$(*) \qquad F L^{n-p+1}(\sigma_i^0, \mathfrak{U}_k) - C^{n-p}(\mathfrak{U}_k) \qquad (0 \leqq k \leqq 2p-1)$$

n'est situé dans $\overline{P_4^0(\sigma_i^0)}$.

34.2. Supposons généralement que, pour une certaine valeur de h $(0 \leqq h \leqq p-2)$, on ait déjà attaché à chaque σ_i^h $(1 \leqq i \leqq \alpha_h)$ une $(n-p+h+1, \mathfrak{U}_{2p-2h-1})$-chaîne dans $\overline{Q^h(\sigma_i^h)}$, soit $L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_{2p-2h-1})$, de manière que, pour $1 \leqq i \leqq \alpha_h$, aucun simplexe de la chaîne[45]

$$(**) \quad F L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_k) - \sum_{j=1}^{\alpha_{h-1}} \eta_{ij}^{h-1} L^{n-p+h}(\sigma_j^{h-1}, \mathfrak{U}_k) \quad (0 \leqq k \leqq 2p-2h-1)$$

ne soit situé dans $\overline{P_4^h(\sigma_i^h)}$. Il s'agit d'attacher à chaque σ_i^{h+1} $(1 \leqq i \leqq \alpha_{h+1})$ une $(n-p+h+2, \mathfrak{U}_{2p-2h-3})$-chaîne $L^{n-p+h+2}(\sigma_i^{h+1}, \mathfrak{U}_{2p-2h-3})$ dans $\overline{Q^{h+1}(\sigma_i^{h+1})}$ de manière qu'aucun simplexe de la chaîne

$$F L^{n-p+h+2}(\sigma_i^{h+1}, \mathfrak{U}_k) - \sum_{j=1}^{\alpha_{h-1}} \eta_{ij}^h L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_k) \quad (0 \leqq k \leqq 2p-2h-3)$$

ne soit situé dans $\overline{P_4^{h+1}(\sigma_i^{h+1})}$. Or la chaîne

$$M^{n-p+h+1}(\sigma_i^{h+1}, \mathfrak{U}_{2p-2h-1}) = \sum_{j=1}^{\alpha_h} \eta_{ij}^h L^{n-p+h+1}(\sigma_j^h, \mathfrak{U}_{2p-2h-1})$$

[45] Pour $h = 0$ on doit remplacer la chaîne (**) par (*).

se trouve située (v. 30) dans

$$\sum_{\substack{\eta_{ij}^h \neq 0 \\ 1 \leq j \leq \alpha_h}} \overline{Q^h(\sigma_j^h)} \subset \overline{P_1^{h+1}(\sigma_i^{h+1})}.$$

En outre, comme $F\sigma_i^{h+1} \to 0$, d'où $\sum_{j=1}^{\alpha_h} \sum_{k=1}^{\alpha_{h-1}} \eta_{ij}^h \eta_{jk}^{h-1} = 0$, aucun simplexe de la chaîne

$$FM^{n-p+h+1}(\sigma_i^{h+1}, \mathfrak{U}_{2p-2h-1})$$
$$= \sum_{j=1}^{\alpha_h} \eta_{ij}^h \Big[FL^{n-p+h+1}(\sigma_j^h, \mathfrak{U}_{2p-2h-1}) - \sum_{k=1}^{\alpha_{h-1}} \eta_{jk}^{h-1} L^{n-p+h}(\sigma_k^{h-1}, \mathfrak{U}_{2p-2h-1}) \Big]$$

n'est situé (v. 30) dans

$$\prod_{\substack{\eta_{ij}^h \neq 0 \\ 1 \leq j \leq \alpha_h}} \overline{P_4^h(\sigma_j^h)} \supset P_2^{h+1}(\sigma_i^{h+1})^{46}$$

Donc le cycle $FM^{n-p+h+1}(\sigma_i^{h+1}, \mathfrak{U}_{2p-2h-1})$ est situé dans $\overline{P_1^{h+1}(\sigma_i^{h+1})} - P_2^{h+1}(\sigma_i^{h+1})$. Par suite (v. 33) $FM^{n-p+h+1}(\sigma_i^{h+1}, \mathfrak{U}_{2p-2h-2})$ est un $(n-p+h, \mathfrak{U}_{2p-2h-2})$-cycle *essentiel* dans $\overline{P_1^{h+1}(\sigma_i^{h+1})} - P_{h+2}(\sigma_i^{h+1})$; d'après la définition du réseau gén. $\mathfrak{P}_3^{h+1}$ (n° 29) et celle de son sommet P_3^{h+1} (n° 30) il existe donc une $(n-p+h+1, \mathfrak{U}_{2p-2h-2})$-chaîne dans $\overline{P^{h+1}(\sigma_i^{h+1})} - P_3^{h+1}(\sigma_i^{h+1})$, soit $N_i^{n-p+h+1}(\mathfrak{U}_{2p-2h-2})$, telle que

$$M^{n-p+h+1}(\sigma_i^{h+1}, \mathfrak{U}_{2p-2h-2}) - N_i^{n-p+h+1}(\mathfrak{U}_{2p-2h-2})$$

soit un $(n-p+h+1, \mathfrak{U}_{2p-2h-2})$-cycle, situé naturellement dans $\overline{P^{h+1}(\sigma_i^{h+1})}$. Par suite (v. 33)

$$(*) \qquad M^{n-p+h+1}(\sigma_i^{h+1}, \mathfrak{U}_{2p-2h-3}) - N_i^{n-p+h+1}(\mathfrak{U}_{2p-2h-3})$$

est un $(n-p+h+1, \mathfrak{U}_{2p-2h-3})$-cycle essentiel dans $\overline{P^{h+1}(\sigma_i^{h+1})}$. Il existe donc une $(n-p+h+2, \mathfrak{U}_{2p-2h-3})$-chaîne dans $\overline{Q^{h+1}(\sigma_i^{h+1})}$, soit $L^{n-p+h+2}(\sigma_i^{h+1}, \mathfrak{U}_{2p-2h-3})$, dont $(*)$ est la frontière. Comme le noyau de

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[46] Pour $h=0$ on a $\sum_{j=1}^{\alpha_0} \eta_{ij}^0 = 0$, d'où

$$FM^{n-p+1}(\sigma_i^1, \mathfrak{U}_{2p-1}) = \sum_{j=1}^{\alpha_0} \eta_{ij}^0 [FL^{n-p+1}(\sigma_j^0, \mathfrak{U}_{2p-1}) - C^{n-p}(\mathfrak{U}_{2p-1})],$$

de manière que, ici encore, aucun simplexe de $FM^{n-p+1}(\sigma_i^1, \mathfrak{U}_{2p-1})$ n'est situé dans

$$\prod_{\substack{\eta_{ij}^0 \neq 0 \\ 1 \leq j \leq \alpha_0}} \overline{P_4^0(\sigma_j^0)} \supset P_2^1(\sigma_i^1).$$

chaque simplexe de la chaîne $N^{n-p+h+1}(\sigma_i^{h+1}, \mathfrak{U}_{2p-2h-3})$ rencontre $P^{h+1}(\sigma_i^{h+1}) - P_3^{h+1}(\sigma_i^{h+1})$ et comme (v. 32) chaque sommet de $\mathfrak{U}$ rencontrant $\overline{P_4^{h+1}(\sigma_i^{h+1})}$ fait partie de $P_3^{h+1}(\sigma_i^{h+1})$, on voit qu'aucun simplexe de la chaîne $N^{n-p+h+1}(\sigma_i^{h+1}, \mathfrak{U}_k)$ $(0 \leqq k \leqq 2p-2h-3)$ n'est situé dans $\overline{P_4^{h+1}(\sigma_i^{h+1})}$. Or

$$-N^{n-p+h+1}(\sigma_i^{h+1}, \mathfrak{U}_k) = FL^{n-p+h+2}(\sigma_i^{h+1}, \mathfrak{U}_k) - \sum_{j=1}^{\alpha_h} \eta_{ij}^h L^{n-p+h+1}(\sigma_j^h, \mathfrak{U}_k)$$
$$(0 \leqq k \leqq 2p-2h-3).$$

34.3. En procédant de cette manière, on finit par attacher à chaque σ_i^{p-1} $(1 \leqq i \leqq \alpha_{p-1})$ une $(n, \mathfrak{U}_1)$-chaîne $L^n(\sigma_i^{p-1}, \mathfrak{U}_1)$ dans $\overline{Q^{p-1}(\sigma_i^{p-1})}$ de manière qu'aucun simplexe de la chaîne[47]

$$FL^n(\sigma_i^{p-1}, \mathfrak{U}_k) - \sum_{j=1}^{\alpha_{p-2}} \eta_{ij}^{p-2} L^{n-1}(\sigma_j^{p-2}, \mathfrak{U}_k) \qquad (k = 0 \text{ ou } 1)$$

ne soit situé dans $\overline{P_4^{p-1}(\sigma_i^{p-1})}$. Alors la chaîne

$$M^n(\sigma_i^p, \mathfrak{U}_1) = \sum_{j=1}^{\alpha_{p-1}} \eta_{ij}^{p-1} L^n(\sigma_j^{p-1}, \mathfrak{U}_1)$$

est située (v. 30) dans

$$\sum_{\substack{\eta_{ij}^{p-1} \neq 0 \\ 1 \leqq j \leqq \alpha_{p-1}}} \overline{Q^{p-1}(\sigma_j^{p-1})} \subset \overline{P_1^p(\sigma_i^p)}.$$

En outre aucun simplexe de la chaîne

$$FM^n(\sigma_i^p, \mathfrak{U}_1) = \sum_{j=1}^{\alpha_{p-1}} \eta_{ij}^{p-1} \left[FL^n(\sigma_j^{p-1}, \mathfrak{U}_1) - \sum_{k=1}^{\alpha_{p-2}} \eta_{jk}^{p-2} L^{n-p}(\sigma_k^{p-1}, \mathfrak{U}_1) \right]$$

n'est situé (v. 30) dans

$$\prod_{\substack{\eta_{ij}^{p-1} \neq 0 \\ 1 \leqq j \leqq \alpha_{p-1}}} \overline{P_4^{p-1}(\sigma_j^{p-1})} \supset \overline{P_2^p(\sigma_i^p)}.$$

Donc $M^n(\sigma_i^p, \mathfrak{U}_1)$ est un $(n, \mathfrak{U}_1)$-cycle mod $\overline{P_1^p(\sigma_i^p)} - P_2^p(\sigma_i^p)$ dans $\overline{P_1^p(\sigma_i^p)}$.

34.4. Si l'indice i $(1 \leqq i \leqq \alpha_0)$ est tel que $A\overline{P_1^0(\sigma_i^0)} = 0$, on voit sans peine (v. 32) que $C^{n-p}(\sigma_i^0, \mathfrak{U}_{2p-1}) = 0$; on peut dans ce cas supposer que $L^{n-p+1}(\sigma_i^0, \mathfrak{U}_{2p-1}) = 0$. Plus généralement, si l'indice i $(1 \leqq i \leqq \alpha_h$, $0 \leqq h \leqq p-1)$ est tel que $A\overline{P_1^0(\sigma_j^0)} = 0$ pour chaque sommet σ_j^0 de σ_i^h, on peut supposer que $L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_{2p-2h-1}) = 0$. On n'a donc

[47] Dans le cas $p = 1$ on doit remplacer $\sum_{j=1}^{\alpha_{p-2}} \eta_{ij}^{p-2} L^{n-1}(\sigma_j^{p-2}, \mathfrak{U}_k)$ par $C^{n-p}(\mathfrak{U}_k)$.

$L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_{2p-2h-1}) \neq 0$ que dans le cas où $A\overline{P_1^0(\sigma_j^0)} \neq 0$ pour un sommet σ_j^0 de σ_i^h. Or la chaîne $L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_{2p-2h-1})$ est située dans $\overline{Q^h(\sigma_i^h)}$. Les réseaux gén. $\mathfrak{P}_1^0$ et $\mathfrak{Q}^h$ étant (v. 29) des affinements de $\mathfrak{P}_1^p$, il existe deux sommets P', P'' de $\mathfrak{P}_1^p$ tels que $P_1^0(\sigma_j^0) \subset P'$, $Q^h(\sigma_i^h) \subset P''$. Or (v. 29) $P_1^0(\sigma_j^0)\, Q^h(\sigma_i^h) \supset \sigma_j^0 \neq 0$, d'où $P'P'' \neq 0$. Comme $A\overline{P_1^0(\sigma_j^0)} \neq 0$, $\overline{P_1^0(\sigma_j^0)} \subset \overline{P'}$, on a $A\overline{P'} \neq 0$, $P'P'' \neq 0$ et par suite $Q^h(\sigma_i^h) \subset \overline{P''} \subset \overline{B}$ d'après la définition de B. Or ceci signifie que toutes les chaînes $L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_{2p-2h-1})$ $(0 \leqq h \leqq p-1,\ 1 \leqq i \leqq \alpha_h)$ sont situées dans $\overline{B}$.

35. Nous allons modifier un nombre fini de fois le cycle $C^{n-p}(\mathfrak{U}_1)$ ainsi que les chaînes $L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_1)$ de manière que, $^*C$ et $^*L$ étant les éléments modifiés, on ait les propriétés suivantes: 1° chaque simplexe de chaque différence $^*C^{n-p}(\mathfrak{U}_1) - C^{n-p}(\mathfrak{U}_1)$ ou $^*L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_1) - L^{n-p+h+1}(\sigma_i^h, \mathfrak{U}_1)$ est une face d'un simplexe de $C^{n-p}(\mathfrak{U}_1)$ ou de $^*L^{n-p+k+1}(\sigma_j^k, \mathfrak{U}_1)$ de manière que les chaînes modifiées restent situées dans $\overline{B}$; 2° les relations 1°, 2°, 3° du n° 34 restent vraies pour les chaînes modifiées $^*C, ^*L$; 3° $^*C^{n-p}(\mathfrak{U}_1) \sim C^{n-p}(\mathfrak{U}_1)$ dans $\overline{B}$.

Toutes ces modifications ayant lieu dans le réseau commode fixe $\mathfrak{U}_1$, nous omettrons $\mathfrak{U}_1$ dans l'écriture.

Pour $0 \leqq q \leqq p-1$, $1 \leqq r \leqq \alpha_q$ nous dirons *qu'une $\mathfrak{U}_1$-chaîne ne contient presque aucun simplexe d'espèce σ_r^q si chaque simplexe τ d'espèce σ_r^q de cette chaîne est une face d'un $\mathfrak{U}_1$-simplexe d'espèce σ^{-1}*.

Ceci posé, le résultat de toutes ces modifications sera que pour $0 \leqq h \leqq p-1$, $0 \leqq k \leqq h$, $0 \leqq q \leqq h$ les chaînes *modifiées* possèderons la propriété $\Theta(k, q, h)$ suivante: Si les simplexes σ_j^k, σ_r^q $(1 \leqq j \leqq \alpha_k,\ 1 \leqq r \leqq \alpha_q)$ sont des faces du simplexe σ_i^h, alors la chaîne $L^{n-p+k+1}(\sigma_j^k)$ (modifiée) ne contiendra presqu'aucun simplexe d'espèce σ_r^q.

Dans le n° 36, nous donnerons un lemme. Dans le n° 37, nous réaliserons la propriété $\Theta(0, 0, 0)$. Dans le n° 38, nous supposerons que, pour une certaine valeur de h $(0 \leqq h \leqq p-2)$, on ait déjà réalisé les propriétés $\Theta(k, q, g)$ pour $0 \leqq g \leqq h$ et nous en déduirons qu'on peut réaliser aussi la propriété $\Theta(k, 0, h+1)$. Dans le n° 39, nous supposerons que pour $0 \leqq h \leqq p-2$, $0 \leqq Q \leqq h$, on ait déjà réalisé les propriétés $\Theta(k, q, g)$ $(0 \leqq g \leqq h)$ ainsi que $\Theta(k, q, h+1)$ $(0 \leqq q \leqq Q)$ et nous en déduirons qu'on peut réaliser aussi la propriété $\Theta(h+1, Q+1, h+1)$. Dans le n° 40, nous supposerons que, pour $0 \leqq h \leqq p-2$, $0 \leqq Q \leqq h$, $h-Q \leqq K \leqq h$, on ait déjà réalisé les propriétés $\Theta(k, q, g)$ $(0 \leqq g \leqq h)$, $\Theta(k, q, h+1)$ $(0 \leqq q \leqq Q)$, $\Theta(k, Q+1, h+1)$ $(K+1 \leqq k \leqq h+1)$ et nous en déduirons qu'on peut réaliser aussi la propriété $\Theta(K, Q+1, h+1)$. Dans le n° 41, nous supposerons que, pour $0 \leqq h \leqq p-2$, $0 \leqq Q \leqq h$, on ait déjà réalisé les propriétés $\Theta(k, q, g)$ $(0 \leqq g \leqq h)$, $\Theta(k, q, h+1)$ $(0 \leqq q \leqq Q)$, $\Theta(k, Q+1, h+1)$

$(h-Q \leqq k \leqq h+1)$ et nous y démontrerons que les propriétés $\Theta(k, Q+1, h+1)$ $(0 \leqq k \leqq h-Q-1)$ se trouvent alors aussi réalisées.

Le résultat de toutes ces modifications sera évidemment qu'on peut réaliser la propriété $\Theta(k, q, h)$ pour $0 \leqq h \leqq p-1$ ce qui était notre but.

36. LEMME. Soit $0 \leqq h \leqq p-1$, $0 \leqq k \leqq h$, $0 \leqq q \leqq h$, $1 \leqq i \leqq \alpha_h$, $1 \leqq j \leqq \alpha_k$, $1 \leqq r \leqq \alpha_q$. Supposons que σ_j^k et σ_r^q soient des faces de σ_i^h. Soit τ un $\mathfrak{U}_1$-simplexe d'espèce σ_r^q. Alors chaque sommet de τ est un sous-ensemble de $P_4^k(\sigma_j^k)$.

Démonstration. Soit τ^0 un sommet de τ. D'après 8.3 il en résulte que $\tau^0 T(\sigma_r^q) \neq 0$. Le réseau $\mathfrak{U}_1$ étant un affinement de $\mathfrak{U}$, il en résulte (v. 32) que $\tau^0 \subset P_4^k(\sigma_j^k)$.

37. Pour $1 \leqq \nu \leqq \alpha_0$, soit $E^{n-p+1}(\sigma_\nu^0)$ la partie de la chaîne $L^{n-p+1}(\sigma_\nu^0)$ dont les simplexes sont d'espèce σ_ν^0 (v. 8.3). Posons

$$*C^{n-p} = C^{n-p} - \sum_{\nu=1}^{\alpha_0} FE^{n-p+1}(\sigma_\nu^0),$$

$$*L^{n-p+1}(\sigma_i^0) = L^{n-p+1}(\sigma_i^0) - \sum_{\nu=1}^{\alpha_0} E^{n-p+1}(\sigma_\nu^0),$$

$$*L^{n-p+h+1}(\sigma_i^h) = L^{n-p+h+1}(\sigma_i^h) \text{ pour } 1 \leqq h \leqq p-1.$$

Les propriétés 1°, 2°, 3° du n° 35 sont évidentes. On voit immédiatement que la chaîne $*L^{n-p+1}(\sigma_i^0)$ $(1 \leqq i \leqq \alpha_0)$ ne contient aucun simplexe d'espèce σ_i^0, donc qu'après notre modification on a la propriété $\Theta(0, 0, 0)$.

38. Soit $0 \leqq h \leqq p-2$ et supposons réalisée la propriété $\Theta(k, q, g)$ pour $0 \leqq g \leqq h$. Soit $1 \leqq \mu \leqq \alpha_0$, $1 \leqq \nu \leqq \alpha_{h+1}$. Lorsque σ_μ^0 n'est pas un sommet de σ_ν^{h+1}, posons $E^{n-p+h+2}(\sigma_\mu^0, \sigma_\nu^{h+1}) = 0$; dans le cas contraire soit $E^{n-p+h+2}(\sigma_\mu^0, \sigma_\nu^{h+1})$ la partie de $L^{n-p+h+2}(\sigma_\nu^{h+1})$ dont les simplexes sont d'espèce σ_μ^0[48]. Pour $1 \leqq j \leqq \alpha_h$, soit

$$D^{n-p+h+2}(\sigma_j^h) = \sum_\mu E^{n-p+h+2}(\sigma_\mu^0, \sigma_\mu^0 \sigma_j^h),$$

μ parcourant toutes les valeurs $(1 \leqq \mu \leqq \alpha_0)$ telles que $\sigma_\mu^0 \sigma_j^h$[49] soit un $(h+1, 3)$-simplexe. Posons

$$*C^{n-p} = C^{n-p}, \quad *L^{n-p+k+1}(\sigma_i^k) = L^{n-p+k+1}(\sigma_i^k) \text{ pour } k \neq h, h+1;$$

$$*L^{n-p+h+1}(\sigma_j^h) = L^{n-p+h+1}(\sigma_j^h) - FD^{n-p+h+2}(\sigma_j^h),$$

$$*L^{n-p+h+2}(\sigma_i^{h+1}) = L^{n-p+h+2}(\sigma_i^{h+1}) - \sum_{j=1}^{\alpha_h} \eta_{ij}^h D^{n-p+h+2}(\sigma_j^h).$$

On voit sans peine que les propriétés 1°, 2°, 3° du n° 35 sont vérifiées.

[48] Si on change l'orientation de σ_ν^{h+1}, la chaîne $E^{n-p+h+2}(\sigma_\mu^0, \sigma_\nu^{h+1})$ se multiplie par -1 (v. la remarque à la fin du n° 34).

[49] Si l'on a (orientation comprise) $\sigma_j^h = (\sigma_{t_0}^0, \sigma_{t_1}^0, \cdots, \sigma_{t_j}^0)$, on a par définition (orientation comprise) $\sigma_\mu^0 \sigma_j^h = (\sigma_\mu^0, \sigma_{t_0}^0, \sigma_{t_1}^0, \cdots, \sigma_{t_j}^0)$. La notation analogue sera employée maintes fois dans ce qui suit.

Il s'agit de voir que la propriété supposée $\Theta(k, q, g)$ $(0 \leqq g \leqq h)$ reste vérifiée après la modification. L'unique cas où ceci n'est pas immédiatement évident est le cas $k = g = h$. Ici il suffit de montrer que si σ_r^q est une face de σ_i^h, la chaîne

$$*L^{n-p+h+1}(\sigma_i^h) - L^{n-p+h+1}(\sigma_i^h) = FD^{n-p+h+2}(\sigma_i^h)$$

ne contient aucun simplexe d'espèce σ_r^q. Or chaque simplexe de la chaîne $D^{n-p+h+2}(\sigma_i^h)$ est d'espèce σ_μ^0, l'indice μ étant tel que $\sigma_\mu^0 \sigma_i^h$ soit un $(h+1, 3)$-simplexe, de manière que σ_μ^0 n'est pas un sommet de σ_i^h. Donc d'après 8.34, aucun simplexe de $FD^{n-p+h+2}(\sigma_i^h)$ ne peut être d'espèce σ_i^h.

On doit enfin démontrer (v. 35) que la propriété $\Theta(k, 0, h+1)$ se trouve réalisée après la modification. Supposons donc que les indices k, j, r, i soient tels que σ_j^k soit une face et que σ_r^0 soit un sommet de σ_i^{h+1}. On doit prouver que presqu'aucun simplexe de la chaîne $*L^{n-p+k+1}(\sigma_j^k)$ n'est d'espèce σ_r^0. Il n'y a que deux cas où ceci n'est pas simplement une conséquence de la propriété $\Theta(k, 0, h)$ qui est, comme nous le savons déjà, conservée après la modification; ce sont: 1° $k = h+1, j = i$; 2° $k = h$, $\sigma_i^{h+1} = \pm \sigma_r^0 \sigma_j^h$.

Premier cas. Soit j un indice tel que $\eta_{ij}^h \neq 0$, c'est-à-dire soit σ_j^h une h-face de σ_i^{h+1}. Les simplexes d'espèce σ_r^0 dans la chaîne $D^{n-p+h+2}(\sigma_j^h)$ ne peuvent exister, comme nous avons vu plus haut, que dans le cas où σ_r^0 n'est pas un sommet de σ_j^h; ceci n'a lieu que pour *une* valeur de j, à savoir pour la valeur $\nu = j$ telle que $\sigma_i^{h+1} = \eta_{i\nu}^h \sigma_r^0 \sigma_\nu^h$. Les simplexes d'espèce σ_r^0 dans $D^{n-p+h+2}(\sigma_\nu^h)$ sont alors évidemment les mêmes que dans la chaîne $E^{n-p+h+2}(\sigma_r^0, \sigma_r^0 \sigma_\nu^h)$. Donc les simplexes d'espèce σ_r^0 dans la chaîne

$$L^{n-p+h+2}(\sigma_i^{h+1}) - *L^{n-p+h+2}(\sigma_i^{h+1}) = \sum_{j=1}^{\alpha_h} \eta_{ij}^h D^{n-p+h+2}(\sigma_j^h)$$

sont les mêmes que dans $\eta_{i\nu}^h E^{n-p+h+2}(\sigma_r^0, \sigma_i^{h+1})$, et par suite que dans la chaîne $L^{n-p+h+2}(\sigma_i^{h+1})$, de manière que presqu'aucun simplexe de $*L^{n-p+h+2}$ n'est d'espèce σ_r^0.

Second cas. Soit $k = h$, $\sigma_i^{h+1} = \pm \sigma_r^0 \sigma_\nu^h$, et par suite $\sigma_i^{h+1} = \eta_{ij}^h \sigma_r^0 \sigma_\nu^h$. Il s'agit de voir que presqu'aucun simplexe de la chaîne $*L^{n-p+h+1}(\sigma_\nu^h)$ n'est d'espèce σ_r^0. Supposons que l'indice j parcoure toutes les valeurs $(1 \leqq j \leqq \alpha_h)$ telles que σ_j^h soit une h-face de σ_i^{h+1}. La propriété $\Theta(h, 0, h)$ étant conservée après la modification, pour $j \neq \nu$ la chaîne $*L^{n-p+h+1}(\sigma_j^h)$ ne contient presqu'aucun simplexe d'espèce σ_r^0. Donc les simplexes d'espèce σ_r^0 sont dans la chaîne $\eta_{i\nu}^h *L^{n-p+h+1}(\sigma_\nu^h)$ presque les mêmes que dans la chaîne $\sum_{j=1}^{\alpha_h} \eta_{ij}^h *L^{n-p+h+1}(\sigma_j^h)$. Or nous savons déjà que presqu'aucun simplexe de

la chaîne $*L^{n-p+h+2}(\sigma_i^{h+1})$ n'est d'espèce $\sigma_{r_1}^0$. Il en résulte sans peine (v. 8.34) que la chaîne $F*L^{n-p+h+2}(\sigma_i^{h+1})$ ne possède non plus presqu'aucun simplexe d'espèce σ_r^0. Il s'agit donc de montrer que la chaîne

$$(*) \qquad F*L^{n-p+h+2}(\sigma_i^{h+1}) - \sum_{j=1}^{\alpha_h} \eta_{ij}^h *L^{n-p+h+1}(\sigma_j^h)$$

ne contient aucun simplexe d'espèce σ_r^0. Or, d'après 36, chaque $\mathfrak{U}_1$-simplexe d'espèce σ_r^0 est situé dans $\overline{P_4^{h+1}(\sigma_i^{h+1})}$, tandis que la chaîne $(*)$ ne contient aucun simplexe dans $\overline{P_4^{h+1}(\sigma_i^{h+1})}$, parce que la propriété 2° du n° 34 reste conservée après la modification.

39. Soit $0 \leqq h \leqq p-2$, $0 \leqq Q \leqq h$. Supposons réalisées les propriétés $\Theta(k, q, g)$ $(0 \leqq g \leqq h)$ ainsi que $\Theta(k, q, h+1)$ $(0 \leqq q \leqq Q)$.

Soit $1 \leqq \mu \leqq \alpha_{Q+1}$, $1 \leqq \nu \leqq \alpha_{h+1}$. Si le simplexe σ_μ^{Q+1} n'est pas une $(Q+1)$-face de σ_ν^{h+1}, posons $E^{n-p+h+2}(\sigma_\mu^{Q+1}, \sigma_\nu^{h+1}) = 0$; dans le cas contraire, soit $E^{n-p+h+2}(\sigma_\mu^{Q+1}, \sigma_\nu^{h+1})$ la partie de la chaîne $L^{n-p+h+2}(\sigma_\nu^{h+1})$ dont les simplexes sont d'espèce σ_μ^{Q+1} [50].

On peut évidemment attacher à chaque indice μ $(1 \leqq \mu \leqq \alpha_{Q+1})$ un indice $\varphi(\mu)$ $[1 \leqq \varphi(\mu) \leqq \alpha_0]$ bien déterminé de manière que $\sigma_{\varphi(\mu)}^0$ soit pour chaque μ $(1 \leqq \mu \leqq \alpha_{Q+1})$ un sommet du simplexe σ_μ^{Q+1}.

Si les indices μ, ν, λ $(1 \leqq \mu \leqq \alpha_{Q+1}, \ 1 \leqq \nu \leqq \alpha_{h+1}, \ 1 \leqq \lambda \leqq \alpha_h)$ sont tels que 1° σ_μ^{Q+1} soit une $(Q+1)$-face de σ_ν^{h+1}; 2° $\sigma_\nu^{h+1} = \pm\sigma_{\varphi(\mu)}^0 \sigma_\lambda^h$, posons

$$\varepsilon(\sigma_\mu^{Q+1}, \sigma_\lambda^h, \sigma_\nu^{h+1}) = \eta_{\nu\lambda}^h;$$

dans tous les autres cas, posons

$$\varepsilon(\sigma_\mu^{Q+1}, \sigma_\lambda^h, \sigma_\nu^{h+1}) = 0.$$

Pour $1 \leqq \lambda \leqq \alpha_h$, posons

$$D^{n-p+h+2}(\sigma_\lambda^h) = \sum_{\mu=1}^{\alpha_{Q+1}} \sum_{\nu=1}^{\alpha_{h+1}} \varepsilon(\sigma_\mu^{Q+1}, \sigma_\lambda^h, \sigma_\nu^{h+1}) \, E^{n-p+h+2}(\sigma_\mu^{Q+1}, \sigma_\nu^{h+1}).$$

Ceci étant, soit

$$*C^{n-p} = C^{n-p}, \qquad *L^{n-p+k+1}(\sigma_i^k) = L^{n-p+k+1}(\sigma_i^k) \text{ pour } k \neq h, \ k \neq h+1;$$

$$*L^{n-p+h+1}(\sigma_j^h) = L^{n-p+h+1}(\sigma_j^h) - FD^{n-p+h+2}(\sigma_j^h),$$

$$*L^{n-p+h+2}(\sigma_i^{h+1}) = L^{n-p+h+2}(\sigma_i^{h+1}) - \sum_{j=1}^{\alpha_h} \eta_{ij}^h D^{n-p+h+2}(\sigma_j^h).$$

On voit sans peine que les propriétés 1°, 2°, 3°, du n° 35 sont vérifiées.

[50] Si l'on change l'orientation de σ_ν^{h+1}, la chaîne $E^{n-p+h+2}(\sigma_\mu^{Q+1}, \sigma_\nu^{h+1})$ se multiplie par -1 (v. la remarque à la fin du n° 34).

Démontrons que les propriétés $\Theta(k, q, g)$ $(0 \leqq g \leqq h)$ restent conservées après la modification. L'unique cas à considérer est évidemment le cas $k = g = h$. Soit donc σ_r^q une q-face $(0 \leqq q \leqq h)$ du simplexe σ_j^h. Il s'agit de montrer que presqu'aucun simplexe de la chaîne

$$-{}^*L^{n-p+h+1}(\sigma_j^h) + L^{n-p+h+1}(\sigma_j^h) = F D^{n-p+h+2}(\sigma_j^h)$$

n'est d'espèce σ_r^q. On voit sans peine (v. 8.34) qu'il suffit de prouver que, si σ_r^q est une face de σ_j^h, presqu'aucun simplexe de la chaîne $D^{n-p+h+2}(\sigma_j^h)$ n'est d'espèce σ_r^q. Or chaque simplexe de $D^{n-p+h+2}(\sigma_j^h)$ est d'espèce σ_u^{Q+1}, où la valeur de μ est telle que, pour une valeur convenable de ν, on ait $\varepsilon(\sigma_\mu^{Q+1}, \sigma_j^h, \sigma_\nu^{h+1}) \neq 0$. Or cette inégalité donne $\sigma_\nu^{h+1} = \pm \sigma^0_{\varphi(\mu)} \sigma_j^h$ de manière que $\sigma^0_{\varphi(\mu)}$ n'est pas un sommet de σ_j^h, tandis que c'est un sommet de σ_μ^{Q+1}; σ_μ^{Q+1} n'est donc pas une face de σ_j^h.

Il est évident que la propriété $\Theta(k, q, h+1)$ $(0 \leqq q \leqq Q)$ reste conservée après la modification, car (v. 34) presque chaque simplexe de chaque ${}^*L - L$ est d'espèce σ_μ^l, où $l \geqq q+1$.

On doit enfin démontrer (v. 35) que la propriété $\Theta(h+1, Q+1, h+1)$ se trouve réalisée après la modification. Supposons donc que les indices i, r soient tels que σ_r^{Q+1} soit une face de σ_i^{h+1}. On doit prouver que presqu'aucun simplexe de la chaîne ${}^*L^{n-p+h+2}(\sigma_i^{h+1})$ n'est d'espèce σ_r^{Q+1}. Supposons que l'indice j parcoure toutes les valeurs $(1 \leqq j \leqq \alpha_h)$ telles que $\eta_{ij}^h \neq 0$. Il existe une seule valeur de j, soit $j = \lambda$, telle que $\sigma_i^{h+1} = \pm \sigma^0_{\varphi(r)} \sigma_\lambda^h$. Pour $j \neq \lambda$, $\sigma^0_{\varphi(r)}$ est un sommet de σ_j^h, d'où il résulte que $\varepsilon(\sigma_r^{Q+1}, \sigma_j^h, \sigma_\nu^{h+1}) = 0$. On en déduit sans peine que la partie de $D^{n-p+h+2}(\sigma_j^h)$ dont les simplexes sont d'espèce σ_r^{Q+1}, est égale à 0 pour $j \neq \lambda$ et égale à $\eta_{i\lambda}^h E^{n-p+h+2}(\sigma_r^{Q+1}, \sigma_i^{h+1})$ pour $j = \lambda$. Donc les simplexes d'espèce σ_r^{Q+1} dans la chaîne

$$\sum_{j=1}^{\alpha_h} \eta_{ij}^h D^{n-p+h+2}(\sigma_j^h) = L^{n-p+h+2}(\sigma_i^{h+1}) - {}^*L^{n-p+h+2}(\sigma_i^{h+1})$$

constituent la chaîne $E^{n-p+h+2}(\sigma_r^{Q+1}, \sigma_i^{h+1})$, qui était égale à la partie de $L^{n-p+h+2}(\sigma_i^{h+1})$ dont les simplexes sont d'espèce σ_r^{Q+1}. Par suite presqu'aucun simplexe de ${}^*L^{n-p+h+2}(\sigma_i^{h+1})$ n'est d'espèce σ_r^{Q+1}.

40. Soit $0 \leqq h \leqq p-2$, $0 \leqq Q \leqq h$, $h - Q \leqq K \leqq h$. Supposons réalisées les propriétés $\Theta(k, q, g)$ $(0 \leqq g \leqq h)$, $\Theta(k, q, h+1)$ $(0 \leqq q \leqq Q)$ et $\Theta(k, Q+1, h+1)$ $(K+1 \leqq k \leqq h+1)$. Il s'agit de réaliser encore la propriété $\Theta(K, Q+1, h+1)$. Ceci sera effectué dans les n^{os} 40.1—40.5.

40.1. Choisissons les indices i et r de manière que le simplexe σ_r^{Q+1} soit une face du simplexe σ_i^{h+1}; ensuite, déterminons l'indice s de manière que l'on ait

(1) $$\sigma_i^{h+1} = \pm \sigma_r^{Q+1} \sigma_s^{h-Q-1}.$$

Pour chaque $(K-h+Q)$-face σ_λ^{K-h+Q} du simplexe σ_r^{Q+1} soit : 1° $E_1^{n-p+K+1}(\sigma_\lambda^{K-h+Q})$ la partie de la chaîne $L^{n-p+K+1}(\sigma_\lambda^{K-h+Q}\,\sigma_s^{h-Q-1})$ [51] dont les simplexes sont d'espèce σ_r^{Q+1}; 2° $E_2^{n-p+K+1}(\sigma_\lambda^{K-h+Q})$ la partie de $E_1^{n-p+K+1}(\sigma_\lambda^{K-h+Q})$ dont chaque simplexe est une face d'un $\mathfrak{U}_1$-simplexe d'espèce σ^{-1}; 3° $E^{n-p+K+1}(\sigma_\lambda^{K-h+Q}) = E_1^{n-p+K+1}(\sigma_\lambda^{K-h+Q}) - E_2^{n-p+K+1}(\sigma_\lambda^{K-h+Q})$.

Soit $\sigma_\mu^{K-h+Q+1}$ une $(K-h+Q+1)$-face de σ_r^{Q+1} de manière que

$$\sigma_\nu^{K+1} = \varepsilon\,\sigma_\mu^{K-h+Q+1}\,\sigma_s^{h-Q-1} \qquad (\varepsilon = \pm 1)$$

est une $(K+1)$-face de σ_i^{h+1}. Nous allons démontrer que

$$\sum_\lambda \eta_{\mu\lambda}^{K-h+Q}\,E^{n-p+K+1}(\sigma_\lambda^{K-h+Q}) = 0. \tag{2}$$

La relation (2) dit que presqu'aucun simplexe de la chaîne

$$\sum_\lambda \eta_{\mu\lambda}^{K-h+Q}\,L^{n-p+K+1}(\sigma_\lambda^{K-h+Q}\,\sigma_s^{h-Q-1}) \tag{3}$$

n'est d'espèce σ_r^{Q+1}.

Lorsque σ_λ^{K-h+Q} parcourt toutes les $(K-h+Q)$-faces de $\sigma_\mu^{K-h+Q+1}$, on peut attacher à chaque λ une valeur de j de manière que

$$\sigma_j^K = \pm\,\sigma_\lambda^{K-h+Q}\,\sigma_s^{h-Q-1}$$

soit une K-face de σ_ν^{K+1} telle que σ_s^{h-Q-1} en soit une face; et on voit sans peine que le terme correspondant de la somme (3) est égal à $\varepsilon\,\eta_{\nu j}^K\,L^{n-p+K+1}(\sigma_j^K)$. Lorsque $s>1$, il y a encore d'autres K-faces σ_j^K de σ_ν^{K+1} qui ne correspondent à aucune valeur de λ; ce sont les faces σ_j^K de σ_ν^{K+1} telles que σ_s^{h-Q-1} n'est pas une face de σ_j^K; à chaque telle valeur de j correspond, comme on le voit sans peine, une valeur de v telle que σ_j^K et σ_r^{Q+1} soient des faces de σ_v^h (tandis que σ_v^h est une face de σ_i^{h+1}); la propriété $\Theta(K, Q+1, h)$ étant supposée vérifiée, il en résulte que, dans le cas actuel, la chaîne $L^{n-p+K+1}(\sigma_j^K)$ ne contient presqu'aucun simplexe d'espèce σ_r^{Q+1}.

Donc, les simplexes d'espèce σ_r^{Q+1} dans la chaîne (3) sont presque les mêmes que dans la chaîne

$$\sum_{j=1}^{a_K} \eta_{\nu j}^K\,L^{n-p+K+1}(\sigma_j^K); \tag{4}$$

on doit donc prouver que presqu'aucun simplexe de (4) n'est d'espèce σ_r^{Q+1}. Or σ_r^{Q+1} et σ_ν^{K+1} étant des faces de σ_i^{h+1}, il résulte de 36 que chaque

[51] Puisque σ_λ^{K-h+Q} est une $(K-h+Q)$-face de σ_r^{Q+1}, $\sigma_\lambda^{K-h+Q}\,\sigma_s^{h-Q-1}$ est, en vertu de (1), une K-face de σ_i^{h+1}.

simplexe d'espèce σ_r^{Q+1} est situé dans $\overline{P_4^{K+1}(\sigma_\nu^{K+1})}$. La propriété 2° du n° 34 étant vraie (v. n° 35, 2°), il en résulte qu'aucun simplexe de la chaîne

$$FL^{n-p+K+2}(\sigma_\nu^{K+1}) - \sum_{j=1}^{\alpha_K} \eta_{\nu j}^K L^{n-p+K+1}(\sigma_j^K)$$

n'est d'espèce σ_r^{Q+1}. Donc il suffit de montrer que presqu'aucun simplexe de la chaîne

$$FL^{n-p+K+2}(\sigma_\nu^{K+1})$$

n'est d'espèce σ_r^{Q+1}. A cet effet, d'après 8.34, il suffit de prouver que, σ_u^q étant une face de σ_r^{Q+1}, la chaîne $L^{n-p+K+2}(\sigma_\nu^{K+1})$ ne contient presqu'aucun simplexe d'espèce σ_u^q. Or ceci est une conséquence de la propriété $\Theta(K+1, q, h+1)$ $(0 \leqq q \leqq Q+1)$, réalisée par hypothèse.

40.2. Considérons le cas $K > h-Q$, d'où $K \geqq 1$. Continuons à supposer qu'on ait choisi les indices r et i de manière que σ_r^{Q+1} soit une face de σ_i^{h+1}. Si l'on attache à chaque $(K-h+Q)$-face σ_λ^{K-h+Q} du simplexe σ_r^{Q+1} un entier a_λ, on obtient une $(K-h+Q)$-sous-chaîne de σ_r^{Q+1} :

$$\tau^{K-h+Q} = \sum_\lambda a_\lambda \sigma_\lambda^{K-h+Q}.$$

Posons

$$E^{n-p+K+1}(\tau^{K-h+Q}) = \sum a_\lambda E^{n-p+K+1}(\sigma_\lambda^{K-h+Q}), \tag{5}$$

les chaînes à droite ayant été définies dans 40.1 ; (5) est une $(n-p+K+1, \mathfrak{U}_1)$-chaîne dont tous les simplexes sont d'espèce σ_r^{Q+1}.

En particulier, si τ^{K-h+Q} est la frontière d'une $(K-h+Q+1)$-face $\sigma_\mu^{K-h+Q+1}$ de σ_r^{Q+1}, la relation (1) du n° 40.1 dit que

$$E^{n-p+K+1}(\tau^{K-h+Q}) = 0. \tag{6}$$

Or on sait que, si $\tau^{K-h+Q} \to 0$, il existe des entiers b_μ tels que $\tau^{K-h+Q} = F\sum b_\mu \sigma_\mu^{K-h+Q+1}$; donc $\tau^{K-h+Q} \to 0$ entraîne (6). On en déduit sans peine que la chaîne (5) dépend seulement de la frontière $F\tau^{K-h+Q}$ de la chaîne τ^{K-h+Q}. On peut donc poser

$$E^{n-p+K+1}(\tau^{K-h+Q}) = \Phi^{n-p+K+1}(F\tau^{K-h+Q}). \tag{7}$$

Si $\tau^{K-h+Q-1}$ est un $(K-h+Q-1)$-sous-cycle[52] arbitraire de σ_r^{Q+1}, il existe, comme on sait, une $(K-h+Q)$-sous-chaîne de σ_r^{Q+1} τ^{K-h+Q} dont la frontière est égale à $\tau^{K-h+Q+1}$. La relation (7) définit donc (comme on voit sans peine, sans ambiguité) la $(n-p+K+1, \mathfrak{U}_1)$-chaîne $\Phi^{n-p+K+1}(\tau^{K-h+Q-1})$ pour chaque $(K-h+Q-1)$-sous-cycle de σ_r^{Q+1}.

[52] Lorsque $K-h+Q-1=0$, on doit convenir d'entendre par un 0-sous-cycle de σ_r^{Q+1} seulement une 0-sous-chaîne de σ_r^{Q+1} dont la somme des coefficients égale à zéro.

Or soit $\tau_1, \tau_2, \cdots, \tau_a$ une base (lin. indépendante) de toutes les $(K-h+Q-1)$-sous-chaînes de σ_r^{Q+1} sousmise à la seule condition que les premiers b élements de cette base constituent une base de toutes les $(K-h+Q-1)$-sous-cycles de σ_r^{Q+1}. A chaque $(K-h+Q-1)$-sous-chaîne $\tau^{K-h+Q-1}$ de σ_r^{Q+1} on peut attacher, et d'une seule manière, des entiers $c_1, c_2, \cdots, c_a$ tels que

$$\tau^{K-h+Q-1} = \sum_{t=1}^{a} c_t \tau_t.$$

On définit alors

$$\Phi^{n-p+K+1}(\tau^{K-h+Q-1}) = \sum_{t=1}^{b} c_t \Phi^{n-p+K+1}(\tau_t), \tag{8}$$

les valeurs de Φ à droite étant déterminées par (7). Dans le cas particulier où $\tau^{K-h+Q-1}$ est un cycle, le premier membre de (8) était déjà défini; or on voit sans peine que l'ancienne définition est dans le cas actuel d'accord avec (8).

40.3. Nous sommes en état d'atteindre, dans le cas où $K > h-Q$, le but proposé dans 40. Si les indices i, r, u sont tels que 1° σ_r^{Q+1} est une face de σ_i^{h+1} de manière qu'il existe une valeur de s telle que $\sigma_i^{h+1} = \pm\sigma_r^{Q+1}\sigma_s^{h-Q-1}$; 2° σ_u^{K-1} est une face de σ_i^{h+1}; 3° chaque sommet de σ_i^{h+1} est un sommet de σ_r^{Q+1} ou de σ_u^{K-1}, posons

$$H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_u^{K-1}, \sigma_i^{h+1}) = \varepsilon \Phi^{n-p+K+1}(\sigma_t^{K-h+Q-1}) \tag{9}$$

l'indice t et le signe $\varepsilon = \pm 1$ étant tels que

$$\sigma_u^{K-1} = \varepsilon \sigma_t^{K-h+Q-1} \sigma_s^{h-Q-1};$$

quant à la valeur de la chaîne Φ, elle est déterminée par l'équation (8) du n° 40.2. Si les conditions 1°, 2°, 3° ne sont pas toutes vérifiées, posons

$$H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_u^{K-1}, \sigma_i^{h+1}) = 0.$$

Posons encore

$$D^{n-p+K+1}(\sigma_u^{K-1}) = \sum_{r=1}^{\alpha_{Q+1}} \sum_{i=1}^{\alpha_{h+1}} H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_u^{K-1}, \sigma_i^{h+1}).$$

Ceci posé, soit

$$^*C^{n-p} = C^{n-p}, \quad ^*L^{n-p+k+1}(\sigma_v^k) = L^{n-p+k+1}(\sigma_v^k) \text{ pour } k \neq K-1, k \neq K,$$

$$^*L^{n-p+K}(\sigma_u^{K-1}) = L^{n-p+K}(\sigma_u^{K-1}) - F D^{n-p+K+1}(\sigma_u^{K-1}),$$

$$^*L^{n-p+K+1}(\sigma_j^K) = L^{n-p+K+1}(\sigma_j^K) - \sum_{u=1}^{\alpha_{K-1}} \eta_{ju}^{K-1} D^{n-p+K+1}(\sigma_u^{K-1}).$$

On voit sans peine que les propriétés 1°, 2°, 3° du no° 35 sont vérifiées.

Démontrons que les propriétés $\Theta(k, q, g)$ $(0 \leqq g \leqq h)$ restent conservées après la modification. Deux cas seulement exigent une considération: 1° $k = K$, $q = Q+1$; 2° $k = K-1$, $q \geqq Q+1$ (v. 8.34).

Supposons donc *en premier lieu* que les indices g, j, r, t soient tels que σ_j^K et σ_r^{Q+1} soient des faces de σ_g^t $(0 \leqq g \leqq h)$. On doit prouver que la chaîne

$$L^{n-p+K+1}(\sigma_j^K) - {}^*L^{n-p+K+1}(\sigma_j^K) = \sum_{u=1}^{\alpha_{K-1}} \eta_{ju}^{K-1} D^{n-p+K+1}(\sigma_u^{K-1})$$

ne contient presqu'aucun simplexe d'espèce σ_r^{Q+1}. Dans le cas contraire il existerait des valeurs des indices u et i telles que 1° σ_u^{K-1} est une face de σ_j^K; 2° $H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_u^{K-1}, \sigma_i^{h+1}) \neq 0$. Or ceci entraînerait que chaque sommet de σ_i^{h+1} serait un sommet de σ_r^{Q+1} ou de σ_u^{K-1}, donc de σ_r^{Q+1} ou de σ_j^K, donc de σ_g^t, ce qui est impossible, car $g < h+1$.

En second lieu, supposons que les indices q, g, u, v, t soient tels que σ_u^{K-1} et σ_v^q $(q \geqq Q+1)$ soient des faces de σ_g^t $(0 \leqq g \leqq h)$. On doit prouver que la chaîne

$$L^{n-p+K}(\sigma_u^{K-1}) - {}^*L^{n-p+K}(\sigma_u^{K-1}) = FD^{n-p+K+1}(\sigma_u^{K-1})$$

ne contient presqu'aucun simplexe d'espèce σ_v^q. Dans le cas contraire, il existerait une valeur de r telle que la chaîne $D^{n-p+K+1}(\sigma_u^{K-1})$ contiendrait un simplexe d'espèce σ_r^{Q+1}, où σ_r^{Q+1} est une face de σ_v^q. Comme plus haut, on en déduirait l'existence d'un indice i tel que chaque sommet σ_i^{h+1} serait un sommet de σ_r^{Q+1} ou de σ_u^{K-1}, donc de σ_g^t, ce qui est impossible, car $g < h+1$.

Il est évident (v. 8.34) que les propriétés $\Theta(k, q, h+1)$ $(0 \leqq q \leqq Q)$ restent conservées après la modification. De même il est évident que les propriétés $\Theta(k, Q+1, h+1)$ $(K+1 \leqq k \leqq h+1)$ restent aussi conservées.

Il ne s'agit donc que de prouver que la propriété $\Theta(K, Q+1, h+1)$ se trouve réalisée après la modification. Supposons donc que les indices j, r, i soient tels que σ_j^K et σ_r^{Q+1} soient des faces de σ_i^{h+1}. On doit démontrer que presqu'aucun simplexe de la chaîne ${}^*L^{n-p+K+1}(\sigma_j^K)$ n'est d'espèce σ_r^{Q+1}. Ceci est évident dans le cas où σ_j^K et σ_r^{Q+1} sont des faces d'une h-face de σ_i^{h+1}, car nous savons que la propriété $\Theta(K, Q+1, h)$ reste conservée après la modification. Supposons donc que chaque sommet de σ_i^{h+1} soit un sommet de σ_j^K ou de σ_r^{Q+1}.

Supposons que l'indice u parcoure toutes les valeurs telles que σ_u^{K-1} soit une $(K-1)$-face de σ_j^K. Les simplexes d'espèce σ_r^{Q+1} dans la chaîne $D^{n-p+K+1}(\sigma_u^{K-1})$ constituent la chaîne

$$\sum_{\nu=1}^{\alpha_{h+1}} H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_u^{K-1}, \sigma_\nu^{h+1}).$$

Or pour $\nu \neq i$ on a

$$H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_u^{K-1}, \sigma_\nu^{h+1}) = 0,$$

car: ou bien les simplexes σ_r^{Q+1} et σ_u^{K-1} ne sont pas tous les deux des faces de σ_ν^{h+1}, ou bien ces deux simplexes sont tous les deux des faces communes de σ_ν^{h+1} et de σ_i^{h+1} de manière qu'il existe un sommet de σ_ν^{h+1} qui n'est sommet ni pour σ_r^{Q+1} ni pour σ_u^{K-1}. Donc les simplexes d'espèce σ_r^{Q+1} dans la chaîne $D^{n-p+K+1}(\sigma_u^{K-1})$ constituent la chaîne

$$(*) \qquad H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_u^{K-1}, \sigma_i^{h+1}).$$

Si l'indice u est tel qu'il existe un sommet de σ_s^{h-Q-1} [v. 40.1 (1)] qui ne soit pas un sommet de σ_u^{K-1}, on a de nouveau le résultat que la chaîne $(*)$ est égale à zéro. Il ne reste donc que des valeurs de u telles qu'il existe une $(K-h+Q-1)$-face $\sigma_t^{K-h+Q-1}$ de σ_u^{K-1} telle que

$$(**) \qquad \sigma_u^{K-1} = \varepsilon_t \sigma_t^{K-h+Q-1} \sigma_s^{h-Q-1} \qquad (\varepsilon_t = \pm 1)$$

et on voit que les simplexes d'espèce σ_r^{Q+1} de la chaîne

$$L^{n-p+K+1}(\sigma_j^K) - {}^*L^{n-p+K+1}(\sigma_j^K) = \sum_{u=1}^{\alpha_{K-1}} \eta_{ju}^{K-1} D^{n-p+K+1}(\sigma_u^{K-1})$$

constituent la chaîne

$$(\dagger) \qquad \sum_u \eta_{ju}^{K-1} H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_u^{K-1}, \sigma_i^{h+1}),$$

où l'indice u parcourt toutes les valeurs déterminées par $(**)$, l'indice t parcourant toutes les valeurs telles que $\eta_{\lambda t}^{K-h+Q-1} \neq 1$, où λ est déterminé par la condition

$$\sigma_j^K = \delta \sigma_\lambda^{K-h+Q} \sigma_s^{h-Q-1} \qquad (\delta = \pm 1).$$

Or pour chacune de ces valeurs de u on a [v. 40.3 (9)]

$$H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_u^{K-1}, \sigma_i^{h+1}) = \varepsilon_t \Phi^{n-p+K+1}(\sigma_t^{K-h+Q-1})$$

ainsi que

$$\eta_{ju}^{K-1} = \delta \varepsilon_t \eta_{\lambda t}^{K-h+Q-1},$$

de manière que la chaîne $(\dagger)$ est égale [v. 40.1 et 40.2] à

$$\delta \sum_t \eta_{\lambda t}^{K-h+Q-1} \Phi(\sigma_t^{K-h+Q-1}) = \delta \Phi\left(\sum_t \eta_{\lambda t}^{K-h+Q-1} \sigma_t^{K-h+Q-1}\right)$$
$$= \delta \Phi(F \sigma_\lambda^{K-h+Q}) = \delta E^{n-p+K+1}(\sigma_\lambda^{K-h+Q}).$$

Or $E^{n-p+K+1}(\sigma_\lambda^{K-h+Q})$ est, abstraction faite des simplexes qui sont des faces d'un $\mathfrak{U}_1$-simplexe d'espèce σ^{-1}, cette partie de la chaîne

$$L^{n-p+K+1}(\sigma_\lambda^{K-h+Q}\,\sigma_s^{h-Q-1}) = \delta\, L^{n-p+K+1}(\sigma_j^K)$$

dont les simplexes sont d'espèce σ_r^{Q+1}. Il en résulte que les simplexes d'espèce σ_r^{Q+1} sont dans la chaîne

$$L^{n-p+K+1}(\sigma_j^K) - {}^*L^{n-p+K+1}(\sigma_j^K)$$

presque les mêmes comme dans $L^{n-p+K+1}(\sigma_j^K)$. Donc presqu'aucun simplexe de la chaîne ${}^*L^{n-p+K+1}(\sigma_j^K)$ n'est d'espèce σ_r^{Q+1}.

40.4. Il reste à considérer le cas $K = h - Q$. Soit d'abord $K > 0$ ou $h > Q$. La relation (2) du n° 40.1 dit que, si σ_λ^0, σ_μ^0 sont les deux sommets d'une 1-face de σ_r^{Q+1}, on a

$$E^{n-p+K+1}(\sigma_\lambda^0) = E^{n-p+K+1}(\sigma_\mu^0).$$

Il en résulte que la chaîne $E^{n-p+K+1}(\sigma_\lambda^0)$ reste la même, si on y remplace σ_λ^0 successivement par tous les sommets de σ_r^{Q+1}.

Donc, si les indices i, r, s sont tels que

$$\sigma_i^{h+1} = \pm\,\sigma_r^{Q+1}\,\sigma_s^{K-1}, \tag{1}$$

on peut définir sans ambiguité la chaîne

$$H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_s^{K-1}, \sigma_s^{h+1})$$

égale, abstraction faite des simplexes qui sont des faces d'un $\mathfrak{U}_1$-simplexe d'espèce σ^{-1}, à la partie de la chaîne $L^{n-p+K+1}(\sigma_\lambda^0\,\sigma_s^{K-1})$ (où σ_λ^0 est un sommet quelconque de σ_r^{Q+1}) dont tous les simplexes sont d'espèce σ_r^{Q+1}. Si les indices i, r, s sont tels que la relation (1) n'est pas vraie, on pose

$$H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_s^{K-1}, \sigma_i^{h+1}) = 0.$$

Posons encore

$$D^{n-p+K+1}(\sigma_s^{K-1}) = \sum_{r=1}^{\alpha_{Q+1}} \sum_{i=1}^{\alpha_{h+1}} H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_s^{K-1}, \sigma_i^{h+1}).$$

Ceci étant, soit

$${}^*C^{n-p} = C^{n-p}, \quad {}^*L^{n-p+k+1}(\sigma_i^k) = L^{n-p+k+1}(\sigma_i^k) \text{ pour } k \neq K-1, k \neq K;$$

$${}^*L^{n-p+K}(\sigma_s^{K-1}) = L^{n-p+K}(\sigma_s^{K-1}) - F\,D^{n-p+K+1}(\sigma_s^{K-1}),$$

$${}^*L^{n-p+K+1}(\sigma_j^K) = L^{n-p+K+1}(\sigma_j^K) - \sum_{s=1}^{\alpha_{K-1}} \eta_{js}^{K-1}\, D^{n-p+K+1}(\sigma_s^{K-1}).$$

On voit sans peine que les propriétés 1°, 2°, 3° du n° 35 sont vérifiées.

On vérifie de la même façon comme dans le n° 40.3 que les propriétés $\Theta(k, q, g)$ $(0 \leqq g \leqq h)$ sont conservées après la modification. Comme au n° cité, il est évident que les propriétés $\Theta(k, q, h+1)$ $(0 \leqq q \leqq Q)$ et $\Theta(k, Q+1, h+1)$ $(K+1 \leqq k \leqq h+1)$ restent aussi conservées.

On doit encore vérifier que la propriété $\Theta(K, Q+1, h+1)$ se trouve réalisée après la modification. Supposons donc que les indices j, r, i soient tels que σ_j^K et σ_r^{Q+1} soient des faces de σ_i^{h+1}. On doit démontrer que presqu'aucun simplexe de la chaîne ${}^*L^{n-p+K+1}(\sigma_j^K)$ n'est d'espèce σ_r^{Q+1}. Comme au n° cité, l'unique cas à considérer est celui où chaque sommet de σ_i^{h+1} est un sommet de σ_j^K ou de σ_r^{Q+1}. Or cette hypothèse entraîne, dans le cas actuel où $K = h - Q > 0$, que le simplexe σ_j^K possède une $(K-1)$-face σ_s^{K-1} telle qu'on ait la relation (1).

σ_u^{K-1} parcourant toutes les $(K-1)$-faces de σ_j^K, on voit comme au n° cité que

$$H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_u^{K-1}, \sigma_\nu^{h+1}) = 0$$

si $\nu \neq i$ ou bien $\nu = i$ et $u \neq s$. Donc la partie de la chaîne

$$L^{n-p+K+1}(\sigma_j^K) - {}^*L^{n-p+K+1}(\sigma_j^K) = \sum_{u=1}^{\alpha_{K-1}} \eta_{ju}^{K-1} D^{n-p+K+1}(\sigma_u^{K-1})$$

dont les simplexes sont d'espèce σ_r^{Q+1}, est presque égale à

$$(*) \qquad \eta_{js}^{K-1} H^{n-p+K+1}(\sigma_r^{Q+1}, \sigma_s^{K-1}, \sigma_i^{h+1}).$$

Or si l'on détermine le signe $\varepsilon = \pm 1$ et la valeur de l'indice λ de manière que $\sigma_j^K = \varepsilon \sigma_\lambda^0 \sigma_s^{K-1}$, on a $\varepsilon = \eta_{js}^{K-1}$ et on voit que $(*)$ est presque égale à la partie de la chaîne $L^{n-p+K+1}(\sigma_j^K)$ dont les simplexes sont d'espèce σ_r^{Q+1}. Par suite presqu'aucun simplexe de la chaîne $L^{n-p+K+1}(\sigma_j^K)$ n'est d'espèce σ_r^{Q+1}.

40.5. On doit enfin considérer le cas où $K = 0$, $Q = h$. La relation (1) du n° 40.1 dit que la partie de la chaîne $L^{n-p+1}(\sigma_\lambda^0)$ dont les simplexes sont d'espèce σ_i^{h+1} est la même pour tous les sommets de σ_i^{h+1}; désignons la par $E^{n-p+1}(\sigma_i^{h+1})$. Posons

$$\begin{aligned}
{}^*C^{n-p} &= C^{n-p} - \sum_{i=1}^{\alpha_{h+1}} F E^{n-p+1}(\sigma_i^{h+1}), \\
{}^*L^{n-p+1}(\sigma_j^0) &= L^{n-p+1}(\sigma_j^0) - \sum_{i=1}^{\alpha_{h+1}} E^{n-p+1}(\sigma_i^{h+1}), \\
{}^*L^{n-p+k+1}(\sigma_\mu^k) &= L^{n-p+k+1}(\sigma_\mu^k) \text{ pour } 1 \leqq k \leqq p-1.
\end{aligned}$$

On voit sans peine que les propriétés 1°, 2°, 3° du n° 35 sont vérifiées.

Il est évident que les propriétés $\Theta(k, q, g)$ $(0 \leqq g \leqq h)$, $\Theta(k, q, h+1)$ $(0 \leqq q \leqq h)$ et $\Theta(k, h+1, h+1)$ restent conservées après la modification et que la propriété $\Theta(0, h+1, h+1)$ se trouve réalisée après la modification.

41. Soit $0 \leqq h \leqq p-2$, $0 \leqq Q \leqq h$. Supposons qu'on ait déjà réalisé les propriétés suivantes: $\Theta(k, q, g)$ pour $0 \leqq g \leqq h$, $\Theta(k, q, h+1)$ pour $0 \leqq q \leqq Q$, $\Theta(k, Q+1, h+1)$ $(h-Q \leqq k \leqq h+1)$. On doit démontrer que les propriétés $\Theta(k, Q+1, h+1)$ $(0 \leqq k \leqq h-Q-1)$ sont aussi réalisées. Supposons donc que les indices k, j, r, i soient tels que $0 \leqq k \leqq h-Q-1$ (d'où $Q<h$) et que les deux simplexes σ_j^k et σ_r^{Q+1} soient des faces de σ_i^{h+1}; on doit prouver que presqu'aucun simplexe de la chaîne $L^{n-p+k+1}(\sigma_j^k)$ n'est d'espèce σ_r^{Q+1}. Ceci est évident si σ_j^k et σ_r^{Q+1} sont des faces d'une h-face de σ_i^{h+1}, car la propriété $\Theta(k, Q+1, h)$ est vérifiée. Supposons donc que chaque sommet de σ_i^{h+1} soit un sommet de σ_j^k ou de σ_r^{Q+1}. On a alors $k+Q+1 \geqq h$; d'autre part on avait $k \leqq h-Q-1$, de manière que $k = h-Q-1$ et on a évidemment

$$\sigma_i^{h+1} = \pm \sigma_r^{Q+1} \sigma_j^{h-Q-1}.$$

Puisque $Q<h$, on peut distinguer deux cas: 1° $Q \leqq h-2$; 2° $Q = h-1$.

Premier cas: $Q \leqq h-2$. Choisissons un sommet σ_λ^0 du simplexe σ_r^{Q+1}. D'après 34.2, 2° la chaîne

$$(*) \qquad FL^{n-p+h-Q+1}(\sigma_\lambda^0 \sigma_j^{h-Q-1}) - L^{n-p+h-Q}(\sigma_j^{h-Q-1}) + \sum_{\nu=1}^{\alpha_{h-Q-2}} \eta_{jk}^{h-Q-2} L^{n-p+h-Q}(\sigma_\lambda^0 \sigma_\nu^{h-Q-2})$$

ne contient aucun simplexe situé dans $\overline{P_4^{h-Q}(\sigma_\lambda^0 \sigma_j^{h-Q-1})}$. D'après 36, il en résulte qu'aucun simplexe de la chaîne (*) n'est d'espèce σ_r^{Q+1}. Puisque notre but était de montrer que la chaîne $L^{n-p+h-Q}(\sigma_j^{h-Q-1})$ ne contient presqu'aucun simplexe d'espèce σ_r^{Q+1}, il suffit donc de prouver que ceci est vrai pour la chaîne $FL^{n-p+h-Q+1}(\sigma_\lambda^0 \sigma_j^{h-Q-1})$ ainsi que pour chaque chaîne $L^{n-p+h-Q}(\sigma_\lambda^0 \sigma_\nu^{h-Q-2})$, où σ_ν^{h-Q-2} parcourt toutes les $(h-Q-2)$-faces de σ_j^{h-Q-1}.

Or des propriétés $\Theta(k, q, h+1)$ $(0 \leqq q \leqq Q)$ et $\Theta(k, Q+1, h+1)$ $(h-Q \leqq k \leqq h+1)$ il résulte que, si σ_ι^q est une face de σ_r^{Q+1} $(0 \leqq g \leqq Q+1)$, la chaîne $L^{n-p+h-Q+1}(\sigma_\lambda^0 \sigma_j^{h-Q-1})$ ne contient presqu'aucun simplexe d'espèce σ_μ^q. D'après 8.34, il en résulte que la chaîne $FL^{n-p+h-Q+1}(\sigma_\lambda^0 \sigma_j^{h-Q-1})$ ne contient presqu'aucun simplexe d'espèce σ_r^{Q+1}. D'autre part, σ_ν^{h-Q-2} étant une $(h-Q-2)$-face de σ_j^{h-Q-1}, les deux simplexes $\sigma_\lambda^0 \sigma_\nu^{h-Q-2}$ et σ_r^{Q+1} étant des faces de la h-face $\sigma_r^{Q+1} \sigma_\nu^{h-Q-2}$ de σ_i^{h+1} il résulte de $\Theta(k, q, h)$ que presqu'aucun simplexe de la chaîne $L^{n-p+h-Q}(\sigma_\lambda^0 \sigma_\nu^{h-Q-2})$ n'est d'espèce σ_r^{Q+1}.

Second cas: $Q = h-1$, d'où $k = 0$. Les indices j, r, i étant tels que $\sigma_i^{h+1} = \pm \sigma_r^h \sigma_j^0$, on doit démontrer que la chaîne $L^{n-p+1}(\sigma_j^0)$ ne contient presqu'aucun simplexe d'espèce σ_r^h. Choisissons un sommet σ_λ^0 du simplexe σ_r^h. D'après 34.2, 2° la chaîne

(*) $$FL^{n-p+2}(\sigma_\lambda^0\,\sigma_j^0) - L^{n-p+1}(\sigma_j^0) + L^{n-p+1}(\sigma_\lambda^0)$$

ne contient aucun simplexe situé dans $\overline{P_4^1(\sigma_\lambda^0\,\sigma_j^0)}$ et par suite (v. 36) aucun simplexe d'espèce σ_r^h. On ne doit donc montrer que ni $FL^{n-p+2}(\sigma_\lambda^0\,\sigma_j^0)$ ni $L^{n-p+1}(\sigma_\lambda^0)$ ne contient presqu'aucun simplexe d'espèce σ_r^h. Si σ_s^g $(0 \leqq g \leqq h)$ est une face de σ_r^h, les deux simplexes $\sigma_\lambda^0\,\sigma_j^0$ et σ_s^g sont des faces de σ_r^{h+1}, de manière qu'il résulte de $\Theta(k, q, h+1)$ $(0 \leqq q \leqq Q = h-1)$ et de $\Theta(k, h, h+1) = \Theta(k, Q+1, h+1)$ $(1 = h-Q \leqq k \leqq h+1)$ que la chaîne $L^{n-p+2}(\sigma_\lambda^0\,\sigma_j^0)$ ne contient presqu'aucun simplexe d'espèce σ_s^g. D'après 8.34 il en résulte que la chaîne $FL^{n-p+2}(\sigma_\lambda^0\,\sigma_j^0)$ ne contient presqu'aucun simplexe d'espèce σ_r^h. D'autre part, il résulte de $\Theta(k, h, h)$ que la chaîne $L^{n-p+1}(\sigma_\lambda^0)$ ne contient presqu'aucun simplexe d'espèce σ_r^h.

42. Le but proposé au n° 35 est atteint. Projetons toutes les $\mathfrak{U}_1$-chaînes obtenues dans le réseau $\mathfrak{U}$. En tenant compte du fait que (v. 33) $\mathfrak{U}_1$ est un affinement de $\mathfrak{U}$ normal par rapport aux cycles mod $\overline{P_1^p(\sigma_i^p)} - P_2^p(\sigma_i^p)$ dans $\overline{P_1^p(\sigma_i^p)}$ $(1 \leqq i \leqq \alpha_p)$, nous avons donc le résultat suivant: 1° La chaîne $C^{n-p}(\mathfrak{U})$ primitive a été remplacée par une nouvelle $(n-p, \mathfrak{U})$-chaîne, que nous continuons d'appeler $C^{n-p}(\mathfrak{U})$ et qui est homologue à l'ancienne dans $\bar{B}$. 2° A chaque $(h, 3)$-simplexe intérieur σ_i^h $(0 \leqq h \leqq p-1,\ 1 \leqq i \leqq \alpha_h)$ on a attaché une $(n-p+h+1, \mathfrak{U})$-chaîne $L^{n-p+h+1}(\sigma_i^h, \mathfrak{U})$ dans $\bar{B}$. 3° Pour $1 \leqq i \leqq \alpha_0$ la chaîne

$$FL^{n-p+1}(\sigma_i^0, \mathfrak{U}) - C^{n-p}(\mathfrak{U})$$

ne contient aucun simplexe situé dans $\overline{P_4^0(\sigma_i^0)}$. 4° Pour $0 \leqq h \leqq p-2$, $1 \leqq i \leqq \alpha_{h+1}$, la chaîne

$$FL^{n-p+h+2}(\sigma_i^h, \mathfrak{U}) - \sum_{j=1}^{\alpha_h} \eta_{ij}^h\, L^{n-p+h+1}(\sigma_j^h, \mathfrak{U})$$

ne contient aucun simplexe situé dans $\overline{P_4^{h+1}(\sigma_i^{h+1})}$. 5° Pour $1 \leqq i \leqq \alpha_p$,

$$\sum_{j=1}^{\alpha_{p-1}} \eta_{ij}^{p-1}\, L^n(\sigma_j^{p-1}, \mathfrak{U})$$

est un (n, R)-cycle essentiel mod $R - P_2^p(\sigma_i^p)$. 6° Pour $0 \leqq h \leqq p-1$, $0 \leqq k \leqq h$, $0 \leqq q \leqq h$, si les simplexes σ_j^k, σ_r^q $(1 \leqq j \leqq \alpha_k,\ 1 \leqq r \leqq \alpha_q)$ sont des faces du simplexe σ_i^h, la chaîne $L^{n-p+k+1}(\sigma_j^k, \mathfrak{U})$ ne contient essentiellement aucun simplexe d'espèce σ_r^q, ce qui veut dire que chaque simplexe d'espèce σ_r^q de la chaîne $L^{n-p+k+1}(\sigma_j^k, \mathfrak{U})$ est égal à zéro mod $\overline{R - R_0}$ [53].

[53] En effet, soit τ un simplexe de la chaîne $L^{n-p+k+1}(\sigma_j^k, \mathfrak{U})$ tel que $\tau \neq 0$ mod $\overline{R-R_0}$. Il existe alors un simplexe τ_1 de $L^{n-p+k+1}(\sigma_j^k, \mathfrak{U}_1)$ tel que (v. 33) $\tau = \pi_0^\# \tau_1$. Evidemment $\tau_1 \neq 0$ mod $\overline{R-R_0}$. Par suite, comme on le voit sans peine, τ_1 n'est pas une face d'un

Or nous en déduirons (dans les n^{os} 42.1–42.5) que la chaîne $C^{n-p}(\mathfrak{U})$ est égale mod $\overline{B}\,\overline{R-R_0}$ à une $(n-p, \mathfrak{U}_1)$-chaîne élémentaire. Avec cela le théorème du n° 31 sera évidemment prouvé.

42.1. Soit τ un $\mathfrak{U}_1$-simplexe dont le noyau rencontre et $\overline{B}$ et $\overline{R-R_0}$. Alors le noyau de $\pi_{20}\tau$ (v.33) et donc aussi celui de $\pi_0^*\tau$ rencontre $\overline{B}\,\overline{R-R_0}$. Or toutes les $\mathfrak{U}$-chaînes que nous considérons ici sont des projections de $\mathfrak{U}_1$-chaînes situées dans $\overline{B}$; donc si un simplexe d'une de nos $\mathfrak{U}$-chaînes est $= 0 \bmod \overline{R-R_0}$, il est aussi $= 0 \bmod \overline{B}\,\overline{R-R_0}$. Les chaînes élémentaires ne contenant que des simplexes $\neq 0 \bmod \overline{R-R_0}$ (v. 19.1, 1°), on voit sans peine qu'il suffit de démontrer que $C^{n-p}(\mathfrak{U})$ est égale mod $\overline{R-R_0}$ à une chaîne élémentaire.

42.2. Lemme $\mathcal{A}_k\,(0 \leqq k \leqq p-1)$: Supposons que les simplexes σ_j^k, σ_r^q soient des faces de $\sigma_i^p\,(1 \leqq j \leqq \alpha_k,\ 1 \leqq r \leqq \alpha_q,\ 1 \leqq i \leqq \alpha_p)$. Soit $E^{n-p+k+1}(\sigma_j^k, \sigma_r^q, \mathfrak{U})$ la partie de la chaîne $L^{n-p+k+1}(\sigma_j^k, \mathfrak{U})$ dont les simplexes sont 1° d'espèce σ_r^q, 2° $\neq 0 \bmod \overline{R-R_0}$. Alors: 1° si $q \neq p-k-1$, on a $E^{n-p+k+1}(\sigma_j^k, \sigma_r^q, \mathfrak{U}) = 0$; 2° si $q = p-k-1$ il existe un nombre $a_{jr} \in \mathfrak{R}$ tel que $E^{n-p+k+1}(\sigma_j^k, \sigma_r^q, \mathfrak{U}) = a_{jr} K^{n-q}(\sigma_r^q)$.

Dans le n° 42.3, nous démontrerons $\mathcal{A}_{p-1}$. Dans le n° 42.4, nous déduirons $\mathcal{A}_k\,(0 \leqq k \leqq p-2)$ de $\mathcal{A}_{k+1}$. Par conséquent, le lemme $\mathcal{A}_0$ sera vrai.

42.3. Soient σ_j^{p-1}, σ_r^q des faces de σ_i^p. Pour $q \geqq 1$, on a $E^n(\sigma_j^{p-1}, \sigma_r^q, \mathfrak{U}) = 0$ d'après 8.33. Soit donc $q = 0$. Dans le cas où σ_r^0 est un sommet de σ_j^{p-1}, on a $E^n(\sigma_j^{p-1}, \sigma_r^0, \mathfrak{U}) = 0$ d'après 42, 6°. Il ne reste que le cas où $\sigma_i^p = \pm\sigma_r^0\,\sigma_j^{p-1}$. Si σ_ν^{p-2} est une $(p-2)$-face quelconque de σ_j^{p-1}, nous savons déjà que $E^n(\sigma_r^0\,\sigma_\nu^{p-2}, \sigma_r^0, \mathfrak{U}) = 0$. Il en résulte que $E^n(\sigma_j^{p-1}, \sigma_r^0, \mathfrak{U})$ est la partie de la chaîne[54]

$$M^n(\sigma_i^p, \mathfrak{U}) = L^n(\sigma_j^{p-1}, \mathfrak{U}) - \sum_{\nu=1}^{\alpha_{p-2}} \eta_{j\nu}^{p-2}\, L^n(\sigma_r^0\,\sigma_\nu^{p-2}, \mathfrak{U})$$

dont les simplexes sont 1° d'espèce σ_r^0, 2° $\neq 0 \bmod \overline{R-R_0}$. D'après 42, 5° $M^n(\sigma_i^p, \mathfrak{U})$ est un (n, R)-cycle essentiel mod $R - P_2^p(\sigma_i^p)$; puisque $S \subset R - P_2^p(\sigma_i^p)$, $G(\mathfrak{U})$ est aussi (v. 17) un tel (n, R)-cycle. D'après la définition du réseau gén. $\mathfrak{P}_3^p$ (n° 29) et celle de $P_2^p(\sigma_i^p)$ (n° 30), il en résulte l'existence de deux nombres $r_1, r_2 \in \mathfrak{R}$, dont un au moins $\neq 0$, tels que $r_1 M^n(\sigma_i^p, \mathfrak{U}) + r_2 G^n(\mathfrak{U}) \sim 0 \bmod R - P_3^p(\sigma_i^p)$. Or (v. 32) $G^n(\mathfrak{U}) \not\sim 0 \bmod R - P_3^p(\sigma_i^p)$ de manière qu'il existe un nombre $a \in \mathfrak{R}$ $(a = -r_2 : r_1)$ tel

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$\mathfrak{U}_1$-simplexe d'espèce σ^{-1}. Soit σ_ν^l l'espèce du simplexe τ_1. On a $l \geqq 0$ et, en vertu de la propriété $\boldsymbol{\theta}(k, q, h)$, σ_ν^l n'est pas une face de σ_i^h. Or si le simplexe τ était d'espèce σ_r^q, on voit tout de suite que σ_ν^l serait une face de σ_r^q, donc de σ_i^h.

[54] Dans le cas $p = 1$, il existe un indice ν tel que $\pm\sigma_i^1 \to \sigma_j^0 - \sigma_\nu^0$ et on doit poser

$$M^n(\sigma_i^1, \mathfrak{U}) = L^n(\sigma_j^0, \mathfrak{U}) - L^n(\sigma_\nu^0, \mathfrak{U}).$$

que $M^n(\sigma_i^p, \mathfrak{U}) \sim a\,G^n(\mathfrak{U}) \bmod R - P_3^p(\sigma_i^p)$. Il existe donc une $(n+1, \mathfrak{U})$-chaîne $H^{n+1}(\mathfrak{U})$ et une $(n, \mathfrak{U})$-chaîne $N^n(\mathfrak{U}) \subset R - P_3^p(\sigma_i^p)$ telles que

$$M^n(\sigma_i^p, \mathfrak{U}) = a\,G^n(\mathfrak{U}) + F H^{n+1}(\mathfrak{U}) + N^n(\mathfrak{U}).$$

Le réseau $\mathfrak{U}$ étant commode, il est d'ordre $\leqq n \bmod S$; par suite chaque $(n+1, \mathfrak{U})$-simplexe possède un sommet U tel que $US \neq 0$. Or si U', U'' sont deux sommets de $\mathfrak{U}$ tels que $U'R_0 \neq 0$, $U''S \neq 0$, on a (n° 32) $U'U'' = 0$. Il en résulte que $F H^{n+1}(\mathfrak{U}) = 0 \bmod \overline{R - R_0}$. Lorsque τ est un sommet de $\mathfrak{U}$ d'espèce σ_r^0, chacun de ses sommets U rencontre T_r et par suite (v. 32) $U \subset P_3^p(\sigma_i^p)$. Il en résulte que la chaîne $N^n(\mathfrak{U})$ ne contient aucun simplexe d'espèce σ_r^0. Donc $E^n(\sigma_j^{p-1}, \sigma_r^0)$, c'est-à-dire la partie de la chaîne $M^n(\sigma_i^p, \mathfrak{U})$ dont les simplexes sont 1° d'espèce σ_r^0, 2° $\neq 0 \bmod \overline{R - R_0}$, coïncide avec la partie pareille de la chaîne $a\,G^n(\mathfrak{U})$, c'est-à-dire (v. 19.1) avec $a\,K^n(\sigma_r^0, \mathfrak{U})$.

42.4. Soit $0 \leqq k \leqq p-2$ et supposons la validité du lemme $\mathcal{A}_{k+1}$. Il s'agit d'en déduire la validité du lemme $\mathcal{A}_k$. Supposons donc que les simplexes σ_j^k, σ_r^q soient des faces de σ_i^p. S'il existe un sommet de σ_i^p qui n'est sommet ni pour σ_j^k, ni pour σ_r^q, il existe une $(p-1)$-face σ_ν^{p-1} de σ_i^p telle que σ_j^k et σ_r^q sont des faces de σ_ν^{p-1}; par suite $E^{n-p+k+1}(\sigma_j^k, \sigma_r^q) = 0$, d'après 42, 6°. Supposons donc que chaque sommet de σ_i^p soit un sommet de σ_j^k ou de σ_r^q. Il en résulte que $k+q \geqq p-1$. Lorsque $q \geqq p-k$, on a $E^{n-p+k+1}(\sigma_j^k, \sigma_r^q) = 0$ d'après 8.33. Il ne reste donc que le cas où $q = p-k-1$, les indices j, r, i étant tels que $\sigma_i^p = \pm\,\sigma_j^k\,\sigma_r^{p-k-1}$. Choisissons un sommet σ_t^0 de σ_r^{p-k-1} et supposons que σ_s^{k-1} parcoure toutes les $(k-1)$-faces de σ_j^k. Alors $\sigma_s^{k-1}\,\sigma_t^0$ [55] et σ_r^{p-k-1} sont des faces d'une $(p-1)$-face de σ_i^p, de manière que $E^{n-p+k+1}(\sigma_s^{k-1}\,\sigma_t^0, \sigma_r^{p-k-1}) = 0$ d'après 42.6. En outre, d'après 42, 4° et 36 la chaîne [56]

$$F L^{n-p+k+2}(\sigma_t^0\,\sigma_j^k, \mathfrak{U}) - L^{n-p+k+1}(\sigma_j^k, \mathfrak{U}) + \sum_{s=1}^{a_{k-2}} \eta_{js}^{k-2}\, L^{n-p+k+1}(\sigma_t^0\,\sigma_s^{k-1}, \mathfrak{U})$$

ne contient aucun simplexe d'espèce σ_r^{p-k-1}. Il en résulte que $E^{n-p+k+1}(\sigma_j^k, \sigma_r^{p-k-1}, \mathfrak{U})$ est la partie de la chaîne

$$(*) \qquad F L^{n-p+k+2}(\sigma_i^0\,\sigma_j^k, \mathfrak{U})$$

dont les simplexes sont 1° d'espèce σ_r^{p-k-1}, 2° $\neq 0 \bmod \overline{R - R_0}$. Soit τ un tel simplexe de la chaîne $(*)$. Il existe alors un simplexe τ_1 de la chaîne

$$L^{n-p+k+2}(\sigma_t^0\,\sigma_j^k, \mathfrak{U})$$

[55] Pour $k = 0$ ce symbole signifie σ_t^0.

[56] Pour $k = 0$:

$$F L^{n-p+2}(\sigma_i^0\,\sigma_j^0, \mathfrak{U}) - L^{n-p+1}(\sigma_j^0, \mathfrak{U}) + L^{n-p+1}(\sigma_i^0, \mathfrak{U}).$$

tel que τ appartienne à la chaîne $F\tau_1$. Puisque $\tau \neq 0 \bmod \overline{R-R_0}$, évidemment $\tau_1 \neq 0 \bmod \overline{R-R_0}$ de manière que (v. 8.32) τ_1 n'est pas d'espèce σ^{-1}. Donc (v. 8.34) le simplexe τ_1 est d'espèce σ_ν^h, où σ_ν^h est une face de σ_r^{p-k-1}. Donc τ_1 est un simplexe de la chaîne $E^{n-p+k+2}(\sigma_t^0\, \sigma_j^k, \sigma_\nu^h, \mathfrak{U})$. Or en vertu du lemme Λ_{k+1}, on a

$$1^\circ \quad E^{n-p+k+2}(\sigma_t^0\, \sigma_j^k, \sigma_\nu^h, \mathfrak{U}) \neq 0 \quad \text{pour } h \neq p-k-2,$$

$$2^\circ \quad E^{n-p+k+2}(\sigma_t^0\, \sigma_j^k, \sigma_\nu^{p-k-2}, \mathfrak{U}) = \eta_{r\nu}^{p-k-2}\, a_\nu\, K^{n-p+k+2}(\sigma_\nu^{p-k-2}, \mathfrak{U}) \quad (a_\nu \in \mathfrak{R}),$$

où σ_ν^{p-k-2} parcourt toutes les $(p-k-2)$-faces de σ_r^{p-k-1}. Il en résulte que $E^{n-p+k+1}(\sigma_j^k, \sigma_r^{p-k-1}, \mathfrak{U})$ est égale à la partie de la chaîne

$$(**) \qquad F \sum_{\nu=1}^{\alpha_{p-k-2}} \eta_{r\nu}^{p-k-2}\, a_\nu\, K^{n-p+k+2}(\sigma_\nu^{p-k-2}, \mathfrak{U})$$

dont les simplexes sont 1° d'espèce σ_r^{p-k-1}, $2^\circ \neq 0 \bmod \overline{R-R_0}$. Or la chaîne $(**)$ est égale, en vertu de 19.1, 3°, à la chaîne

$$\sum_{\nu=1}^{\alpha_{p-k-2}} \sum_{\mu=1}^{\alpha_{p-k-1}} \eta_{r\nu}^{p-k-2}\, \eta_{\mu\nu}^{p-k-1}\, a_\nu\, K^{n-p+k+1}(\sigma_\mu^{p-k-1}, \mathfrak{U}),$$

de manière que, d'après 19.1, 1° et 2°,

$$E^{n-p+k+1}(\sigma_j^k, \sigma_r^{p-k-1}, \mathfrak{U}) = a\, K^{n-p+k+1}(\sigma_r^{p-k-1}, \mathfrak{U}).$$

42.5. Le lemme Λ_0 étant démontré, passons à la démonstration du fait que (v. 42.1) la chaîne $C^{n-p}(\mathfrak{U})$ est égale $\bmod \overline{R-R_0}$ à une chaîne élémentaire.

Soit τ^{n-p} un simplexe $\neq 0 \bmod \overline{R-R_0}$. D'après 8.32 et 8.33, τ est d'espèce σ_i^k, où $0 \leqq k \leqq p$. Choisissons un sommet σ_t^0 de σ_i^k. D'après 42, 3° et 36, la partie dont les simplexes sont d'espèce σ_i^k est dans $C^{n-p}(\mathfrak{U})$ la même que dans $FL^{n-p+1}(\sigma_t^0, \mathfrak{U})$ et par suite, comme on voit par le raisonnement employé dans le n° 42.4, la même que dans

$$(*) \qquad F \sum_{h,j} E^{n-p+1}(\sigma_t^0, \sigma_j^h, \mathfrak{U}),$$

où σ_j^h parcourt toutes les faces de σ_i^k. Or d'après 42, 6° et d'après le lemme Λ_0, on a

$$1^\circ \quad E^{n-p+1}(\sigma_t^0, \sigma_j^h) = 0 \quad \text{pour} \quad h \neq p-1$$

ainsi que pour $h = p-1$, si σ_t^0 est un sommet de σ_j^{p-1};

$$2^\circ \quad E^{n-p+1}(\sigma_t^0, \sigma_j^{p-1}, \mathfrak{U}) = a_{ij}\, K^{n-p+1}(\sigma_j^{p-1}, \mathfrak{U}),$$

si $k = p$ et si l'indice j est tel que $\sigma_i^p = \pm \sigma_t^0 \sigma_j^{p-1}$. Donc, si $k \leq p-1$, la chaîne (*) est égale à zéro ; si $k = p$, la chaîne (*) est, d'après 19.1, 3°, égale à $\eta_{ij}^{p-1} a_{ij} K^{n-p}(\sigma_i^p, \mathfrak{U})$. Par suite

$$C^{n-p}(\mathfrak{U}) = \sum_{i=1}^{\alpha_p} a_i K^{n-p}(\sigma_i^p, \mathfrak{U}) \mod \overline{R - R_0}.$$

V.

43. Dans tout ce Chapitre, supposons donnée une valeur fixe de p ($1 \leq p \leq n$). On suppose la validité de tous les axiomes des Chap. I et II ainsi que celle des axiomes G^k (v. 21) pour $n-p \leq k \leq n-1$ et des axiomes H^k (v. 27) pour $n-p \leq k \leq n-2$.

44. *Soit $\mathfrak{Z} \in \Xi$ (v. 23). Soit S_1 la somme de tous les sommets extérieurs de $\mathfrak{Z}$; soit Ω un entourage donné de $\overline{S_1}$. Soit A un sous-ensemble bicompact de R ; soit B un entourage donné de A. Il existe un affinement $\mathfrak{V}$ de $\mathfrak{Z}$ et un réseau gén. $\mathfrak{M}$ tels que l'énoncé suivant soit vrai : Soit $\mathfrak{z} \in \Xi$ un affinement de $\mathfrak{V}$ et un affinement mod S de $\mathfrak{M}$. Soit $\mathfrak{t}$ un réseau fermé correspondant à $\mathfrak{z}$ (n° 8) et possédant par rapport à $\mathfrak{z}$ la propriété du n° 25. On peut choisir la projection $\pi = Pr.(\mathfrak{z}, \mathfrak{Z})$ de manière que, le réseau fermé $\mathfrak{T}$ correspondant à $\mathfrak{Z}$ étant construit selon le n° 24 en y faisant usage de $\mathfrak{z}$, $\mathfrak{t}$ et π, on ait la propriété du n° 25 ainsi que la propriété suivante, valable pour chaque réseau $\mathfrak{U}$ suffisamment fin et commode relativement à $\mathfrak{z}+\mathfrak{t}$ (et par suite aussi par rapport à $\mathfrak{Z}+\mathfrak{T}$ (v. 24)) : $C^{n-p}(\mathfrak{U})$ étant un $(n-p, \mathfrak{U})$-cycle mod $A\,\overline{R-R_0}$ élémentaire (v. 19.1) par rapport à $\mathfrak{Z}+\mathfrak{T}$ et tel que $C^{n-p}(\mathfrak{U}) \sim 0$ mod $A\,\overline{R-R_0}$ dans A, il existe une $(n-p+1, \mathfrak{U})$-chaîne $E^{n-p+1}(\mathfrak{U})$ dans $\overline{B}$ élémentaire par rapport à $\mathfrak{z}+\mathfrak{t}$ et telle que $E^{n-p+1}(\mathfrak{U}) \to C^{n-p}(\mathfrak{U})$ mod $\overline{\Omega}$.*

La démonstration fera l'objet des n^{os} 45.1—54.3.

45.1. Soit $\mathfrak{T}^*$ un réseau fermé correspondant à $\mathfrak{Z}$ (v. 8) et choisi arbitrairement ; soient T_i^* ($1 \leq i \leq \alpha_0$) les sommets de $\mathfrak{T}^*$ (v. 8) de manière que $T_i^* \subset \sigma_i^0$ ($1 \leq i \leq \alpha_0$) ; posons $R_0^* = \sum_{i=1}^{\alpha_0} T_i^* \subset R - S$ (v. 8).

A chaque point $a \in R$ attachons un voisinage $W(a)$ tel que pour $1 \leq i \leq \alpha_0$: 1° $a \in T_i^*$ entraîne $W(a) \subset \sigma_i^0$; 2° $a \in R - T_i^*$ entraîne $W(a)\,T_i^* = 0$. Soit $W'(a)$ un entourage de a si petit que $W'(a) \Subset W(a)$. L'espace R étant bicompact, il existe un nombre fini de points $a = a_1, a_2, \cdots, a_m$ tels que les entourages correspondants $W_r' = W'(a_r)$ constituent un réseau $\mathfrak{W}'$ [57] ; soit encore $W_r = W(a_r)$ ($1 \leq r \leq m$). On peut supposer qu'il existe un entier m'

[57] On voit sans peine qu'on peut s'arranger de façon que le réseau $\mathfrak{W}'$ soit arbitrairement fin.

$(0 \leqq m' \leqq m)$ tel que $a_r \in R_0^*$ pour $1 \leqq r \leqq m'$, $a_r \in R - R_0^*$ pour $m'+1 \leqq r \leqq m$. Pour $1 \leqq r \leqq m'$ il existe un indice i $(1 \leqq i \leqq \alpha_0)$ tel que $a_r \in T_i^*$, d'où $\bar{W}_r' \subset W_r \subset \sigma_i^0 \subset R - S$. Pour $1 \leqq i \leqq \alpha_0$, il existe évidemment un entourage H_i de T_i^* si petit que: 1° $T_i^* \subset H_i \subset \sigma_i^0$; 2° $\bar{W}_r' T_i^* = 0$ entraîne $\bar{W}_r' H_i = 0$. Pour $1 \leqq r \leqq m$, soit W_r'' un entourage de a_r si petit que: 1° $W_r'' \Subset W_r'$; 2° $a_r \in T_i^*$ entraîne $W_r'' \Subset H_i$ pour $1 \leqq i \leqq \alpha_0$; 3° $\bar{W}_r'' \bar{W}_s'' = 0$ pour $1 \leqq r < s \leqq m$. Pour $1 \leqq r \leqq m$, soit W_r''' un entourage de a_r si petit que $W_r''' \Subset W_r''$.

Il existe évidemment un affinement $\mathfrak{V}_1$ de $\mathfrak{Z}$ jouissant des propriétés suivantes: 1° Pour $V \in \mathfrak{V}_1$, $1 \leqq r < s \leqq m$, on ne peut avoir simultanément $V \bar{W}_r'' \neq 0$, $V \bar{W}_s'' \neq 0$. 2° Pour $V \in \mathfrak{V}_1$, $1 \leqq r \leqq m$, la relation $V \bar{W}_r' \neq 0$ entraîne $V \subset W_r$. 3° Pour $V \in \mathfrak{V}_1$, $1 \leqq r \leqq m$, la relation $V \bar{W}_r''' \neq 0$ entraîne $V \subset W_r''$. 4° $\mathfrak{V}_1$ est un affinement du réseau $\mathfrak{W}'$. 5° $\mathfrak{V}_1$ est un affinement du réseau $\mathfrak{H}$ dont les sommets sont les ensembles H_i $(1 \leqq i \leqq \alpha_0)$ et l'ensemble $R - R_0^*$. 6° Pour $1 \leqq r \leqq m'$ soit $\mathfrak{M}_r$ un réseau fermé dans l'espace $\bar{W}_r'''$ tel que chaque sommet de $\mathfrak{M}_r$ soit un sous-ensemble d'un sommet de $\mathfrak{V}_1$; alors l'ordre (v. 7.2) du réseau $\mathfrak{M}_r$ est $\geqq n$. Il résulte de 2.24 que la condition 6° est réalisable, car $\dim \bar{W}_r''' = n$, d'après 12.2[58].

45.2 Dorénavant, supposons que le réseau $\mathfrak{z} \in \Xi$ soit un affinement de $\mathfrak{V}_1$ et par suite de $\mathfrak{Z}$. Nous allons choisir une projection $\pi = Pr.(\mathfrak{z}, \mathfrak{Z})$. Soit z un sommet de $\mathfrak{z}$. Deux cas sont à distinguer. *En premier lieu*, soit $z \subset R - R_0^*$. Dans ce cas on choisira le sommet $\pi z = Z$ de $\mathfrak{Z}$ de manière que $ZS \neq 0$; nous savons (v. 8) que c'est possible. *En second lieu*, soit $z R_0^* \neq 0$. En vertu de la propriété 5° du réseau $\mathfrak{V}_1$, on peut choisir l'indice i $(1 \leqq i \leqq \alpha_0)$ de telle façon que $z \subset H_i$; on peut donc poser $\pi z = \sigma_i^0$, car $H_i \subset \sigma_i^0$. La projection π n'est pas encore complètement determinée et nous poserons tout de suite des conditions ultérieures.

Choisissons un réseau fermé $\mathfrak{t}$ correspondant à $\mathfrak{z}$ (v. 8). Comme dans le n° 24, désignons par t_ν^0 $(1 \leqq \nu \leqq \beta_0)$ les sommets intérieurs de $\mathfrak{z}$ et par t_ν les sommets correspondants de $\mathfrak{t}$ et posons $R_0' = \sum_{\nu=1}^{\beta_0} t_\nu$. D'après 8, on a $R_0' + g = R$, où g désigne la somme de tous les sommets extérieurs de $\mathfrak{z}$. Soit $b \in \bar{W}_r'''$ $(1 \leqq r \leqq m')$ et soit z un sommet de $\mathfrak{z}$ contenant b. On a $z \bar{W}_r''' \neq 0$, d'où $z \subset W_r''$ d'après la propriété 3° du réseau $\mathfrak{V}_1$. Or $W_r'' \subset R - S$, car $1 \leqq r \leqq m'$; donc $z \subset R - S$ est un sommet intérieur de $\mathfrak{z}$. Donc $b \in R - g \subset R_0'$, c'est-à-dire $\bar{W}_r''' \subset R_0'$ pour $1 \leqq r \leqq m'$. Il en résulte que les ensembles $\bar{W}_r''' t_\nu$ $(1 \leqq \nu \leqq \beta_0)$ constituent (si $1 \leqq r \leqq m'$) un réseau fermé dans $\bar{W}_r'''$. D'après la propriété 6° du réseau $\mathfrak{V}_1$, on en déduit la validité de la remarque suivante: Si $1 \leqq r \leqq m'$, on peut indiquer un

[58] En effet, puisque $1 \leqq r \leqq m'$, on a $\bar{W}_r''' \subset \bar{W}_r' \subset R - S$.

point $b_r \in \overline{W}_r'''$ tel que $b_r \in t_\nu$ pour au moins $n+1$ valeurs différentes de l'indice ν $(1 \leqq \nu \leqq \beta_0)$. Pour $m'+1 \leqq r \leqq m$, choisissons arbitrairement un point $b_r \in \overline{W}_r'''$.

Pour $1 \leqq r \leqq m'$, il existe des indices i $(1 \leqq i \leqq \alpha_0)$ tels que $a_r \in T_i$; soient $i_0, i_1, \cdots, i_q$ tous ces indices. D'après 8, on a $0 \leqq q \leqq n$. D'après la remarque que nous venons de faire, on peut donc indiquer $q+1$ valeurs différentes de ν $(1 \leqq \nu \leqq \beta_0)$ telles que $b_r \in t_\nu$; soient $\nu_0, \nu_1, \cdots, \nu_q$ ces valeurs. Posons pour $0 \leqq h \leqq q$: $\pi \tau^0_{\nu_h} = \sigma^0_{i_h}$. C'est possible et c'est aussi d'accord avec les conditions déjà posées pour π. En effet: 1° Une valeur donnée de ν $(1 \leqq \nu \leqq \beta_0)$ ne peut appartenir à la suite $\nu_0, \nu_1, \cdots, \nu_q$ que pour *une* valeur de r $(1 \leqq r \leqq m')$ au plus; ceci résulte de la propriété 1° du réseau $\mathfrak{B}_1$, car l'inclusion $b_r \in t_\nu$ entraîne que $\tau^0_\nu \overline{W}_r''' \neq 0$, d'où $\tau^0_\nu \subset W_r''$ d'après la propriété 3° de $\mathfrak{B}_1$. 2° Pour $1 \leqq r \leqq m'$, $0 \leqq h \leqq q$, on a $\tau^0_{\nu_h} \subset W_r''$, comme nous venons de voir; d'autre part on a $a_r \in T^*_{i_h}$, d'où $W_r'' \subset H_{i_h}$ d'après la propriété 2° des ensembles W_r''; donc $\tau^0_{\nu_h} \subset H_{i_h}$ de manière qu'il est permis de poser $\pi \tau^0_{\nu_h} = \sigma^0_{i_h}$.

45.3. A l'aide de $\mathfrak{z}$, $\mathfrak{t}$ et π, construisons le réseau fermé $\mathfrak{T}$ correspondant à $\mathfrak{z}$ suivant la manière expliquée au n° 24 et faisons usage des notations de ce n°. En particulier, on a pour $1 \leqq i \leqq \alpha_0$: $T_i = \sum t_\nu$, où ν parcourt toutes les valeurs $(1 \leqq \nu \leqq \gamma_0)$ telles que $\pi \tau^0_\nu = \sigma^0_i$. Il en résulte que pour $1 \leqq r \leqq m'$, $0 \leqq h \leqq q$, on a (dans les notations du n° 45.2) $t_{\nu_h} \subset T_{i_h}$, d'où $b_r \in T_{i_h}$ pour $0 \leqq h \leqq q$. Inversement, supposons que, pour de certaines valeurs de i et de r $(1 \leqq i \leqq \alpha_0,\ 1 \leqq r \leqq m)$ on ait l'inclusion $b_r \in T_i$. Il existe alors une valeur de ν $(1 \leqq \nu \leqq \gamma_0)$ telle que $b_r \in t_\nu$, $\pi \tau^0_\nu = \sigma^0_i$. D'après nos conventions relatives à la projection π, on a donc l'inclusion $\tau^0_\nu \subset H_i$. Or $b_r \in t_\nu \subset \tau^0_\nu$, $b_r \in \overline{W}_r''' \subset \overline{W}_r'$; donc $H_i \overline{W}_r' \neq 0$ et par suite $T_i^* \overline{W}_r' \neq 0$ d'après la propriété 2° de H_i. Or $\overline{W}_r' \subset W_r$, d'où $T_i^* W_r \neq 0$ ce qui entraîne $a_r \in T_i^* \subset R_0^*$. Donc $1 \leqq r \leqq m'$ et $i = i_h$ pour une certaine valeur de h $(0 \leqq h \leqq q)$.

Nous avons donc prouvé que pour $1 \leqq r \leqq m$, $1 \leqq i \leqq \alpha_0$ les deux inclusions $a_r \in T_i^*$ et $b_r \in T_i$ sont équivalentes. On en déduit le LEMME: Le réseau $\mathfrak{W}'$ (aux sommets $W_1', \cdots, W_m'$) possède la propriété suivante: pour $1 \leqq i \leqq \alpha_0$, $1 \leqq r \leqq m$: 1° $b_r \in T_i$ entraîne $W_r' \subset \sigma^0_i$; 2° $b_r \in R - T_i$ entraîne $W_r' T_i = 0$.

Démonstration. 1° Soit $b_r \in T_i$; alors $a_r \in T_i^*$, donc $W_r' \subset W_r \subset \sigma^0_i$. 2° Soit $b_r \in R - T_i$; alors $a_r \in R - T_i^*$, donc $W_r T_i^* = 0$. Supposons, par impossible, qu'il existe un point $c \in W_r' T_i$. D'après la définition de $\mathfrak{T}$, il existe une valeur de ν $(1 \leqq \nu \leqq \alpha_0)$ telle que $c \in t_\nu \subset \tau^0_\nu \subset H_i$. On a donc $W_r' H_i \neq 0$, d'où $\overline{W}_r' T_i^* \neq 0$, ce qui donne la contradiction $W_r T_i^* \neq 0$.

45.31. Nous avons remarqué (v. [57]) que le réseau $\mathfrak{W}'$ peut être supposé arbitrairement fin. Nous voulons profiter de cette remarque. Soit B_1 un ensemble ouvert tel que

$$A \subset B_1 \Subset B. \tag{1}$$

Nous supposerons que le réseau $\mathfrak{W}'$ soit choisi de manière qu'il jouisse de la propriété suivante (v. 1.2): Si $W'_r A \neq 0$, $W'_r W'_s \neq 0$, on a $W'_s \subset B_1$.

45.4. De la démonstration faite au n° 25.2 du théorème du n° 25 on déduit sans peine qu'il existe un affinement $\mathfrak{V}_2$ de $\mathfrak{V}_1$ jouissant de la propriété suivante: Supposons que le réseau $\mathfrak{z} \in \Xi$ soit un affinement de $\mathfrak{V}_2$. On peut choisir la projection $\pi = Pr.(\mathfrak{z}, \mathfrak{Z})$ (satisfaisant aux conditions du n° 45.2) de manière que $K^{n-p}(\sigma_i^p, \mathfrak{U}) \neq 0$ pour $1 \leqq i \leqq \alpha_p$ dans chaque réseau commode suffisamment fin.

Dorénavant on suppose $\mathfrak{z}$ et π choisis de manière que cette propriété soit vérifiée, ainsi que le lemme du n° 45.3.

46. Lemme. Soit $\mathfrak{U}$ un réseau commode (relativement à $\mathfrak{Z}+\mathfrak{T}$). Soit $C^{n-p}(\mathfrak{U})$ une $(n-p, \mathfrak{U})$-chaîne élémentaire (v. 19.1) telle que $C^{n-p}(\mathfrak{U}) \to 0 \bmod \overline{R - R_0}$. On peut attacher à chaque combinaison $r_0, r_1, \cdots, r_h$ $(1 \leqq h \leqq p-1)$, d'indices $1, 2, \cdots, m$ une $(n-p+1, \mathfrak{U})$-chaîne élémentaire $D^{n-p+h+1}_{r_0 r_1 \cdots r_h}(\mathfrak{U})$ de manière que: 1° chaque simplexe $\neq 0 \bmod \overline{R - R_0}$ de la chaîne $FD^{n-p+1}_r(\mathfrak{U}) - C^{n-p}(\mathfrak{U})$ $(1 \leqq r \leqq m)$ est situé dans $R - W'_r$; 2° pour $1 \leqq h \leqq p-1$, $D^{n-p+h+1}_{r_0 r_1 \cdots r_h}(\mathfrak{U})$ est une fonction alternée des indices $r_0, r_1, \cdots, r_h$; 3° pour chaque combinaison $r_0, r_1, \cdots, r_h$ $(1 \leqq h \leqq p-1)$, chaque simplexe $\neq 0 \bmod \overline{R - R_0}$ de la chaîne

$$FD^{n-p+h+1}_{r_0 r_1 \cdots r_h}(\mathfrak{U}) - \sum_{u=0}^{h} (-1)^u D^{n-p+h}_{r_0 \cdots r_{u-1} r_{u+1} \cdots r_h}(\mathfrak{U})$$

est situé dans $R - \prod_{u=0}^{h} W'_{r_u}$.

Démonstration. Soit $C^{n-p}(\mathfrak{U}) = \sum_{i=1}^{\alpha_p} c_i K^{n-p}(\sigma_i^p, \mathfrak{U})$. D'après 19.1, 1° et 3°

$$\sum_{i=1}^{\alpha_p} \eta^p_{ji} c_i = 0 \quad \text{pour} \quad 1 \leqq j \leqq \alpha_{p+1} \text{ [59]}. \tag{1}$$

Pour $1 \leqq r \leqq m$, soit N_r l'ensemble de toutes les valeurs de i $(1 \leqq i \leqq \alpha_0)$ telles que $b_r \in T_i$ (b_r étant les points du lemme du n° 45.3); les σ^0_i, $i \in N(r)$ sont les sommets d'un $\mathfrak{Z}$-simplexe intérieur que nous désignerons par $\sigma(r)$. Plus généralement, pour chaque combinaison $r_0, r_1, \cdots, r_h$ $(0 \leqq h \leqq p-1)$ soit $\sigma(r_0, r_1, \cdots, r_h)$ la face commune de dimension maxima des $h+1$ $\mathfrak{Z}$-simplexes $\sigma(r_0), \sigma(r_1), \cdots, \sigma(r_h)$.

[59] Ces équations ne disent *rien* dans le cas $p = n$.

Ceci étant, posons pour chaque combinaison $r_0, r_1, \cdots, r_h$ $(0 \leqq h \leqq p-1)$

$$D^{n-p+h+1}_{r_0 r_1 \cdots r_h}(\mathfrak{U}) = \sum_i x_i^{r_0 r_1 \cdots r_h} K^{n-p+h+1}(\sigma_i^{p-h-1}, \mathfrak{U}),$$

où la sommation se rapporte à toutes les valeurs de i $(1 \leqq i \leqq \alpha_{p-h-1})$ telles que σ_i^{p-h-1} soit une face du simplexe $\sigma(r_0, r_1, \cdots, r_h)$ et où les nombres $x_i^{r_0 r_1 \cdots r_h} \in \mathfrak{R}$ sont des fonctions alternées des indices supérieurs.

Or on déduit sans peine du lemme du n° 45.3 que, si le simplexe σ_i^k $(0 \leqq k \leqq n,\ 1 \leqq i \leqq \alpha_k)$ n'est pas une face de $\sigma(r_0, r_1, \cdots, r_h)$, la chaîne $K^{n-k}(\sigma_i^k, \mathfrak{U})$ est située dans $R - \prod_{u=0}^{h} W'_{r_u}$. Il en résulte aisément (v. 19.1, 1° et 3°) qu'il suffit de déterminer les nombres $x_i^{r_0 r_1 \cdots r_h}$ de manière que l'on ait

$$1° \quad \sum_{i=1}^{\alpha_{p-1}} \eta_{ji}^{p-1} x_i^r = c_j$$

pour $1 \leqq r \leqq m$ et pour toutes les valeurs de j $(1 \leqq j \leqq \alpha_p)$ telles que σ_j^p soit une face de $\sigma(r)$;

$$2° \quad \sum_{i=1}^{\alpha_{p-h-1}} \eta_{ji}^{p-h-1} x_i^{r_0 r_1 \cdots r_h} = \sum_{u=0}^{h} (-1)^u x_j^{r_0 \cdots r_{u-1} r_{u+1} \cdots r_h}$$

pour chaque combinaison $r_0, r_1, \cdots, r_h$ $(1 \leqq h \leqq p-1)$ d'indices $1, 2, \cdots, m$ et pour toutes les valeurs de j $(1 \leqq j \leqq \alpha_{p-h})$ telles que σ_j^{p-h} soit une face de $\sigma(r_0, r_1, \cdots, r_h)$.

Pour voir que ces conditions sont réalisables, choisissons une combinaison $r_0, r_1, \cdots, r_h$ $(0 \leqq h \leqq p-1)$ et supposons que les indices i, j, k parcourent resp. toutes les valeurs $(1 \leqq i \leqq \alpha_{p-h-1},\ 1 \leqq j \leqq \alpha_{p-h},\ 1 \leqq k \leqq \alpha_{p-h+1})$ telles que σ_i^{p-h-1}, σ_j^{p-h}, σ_k^{p-h+1} soient des faces de $\sigma(r_0, r_1, \cdots, r_h)$. Considérons le système d'équations linéaires pour les inconnues $z_i \in \mathfrak{R}$

$$(*) \qquad \sum_i \eta_{ji}^{p-h-1} z_i = u_j.$$

Pour que ce système possède une solution, il faut et il suffit que les nombres $u_j \in \mathfrak{R}$ satisfassent à $\delta_{p-h} - \varrho_{p-h-1}$ conditions lin. indépendantes, où δ_{p-h} désigne le nombre des $(p-h)$-faces du simplexe $\sigma(r_0, r_1, \cdots, r_h)$ et ϱ_{p-h-1} est le rang de la matrice (η_{ji}^{p-h-1}). Or le $(p-h)^{\text{ème}}$ nombre de Betti du simplexe $\sigma(r_0, r_1, \cdots, r_h)$ étant égal à zéro, on a $\delta_{p-h} - \varrho_{p-h-1} = \varrho_{p-h}$ = rang de la matrice (η_{kj}^{p-h}). D'autre part, on voit sans peine que (*) entraîne $\sum_j \eta_{kj}^p u_j = 0$, car $\sum_j \eta_{kj}^p \eta_{ji}^{p-1} = 0$. Le rang de la matrice (η_{k-j}^{p-h}) étant égal à $\delta_{p-h} - \varrho_{p-h-1}$, on voit que le système

$$(**) \qquad \sum_j \eta_{kj}^p u_j = 0$$

donne la condition nécéssaire et suffisante pour la résolubilité du système (*). En faisant usage de ce critère et en s'appuyant sur les équations (1) on voit sans peine qu'il est possible de construire successivement les nombres $x_i^{r_0 r_1 \cdots r_h}$ [60].

46.1. *Remarque.* On voit sans peine que l'on peut s'arranger de façon qu'on ait $D^{n-p+h+1}_{r_0 r_1 \cdots r_h}(\mathfrak{U}) \neq 0$ seulement si un des indices $r_0, r_1, \cdots, r_h$, soit r, jouit de la propriété que $b_r \in T(\sigma_i^p)$ pour une valeur de i $(1 \leqq i \leqq \alpha_0)$ telle que le coefficient c_i de $K^{n-p}(\sigma_i^p, \mathfrak{U})$ dans la chaîne $C^{n-p}(\mathfrak{U})$ soit $\neq 0$.

47.1. Nous allons démontrer le théorème du n° 44 dans le cas $p = 1$. L'espace R étant normal, il existe deux ensembles ouverts Ω_1 et Ω_2 tels que

$$S_1 \Subset \Omega_2 \Subset \Omega_1 \Subset \Omega.$$

Soit $\mathfrak{P}$ un réseau gén. tel que 1° chaque sommet de $\mathfrak{P}$ est un sous-ensemble d'un sommet W_r' du réseau $\mathfrak{W}'$ (v. 45.1); 2° pour chaque sommet P de $\mathfrak{P}$ on a $P\Omega_2 = 0$ ou bien $P \subset \Omega_1$. On voit sans peine que $\mathfrak{P}$ existe. Déterminons un affinement $\mathfrak{Q}$ de $\mathfrak{P}$ ainsi qu'une projection $\pi' = Pr.(\mathfrak{Q}, \mathfrak{P})$ jouissant de la propriété du n° 14.2. Soit $\mathfrak{V}$ un affinement du réseau $\mathfrak{V}_2$ (v. 45.4) et donc aussi de $\mathfrak{V}_1$ (v. 45.1) tel que 1° $V \in \mathfrak{V}$, $V\overline{S_1} \neq 0$ entraîne $V \subset \Omega_1$; 2° chaque sommet V de $\mathfrak{V}$ tel que $V\overline{S_1} = 0$ est un sous-ensemble d'un sommet de $\mathfrak{Q}$.

Supposons que $\mathfrak{z}$ soit un affinement de $\mathfrak{V}$. La suite $1, 2, \cdots, \beta_0$ des indices ν se divise en deux classes N', N'' d'après la convention suivante: $\nu \in N'$ $(\nu \in N'')$ signifie que $\tau_\nu^0 \subset \Omega_1$ $(\tau_\nu^0 - \Omega_1 \neq 0)$. Soit $\nu \in N''$. Alors il existe un sommet $Q(\tau_\nu^0)$ de $\mathfrak{Q}$ tel que $\tau_\nu^0 \subset Q(\tau_\nu^0)$. Posons $P(\tau_\nu^0) = \pi' Q(\tau_\nu^0)$. Il existe un indice $r(\nu)$ $(1 \leqq r(\nu) \leqq m)$ tel que $P(\tau_\nu^0) \subset W'_{r(\nu)}$. On a $P(\tau_\nu^0) \subset \Omega_1$ ou bien $P(\tau_\nu^0)\Omega_2 = 0$.

Pour chaque réseau $\mathfrak{U}$ commode relativement à $\mathfrak{z} + \mathfrak{t}$ considérons l'ensemble $M(\mathfrak{U})$ de toutes les $(n-1, \mathfrak{U})$-chaînes $C^{n-1}(\mathfrak{U})$ dans A élémentaires par rapport à $\mathfrak{Z} + \mathfrak{T}$ et telles que $H^n(\mathfrak{U}) \to C^{n-1}(\mathfrak{U}) \bmod \overline{R - R_0}$ pour une $(n, \mathfrak{U})$-chaîne $H^n(\mathfrak{U}) \subset A$. L'ensemble $M(\mathfrak{U})$ est un module fini; désignons par $s(\mathfrak{U})$ son rang. On a $0 \leqq s(\mathfrak{U}) \leqq n$. Par suite l'ensemble de toutes les valeurs de $s(\mathfrak{U})$ admet un minimum s_0. Lorsque $\mathfrak{U}_1$ est un affinement de $\mathfrak{U}$, on voit sans peine (v. 19.1, 3° et 19.2) que $s(\mathfrak{U}_1) \leqq s(\mathfrak{U})$; par suite l'égalité $s(\mathfrak{U}) = s_0$ entraîne $s(\mathfrak{U}_1) = s_0$.

Désignons par Φ la famille de tous les réseaux commodes (relativement à $\mathfrak{z} + \mathfrak{t}$) $\mathfrak{U}$ jouissant des propriétés suivantes: 1° $U \in \mathfrak{U}$, $U\Omega_1 \neq 0$ entraîne $U \subset \Omega$; 2° $U \in \mathfrak{U}$, $U - \Omega_2 \neq 0$ entraîne $U\overline{R - R_0} = 0$ [61]; 3° $U \in \mathfrak{U}$, $U\tau_\nu^0 \neq 0$ $(1 \leqq \nu \leqq \beta_0)$ entraîne $U \subset \tau_\nu^0$; 4° $G^n(\mathfrak{U}) \not\sim 0 \bmod [R - Q(\tau_\nu^0)]$ pour $\nu \in N''$

[60] Il faut tenir compte de ce que $\sigma(r_0, r_1, \cdots, r_h)$ est une face de $\sigma(r_1, \cdots, r_h)$.

[61] D'après 8, on a $\overline{R - R_0} \subset \overline{S_1}$, de sorte que Ω_2 est un entourage de $\overline{R - R_0}$.

(v. 17.5); 5° $s(\mathfrak{U}) = s_0$. On voit sans peine que $\mathfrak{U} \in \Phi$ entraîne que $\mathfrak{U}_1 \in \Phi$ pour chaque affinement commode $\mathfrak{U}_1$ de $\mathfrak{U}$.

47.2. Pour démontrer le théorème du n° 44 dans le cas actuel $p = 1$, supposons que $C^{n-1}(\mathfrak{U}) \in M(\mathfrak{U})$, $\mathfrak{U}$ étant un réseau de la famille Φ. Soit $H^n(\mathfrak{U})$ une $(n, \mathfrak{U})$-chaîne dans A telle que $H^n(\mathfrak{U}) \to C^{n-1}(\mathfrak{U}) \bmod \overline{R - R_0}$. On doit démontrer qu'il existe une $(n, \mathfrak{U})$-chaîne $E^n(\mathfrak{U})$ dans B élémentaire relativement à $\mathfrak{z} + \mathfrak{t}$ et telle que $E^n(\mathfrak{U}) \to C^{n-1}(\mathfrak{U}) \bmod \bar{\Omega}$. Il suffit donc de prouver qu'il existe une $(n, \mathfrak{U})$-chaîne $E^n(\mathfrak{U})$ élémentaire relativement à $\mathfrak{z} + \mathfrak{t}$ et telle que $H^n(\mathfrak{U}) = E^n(\mathfrak{U}) \bmod \bar{\Omega}$.

D'après 8.33, chaque simplexe φ^n de $H^n(\mathfrak{U})$ est (relativement à $\mathfrak{z} + \mathfrak{t}$) d'espèce τ^{-1} ou d'espèce τ^0_ν, où $1 \leqq \nu \leqq \beta_0$. Si φ^n est d'espèce τ^{-1}, on a (v. 8.32) $\varphi^n = 0 \bmod \overline{R - R'_0} \subset \overline{R - R_0} \subset \bar{\Omega}$. Si φ^n est d'espèce τ^0_ν, où $\nu \in N'$, on a $\tau^0_\nu \subset \Omega_1$; or chaque sommet U de φ^n rencontre τ^0_ν, de manière que $U \subset \Omega$ d'après la propriété 1° de la famille Φ; donc, ici encore, on a $\varphi^n = 0 \bmod \bar{\Omega}$.

Considérons le cas où le simplexe φ^n de $H^n(\mathfrak{U})$ est d'espèce τ^0_ν, où $\nu \in N''$. On a $\tau^0_\nu \subset Q(\tau^0_\nu) \subset P(\tau^0_\nu) \subset W'_{r(\nu)}$. Lorsque $P(\tau^0_\nu) \subset \Omega$, on a de nouveau $\varphi^n = 0 \bmod \bar{\Omega}$; supposons donc que $P(\tau^0_\nu) - \Omega_1 \neq 0$ ce qui entraîne que $P(\tau^0_\nu)\, \Omega_2 = 0$.

Il suffit donc de démontrer l'énoncé suivant: Soit $\nu \in N''$, $P(\tau^0_\nu)\, \Omega_2 = 0$. Soit $E^n_\nu(\mathfrak{U})$ cette partie de la chaîne $H^n(\mathfrak{U})$ dont les simplexes sont d'espèce τ^0_ν. Il existe un nombre $c \in \mathfrak{R}$ tel que $E^n_\nu(\mathfrak{U}) = c k^n(\tau^0_\nu, \mathfrak{U})$.

Soit $\mathfrak{U}_1 \in \Phi$ un affinement de $\mathfrak{U}$ normal par rapport aux cycles mod $R - P(\tau^0_\nu)$, pour chaque $\nu \in N''$. Comme $s(\mathfrak{U}) = s_0$, on voit sans peine qu'il existe une $(n-1, \mathfrak{U}_1)$-chaîne $C^{n-1}(\mathfrak{U}_1) \in M(\mathfrak{U}_1)$ telle que $C^{n-1}(\mathfrak{U}) = \pi C^{n-1}(\mathfrak{U}_1) \bmod \overline{R - R_0}$, $\pi = Pr.(\mathfrak{U}_1, \mathfrak{U})$. Il existe une $(n, \mathfrak{U}_1)$-chaîne $H^n(\mathfrak{U}_1)$ dans A telle que $H^n(\mathfrak{U}_1) \to C^{n-1}(\mathfrak{U}_1) \bmod \overline{R - R_0}$. On peut donc supposer que $H^n(\mathfrak{U}) = \pi H^n(\mathfrak{U}_1)$.

Ceci étant, d'après le lemme du n° 46, il existe une $(n, \mathfrak{U}_1)$-chaîne $D^n(\mathfrak{U}_1)$ élémentaire [relativement à $\mathfrak{Z} + \mathfrak{T}$ et par suite aussi relativement à $\mathfrak{z} + \mathfrak{t}$, v. 24 (3)] telle que $C^{n-1}(\mathfrak{U}_1) - F D^n(\mathfrak{U}_1) \subset \overline{R - R_0} + [R - W'_{r(\nu)}]$. Comme $P(\tau^0_\nu) \subset W'_{r(\nu)}$, on a $P(\tau^0_\nu)[R - W'_{r(\nu)}] = 0$. Comme $P(\tau^0_\nu)\, \Omega_2 = 0$, $\Omega_2 \supset S_1 \supset \overline{R - R_0}$, on a $\overline{R - R_0} \subset R - P(\tau^0_\nu)$. Par suite $C^{n-1}(\mathfrak{U}_1) - F D^n(\mathfrak{U}_1) \subset R - P(\tau^0_\nu)$. D'autre part $H^n(\mathfrak{U}_1) \to C^{n-1}(\mathfrak{U}) \bmod \overline{R - R_0} \subset R - P(\tau^0_\nu)$. Par suite $H^n(\mathfrak{U}_1) - D^n(\mathfrak{U}_1)$ est un $(n, \mathfrak{U}_1)$-cycle mod $R - P(\tau^0_\nu)$. Or soit (v. 19.2) $D^n(\mathfrak{U})$ une $(n, \mathfrak{U})$-chaîne élémentaire telle que $D^n(\mathfrak{U}) = \pi D^n(\mathfrak{U}_1) \bmod \overline{R - R_0}$. Evidemment $H^n(\mathfrak{U}) - D^n(\mathfrak{U})$ est un $(n, \mathfrak{U})$-cycle essentiel mod $[R - P(\tau^0_\nu)]$. D'après 14.2, il existe deux nombres $c_1, c_2 \in \mathfrak{R}$ dont un au moins $\neq 0$, tels que $c_1 [H^n(\mathfrak{U}) - D^n(\mathfrak{U})] + c_2 G^n(\mathfrak{U}) \sim 0 \bmod [R - Q(\tau^0_\nu)]$. On a $c_1 \neq 0$ d'après la propriété 4° de la famille Φ. Donc il existe un nombre $c_0 \in \mathfrak{R}$ tel que

$H^n(\mathfrak{U}) - D^n(\mathfrak{U}) - c_0 G^n(\mathfrak{U}) \sim 0 \bmod [R - Q(\tau_\nu^0)]$. Il existe donc une $(n, \mathfrak{U})$-chaîne $X^n(\mathfrak{U}) \subset R - Q(\tau_\nu^0)$ et une $(n+1, \mathfrak{U})$-chaîne $Y^{n+1}(\mathfrak{U})$ telles que

$$(*) \qquad H^n(\mathfrak{U}) = D^n(\mathfrak{U}) + c_0 G^n(\mathfrak{U}) + X^n(\mathfrak{U}) + F Y^{n+1}(\mathfrak{U}).$$

Or soit φ^n un $(n, \mathfrak{U})$-simplexe d'espèce τ_ν^0 et soit U un sommet de φ^n. On a $U t_\nu \neq 0$, d'où $U \subset \tau_\nu^0$ d'après la propriété 3° de la famille Φ. Puisque $\tau_\nu^0 \subset Q(\tau_\nu^0)$, la chaîne $X^n(\mathfrak{U})$ ne contient aucun simplexe d'espèce τ_ν^0. Puisque $U \subset \tau_\nu^0 \subset Q(\tau_\nu^0) \subset P(\tau_\nu^0) \subset R - \Omega_2$, U ne peut rencontrer aucun sommet U' de $\mathfrak{U}$ tel que $U' S \neq 0$ (v. 8.1, 2°). D'après 8.1, 3° il en résulte qu'aucun simplexe de la chaîne $F Y^{n+1}(\mathfrak{U})$ n'est d'espèce τ_ν^0. L'inclusion $U \subset R - \Omega_2$ montre encore que $\varphi^n \neq 0 \bmod \overline{R - R_0}$. De (*) il résulte maintenant que $E_\nu^n(\mathfrak{U})$ est cette partie de la chaîne $D^n(\mathfrak{U}) + c_0 G^n(\mathfrak{U})$ dont les simplexes sont d'espèce ν. Donc [v. 19.1, 2°, 3° et 4° ainsi que 24, (3)] il existe un nombre $c \in \mathfrak{R}$ tel que $E_\nu^n(\mathfrak{U}) = c k^n(\tau_\nu^0, \mathfrak{U})$.

48. Le théorème du n° 44 étant démontré pour $p = 1$, supposons dorénavant que $2 \leqq p \leqq n$.

49.1. L'espace R étant normal, il existe un ensemble ouvert Ω_1 tel que $S_1 \Subset \Omega_1 \Subset \Omega$. Soit $\mathfrak{P}_1^{p-1}$ un réseau gén. jouissant des propriétés suivantes : 1° $P \in \mathfrak{P}_1^{p-1}$, $P \bar{B}_1 \neq 0$ entraîne $P \subset B$ [v. 45.31 (1)] ; 2° a étant un point quelconque de $R - S$, il existe un sommet du réseau $\mathfrak{W}'$ du n° 45.1 contenant la fermeture de tous les sommets de $\mathfrak{P}_1^{p-1}$ qui passent par a (v. 4.3 et 4.4) ; 3° la fermeture d'aucun sommet de $\mathfrak{P}_1^{p-1}$ ne rencontre simultanément $\bar{S}_1$ et $R - \Omega_1$.

49.2. En partant du réseau gén. $\mathfrak{P}_1^{p-1}$, construisons les réseaux gén. $\mathfrak{Q}^h$, $\mathfrak{P}^h$, $\mathfrak{P}_1^h$, $\mathfrak{P}_2^h$, $\mathfrak{P}_3^h$, $\mathfrak{P}_4^h$ $(0 \leqq h \leqq p-2)$ ainsi que $\mathfrak{P}_2^{p-1}$, $\mathfrak{P}_3^{p-1}$, $\mathfrak{P}_4^{p-1}$, $\mathfrak{P}_5^{p-1}$ selon la manière expliquée dans le n° 29, en y remplaçant p par $p-1$.

50.1. Supposons que $\mathfrak{z}$ soit un affinement du réseau $\mathfrak{V} = \mathfrak{V}_2$ du n° 45.4 (et donc aussi du réseau $\mathfrak{V}_1$ du n° 45.1) et que $\mathfrak{z}$ soit un affinement mod S de $\mathfrak{M} = \mathfrak{P}_5^{p-1}$.

Pour $0 \leqq h \leqq p-2$, $1 \leqq \nu \leqq \beta_h$ attachons à τ_ν^h des sommets $P_4^h(\tau_\nu^h)$, $P_3^h(\tau_\nu^h)$, $P_2^h(\tau_\nu^h)$, $P_1^h(\tau_\nu^h)$, $P^h(\tau_\nu^h)$, $Q^h(\tau_\nu^h)$ resp. de $\mathfrak{P}_4^h$, $\mathfrak{P}_3^h$, $\mathfrak{P}_2^h$, $\mathfrak{P}_1^h$, $\mathfrak{P}^h$, $\mathfrak{Q}^h$; pour $1 \leqq \nu \leqq \beta_{p-1}$ attachons à τ_ν^{p-1} des sommets $P_3^{p-1}(\tau_\nu^{p-1})$, $P_2^{p-1}(\tau_\nu^{p-1})$, $P_1^{p-1}(\tau_\nu^{p-1})$ resp. de $\mathfrak{P}_3^{p-1}$, $\mathfrak{P}_2^{p-1}$, $\mathfrak{P}_1^{p-1}$. Ceci soit fait selon la manière expliquée au n° 30, en y remplaçant p, $\mathfrak{Z}$, σ_i^h resp. par $p-1$, $\mathfrak{z}$, τ_ν^h.

50.2. Les réseaux gén. $\mathfrak{P}_1^h$ $(0 \leqq h \leqq p-1)$ étant des affinements de $\mathfrak{P}_1^{p-1}$, il résulte de 49.1, 2° qu'on peut attacher à chaque ν $(1 \leqq \nu \leqq \beta_0)$ un sommet $W'_{r(\nu)}$ du réseau $\mathfrak{W}'$ de manière que $\overline{P_1^h(\tau_\lambda^h)} \subset W'_{r(\nu)}$ pour $0 \leqq h \leqq p-1$ et pour chaque valeur de λ $(1 \leqq \lambda \leqq \beta_h)$ telle que τ_ν^0 soit un sommet de τ_λ^h.

50.3. Les réseaux gén. $\mathfrak{P}_1^h$ $(0 \leqq h \leqq p-1)$ étant des affinements de $\mathfrak{P}_1^{p-1}$, on ne peut pas (v. 49, 3°) avoir simultanément $\overline{P_1^h(\tau_\nu^h)} - \Omega_1 \neq 0$, $\overline{P_1^h(\tau_\nu^h)} \cdot \overline{R - R_0} \neq 0$ $(0 \leqq h \leqq p-1,\ 1 \leqq \nu \leqq \beta_h)$.

51. Désignons par Φ la famille de tous les réseaux $\mathfrak{U}$ commodes (relativement à $\mathfrak{z}+\mathfrak{t}$) tels que : 1° pour $0 \leqq h \leqq p-2$, $1 \leqq \nu \leqq \beta_h$, $U \in \mathfrak{U}$, la relation $U\overline{P_4^h(\tau_\nu^h)} \neq 0$ entraîne $U \subset P_3^h(\tau_\nu^h)$; 2° pour $0 \leqq h \leqq p-2$, $1 \leqq \nu \leqq \beta_h$, $U \in \mathfrak{U}$, la relation $U\overline{P_2^h(\tau_\nu^0)} \neq 0$ entraîne $U \subset P_1^h(\tau_\nu^0)$; 3° pour $1 \leqq \nu \leqq \beta_{p-1}$, $U \in \mathfrak{U}$, la relation $UP_3^{p-1}(\tau_\nu^{p-1}) \neq 0$ entraîne $U \subset P_2^{p-1}(\tau_\nu^{p-1})$; 4° pour $1 \leqq \nu \leqq \beta_0$, $U \in \mathfrak{U}$, les relations $UA \neq 0$, $U\overline{P_1^0(\tau_\nu^0)} \neq 0$ entraînent $A\overline{P_1^0(\tau_\nu^0)} \neq 0$; 5° pour $1 \leqq \nu \leqq \beta_{p-1}$, on a $G^n(\mathfrak{U}) \not\equiv 0 \bmod [R - P_3^{p-1}(\tau_\nu^{p-1})]$; 6° pour $0 \leqq h \leqq p-2$, $0 \leqq k \leqq p-2$, $1 \leqq \nu \leqq \beta_h$, $1 \leqq \mu \leqq \beta_k$, les indices ν et μ étant tels que les noyaux de τ_ν^h et de τ_μ^h aient un point commun, la relation $Ut(\tau_\mu^k) \neq 0$ (où $U \in \mathfrak{U}$) entraîne $U \subset P_4^h(\tau_\nu^h)$; 7° pour $1 \leqq \lambda \leqq \beta_0$, $1 \leqq \nu \leqq \beta_{p-1}$, les indices λ et ν étant tels que τ_λ^0 est un sommet de τ_ν^{p-1}, la relation $Ut_\lambda \neq 0$ (où $U \in \mathfrak{U}$) entraîne $U \in P_3^{p-1}(\tau_\nu^{p-1})$; 8° pour $U \in \mathfrak{U}$, la relation $US \neq 0$ entraîne $UP_1^0(\tau_\nu^0) = 0$ pour $1 \leqq \nu \leqq \beta_0$; 9° pour U', $U'' \in \mathfrak{U}$, les relations $U'S \neq 0$, $U''R_0' \neq 0$ entraînent $U'U'' = 0$; 10° on a $s(\mathfrak{U}) =$ minimum, $s(\mathfrak{U})$ désignant le rang du module $M(\mathfrak{U})$ de toutes les $(n-p, \mathfrak{U})$-chaînes $C^{n-p}(\mathfrak{U})$ dans A élémentaires (v. 19.1) par rapport à $\mathfrak{Z}+\mathfrak{T}$ et telles qu'il existe une $(n-p+1, \mathfrak{U})$-chaîne $H^{n-p+1}(\mathfrak{U})$ dans A telle que $H^{n-p+1}(\mathfrak{U}) \to C^{n-p}(\mathfrak{U}) \bmod \overline{R - R_0}$; 11° pour $0 \leqq h \leqq p-1$, $1 \leqq \nu \leqq \beta_h'$, $U \in \mathfrak{U}$ la relation $U\overline{P_1^h(\tau_\nu^h)} \neq 0$ entraîne $U \subset W'_{r(\lambda)}$ pour chaque sommet τ_λ^0 de τ_ν^h; 12° $U \in \mathfrak{U}$, $U\Omega_1 \neq 0$ entraîne $U \subset \Omega$.

On voit sans peine (v. 32 et 47.1) que la famille Φ est parfaitement complète.

Nous démontrerons que l'énoncé du n° 44 est vrai pour chaque $\mathfrak{U} \in \Phi$.

52.1. Choisissons un réseau $\mathfrak{U} = \mathfrak{U}_0 \in \Phi$ et déterminons les affinements successifs $\mathfrak{U}_h \in \Phi$ $(1 \leqq h \leqq 2p-1)$ selon la manière expliquée au n° 33, en y remplaçant p, $\mathfrak{Z}$, σ_i^h resp. par $p-1$, $\mathfrak{z}$, τ_ν^h. Soit $\pi_h = Pr.(\mathfrak{U}_{h+1}, \mathfrak{U}_h)$ $(0 \leqq h \leqq 2p-2)$.

52.2. Supposons donnée une chaîne $C^{n-p}(\mathfrak{U}_0) \in M(\mathfrak{U}_0)$ (v. 51, 10°). On doit démontrer (v. 44) qu'il existe une $(n-p+1)$-chaîne $E^{n-p+1}(\mathfrak{U}_0)$ dans $\overline{B}$ élémentaire par rapport à $\mathfrak{z}+\mathfrak{t}$ et telle que $E^{n-p+1}(\mathfrak{U}_0) \to C^{n-p}(\mathfrak{U}_0) \bmod \overline{\Omega}$. Comme $s(\mathfrak{U}_0) =$ minimum, il existe pour $1 \leqq h \leqq 2p-1$ une chaîne $C^{n-p}(\mathfrak{U}_h) \in M(\mathfrak{U}_h)$ telle que $\pi_h C^{n-p}(\mathfrak{U}_{h+1}) = C^{n-p}(\mathfrak{U}_h) \bmod \overline{R - R_0}$ pour $0 \leqq h \leqq 2p-2$. Puisque $C^{n-p}(\mathfrak{U}_{2p-1}) \in M(\mathfrak{U}_{2p-1})$, il existe une $(n-p+1, \mathfrak{U}_{2p-1})$-chaîne $H^{n-p+1}(\mathfrak{U}_{2p-1})$ dans A telle que $H^{n-p+1}(\mathfrak{U}_{2p-1}) \to C^{n-p}(\mathfrak{U}_{2p-1}) \bmod \overline{R - R_0}$. Convenons généralement, si on a déterminé (pour $0 \leqq h \leqq 2p-2$) une certaine $\mathfrak{U}_{h+1}$-chaîne $D(\mathfrak{U}_{h+1})$, de désigner par $D(\mathfrak{U}_h)$ la chaîne $\pi_h D(\mathfrak{U}_{h+1})$. Alors, pour $0 \leqq h \leqq 2p-1$, $H^{n-p+1}(\mathfrak{U}_h)$ est une $(n-p+1, \mathfrak{U}_h)$-chaîne dans A telle que $H^{n-p+1}(\mathfrak{U}_h) \to C^{n-p}(\mathfrak{U}_h) \bmod \overline{R - R_0}$.

53.1. Rangeons les sommets intérieurs de $\mathfrak{z}$ dans la suite $\tau_1^0, \tau_2^0, \cdots, \tau_{\beta_0}^0$ de telle façon qu'il existe un entier β_0' $(0 \leqq \beta_0' \leqq \beta_0)$ tel que $P_1^0(\tau_\nu^0) - \Omega_1 \neq 0$ pour $1 \leqq \nu \leqq \beta_0'$; $\overline{P_1^0(\tau_\nu^0)} \subset \Omega_1$ pour $\beta_0' + 1 \leqq \nu \leqq \beta_0$. Pour $1 \leqq h \leqq n$, rangeons les h-simplexes intérieurs de $\mathfrak{z}$ dans la suite $\tau_1^h, \tau_2^h, \cdots, \tau_{\beta_h}^h$ de telle façon qu'il existe un entier β_h' $(0 \leqq \beta_h' \leqq \beta_h)$ tel que 1° pour $1 \leqq \nu \leqq \beta_h'$, on a $\lambda \leqq \beta_0'$ pour chaque sommet τ_λ^0 de τ_ν^h; 2° pour $\beta_h' + 1 \leqq \nu \leqq \beta_h$, le simplexe τ_ν^h possède un sommet τ_λ^0 tel que $\beta_0' < \lambda$. Si $0 \leqq h \leqq p-1$, $1 \leqq \nu \leqq \beta_h'$, on a $\overline{P_1^h(\tau_\nu^h)} - \Omega_1 \neq 0$; en effet, d'après 51 (v. 30) on a $P_1^0(\tau_\lambda^0) \subset P_1^h(\tau_\nu^h)$ pour chaque sommet τ_λ^0 de τ_ν^h.

53.2. Lemme. On peut attacher à chaque couple d'indices h, ν, où $0 \leqq h \leqq p-2$, $1 \leqq \nu \leqq \beta_h'$, une $(n-p+h+2)$-chaîne $L^{n-p+h+2}(\tau_\nu^h, \mathfrak{U}_1)$ dans $\bar{B}$ de manière que: 1° pour $1 \leqq \nu \leqq \beta_0'$ il existe une $(n-p+1, \mathfrak{U}_1)$-chaîne élémentaire (rel. à $\mathfrak{z}+\mathfrak{t}$) $E_\nu^{n-p+1}(\mathfrak{U}_1)$ telle que la chaîne

$$FL^{n-p+2}(\tau_\nu^0, \mathfrak{U}_1) - H^{n-p+1}(\mathfrak{U}_1) + E_\nu^{n-p+1}(\mathfrak{U}_1)$$

ne contient pas des simplexes situés dans $\overline{P_4^0(\tau_\nu^0)}$; 2° pour $0 \leqq h \leqq p-3$, $1 \leqq \nu \leqq \beta_{h+1}'$, il existe une $(n-p+h+2, \mathfrak{U}_1)$-chaîne élémentaire (rel. à $\mathfrak{z}+\mathfrak{t}$) $E_\nu^{n-p+h+2}(\mathfrak{U}_1)$ telle que la chaîne

$$FL^{n-p+h+3}(\tau_\nu^{h+1}, \mathfrak{U}_1) - \sum_{\mu=1}^{\beta_h} \zeta_{\nu\mu}^h L^{n-p+h+2}(\tau_\mu^h, \mathfrak{U}_1) + E_\nu^{n-p+h+2}(\mathfrak{U}_1)$$

ne contient pas des simplexes situés dans $\overline{P_4^{h+1}(\tau_\nu^{h+1})}$; 3° pour $1 \leqq \nu \leqq \beta_{p-1}'$ il existe une $(n, \mathfrak{U}_1)$-chaîne élémentaire (rel. à $\mathfrak{z}+\mathfrak{t}$) $E_\nu^n(\mathfrak{U}_1)$ telle que la chaîne

$$\sum_{\mu=1}^{\beta_{p-2}} \zeta_{\nu\mu}^{p-2} L^n(\tau_\mu^{p-2}, \mathfrak{U}_1) + E_\nu^n(\mathfrak{U}_1)$$

soit un (n, R)-cycle mod $R - P_2^{p-1}(\tau_\nu^{p-1})$.

La démonstration sera donnée dans les n^{os} 53.21—53.25.

53.21. D'après le lemme du n° 46 [v. aussi 24 (3)], on peut attacher à chaque combinaison $r_0, r_1, \cdots, r_h$ $(0 \leqq h \leqq p-1)$ d'indices $1, 2, \cdots, m$ une chaîne $\Delta_{r_0 r_1 \cdots r_h}^{n-p+h+1}(\mathfrak{U}_{2p-1})$, fonction alternée de $r_0, r_1, \cdots, r_h$, de manière que: 1°

(1) $$F\Delta_r^{n-p+1}(\mathfrak{U}_{2p-1}) - C^{n-p}(\mathfrak{U}_{2p-1}) \subset \overline{R - R_0} + R - W_r'$$

pour $1 \leqq r \leqq m$; 2°

(2) $$F\Delta_{r_0 r_1 \cdots r_h}^{n-p+h+1}(\mathfrak{U}_{2p-1}) - \sum_{u=0}^{h} (-1)^u \Delta_{r_0 \cdots r_{u-1} r_{u+1} \cdots r_h}^{n-p+h}(\mathfrak{U}_{2p-1}) \subset \overline{R - R_0} + R - \prod_{u=0}^{h} W_{r_u}'$$

pour chaque combinaison $r_0, r_1, \cdots, r_h$ $(1 \leqq h \leqq p-1)$ de $1, 2, \cdots, m$.

Ceci étant, soit $0 \leqq h \leqq p-1$, $1 \leqq \nu \leqq \beta'_h$ et soient $\tau^0_{\lambda_0}, \tau^0_{\lambda_1}, \cdots, \tau^0_{\lambda_h}$ tous les sommets de τ^h_ν écrits dans un tel ordre que

$$\tau^h_\nu = +(\tau^0_{\lambda_0}, \tau^0_{\lambda_1}, \cdots, \tau^0_{\lambda_h}).$$

Distinguons deux cas. En premier lieu, supposons que les indices $r(\lambda_0)$, $r(\lambda_1), \cdots, r(\lambda_h)$ ne soient pas tous distincts; on pose dans ce cas $E^{n-p+h+1}_\nu (\mathfrak{U}_{2p-1}) = 0$. En second lieu, les indices $r(\lambda_0), r(\lambda_1), \cdots, r(\lambda_h)$ soient distincts l'un de l'autre; on pose dans ce cas $E^{n-p+h+1}_\nu(\mathfrak{U}_{2p-1}) = \Delta^{n-p+h+1}_{r(\nu'_0), r(\nu'_1), \cdots, r(\nu'_h)}(\mathfrak{U}_{2p-1})$.

Or je dis que: 1° pour $1 \leqq \nu \leqq \beta'_0$

$$FE^{n-p+1}_\nu(\mathfrak{U}_{2p-1}) - C^{n-p}(\mathfrak{U}_{2p-1}) \subset R - P^0_1(\tau^0_\nu); \tag{3}$$

2° pour $1 \leqq h \leqq p-1$, $1 \leqq \nu \leqq \beta'_h$

$$FE^{n-p+h+1}_\nu(\mathfrak{U}_{2p-1}) - \sum_{\mu=1}^{\beta'_{h-1}} \zeta^{h-1}_{\nu\mu} E^{n-p+h}_\mu(\mathfrak{U}_{2p-1}) \subset R - P^h_1(\tau^h_\nu). \tag{4}$$

Supposons, par impossible, qu'il existe un couple h, ν ($0 \leqq h \leqq p-1$, $1 \leqq \nu \leqq \beta'_h$) tel que la relation (3) ou (4) correspondante ne soit pas vraie. La chaîne considérée contient alors un simplexe $\varphi^{n-p+h+1}$ qui n'est pas contenu dans $R - P^h_1(\tau^h_\nu)$, de manière que $\mathfrak{F} \subset P^h_1(\tau^h_\nu)$, $\mathfrak{F}$ étant le noyau de $\varphi^{n-p+h+1}$. Le simplexe τ^h_ν tombe nécessairement sous le second des deux cas distingués plus haut, car on voit sans peine qu'autrement la chaîne considérée serait égale à zéro. D'après (1) et (2), on a donc

$$\mathfrak{F}\,\overline{R-R_0} + \left[\mathfrak{F} - \prod_{u=0}^{h} W'_{r(\lambda_u)}\right] \neq 0.$$

Mais la supposition $1 \leqq \nu \leqq \beta'_h$ donne (v. 53.1) $\overline{P^h_1(\tau^h_\nu)} - \Omega_1 \neq 0$, d'où (v. 50.3) $\overline{P^h_1(\tau^h_\nu)} \cdot \overline{R-R_0} = 0$ et donc $\mathfrak{F}\,\overline{R-R_0} = 0$; d'autre part, puisque $\mathfrak{F} \subset \overline{P^h_1(\tau^h_\nu)}$, on a $\mathfrak{F} \subset \prod_{u=0}^{h} W'_{r(\lambda_u)}$ d'après 51, 11°. Incidemment, nous avons remarqué que

$$\overline{P^h_1(\tau^h_\nu)} \cdot \overline{R-R_0} = 0 \quad \text{pour} \quad 0 \leqq h \leqq p-1,\ 1 \leqq \nu \leqq \beta'_h. \tag{5}$$

D'après 19.1, 3° il existe des chaînes élémentaires rel. à $\mathfrak{z}+\mathfrak{t}$: $E^{n-p+h+1}_\nu(\mathfrak{U}_k)$ ($0 \leqq h \leqq p-1$, $0 \leqq k \leqq 2p-2$, $1 \leqq \nu \leqq \beta'_h$) telles que $\pi_h E^{n-p+h+1}_\nu(\mathfrak{U}_{k+1}) = E^{n-p+h+1}_\nu(\mathfrak{U}_k)$ pour $0 \leqq k \leqq 2p-2$. Les relations (3) et (4) restent naturellement vraies si on y remplace $\mathfrak{U}_{2p-1}$ par $\mathfrak{U}_k$ ($0 \leqq k \leqq 2p-2$).

53.211. Il résulte de la remarque du n° 46.1 qu'on peut s'arranger de façon qu'on ait $E^{n-p+h+1}_\nu(\mathfrak{U}_{2p-1}) \neq 0$ seulement s'il existe un sommet τ^0_λ de τ^h_ν tel que $b_{r(\lambda)} \in t(\tau^p_\mu)$, l'indice μ ($1 \leqq \mu \leqq \beta_p$) étant tel que le coefficient de

$k^{n-p}(\tau_\mu^p, \mathfrak{U}_{2p-1})$ dans la chaîne $C^{n-p}(\mathfrak{U}_{2p-1})$ soit $\neq 0$ [v. 40 (3)]. On a alors la propriété suivante: $E_\nu^{n-p+h+1}(\mathfrak{U}_{2p-1}) \neq 0$ $(0 \leqq h \leqq p-1,\ 1 \leqq \nu \leqq \beta_h')$ entraîne que $\overline{P_1^h(\tau_\nu^h)} \subset B_1$.

Démonstration. Soit τ_λ^0 le sommet de τ_ν^h tel que $b_{r(\lambda)} \in t(\tau_\mu^p)$ et soit τ_ϱ^0 un sommet arbitraire de τ_μ^p de manière que $b_{r(\lambda)} \in P_1^0(\tau_\varrho^0)$. Soit U un sommet d'un simplexe de la chaîne $k^{n-p}(\tau_\mu^p, \mathfrak{U}_{2p-1})$. La chaîne $C^{n-p}(\mathfrak{U})$ étant située dans A, on a $UA \neq 0$; d'autre part $UP_1^0(\tau_\varrho^0) \neq 0$ d'après 19.3. Comme $\mathfrak{U}_{2p-1} \in \Phi$, il en résulte (v. 51, 4°) que $A\,\overline{P_1^0(\tau_\varrho^0)} \neq 0$. Le réseau gén. $\mathfrak{P}_1^0$ étant un affinement de $\mathfrak{P}_1^{p-1}$, on déduit de 49.1, 2° qu'il existe un indice s tel que $\overline{P_1^0(\tau_\varrho^0)} \subset W_s'$. Comme $b_{r(\lambda)} \in P_1^0(\tau_\varrho^0)$, $A\,\overline{P_1^0(\tau_\varrho^0)} \neq 0$, on a $W_{r(\lambda)}' \cdot W_s' \neq 0$, $A\,W_s' \neq 0$, d'où $W_{r(\lambda)}' \subset B_1$ d'après 45.31. Or on a $\overline{P_1^h(\tau_\nu^h)} \subset W_{r(\lambda)}'$ d'après 50.2.

53.22. (Cf. 34.1). Soit $1 \leqq \nu \leqq \beta_0'$. Comme

$$H^{n-p+1}(\mathfrak{U}_{2p-1}) \to C^{n-p}(\mathfrak{U}_{2p-1}) \mod \overline{R-R_0},$$

on déduit de (3) et (5) que

$$F[H^{n-p+1}(\mathfrak{U}_{2p-1}) - E_\nu^{n-p+1}(\mathfrak{U}_{2p-1})] \subset R - P_1^0(\tau_\nu^0). \tag{6}$$

Désignons par $H^{n-p+1}(\tau_\nu^0, \mathfrak{U}_{2p-1})$ cette partie de la chaîne $H^{n-p+1}(\mathfrak{U}_{2p-1}) - E_\nu^{n-p+1}(\mathfrak{U}_{2p-1})$ dont les simplexes sont situés dans $\overline{P_1^0(\tau_\nu^0)}$ et soit

$$\Gamma^{n-p}(\mathfrak{U}_{2p-1}) = FH^{n-p+1}(\tau_\nu^0, \mathfrak{U}_{2p-1}). \tag{7}$$

Soit φ^{n-p} un simplexe de la chaîne $\Gamma^{n-p}(\mathfrak{U}_{2p-1})$. Evidemment φ^{n-p} est situé dans $\overline{P_1^0(\tau_\nu^0)}$. Plus précisément on peut prouver que φ^{n-p} est situé dans $\overline{P_1^0(\tau_\nu^0)} - P_2^0(\tau_\nu^0)$. A cet effet, il suffit de démontrer qu'aucun sommet U de φ^{n-p} ne peut rencontrer $P_2^0(\tau_\nu^0)$. Supposons le contraire. D'après 51, 2° on a alors $U \subset P_1^0(\tau_\nu^0)$. Il résulte donc de (6) que U est un sommet de

$$H^{n-p+1}(\mathfrak{U}_{2p-1}) - E_\nu^{n-p+1}(\mathfrak{U}_{2p-1}) - H^{n-p+1}(\tau_\nu^0, U_{2p-1}), \tag{*}$$

ce qui est une contradiction, car $U \subset P_1^0(\tau_\nu^0)$, tandis qu'aucun simplexe de la chaîne (*) n'est situé dans $\overline{P_1^0(\tau_\nu^0)}$. Il est ainsi démontré que $\Gamma^{n-p}(\mathfrak{U}_{2p-1})$ est un $(n-p, \mathfrak{U}_{2p-1})$-cycle dans $\overline{P_1^0(\tau_\nu^0)} - P_2^0(\tau_\nu^0)$. Donc $\Gamma^{n-p}(\mathfrak{U}_{2p-2})$ est un $(n-p, \mathfrak{U}_{2p-2})$-cycle essentiel dans $\overline{P_1^0(\tau_\nu^0)} - P_2^0(\tau_\nu^0)$. Par suite, il existe une $(n-p+1, \mathfrak{U}_{2p-2})$-chaîne

$$D_\nu^{n-p+1}(\mathfrak{U}_{2p-2}) \subset \overline{P^0(\tau_\nu^0)} - P_3^0(\tau_\nu^0) \tag{8}$$

telle que

$$H^{n-p+1}(\tau_\nu^0, \mathfrak{U}_{2p-2}) - D_\nu^{n-p+1}(\mathfrak{U}_{2p-2}) \tag{9}$$

est un $(n-p+1, \mathfrak{U}_{2p-2})$-cycle, situé nécessairement dans $\overline{P^0(\tau_\nu^0)}$. Donc

$$H^{n-p+1}(\tau_\nu^0, \mathfrak{U}_{2p-3}) - D_\nu^{n-p+1}(\mathfrak{U}_{2p-3})$$

est un $(n-p+1, \mathfrak{U}_{2p-3})$-cycle essentiel dans $\overline{P^0(\tau_\nu^0)}$, de manière qu'il existe une $(n-p+2, \mathfrak{U}_{2p-3})$-chaîne $L^{n-p+2}(\tau_\nu^0, \mathfrak{U}_{2p-3})$ dans $\overline{Q^0(\tau_\nu^0)}$ telle que

$$L^{n-p+2}(\tau_\nu^0, \mathfrak{U}_{2p-3}) \to H^{n-p+1}(\tau_\nu^0, \mathfrak{U}_{2p-3}) - D_\nu^{n-p+1}(\mathfrak{U}_{2p-3}).$$

D'après (9) on a pour $0 \leqq k \leqq 2p-3$ (v. la fin du n° 53.21)

$$\begin{aligned} &FL^{n-p+2}(\tau_\nu^0, \mathfrak{U}_k) - H^{n-p+1}(\mathfrak{U}_k) + E_\nu^{n-p+1}(\mathfrak{U}_k) \\ &= [H^{n-p+1}(\tau_\nu^0, \mathfrak{U}_k) + E_\nu^{n-p+1}(\mathfrak{U}_k) - H^{n-p+1}(\mathfrak{U}_k)] - D_\nu^{n-p+1}(\mathfrak{U}_k). \end{aligned} \tag{10}$$

Chaque simplexe φ^{n-p+1} de la chaîne $[\cdots]$ à droite de (10) est (v. 19.1, 1°) la projection d'un $\mathfrak{U}_{2p-1}$-simplexe ψ^{n-p+1} et, en vertu de la définition de la chaîne $H^{n-p+1}(\tau_\nu^0, \mathfrak{U}_{2p-1})$, le simplexe ψ^{n-p+1} n'est pas situé dans $\overline{P_1^0(\tau_\nu^0)}$. Par suite le noyau de ψ^{n-p+1} est un sous-ensemble de $R - P_1^0(\tau_\nu^0)$; donc le noyau de φ^{n-p+1} rencontre $R - P_1^0(\tau_\nu^0)$; il en résulte qu'aucun sommet U de φ^{n-p+1} n'est un sous-ensemble de $P_1^0(\tau_\nu^0)$, d'où (v. 51, 2°) $U\overline{P_2^0(\tau_\nu^0)} = 0$ pour chaque sommet U de φ^{n-p+1}. Donc φ^{n-p+1} n'est pas situé dans $\overline{P_2^0(\tau_\nu^0)} \supset \overline{P_4^0(\tau_\nu^0)}$. D'après (8), chaque sommet U de $D_\nu^{n-p+1}(\mathfrak{U}_k)$ rencontre $R - P_3^0(\tau_\nu^0)$, d'où $U\overline{P_4^0(\tau_\nu^0)} = 0$ d'après 51, 1°. Il est ainsi démontré qu'aucun simplexe de la chaîne (10) n'est situé dans $\overline{P_4^0(\tau_\nu^0)}$, ce qui est d'accord avec 53.2, 1°.

53.23. (Cf. 34.2.) Supposons généralement que, pour une certaine valeur de h $(0 \leqq h \leqq p-3)$ on ait déjà attaché à chaque τ_ν^h $(1 \leqq \nu \leqq \beta_h')$ une $(n-p+h+2, \mathfrak{U}_{2p-2h-3})$-chaîne $L^{n-p+h+2}(\tau_\nu^h, \mathfrak{U}_{2p-2h-3})$ dans $\overline{Q^h(\tau_\nu^h)}$ de manière que pour $1 \leqq \nu \leqq \beta_h'$ aucun simplexe de la chaîne[62]

$$FL^{n-p+h+2}(\tau_\nu^h, \mathfrak{U}_k) - \sum_{\mu=1}^{\beta_{h-1}'} \zeta_{\nu\mu}^{h-1} L^{n-p+h+1}(\tau_\mu^{h-1}, \mathfrak{U}_k) + E_\nu^{n-p+h+1}(\mathfrak{U}_k) \tag{11}$$
$$(0 \leqq k \leqq 2p-2h-3)$$

ne soit situé dans $\overline{P_4^h(\tau_\nu^h)}$. Il s'agit d'attacher à chaque τ_ν^{h+1} $(1 \leqq \nu \leqq \beta_{h+1}')$ une $(n-p+h+3, \mathfrak{U}_{2p-2h-5})$-chaîne $L^{n-p+h+3}(\tau_\nu^{h+1}, \mathfrak{U}_{2p-2h-5})$ dans $\overline{Q^{h+1}(\tau_\nu^{h+1})}$ de manière qu'aucun simplexe de la chaîne

$$FL^{n-p+h+3}(\tau_\nu^{h+1}, \mathfrak{U}_k) - \sum_{\mu=1}^{\beta_h'} \zeta_{\nu\mu}^h L^{n-p+h+2}(\tau_\mu^h, \mathfrak{U}_k) + E_\nu^{n-p+h+2}(\mathfrak{U}_k)$$
$$(0 \leqq k \leqq 2p-2h-5)$$

ne soit situé dans $\overline{P_4^{h+1}(\tau_\nu^{h+1})}$. On arrive à ce but par un procédé presque identique à celui du n° 34.2 de manière qu'il ne semble pas nécessaire de répéter ici cette construction. Au lieu de $M^{n-p+h+1}(\sigma_i^{h+1}, \mathfrak{U}_{2p-2h-1})$, p. ex., on doit maintenant poser

[62] Pour $h = 0$ on doit remplacer (11) par le premier membre de (10).

$$M^{n-p+h+2}(\tau_\nu^{h+1}, \mathfrak{U}_{2p-2h-3})$$
$$= \sum_{\mu=1}^{\beta'_h} \zeta_{\nu\mu}^h \, L^{n-p+h+2}(\tau_\mu^h, \mathfrak{U}_{2p-2h-3}) + E_\nu^{n-p+h+2}(\mathfrak{U}_{2p-2h-3}).$$

Pour démontrer qu'aucun simplexe de la chaîne $FM^{n-p+h+2}(\tau_\nu^{h+1}, \mathfrak{U}_{2p-2h-3})$ n'est situé dans $\overline{P_2^{h+1}(\tau_\nu^{h+1})}$, on doit s'appuyer sur le fait que la chaîne

$$FE_\nu^{n-p+h+2}(\mathfrak{U}_{2p-2h-3}) - \sum_{\mu=1}^{\beta'_h} \zeta_{\nu\mu}^h \, E_\mu^{n-p+h+1}(\mathfrak{U}_{2p-2h-3}) \tag{12}$$

possède la même propriété; la validité de ce fait résulte de (4) en vertu de 51, 2°.

53.24. (Cf. 34.3.) En procédant de la manière indiquée on finit par construire les chaînes $L^n(\tau_\nu^{p-2}, \mathfrak{U}_1)$ $(1 \leqq \nu \leqq \beta'_{p-2})$ et on démontre comme dans 34.3, en s'appuyant de nouveau sur (4), que la chaîne

$$M^n(\tau_\nu^{p-1}, \mathfrak{U}_1) = \sum_{\mu=1}^{\beta'_{p-2}} \zeta_{\nu\mu}^{p-2} \, L^n(\tau_\mu^{p-2}, \mathfrak{U}_1) + E^n(\tau_\nu^{p-1}, \mathfrak{U}_1) \quad (1 \leqq \nu \leqq \beta'_{p-1})$$

est un $(n, \mathfrak{U}_1)$-cycle mod $\overline{P_1^{p-1}(\tau_\nu^{p-1})} - P_2^{p-1}(\tau_\nu^{p-1})$ dans $P_1^{p-1}(\tau_\nu^{p-1})$.

53.25. (Cf. 34.4.) Si l'indice ν $(1 \leqq \nu \leqq \beta'_0)$ est tel que $\overline{P_1^0(\tau_\nu^0)}$ n'est pas un sous-ensemble de B_1, on a $E_\nu^{n-p+1}(\mathfrak{U}_{2p-1}) = 0$ d'après 53.211. En outre on a $A\,\overline{P_1^0(\tau_\nu^0)} = 0$; en effet, puisque (v. 50.2) $\overline{P_1^0(\tau_\nu^0)} \subset W_{r(\lambda)}$, dans le cas contraire on aurait $A\,W_{r(\lambda)} \neq 0$, d'où (v. 45.31) $W_{r(\lambda)} \subset B_1$, ce qui donnerait la contradiction $\overline{P_1^0(\tau_\nu^0)} \subset B_1$. Il en résulte que (v. 53.22) $H^{n-p+1}(\tau_\nu^0, \mathfrak{U}_{2p-1}) = 0$; en effet, U étant un sommet d'un simplexe φ de cette chaîne supposée non vide, φ est (puisque $E_\nu^{n-p+1}(\mathfrak{U}_{2p-1}) = 0$) un simplexe de la chaîne $H^{n-p+1}(\mathfrak{U}_{2p-1})$ située dans A, d'où $UA \neq 0$; d'autre part, $U\,\overline{P_1^0(\tau_\nu^0)} \neq 0$ d'après la définition même de $H^{n-p+1}(\tau_\nu^0, \mathfrak{U}_{2p-1})$; comme $\mathfrak{U}_{2p-1} \in \varphi$, on arrive (v. 51.4) à la contradiction $A\,\overline{P_1^0(\tau_\nu^0)} \neq 0$. Puisque $H^{n-p+1}(\tau_\nu^0, \mathfrak{U}_{2p-1}) = 0$, on voit sans peine qu'il est permis de poser $L^{n-p+2}(\tau_\nu^0, \mathfrak{U}_{2p-3}) = 0$.

On démontre de la même manière qu'on peut supposer que $L^{n-p+h+2}(\tau_\nu^h, \mathfrak{U}_{2p-2h-3}) \neq 0$ seulement dans le cas où le simplexe τ_ν^h possède un sommet τ_λ^0 tel que $\overline{P_1^0(\tau_\lambda^0)} \subset B_1$. Or la chaîne $L^{n-p+h+2}(\tau_\nu^h, \mathfrak{U}_{2p-2h-3})$ est située dans $\overline{Q^h(\tau_\nu^h)}$; comme $\overline{P_1^0(\tau_\lambda^0)} \cdot \overline{Q^h(\tau_\nu^h)} \supset \tau_\lambda^0 \neq 0$, l'inclusion $\overline{P_1^0(\tau_\lambda^0)} \subset B_1$ entraîne $B_1\,\overline{Q^h(\tau_\nu^h)} \neq 0$, ce qui donne $Q^h(\tau_\nu^h) \subset B$ (v. 49.1, 1°), car $\mathfrak{Q}^h$ est un affinement de $\mathfrak{P}_1^{h-1}$. Donc les chaînes $L^{n-p+h+2}(\tau_\nu^h, \mathfrak{U}_{2p-2h-3})$ et par suite aussi les chaînes $L^{n-p+h+2}(\tau_\nu^h, \mathfrak{U}_1)$ sont situées dans B.

54.1. On peut (cf. 35) modifier un nombre fini de fois la chaîne $H^{n-p+1}(\mathfrak{U}_1)$ ainsi que les chaînes $L^{n-p+h+2}(\tau_\nu^h, \mathfrak{U}_1)$ $(0 \leqq h \leqq p-2,\ 1 \leqq \nu \leqq \beta'_h)$ de manière que $^*H$ et $^*L$ étant les éléments modifiés, on ait les propriétés

suivantes: 1° chaque simplexe de chaque différence $^*H^{n-p+1}(\mathfrak{U}_1)-H^{n-p+1}(\mathfrak{U}_1)$ ou $^*L^{n-p+h+2}(\tau_\nu^h, \mathfrak{U}_1)-L^{n-p+h+2}(\tau_\nu^h, \mathfrak{U}_1)$ est une face d'un simplexe de $H^{n-p+1}(\mathfrak{U}_1)$ ou de $^*L^{n-p+k+2}(\tau_\mu^k, \mathfrak{U}_1)$ de manière que les chaînes modifiées restent situées dans $\bar{B}$; 2° les relations 1°, 2°, 3° du n° 53.2 restent vraies [sans qu'on y change les chaînes élémentaires $E_\nu^{n-p+h+1}(\mathfrak{U}_1)$] pour les chaînes modifiées $^*H$, $^*L$; 3° $^*H^{n-p+1}(\mathfrak{U}_1)-H^{n-p+1}(\mathfrak{U}_1)$ est un $(n-p+1, \mathfrak{U}_1)$-cycle homologue à zéro dans $\bar{B}$.

Le résultat de toutes ces modifications est que pour $0 \leqq h \leqq p-2$, $0 \leqq u \leqq h$, $0 \leqq q \leqq h$ les chaînes modifiées possèdent la propriété suivante: Si les simplexes τ_μ^u, τ_ϱ^q $(1 \leqq \mu \leqq \beta'_u,\ 1 \leqq \varrho \leqq \beta'_q)$ sont des faces du simplexe τ_ν^h, alors les simplexes d'espèce τ_ϱ^q (rel. à $\mathfrak{z}+\mathfrak{t}$) de la chaîne $L^{n-p+u+2}(\tau_\mu^k)$ (modifiée) constituent une chaîne qui est $1^\circ = 0$ pour $q \neq p-u-2$, $2^\circ = a k^{n-q}(\tau_\varrho^q)$ pour $q = p-u-2$ $(a \in \mathfrak{R})$.

Pour voir que ces modifications sont réalisables on procède exactement comme dans les n^{os} 36–41[63]. Il faut seulement remplacer p, C, $\mathfrak{Z}$, σ, α_i resp. par $p-1$, H, $\mathfrak{z}$, τ, β'_i et dire qu'une $\mathfrak{U}_1$-chaîne ne contient presqu'aucun simplexe d'espèce τ_ϱ^q $(1 \leqq \varrho \leqq \beta'_q)$ si elle a la forme $a k^{n-q}(\tau_\varrho^q, \mathfrak{U}_1)$ $(a \in \mathfrak{R})$.

54.2. Ces modifications étant effectuées, pour la démonstration du théorème du n° 44 il suffit évidemment de démontrer que la chaîne $H^{n-p+1}(\mathfrak{U}_0)$ modifiée est égale mod $\bar{\Omega}$ à une chaîne élémentaire (rel. à $\mathfrak{z}+\mathfrak{t}$).

On commence par le lemme $\mathcal{A}_k$ $(0 \leqq k \leqq p-2)$: Supposons que τ_μ^u, τ_ϱ^q soient des faces du τ_ν^p $(1 \leqq \mu \leqq \beta'_u,\ 1 \leqq \varrho \leqq \beta'_q,\ 1 \leqq i \leqq \beta'_p)$. Soit $e^{n-p+u+2}(\tau_\mu^u, \tau_\varrho^q, \mathfrak{U}_0)$ cette partie de la chaîne $L^{n-p+u+2}(\tau_\mu^u, \mathfrak{U}_0)$ (modifiée) dont les simplexes sont d'espèce τ_ϱ^q. Alors la chaîne $e^{n-p+u+2}(\tau_\mu^u, \tau_\varrho^q, \mathfrak{U}_0)$ est $1^\circ = 0$ pour $q \neq p-u-2$; $2^\circ = a\, k^{n-q}(\tau_\varrho^q, \mathfrak{U}_0)$ $(a \in \mathfrak{R})$ pour $q = p-u-2$.

La démonstration par récurrence du lemme $\mathcal{A}_k$ étant parfaitement analogue à celle du lemme du n° 42.2, nous pouvons la laisser au soin du lecteur.

54.3. Le lemme $\mathcal{A}_0$ étant vrai, il est facile de compléter la démonstration du théorème du n° 44 en démontrant que la chaîne $H^{n-p+1}(\mathfrak{U}_0)$ est égale à une chaîne élémentaire:

Soit φ^{n-p+1} un simplexe $\neq 0$ mod $\bar{\Omega}$ de la chaîne $H^{n-p+1}(\mathfrak{U}_0)$. D'après 8.32 et 8.33, φ^{n-p+1} est d'espèce τ_ϱ^q, où $0 \leqq q \leqq p-1$. D'après 51, 6° on a $U \subset P_4^0(\tau_\lambda^0) \subset P_1^0(\tau_\lambda^0)$ pour chaque sommet τ_λ^0 de τ_ϱ^q; or $\varphi^{n-p+1} \neq 0$

[63] Si $\mathfrak{U} \in \Phi$, si le $\mathfrak{U}$-simplexe φ d'espèce τ_ϱ^q $(1 \leqq \varrho \leqq \beta'_q)$ est une face du $\mathfrak{U}$-simplexe ψ, ψ ne peut pas être d'espèce τ^{-1}. Il suffit (v. 8.32) de démontrer que le noyau $\mathfrak{Y}$ de ψ rencontre $R_0 \subset R'_0$. Dans le cas contraire, chaque sommet U de φ rencontrerait $R - R_0 \subset S_1$ (v. 8 et 44). Or, puisque $\varrho \leqq \beta'_q$, on a (v. 53.1) $\overline{P_1^0(\tau_\lambda^0)} - \Omega_1 \neq 0$ et par suite (v. 49.1, 3°) $\overline{P_1^0(\tau_\lambda^0)}\ S_1 = 0$ pour chaque sommet τ_λ^0 de τ_ϱ^q; d'autre part, $U \subset P_4^0(\tau_\lambda^0) \subset \overline{P_1^0(\tau_\lambda^0)}$ d'après 51, 6°.

$\mathrm{mod}\,\bar{\Omega}$, de manière que $U-\bar{\Omega}\neq 0$, d'où $\overline{P_1^0(r_\lambda^0)}-\Omega_1\neq 0$. On a donc (v. 53.1) $1\leqq\varrho\leqq\beta'_q$.

Il suffit donc de démontrer que, $e^{n-p+1}(r_\varrho^q, \mathfrak{U}_0)$ $(0\leqq q\leqq p-1,\ 0\leqq\varrho\leqq\beta'_q)$ étant cette partie de $H^{n-p+1}(\mathfrak{U}_0)$ dont les simplexes sont d'espèce r_ϱ^q, la chaîne $e^{n-p+1}(r_\varrho^q, \mathfrak{U}_\nu)$ est: $1^\circ = 0$ pour $q\neq p-1$; $2^\circ = a_\varrho k^{n-q}(r_\varrho^q, \mathfrak{U}_0)$ $(a_\varrho\in\Re)$ pour $q=p-1$. Cette démonstration, analogue à celle du n° 42.5, sera aussi laissée au soin du lecteur.

VI.

55. Dans tout ce Chapitre, supposons donnée une valeur fixe de p $(0\leqq p\leqq n)$. On suppose la validité de tous les axiomes des Chap. I et II ainsi que celle des axiomes G^k (v. 21) pour $n-p\leqq k\leqq n-1$ et des axiomes H^k (v. 27) pour $\max(0, n-p-1)\leqq k\leqq n-2$.

56. Soient A_1 et A_2 deux sous-ensembles bicompacts donnés de R, assujettis à la condition $A_1A_2S=0$. Soit $\Gamma_1(\Gamma_2)$ la famille de tous les p-cycles [$(n-p)$-cycles] $\mathrm{mod}\,A_1S$ dans A_1 ($\mathrm{mod}\,A_2S$ dans A_2). Le but de ce Chapitre est d'attacher à chaque couple $C^p\in\Gamma_1$, $C^{n-p}\in\Gamma_2$ un nombre $C^pC^{n-p}\in\Re$ jouissant des propriétés suivantes: 1° $(rC^p)C^{n-p}=C^p(rC^{n-p})=r(C^pC^{n-p})$ pour $r\in\Re$, $C^p\in\Gamma_1$, $C^{n-p}\in\Gamma_2$; 2° $(C_1^p+C_2^p)C^{n-p}=C_1^pC^{n-p}+C_2^pC^{n-p}$ pour $C_1^p\in\Gamma_1$, $C_2^p\in\Gamma_1$, $C^{n-p}\in\Gamma_2$; 3° $C^p(C_1^{n-p}+C_2^{n-p})=C^pC_1^{n-p}+C^pC_2^{n-p}$ pour $C^p\in\Gamma_1$, $C_1^{n-p}\in\Gamma_2$, $C_2^{n-p}\in\Gamma_2$; 4° $C^pC^{n-p}=0$ pour $C^p\in\Gamma_1$, $C^{n-p}\in\Gamma_2$ si l'on a soit $C^p\sim 0\ \mathrm{mod}\,A_1S$ dans A_1, soit $C^{n-p}\sim 0\ \mathrm{mod}\,A_2S$ dans A_2. Remarquons qu'il résulte de 1°, 2°, 3° et 4° que $C_1^pC_1^{n-p}=C_2^pC_2^{n-p}$ pour C_1^p, $C_2^p\in\Gamma_1$, C_1^{n-p}, $C_2^{n-p}\in\Gamma_2$, si l'on a $C_1^p\sim C_2^p\ \mathrm{mod}\,A_1S$ dans A_1 et $C_1^{n-p}\sim C_2^{n-p}\ \mathrm{mod}\,A_2S$ dans A_2.

57. Commençons par le cas $p=0$. Soit Ξ_2 la famille de tous les réseaux $\mathfrak{Z}\in\Xi$ (v. 23) jouissant de la propriété suivante: Z_1, Z_2, Z_3 étant trois sommets de $\mathfrak{Z}$ tels que $Z_1Z_2\neq 0$, $Z_1Z_3\neq 0$, $Z_1A_2\neq 0$, $Z_3A_1\neq 0$ on a $Z_2S=0$. Puisque $A_1A_2S=0$, on voit sans peine (v. 1.3) que la famille Ξ_1 est parfaitement complète. Pour $\mathfrak{Z}\in\Xi_2$ désignons par $\Delta(\mathfrak{Z})$ la famille de tous les réseaux fermés $\mathfrak{T}$ correspondant (v. 8) à $\mathfrak{Z}$ et vérifiant la condition du n° 25, c'est-à-dire tels que (dans les notations habituelles) $K^n(\sigma_i^0, \mathfrak{U})\neq 0$ $(1\leqq i\leqq\alpha_0)$ et $K^{n-1}(\sigma_i^1, \mathfrak{U})\neq 0$ $(1\leqq i\leqq\alpha_1)$ pour chaque réseau commode $\mathfrak{U}$ suffisamment fin. $\mathfrak{Z}\in\Xi_2$ et $\mathfrak{T}\in\Delta(\mathfrak{Z})$ étant choisi, désignons par $\Phi(\mathfrak{Z}, \mathfrak{T})$ la famille de tous les réseaux commodes $\mathfrak{U}$ tels que, outre les conditions $K^n(\sigma_i^0, \mathfrak{U})\neq 0$, $K^{n-1}(\sigma_i^1, \mathfrak{U})\neq 0$ on ait encore $\mathfrak{U}\in\Psi_1$ (v. 20.2) et que $\mathfrak{U}$ soit un affinement de $\mathfrak{Z}$; la famille $\Phi(\mathfrak{Z}, \mathfrak{T})$ est évidemment parfaitement complète.

Ceci étant, soient donnés les cycles $C^0\in\Gamma_1$ et $C^n\in\Gamma_2$. Choisissons $\mathfrak{Z}\in\Xi_2$, $\mathfrak{T}\in\Delta(\mathfrak{Z})$, $\mathfrak{U}\in\Phi(\mathfrak{Z}, \mathfrak{T})$. Il existe des nombres $a_i\in\Re$ tels que

$$C^0(\mathfrak{Z}) = \sum_{i=1}^{\alpha_0} a_i \sigma_i^0 \mod S;$$

d'après 20.3, il existe des nombres $b_i \in \mathfrak{R}$ tels que

$$C^n(\mathfrak{U}) = \sum_{i=1}^{\alpha_0} b_i K^n(\sigma_i^0, \mathfrak{U}) \mod \overline{R - R_0}.$$

Posons $C^0 C^n = \sum_{i=1}^{\alpha_0} a_i b_i$. Il s'agit de prouver que ce nombre est déterminé sans ambiguité par les deux cycles C^0 et C^n et que les propriétés 1°—4° du n° 56 sont vérifiées.

L'égalité $\sum_{i=1}^{\alpha_0} b_i K^n(\sigma_i^0, \mathfrak{U}) = 0 \mod \overline{R - R_0}$ est évidemment (v. 19.1, 1°, 2°) possible seulement si $b_i = 0$ pour chaque i $(1 \leqq i \leqq \alpha_0)$. L'égalité $\sum_{i=1}^{\alpha_0} a_i \sigma_i^0 = 0 \mod S$ est évidemment possible seulement si $a_i = 0$ pour chaque i $(1 \leqq i \leqq \alpha_0)$. Donc le nombre $\sum_{i=1}^{\alpha_0} a_i b_i$ est bien déterminé, si on a choisi les réseaux $\mathfrak{Z} \in \Xi_2$, $\mathfrak{T} \in \Delta(\mathfrak{Z})$, $\mathfrak{U} \in \Phi(\mathfrak{Z}, \mathfrak{T})$.

Nous venons de remarquer que les nombres b_i $(1 \leqq i \leqq \alpha_0)$ sont bien déterminés ($\mathfrak{Z}$, $\mathfrak{T}$ et $\mathfrak{U}$ étant choisis). Or il résulte de la démonstration du n° 20.3 que $C^n - b_i G^n \sim 0 \mod (R - P_i)$ pour $1 \leqq i \leqq \alpha_0$ et cette condition détermine les nombres b_i sans ambiguité, en vertu de 20.2, 1°. Il en résulte que, $\mathfrak{Z}$ étant donné, le nombre $\sum_{i=1}^{\alpha_0} a_i b_i$ est indépendant du choix de $\mathfrak{T}$ et de $\mathfrak{U}$. De plus on voit que ce nombre reste inaltéré, si on remplace C^n par un cycle $C_1^n \sim C^n \mod A_2 S$ dans A_2.

Remarque. Pour $1 \leqq j \leqq \alpha_1$, on a $\sum_{i=1}^{\alpha_0} \eta_{ji}^0 b_i = 0$.

Démonstration. Puisque $C^n \to 0 \mod S$, on a

$$\sum_{i=1}^{\alpha_0} b_i K^n(\sigma_i^0, \mathfrak{U}) \to 0 \mod \overline{R - R_0},$$

d'où (v. 19.1, 3°)

$$\sum_{j=1}^{\alpha_1} \sum_{i=1}^{\alpha_0} \eta_{ji}^0 b_i K^{n-1}(\sigma_j^1, \mathfrak{U}) = 0 \mod \overline{R - R_0}$$

et par suite (v. 19.1, 1° et 2°)

$$\sum_{i=1}^{\alpha_0} \eta_{ji}^0 b_i = 0.$$

Remplaçons $C^0(\mathfrak{Z})$ par $C_1^0(\mathfrak{Z}) \sim C^0(\mathfrak{Z}) \mod A_1 S$ dans A_1. On a

$$C_1^0(\mathfrak{Z}) = \sum_{i=1}^{\alpha_0} a_0' \sigma_i^0 \mod S.$$

De plus, il existe des nombres $c_j \in \mathfrak{R}$ tels que

$$\sum_{j=1}^{\alpha_1} c_j \sigma_j^1 \to \sum_{i=1}^{\alpha_0} (a_i' - a_i) \sigma_i^0 \bmod S,$$

d'où

$$a_i' - a_i = \sum_{j=1}^{\alpha_1} \eta_{ji}^0 c_j.$$

Donc

$$\sum_{i=1}^{\alpha_0} a_i' b_i - \sum_{i=1}^{\alpha_0} a_i b_i = \sum_{j=1}^{\alpha_1} c_j \sum_{i=1}^{\alpha_0} \eta_{ji}^0 b_i = 0$$

d'après la *remarque*. Donc le nombre $\sum_{i=1}^{\alpha_0} a_i b_i$ reste inaltéré en remplaçant $C^0(\mathfrak{Z})$ par $C_1^0(\mathfrak{Z})$. La propriété 4° du n° 57 est donc vraie; les propriétés 1°, 2°, 3° du n° 57 sont banales.

On doit encore prouver que le nombre $\sum_{i=1}^{\alpha_0} a_i b_i$ reste inaltéré en remplaçant $\mathfrak{Z}$ par un autre réseau $\mathfrak{z} \in \Xi_2$. La famille Ξ_2 étant complète, il suffit de faire cette démonstration en supposant que $\mathfrak{z}$ soit un affinement de $\mathfrak{Z}$. Choisissons un réseau fermé $\mathfrak{t}$ correspondant à $\mathfrak{z}$ (et satisfaisant rel. à $\mathfrak{z}$ à la condition du n° 25). Choisissons une projection $\pi = Pr.(\mathfrak{z}, \mathfrak{Z})$ et déterminons le réseau fermé $\mathfrak{T}$ correspondant à $\mathfrak{Z}$ selon le n° 24; c'est permis, car $\mathfrak{T}$ vérifie évidemment la condition du n° 25 [v. 24, (3)]. Faisons usage des notations du n° 24. Posons

$$C^0(\mathfrak{z}) = \sum_{\nu=1}^{\beta_0} a_\nu' \tau_\nu^0 \bmod S,$$

$$C^n(\mathfrak{U}) = \sum_{\nu=1}^{\beta_0} b_\nu' k^n(\tau_\nu^0, \mathfrak{U}) \bmod \overline{R - R_0},$$

$\mathfrak{U}$ étant un réseau commode (rel. à $\mathfrak{z} + \mathfrak{t}$ et par suite aussi rel. à $\mathfrak{Z} + \mathfrak{T}$) suffisamment fin. Il s'agit de prouver que $\sum_{i=1}^{\alpha_0} a_i b_i = \sum_{\nu=1}^{\beta_0} a_\nu' b_\nu'$. Pour $1 \leqq i \leqq \alpha_0$ posons $a_i'' = \sum_{\nu} a_\nu'$, la sommation étant étendue à toutes les valeurs de ν $(1 \leqq \nu \leqq \beta_0')$ telles que $\pi \tau_\nu^0 = \sigma_i^0$. On a alors $\pi C^0(\mathfrak{z}) = \sum_{i=1}^{\alpha_0} a_i'' \sigma_i^0 \sim C^0(\mathfrak{Z})$ mod $A_1 S$ dans A_1 de manière que, comme nous avons vu plus haut, $\sum_{i=1}^{\alpha_0} a_i b_i = \sum_{i=1}^{\alpha_0} a_i'' b_i$. Pour $1 \leqq \nu \leqq \beta_0'$, $\pi \tau_\nu^0 = \sigma_i^0$, posons $b_\nu'' = b_i$ de manière que $\sum_{i=1}^{\alpha_0} a_i'' b_i = \sum_{\nu=1}^{\beta_0} a_\nu' b_\nu''$, où $b_\nu'' = 0$ pour $\beta_0' + 1 \leqq \nu \leqq \beta_0$. Il suffit donc de montrer que l'on a $a_\nu'(b_\nu' - b_\nu'') = 0$ pour $1 \leqq \nu \leqq \beta_0$. D'après 24 (3)

$$C^n(\mathfrak{U}) = \sum_{i=1}^{\alpha_0} b_i K^n(\sigma_i^0, \mathfrak{U}) = \sum_{\nu=1}^{\beta_0} b_\nu'' k^n(\tau_\nu^0, \mathfrak{U}) \mod \overline{R - R_0},$$

de sorte que

$$(*) \qquad \sum_{\nu=1}^{\beta_0} (b_\nu' - b_\nu'') k^n(\tau_\nu^0, \mathfrak{U}) = 0 \mod \overline{R - R_0}.$$

Supposons, par impossible, que $a_\nu'(b_\nu' - b_\nu'') \neq 0$ pour une valeur donnée de ν. Soit U un sommet d'un simplexe de la chaîne $k^n(\tau_\nu^0, \mathfrak{U})$; on a $U\tau_\nu^0 \neq 0$ d'après 19.3; d'autre part, $UA_2 \neq 0$, car $b_\nu' - b_\nu'' \neq 0$ et la chaîne (*) est évidemment située dans A_2; enfin, $U\overline{R - R_0} \neq 0$ d'après (*). $\mathfrak{U}$ étant un affinement de $\mathfrak{Z}$, il existe un sommet Z_1 de $\mathfrak{Z}$ tel que $Z_1 \subset U$ et donc $Z_1 A_2 \neq 0$, $Z_1 \tau_\nu^0 \neq 0$, $Z_1 \overline{R - R_0} \neq 0$. De la dernière de ces relations on déduit sans peine (v. 8) qu'il existe un sommet Z_2 de $\mathfrak{Z}$ tel que $Z_1 Z_2 \neq 0$, $Z_2 S \neq 0$. La chaîne $C^0(\mathfrak{z})$ étant située dans A_1, l'inégalité $a_\nu' \neq 0$ entraîne $\tau_\nu^0 A_1 \neq 0$. Le réseau $\mathfrak{z}$ étant un affinement de $\mathfrak{Z}$, il existe un sommet Z_3 de $\mathfrak{Z}$ tel que $\tau_\nu^0 \subset Z_3$. On a donc $Z_1 Z_2 \neq 0$, $Z_1 Z_3 \neq 0$, $Z_1 A_2 \neq 0$, $Z_2 S \neq 0$, $Z_3 A_1 \neq 0$, ce qui est une contradiction, car $\mathfrak{Z} \in \Xi_2$.

58. Reste à considérer le cas $1 \leqq p \leqq n$. Puisque $A_1 S A_2 = 0$, il existe un entourage B de A_2 tel que $A_1 \overline{B} S = 0$. Ensuite, il existe un entourage Ω de S tel que $A_1 \overline{B}\,\overline{\Omega} = 0$. Soit encore B' un entourage de A_2 tel que $A \subset B' \Subset B$.

Soit Ξ_1 la famille de tous les réseaux $\mathfrak{Z} \in \Xi$ (v. 23) pour lesquels vaut l'énoncé du n° 31 en y remplaçant A, B resp. par A_2, B'. La famille Ξ_1 est évidemment parfaitement complète. Soit Ξ_2 la famille de tous les réseaux $\mathfrak{Z} \in \Xi_1$ jouissant de la propriété suivante: 1° Si Z_1, $Z_2 \in \mathfrak{Z}$, $Z_1 A_1 \neq 0$, $Z_2 \overline{B} \neq 0$, $Z_1 Z_2 \neq 0$, on a $(Z_1 + Z_2)\overline{\Omega} = 0$; 2° pour chaque sommet extérieur de $\mathfrak{Z}$ on a $Z \Subset \Omega$. On déduit sans peine de 1.3 et de 5.4 que la famille Ξ_2 est parfaitement complète, car $A_1 \overline{B}\,\overline{\Omega} = 0$.

Pour chaque $\mathfrak{Z} \in \Xi_2$, soit $\Delta(\mathfrak{Z})$ la famille de tous les réseaux fermés correspondant à $\mathfrak{Z}$ (v. 8) et choisis selon l'énoncé du n° 44 où on choisit $\mathfrak{z} \in \Xi_2$ et où on remplace A par $\overline{B}'$, tandis que Ω a la valeur que nous venons de déterminer.

Pour $\mathfrak{Z} \in \Xi_2$, $\mathfrak{T} \in \Delta(\mathfrak{Z})$ soit $\Phi(\mathfrak{Z}, \mathfrak{T})$ la famille de tous les réseaux commodes $\mathfrak{U}$ si fins que: 1° on puisse appliquer le théorème du n° 44 avec $\overline{B}'$ au lieu de A; 2° l'énoncé du n° 26 soit vrai pour chaque $\mathfrak{U}_0 \in \Phi(\mathfrak{Z}, \mathfrak{T})$; 3° $\mathfrak{U}$ soit un affinement de $\mathfrak{Z}$; 4° on puisse appliquer l'énoncé du n° 25 non seulement rel. à $\mathfrak{Z}$ et $\mathfrak{T}$, mais aussi rel. à $\mathfrak{z}$ et t dont on construit (cf. 44) le réseau fermé $\mathfrak{T} \in \Delta(\mathfrak{Z})$ correspondant à $\mathfrak{Z}$.

Ceci étant, supposons qu'on ait des cycles $C^p \in \Gamma_1$ et $C^{n-p} \in \Gamma_2$ (v. 56). Choisissons $\mathfrak{Z} \in \Xi_2$, $\mathfrak{T} \in \Delta(\mathfrak{Z})$, $\mathfrak{U} \in \Phi(\mathfrak{Z}, \mathfrak{T})$. D'après la propriété 2° de la famille Ξ_2, il existe des nombres $a_i \in \mathfrak{R}$ tels que

$$C^p(\mathfrak{Z}) = \sum_{i=1}^{\alpha_p} a_i \sigma_i^p \mod \bar{\Omega}. \tag{1}$$

D'après 31, il existe des nombres $b_i \in \mathfrak{R}$ tels que

$$C^{n-p}(\mathfrak{U}) \sim \sum_{i=1}^{\alpha_p} b_i K^{n-p}(\sigma_i^p, \mathfrak{U}) \mod \bar{B}' \overline{R - R_0} \text{ dans } \bar{B}'. \tag{2}$$

Posons

$$C^p C^{n-p} = \sum_{i=1}^{\alpha_p} a_i b_i. \tag{3}$$

Pour justifier cette définition, nous procéderons comme il suit: 1° dans 58.1 nous démontrerons que le nombre (3) reste inaltéré en remplaçant $C^p(\mathfrak{Z})$ par $C_1^p(\mathfrak{Z}) \sim C^p(\mathfrak{Z}) \mod A_1 S$ dans A_1; 2° dans 58.2 nous démontrerons que le nombre (3) reste inaltéré en remplaçant $C^{n-p}(\mathfrak{U})$ par $C_1^{n-p}(\mathfrak{U}) \sim C^{n-p}(\mathfrak{U}) \mod A_2 S$ dans A_2; 3° dans 58.3 nous démontrerons que, $\mathfrak{Z}$, $\mathfrak{T}$ et $\mathfrak{U}$ étant choisis, le nombre (3) est bien déterminé; 4° dans 58.4 nous démontrerons que, $\mathfrak{Z}$ et $\mathfrak{T}$ étant choisis, le nombre (3) ne dépend pas du choix de $\mathfrak{U} \in \Phi(\mathfrak{Z}, \mathfrak{T})$; 5° dans 58.5 nous donnons un lemme; 6° dans 58.6 nous démontrerons que, $\mathfrak{Z}$ étant choisi, le nombre (3) ne dépend pas du choix de $\mathfrak{T} \in \Delta(\mathfrak{Z})$; 7° dans 58.7 nous démontrerons que le nombre (3) ne dépend non plus du choix $\mathfrak{Z} \in \Xi_2$. Il en résulte que le nombre (3) est déterminé sans ambiguité par les deux cycles $C^p \in \Gamma_1$, $C^{n-p} \in \Gamma_2$ et qu'il jouit de la propriété 4° du n° 56; les propriétés 1°, 2°, 3°, du n° 56 sont banales.

58.1. Puisque $C^{n-p}(\mathfrak{U}) \to 0 \mod S$, il résulte de (2) que $\sum_{i=1}^{\alpha_p} b_i K^{n-p}(\sigma_i^p, \mathfrak{U}) \to 0 \mod \overline{R - R_0}$. Or pour $p < n$ on a (v. 19.1, 3°)

$$\sum_{i=1}^{\alpha_p} b_i K^{n-p}(\sigma_i^p, \mathfrak{U}) \to \sum_{i=1}^{\alpha_p} \sum_{j=1}^{\alpha_{p+1}} \eta_{ji}^p b_i K^{n-p-1}(\sigma_j^{p+1}, \mathfrak{U}) \mod \overline{R - R_0},$$

de sorte que dans le cas $p < n$ on a [v. 19.1, 1°, 2°, et la propriété 4° de la famille $\Phi(\mathfrak{Z}, \mathfrak{T})$]

$$(*) \qquad \sum_{i=1}^{\alpha_p} \eta_{ji}^p b_i = 0 \quad \text{pour } 1 \leqq j \leqq \alpha_{p+1}.$$

Ceci étant, soit $C_1^p(\mathfrak{Z}) \sim C^p(\mathfrak{Z}) \mod A_1 S$ dans A_1. Puisque $\mathfrak{Z} \in \Xi_1$, il existe des nombres $a_i' \in \mathfrak{R}$ tels que

$$C_1^p(\mathfrak{Z}) = \sum_{i=1}^{\alpha_p} a_i' \sigma_i^p \mod \bar{\Omega}$$

et pour $p < n$ il existe des nombres $c_j \in \mathfrak{R}$ tels que

$$\sum_{j=1}^{\alpha_{p+1}} c_j \sigma_j^{p+1} \to \sum_{i=1}^{\alpha_p} (a_i' - a_i) \sigma_i^p \mod \bar{\Omega} \text{ dans } A_1,$$

tandis que pour $p = n$ on a $\sum_{i=1}^{\alpha_n} (a_i' - a_i)\sigma_i^n = 0 \mod \bar{\Omega}$. On a donc dans les deux cas

$$\sum_{i=1}^{\alpha_p} \left(a_i' - a_i - \sum_{j=1}^{\alpha_{p+1}} \eta_{ji}^p c_j \right) \sigma_i^p = 0 \mod \bar{\Omega},$$

en convenant que $\sum_{1}^{\alpha_{p+1}} = 0$ pour $p = n$. Il en résulte que

$$a_i' - a_i = \sum_{j=1}^{\alpha_{p+1}} \eta_{ji}^p c_j \qquad (= 0 \text{ pour } p = n)$$

pour chaque valeur de i telle que $\sigma_i^p \neq 0 \mod \bar{\Omega}$. Or on a, en vertu de (*),

$$\sum_{i=1}^{\alpha_p} b_i \sum_{j=1}^{\alpha_{p+1}} \eta_{ji}^p c_j = 0$$

de sorte que, pour arriver au but $\sum_{i=1}^{\alpha_p} a_i b_i = \sum_{i=1}^{\alpha_p} a_i' b_i$ il suffit de montrer que pour $\sigma_i^p = 0 \mod \bar{\Omega}$ on a soit $a_i' = a_i = \sum_{j=1}^{\alpha_{p+1}} \eta_{ji}^p c_j = 0$, soit $b_i = 0$. Supposons que ceci ne soit pas vrai pour une certaine valeur de i telle que $\sigma_i^p = 0 \mod \bar{\Omega}$. Les chaînes $\sum a_i \sigma_i^p$, $\sum a_i' \sigma_i^p$, $\sum c_j \sigma_j^{p+1}$ étant situées dans A_1, on aurait $\sigma_i^p = 0 \mod A_1$; la chaîne $\sum b_i K^{n-p}(\sigma_i^p, \mathfrak{U})$ étant située dans $\bar{B}' \subset \bar{B}$, on aurait $K^{n-p}(\sigma_i^p, \mathfrak{U}) = 0 \mod \bar{B}$. Soit Z_1 un sommet de σ_i^p et soit U un sommet d'un simplexe de $K^{n-p}(\sigma_i^p, \mathfrak{U})$ [v. la propriété 4° de la famille $\Phi(\mathfrak{Z}, \mathfrak{T})$]; on aurait $Z_1 \bar{\Omega} \neq 0$, $Z_1 A_1 \neq 0$, $U\bar{B} \neq 0$ et aussi $UZ_1 \neq 0$ en vertu de 19.3. Or d'après la propriété 3° de la famille $\Phi(\mathfrak{Z}, \mathfrak{T})$, il existe un sommet Z_2 de $\mathfrak{Z}$ tel que $U \subset Z_2$. On aurait donc $Z_1 Z_2 \neq 0$, $Z_1 A_1 \neq 0$, $Z_2 \bar{B} \neq 0$, $Z_1 \bar{\Omega} \neq 0$, ce qui est une contradiction, car $\mathfrak{Z} \in \Xi_2$.

58.2. D'après la définition de la famille $\mathfrak{Z}$, il existe un affinement $\mathfrak{z} \in \Xi_2$ de $\mathfrak{Z}$ et une projection $\pi = Pr.(\mathfrak{z}, \mathfrak{Z})$ dont on obtient $\mathfrak{T}$ moyennant la construction du n° 24 (après avoir choisi le réseau fermé t correspondant à $\mathfrak{z}$). Faisons usage des notations du n° 24. On peut poser [v. 58 (1)]

$$C^p(\mathfrak{z}) = \sum_{\nu=1}^{\beta_p} a_\nu' \tau_\nu^p \mod \bar{\Omega}.$$

Or on a $C^p(\mathfrak{z}) \to 0 \mod S$, d'où $\sum_{\nu=1}^{\beta_p} a_\nu' \tau_\nu^p \to 0 \mod \bar{\Omega}$, donc

$$\sum_{\nu=1}^{\beta_p} \sum_{\mu=1}^{\beta_{p-1}} \zeta_{\nu\mu}^{p-1} a_\nu' \tau_\mu^{p-1} = 0 \mod \bar{\Omega}.$$

Il en résulte que

(*) $$\sum_{\nu=1}^{\beta_p} \zeta_{\nu\mu}^{p-1} a_\nu' = 0$$

pour chaque valeur de μ telle que $\tau_\mu^{p-1} \not\equiv 0 \bmod \bar{\Omega}$. Pour $1 \leqq i \leqq \alpha_p$ posons $a_i'' = \sum_\nu a_\nu'$, la sommation étant étendue à toutes les valeurs de ν telles que $\pi\tau_\nu^p = \sigma_i^p$. On a

$$\pi C^p(\mathfrak{z}) = \sum_{i=1}^{\alpha_p} a_i'' \sigma_i^p \mod \bar{\Omega}.$$

Ceci étant, remplaçons $C^{n-p}(\mathfrak{U})$ par $C_1^{n-p}(\mathfrak{U}) \sim C^{n-p}(\mathfrak{U}) \bmod A_2 S$ dans A_2. Il existe [v. 58 (2)] des nombres $b_i' \in \mathfrak{R}$ tels que

$$(3') \qquad C_1^{n-p}(\mathfrak{U}) \sim \sum_{i=1}^{\alpha_p} b_i' K^{n-p}(\sigma_i^p, \mathfrak{U}) \mod \bar{B}' \overline{R - R_0} \text{ dans } \bar{B}'.$$

Il s'agit de montrer que $\sum_{i=1}^{\alpha_p} a_i b_i = \sum_{i=1}^{\alpha_p} a_i b_i'$. Or d'après 58.1 on a $\sum_{i=1}^{\alpha_p} a_i b_i = \sum_{i=1}^{\alpha_p} a_i'' b_i$, $\sum_{i=1}^{\alpha_p} a_i b_i' = \sum_{i=1}^{\alpha_p} a_i'' b_i'$, car $\pi C^p(\mathfrak{z}) \sim C^p(\mathfrak{Z}) \bmod A_1 S$ dans A_1. Il suffit donc de prouver que $\sum_{i=1}^{\alpha_p} a_i''(b_i' - b_i) = 0$. Posons $b_\nu'' = 0$ pour $\gamma_p + 1 \leqq \nu \leq \beta_p$ ainsi que pour toutes les valeurs de ν telles que $1 \leqq \nu \leqq \gamma_p$, $\pi\tau_\nu^p = 0$; pour $1 \leqq \nu \leqq \gamma_p$, $\pi\tau_\nu^p = \sigma_i^p$ posons $b_\nu'' = b_i' - b_i$. D'après la définition de a_i'' on a alors

$$\sum_{i=1}^{\alpha_p} a_i''(b_i' - b_i) = \sum_{\nu=1}^{\beta_p} a_\nu' b_\nu'';$$

on doit donc démontrer que $\sum_{\nu=1}^{\beta_p} a_\nu' b_\nu'' = 0$. On a [v. (3), (3') ainsi que 40, (3)]

$$\sum_{\nu=1}^{\beta_p} b_\nu'' k^{n-p}(\tau_\nu^p, \mathfrak{U}) = \sum_{i=1}^{\alpha_p} (b_i' - b_i) K^{n-p}(\sigma_i^p, \mathfrak{U}) \mod \bar{B}' \overline{R - R_0} \text{ dans } \bar{B}',$$

et les deux membres de cette égalité sont $\sim 0 \bmod \bar{B}' \overline{R - R_0}$ dans $\bar{B}'$. D'après 44, il existe donc des nombres $d_\mu \in \mathfrak{R}$ tels que

$$\sum_{\mu=1}^{\beta_{p-1}} d_\mu k^{n-p+1}(\tau_\mu^{p-1}, \mathfrak{U}) \to \sum_{\nu=1}^{\beta_p} b_\nu'' k^{n-p}(\tau_\nu^p, \mathfrak{U}) \mod \bar{\Omega} \text{ dans } \bar{B},$$

de manière que (v. 19.1, 3°)

$$\sum_{\nu=1}^{\beta_p} \left(b_\nu'' - \sum_{\mu=1}^{\beta_{p-1}} \zeta_{\mu\nu}^{p-1} d_\mu \right) k^{n-p}(\tau_\nu^p, \mathfrak{U}) = 0 \mod \bar{\Omega} \text{ dans } \bar{B},$$

d'où il résulte (v. 19.1, 2° et la propriété 4° de la famille Φ) que

$$(**) \qquad b_\nu'' = \sum_{\mu=1}^{\beta_p} \zeta_{\mu\nu}^{p-1} d_\mu$$

pour chaque valeur de ν telle que $k^{n-p}(\tau_\nu^p, \mathfrak{U}) \not\equiv 0 \bmod \bar{\Omega}$.

Remarque 1. Si $\tau_\mu^{p-1} = 0 \bmod \bar{\Omega}$ on a soit $d_\mu = 0$ soit $\sum_{\nu=1}^{\beta_p} \zeta_{\nu\mu}^{p-1} a'_\nu = 0$.

Remarque 2. Si $k^{n-p}(\tau_\nu^p, \mathfrak{U}) = 0 \bmod \bar{\Omega}$ on a soit $a'_\nu = 0$ soit $\sum_{\mu=1}^{\beta_{p-1}} \zeta_{\mu\nu}^{p-1} d_\mu = 0$.

En supposant la validité de ces dex remarques, on obtient de (*) et de (**) que

$$\sum_{\nu=1}^{\beta_p} a'_\nu b''_\nu = \sum_{\nu=1}^{\beta_p} \sum_{\mu=1}^{\beta_{p-1}} \zeta_{\nu\mu}^{p-1} a'_\nu d_\mu = 0, \quad \text{q. f. d.}$$

Il ne reste donc qu'à démontrer les deux remarques. Supposons donc que les indices ν et μ soient tels que $\zeta_{\nu\mu}^{p-1} \neq 0$ et que soit $\tau_\mu^{p-1} = 0 \bmod \bar{\Omega}$ soit $k^{n-p}(\tau_\nu^p, \mathfrak{U}) = 0 \bmod \bar{\Omega}$. Puisque la chaîne $\sum_{\nu=1}^{\beta_p} a'_\nu \tau_\nu^p$ est située dans A_1 et puisque les chaînes $\sum_{\nu=1}^{\beta_p} b''_\nu k^{n-p}(\tau_\nu^p, \mathfrak{U})$ et $\sum_{\mu=1}^{\beta_{p-1}} d_\mu k^{n-p-1}(\tau_\mu^{p-1}, \mathfrak{U})$ sont situées dans $\bar{B}$, dans le cas $a'_\nu \neq 0$ ou $\sum_{\nu=1}^{\beta_p} \zeta_{\nu\mu}^{p-1} a'_\nu \neq 0$ on a $Z_1 A_1 \neq 0$ pour chaque sommet Z_1 de τ_μ^{p-1} et dans le cas $d_\mu \neq 0$ ou $\sum_{\mu=1}^{\beta_{p-1}} \zeta_{\nu\mu}^{p-1} d_\mu \neq 0$ on a $U\bar{B} \neq 0$ pour chaque sommet U de chaque simplexe de $k^{n-p}(\tau_\nu^p, \mathfrak{U})$[64]. Puisque soit $\tau_\mu^{p-1} = 0 \bmod \bar{\Omega}$, soit $k^{n-p}(\tau_\nu^p, \mathfrak{U}) = 0 \bmod \bar{\Omega}$, on a $(Z_1 + U)\bar{\Omega} \neq 0$. D'autre part on a $UZ_1 \neq 0$ d'après 19.3. D'après la propriété 3° de la famille $\Phi(\mathfrak{Z}, \mathfrak{T})$ il existe un sommet Z_2 de $\mathfrak{Z}$ tel que $U \subset Z_2$. Donc $Z_1 Z_2 \neq 0$, $(Z_1 + Z_2)\bar{\Omega} \neq 0$ et, si une de nos deux remarques n'était pas vraie, on aurait encore $Z_1 A_1 \neq 0$, $Z_2 \bar{B} \neq 0$ ce qui donnerait une contradiction, car $\mathfrak{Z} \in \Xi_2$.

58.3. Il résulte de 58.1 que le nombre (3) est indépendant du choix des nombres a_i pourvu que ces nombres satisfassent à (1). Il résulte de 58.2 que le nombre (3) est indépendant du choix des nombres b_i pourvu que ces nombres satisfassent à (2). Donc, $\mathfrak{Z}$, $\mathfrak{T}$ et $\mathfrak{U}$ étant choisis, le nombre (3) est bien déterminé.

58.4. Il est presque évident que $\mathfrak{Z}$ et $\mathfrak{T}$ étant choisis, le nombre (3) ne dépend pas du choix de $\mathfrak{U} \in \Phi(\mathfrak{Z}, \mathfrak{T})$. La famille $\Phi(\mathfrak{Z}, \mathfrak{T})$ étant complète, il suffit de prouver que le nombre (3) est le même pour $\mathfrak{U} = \mathfrak{U}_1$ et pour $\mathfrak{U} = \mathfrak{U}_2$, $\mathfrak{U}_2 \in \Phi(\mathfrak{Z}, \mathfrak{T})$ étant un affinement de $\mathfrak{U}_1 \in \Phi(\mathfrak{Z}, \mathfrak{T})$. A cet effet il suffit de remarquer que, si les nombres $b_i \in \mathfrak{R}$ sont telles qu'on ait (2) pour $\mathfrak{U} = \mathfrak{U}_2$, on a aussi (2) pour $\mathfrak{U} = \mathfrak{U}_1$; et cette remarque est évidente en vertu de 19.2 (v. aussi 19.1, 1°).

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[64] Il faut tenir compte de ce que chaque simplexe de $k^{n-p-1}(\tau_\mu^{p-1}, \mathfrak{U})$ est (v. la démonstration de 19.1) une face d'un simplexe de $k^{n-p}(\tau_\nu^p, \mathfrak{U})$.

58.5. LEMME. Soit $\mathfrak{z} \in \Xi_2$ un affinement de $\mathfrak{Z} \in \Xi_2$; soit $\pi = Pr.(\mathfrak{z}, \mathfrak{Z})$. Soit $\mathfrak{t} \in \Delta(\mathfrak{z})$; construisons le réseau fermé $\mathfrak{T}$ correspondant à $\mathfrak{Z}$ selon la manière du n° 24, en faisant usage de $\mathfrak{z}$, $\mathfrak{t}$ et π. On a $\mathfrak{T} \in \Delta(\mathfrak{Z})$ et on obtient la même valeur pour $C^p C^{n-p}$, en le calculant soit relativement à $\mathfrak{Z}$ et $\mathfrak{T}$, soit relativement à $\mathfrak{z}$ et $\mathfrak{t}$.

Démonstration. On voit sans aucune difficulté que $\mathfrak{T}, \mathfrak{Z} \in \Delta(\mathfrak{Z})$. Faisons usage des notations du n° 24. Déterminons les nombres $a_\nu, b_i \in \mathfrak{R}$ de manière que

$$C^p(\mathfrak{z}) = \sum_{\nu=1}^{\beta_p} a_\nu \tau_\nu^p \mod \bar{\Omega},$$

$$C^{n-p}(\mathfrak{U}) \sim \sum_{i=1}^{\alpha_p} b_i K^{n-p}(\sigma_i^p, \mathfrak{U}) \mod \bar{B}' \overline{R - R_0} \text{ dans } \bar{B}',$$

où le réseau $\mathfrak{U}$ commode rel. à $\mathfrak{z}+\mathfrak{t}$ (et donc aussi rel. à $\mathfrak{Z}+\mathfrak{T}$, v. 24) est choisi (v. 58.4) si fin qu'il appartienne à $\Phi(\mathfrak{Z}, \mathfrak{T})$ et à $\Phi(\mathfrak{z}, \mathfrak{t})$. Pour $1 \leqq i \leqq \alpha_p$, soit $a_i' = \sum a_\nu$, la sommation se rapportant à toutes les valeurs de ν telles que $\pi \tau_\nu^p = \sigma_i^p$. On a alors

$$\pi C^p(\mathfrak{z}) = \sum_{i=1}^{\alpha_p} a_i' \sigma_i^p \mod \bar{\Omega},$$

de sorte que $C^p C^{n-p}$, calculé rel. à $\mathfrak{Z}+\mathfrak{T}$ a la valeur $\sum_{i=1}^{\alpha_p} a_i' b_i$. D'autre part on a, d'après 24 (3)

$$C^{n-p}(\mathfrak{U}) \sim \sum_{\nu=1}^{\beta_p} b_\nu' k^{n-p}(\tau_\nu^p, \mathfrak{U}) \mod \bar{B}' \overline{R - R_0} \text{ dans } \bar{B}',$$

où l'on a posé: 1° $b_\nu' = 0$ pour $\gamma_p + 1 \leqq \nu \leqq \beta_p$ ainsi que pour toutes les valeurs de ν telles que $1 \leqq \nu \leqq \gamma_p$, $\pi \tau_\nu^p = 0$; 2° $b_\nu' = b_i$ pour $1 \leqq \nu \leqq \gamma_p$, $\pi \tau_\nu^p = \sigma_i^p$. Donc le nombre $C^p C^{n-p}$, calculé rel. à $\mathfrak{z}+\mathfrak{t}$, a la valeur $\sum_{\nu=1}^{\beta_p} a_\nu b_\nu'$. On doit donc seulement démontrer que $\sum_{i=1}^{\alpha_p} a_i' b_i = \sum_{\nu=1}^{\beta_p} a_\nu b_\nu'$ ce qui est évident d'après la définition même des nombres a_i' et b_ν'.

58.6. Supposons donné $\mathfrak{Z} \in \Xi_2$, $\mathfrak{T} \in \Delta(\mathfrak{Z})$, $\mathfrak{U} \in \Phi(\mathfrak{Z}, \mathfrak{T})$. Soient U_ν $(1 \leqq \nu \leqq m)$ tous les sommets de $\mathfrak{U}$ tels que $UR_0 \neq 0$ (v. 8). Pour chaque ν $(1 \leqq \nu \leqq m)$ on peut (v. 8.1, 6°) indiquer un $\mathfrak{Z}$-simplexe intérieur $\sigma(\nu) = \sigma_i^h$ tel que 1° $U_\nu T(\sigma_i^h) \neq 0$; 2° $U_\nu T_j = 0$ pour chaque valeur de j $(1 \leqq j \leqq \alpha_0)$ telle que σ_j^0 n'est pas un sommet de $\sigma(\nu)$. Choisissons un point $a_\nu \in UT(\sigma_i^h)$ $(1 \leqq \nu \leqq m)$. On ne peut avoir $a_\nu = a_\mu$ que si $\sigma(\nu) = \sigma(\mu)$. Soit V_ν $(1 \leqq \nu \leqq m)$ un entourage de a_ν si petit que 1° $V_\nu \subset U_\nu$; 2° $V_\nu V_\mu = 0$ pour $a_\nu \neq a_\mu$; soit V_ν' un entourage de a_ν si petit que $V_\nu' \Subset V_\nu$. Soit N l'ensemble de tous les couples (i, U) $(1 \leqq i \leqq \alpha_0,\ U \in \mathfrak{U})$ tels que $UT_i = 0$

et par suite aussi (v. 8.1, 5°) $\bar{U}T_i = 0$. Soit $\mathfrak{B}$ un réseau si fin que pour $(i, U) \in N$ aucun sommet de $\mathfrak{B}$ ne rencontre simultanément $\bar{U}$ et T_i. Soit $\mathfrak{z} \in \Xi_2$ un affinement de $\mathfrak{Z}$ si petit que 1° z_1, $z_2 \in \mathfrak{z}$, $z_1 z_2 \neq 0$, $z_1 \bar{V}'_\nu \neq 0$ $(1 \leqq \nu \leqq m)$ entraîne $z_2 \subset V_\nu$ (v. 1.2); 2° $\mathfrak{z}$ est un affinement de $\mathfrak{B}$; 3° la chaîne $G^n(\mathfrak{z})$ n'est située dans $R - V'_\nu$ pour aucune valeur de ν $(1 \leqq \nu \leqq m$; v. 17.5). En vertu de la propriété 3° de $\mathfrak{z}$, il existe pour chaque ν $(1 \leqq \nu \leqq m)$ un $(n, \mathfrak{z})$-simplexe intérieur ϱ^n_ν dont le noyau rencontre V'_ν; en vertu de la propriété 1° de $\mathfrak{z}$, chaque sommet de ϱ^n_ν fait partie de V_ν; il existe donc une h-face [h étant déterminé par l'égalité $\sigma^h_i = \sigma(\nu)$] ϱ^h_ν de ϱ^n_ν. On peut supposer que $\varrho^h_\nu = \varrho^h_\mu$ pour $a_\nu = a_\mu$. Puisque $V_\mu V_\nu = 0$ pour $a_\mu \neq a_\nu$, on peut évidemment choisir la projection $\pi = Pr.(\mathfrak{z}, \mathfrak{Z})$ de manière que 1° $\pi \varrho^h_\nu = \sigma(\nu)$ pour $1 \leqq \nu \leqq m$; 2° $\tau^0 \in \mathfrak{z}$, $\pi\tau^0 = \sigma^0_i$ $(1 \leqq i \leqq \alpha_0)$ entraîne que $\tau^0 T_i \neq 0$. Choisissons $\mathfrak{t} \in \Delta(\mathfrak{z})$. Puisque chaque sommet de ϱ^h_ν est un sous-ensemble de U_ν, on a $t(\varrho^h_\nu) \subset U_\nu$ pour $1 \leqq \nu \leqq m$. A l'aide de $\mathfrak{z}$, $\mathfrak{t}$ et π, construisons le réseau fermé $\mathfrak{T}^*$ correspondant à $\mathfrak{Z}$ (v. 8) suivant la manière expliquée au n° 24. Supposons que $T^*(\sigma^h_i)$ ait la même signification rel. à $\mathfrak{T}^*$ que $T(\sigma^h_i)$ rel. à $\mathfrak{T}$. Puisque $t(\varrho^h_\nu) \subset T^*(\sigma^h_i)$ $[\sigma^h_i = \sigma(\nu)]$ [v. 24 (2)] et puisque $t(\varrho^h_\nu) \subset U_\nu$, on a $U_\nu T^*(\sigma^h_i) \neq 0$[65]. Autrement dit: $0 \leqq h \leqq n$, $1 \leqq i \leqq \alpha_h$, $U \in \mathfrak{U}$, $UT(\sigma^h_i) \neq 0$ entraîne $UT^*(\sigma^h_i) \neq 0$. Réciproquement supposons que, pour de certaines valeurs de i, h, U $(0 \leqq h \leqq n$, $1 \leqq i \leqq \alpha_h$, $U \in \mathfrak{U})$ on ait $UT(\sigma^h_i) = 0$ et par suite $\bar{U}T(\sigma^h_i) = 0$. Nous prouverons que $\bar{U}T^*(\sigma^h_i) = 0$. D'après 8.1, 5° et 6°, il existe un sommet σ^0_j de σ^h_i tel que $\bar{U}T_j = 0$. Il suffit de prouver que $\bar{U}T^*_j = 0$. Dans le cas contraire, il existerait un sommet τ^0 de $\mathfrak{z}$ tel que $\bar{U}\tau^0 \neq 0$, $\pi\tau^0 = \sigma^0_j$. D'après la propriété 2° de la projection π, on aurait $\tau^0 T_j \neq 0$ d'où la contradiction $\bar{U}\tau^0 = 0$ d'après la propriété 2° du réseau $\mathfrak{z}$. Maintenant on voit sans peine que le réseau $\mathfrak{U}$ est commode non seulement rel. à $\mathfrak{Z} + \mathfrak{T}$, mais aussi rel. à $\mathfrak{Z} + \mathfrak{T}^*$. Puisque $\mathfrak{t} \in \Delta(\mathfrak{z})$, on a $\mathfrak{T}^* \in \Delta(\mathfrak{Z})$ (v. 58.5). Choisissons un affinement $\mathfrak{U}_1$ de $\mathfrak{U}$ de manière que $\mathfrak{U}_1 \in \Phi(\mathfrak{Z}, \mathfrak{T}^*)$. Déterminons les nombres $a_i \in \mathfrak{R}$ d'après 58 (1). Déterminons les nombres $b_i \in \mathfrak{R}$ de manière que

$$C^{n-p}(\mathfrak{U}_1) \sim \sum_{i=1}^{\alpha_p} b_i \, {}^*K^{n-p}(\sigma^p_i, \mathfrak{U}_1) \mod \bar{B}' \, \overline{R - R_0} \text{ dans } \bar{B}',$$

$^*K$ désignant les chaînes fondamentales (v. 19.1) relatives à $\mathfrak{Z} + \mathfrak{T}^*$. On voit sans peine (v. 58.4) que la relation (*) reste vraie en y remplaçant $\mathfrak{U}_1$ par $\mathfrak{U}$. Or on déduit facilement de 19.1 que $^*K^{n-h}(\sigma^h_i, \mathfrak{U}) = K^{n-h}(\sigma^h_i, \mathfrak{U})$. Par suite, la relation 58 (2) est vraie. Donc on obtient la même valeur

[65] $t(\varrho^h_\nu) \neq 0$ car $\mathfrak{z} \in \Xi$ et par suite (v. 23) le noyau de ϱ^h_ν contient un point b n'appartenant à la fermeture d'aucun sommet de $\mathfrak{z}$ qui ne soit pas un sommet de ϱ^h_ν, d'où il résulte sans peine que $b \in t(\varrho^h_\nu)$.

pour le nombre $C^p\,C^{n-p}$ en le calculant soit rel. à $\mathfrak{Z}+\mathfrak{T}$, soit rel. à $\mathfrak{Z}+\mathfrak{T}^*$ et par suite, d'après le lemme du n° 58.5, aussi en le calculant rel. à $\mathfrak{z}+\mathfrak{t}$.

Ceci étant, supposons qu'on ait donné $\mathfrak{Z}\in\Xi_2$ et $\mathfrak{T}_1, \mathfrak{T}_2\in\Delta(\mathfrak{Z})$. On voit sans peine que l'on peut déterminer $\mathfrak{z}$ et $\mathfrak{t}$ de manière que les conditions qui ont été énoncées plus haut soient vérifiées simultanément pour $\mathfrak{T}=\mathfrak{T}_1$ et pour $\mathfrak{T}=\mathfrak{T}_2$. Il en résulte que le nombre $C^p\,C^{n-p}$ reste le même en le calculant soit rel. à $\mathfrak{Z}+\mathfrak{T}_1$, soit rel. à $\mathfrak{Z}+\mathfrak{T}_2$.

58.7. Le nombre $C^p\,C^{n-p}$ dépend donc tout au plus du choix de $\mathfrak{Z}\in\Xi$, le choix de $\mathfrak{T}\in\Delta(\mathfrak{Z})$ et celui de $\mathfrak{U}\in\Phi(\mathfrak{Z},\mathfrak{T})$ étant sans influence sur lui. Or supposons donné $\mathfrak{Z}_1, \mathfrak{Z}_2\in\Xi_2$. La famille Ξ_2 étant complète, il existe un affinement simultané $\mathfrak{z}$ de $\mathfrak{Z}_1$ et de $\mathfrak{Z}_2$. D'après 58.5 le nombre $C^p\,C^{n-p}$, calculé rel. à $\mathfrak{z}$ est le même que si on le calcule rel. à $\mathfrak{Z}_1$ ou rel. à $\mathfrak{Z}_2$. Donc $C^p\,C^{n-p}$ peut être calculé indifféremment soit rel. à $\mathfrak{Z}_1$ soit rel. à $\mathfrak{Z}_2$.

59.1. Si les deux ensembles A_1, A_2 du n° 56 sont sans point commun, on a $C^p\,C^{n-p}=0$ pour chaque $C^p\in\Gamma_1$, $C^{n-p}\in\Gamma_2$. La facile démonstration sera laissée au soin du lecteur.

59.2. Soient A_1 et A_2 deux sous-ensembles bicompacts donnés de R, assujettis à la condition $A_1\,A_2\,S=0$. Définissons Γ_1 et Γ_2 comme dans 56. Soit F un sous-ensemble bicompact de R tel que $F\supset S$, $A_1\,A_2\,F=0$. Tous les axiomes supposés au n° 55 restent évidemment vérifiés en remplaçant S par F. Désignons par Γ_1' et Γ_2' ce que deviennent les familles Γ_1 et Γ_2 si on remplace S par F. Soit $C^p\in\Gamma_1$, $C^{n-p}\in\Gamma_2$. Il existe évidemment des cycles $C_1^p\in\Gamma_1'$, $C_1^{n-p}\in\Gamma_2'$ tels que $C^p(\mathfrak{Z})=C_1^p(\mathfrak{Z})$ mod F, $C^{n-p}(\mathfrak{Z})=C_1^{n-p}(\mathfrak{Z})$ mod F dans chaque réseau $\mathfrak{Z}$. On voit sans peine que $C^p\,C^{n-p}=C_1^p\,C_1^{n-p}$.

VII.

60. Dans ce Chapitre, nous ferons de nouveau les hypothèses énoncées au n° 55.

61. Supposons donné un sous-ensemble bicompact S_2 de S. Posons $S_1=S-S_2$. Soit Γ_1 l'ensemble de tous les (p,R)-cycles mod SA_1 dans A_1, où A_1 parcourt tous les sous-ensembles bicompacts de R tels que $SA_1\subset S_1$; deux éléments C_1^p, C_2^p de Γ_1 seront considérés comme égaux si l'on peut attacher à chaque entourage Ω_1 de S_1[66] un sous-ensemble bicompact A_1 de R tel que 1° $SA_1\subset\Omega_1$, 2° $C_1^p\sim C_2^p$ mod $\overline{\Omega}_1 A_1$ dans A_1. L'ensemble Γ_1 est évidemment un module. Soit Γ_2 l'ensemble de tous les $(n-p,R)$-cycles mod SA_2 dans A_2 où A_2 parcourt tous les sous-ensembles bicompacts de R tels que $SA_2\subset S_2$; deux éléments C_1^{n-p}, C_2^{n-p} de Γ_2 seront considérés comme égaux si l'on peut attacher à chaque entourage Ω_2 de S_2 un sous-ensemble

[66] On voit sans peine qu'on peut supposer que $\Omega_1 S_2=0$.

bicompact A_2 de R tel que 1° $SA_2 \subset \Omega_2$; 2° $C_1^{n-p} \sim C_2^{n-p} \bmod \bar{\Omega}_2 A_2$ dans A_2. L'ensemble Γ_2 est aussi un module.

D'après le n° 56, on peut attacher à chaque couple $C^p \in \Gamma_1$, $C^{n-p} \in \Gamma_2$ un nombre $C^p C^{n-p} \in \mathfrak{R}$ de manière que :

$$1^\circ \quad (rC^p)C^{n-p} = C^p(rC^{n-p}) = r(C^p C^{n-p}) \quad \text{pour } r \in \mathfrak{R};$$

$$2^\circ \quad (C_1^p + C_2^p)C^{n-p} = C_1^p C^{n-p} + C_2^p C^{n-p};$$

$$3^\circ \quad C^p(C_1^{n-p} + C_2^{n-p}) = C^p C_1^{n-p} + C^p C_2^{n-p}.$$

En vertu de 56, 4° (v. aussi 59) on voit sans peine que le nombre $C^p C^{n-p}$ est toujours bien déterminé, malgré les conventions que nous venons de faire sur l'égalité de deux éléments de Γ_1 et de Γ_2.

Le but de ce Chapitre est de montrer[67] que, relativement à la multiplication $C^p C^{n-p}$ considérée, *les deux modules* Γ_1 *et* Γ_2 *sont duels* (primitifs)[68], c'est-à-dire, outre les propriétés 1°, 2°, 3° déjà énoncées, on a encore les deux suivantes (les égalités $C^p = 0$, $C^{n-p} = 0$ y ont le sens conventionnel adopté) : 4° lorsque le cycle $C^p \in \Gamma_1$ est tel que $C^p C^{n-p} = 0$ pour chaque choix de $C^{n-p} \in \Gamma_2$, on a $C^p = 0$; 5° lorsque le cycle $C^{n-p} \in \Gamma_2$ est tel que $C^p C^{n-p} = 0$ pour chaque choix de $C^p \in \Gamma_1$, on a $C^{n-p} = 0$. La propriété 4° sera démontrée dans le n° 62, la propriété 5° dans le n° 63.

62. Soit A_1 un sous-ensemble bicompact de R tel que $A_1 S \subset S_1$. Soit C^p un (p, R)-cycle $\bmod A_1 S$ dans A_1. Supposons que C^p, considéré comme élément du module Γ_1, soit $\neq 0$. Il existe alors un entourage $\Omega_1 \subset R - S_2$ de S_1 tel que, A_1' étant un sous-ensemble bicompact de R assujetti à la condition $A_1' S \subset S_1$, on n'ait jamais $C^p \sim 0 \bmod A_1' \bar{\Omega}_1$ dans A_1'. On doit prouver qu'il existe un élément C^{n-p} de Γ_2 tel que $C^p C^{n-p} = 1$.

Soit Ω_1' un entourage de S_1 si petit que $\Omega_1' \subset \Omega_1$, $\bar{\Omega}_1' - S \subset \Omega_1$[69]. Posons $A_2 = R - \Omega_1'$ de sorte que A_2 est un sous-ensemble bicompact de R tel que $A_2 S = S_2$. Soit Θ la famille de tous les entourages Ω_2 de S_2 si petits que $A_1 \bar{\Omega}_2 = 0$. Pour chaque $\Omega_2 \in \Theta$, désignons par $\Pi(\Omega_2)$ la famille de tous les $(n-p, R)$-cycles $C^{n-p} \bmod A_2 \bar{\Omega}_2$ dans A_2 et tels que $C^p C^{n-p} = 1$ (v. 59.2 où on remplace F par $A_2 \bar{\Omega}_2$). Lorsque Ω_2, $\Omega_2^* \in \Theta$; $\Omega_2^* \subset \Omega_2$, on voit sans peine (v. 59.2) qu'avec chaque $C^{n-p} \in \Pi(\Omega_2^*)$ il existe un ${}^*C^{n-p} \in \Pi(\Omega_2)$ tel que $C^{n-p} = {}^*C^{n-p} \bmod A_2 \bar{\Omega}_2$. Si $\Pi(\Omega_2) \neq 0$, c'est évidemment un système linéaire (v. *Homologie*, I, 14). D'après le théorème du n° 6.1 (v. aussi 59.2), où on remplace R par A_2 et S par S_2 (ce qui est évidemment permis) il suffit donc de montrer que $\Pi(\Omega_2) \neq 0$ pour chaque $\Omega_2 \in \Theta$.

[67] C'est essentiellement le premier théorème de dualité de M. Lefschetz [v. S. Lefschetz, Topology, p. 142, formule (7)].

[68] V. Pontrjagin, Math. Ann., t. 105, 1931, pp. 165–205.

[69] Ω_1' existe, car $R - S_2$ est un espace normal.

Soit donc $\Omega_2 \in \Theta$ et choisissons un entourage Ω_2' de S_2 si petit que $\Omega_2' \Subset \Omega_2$. Choisissons un réseau $\mathfrak{Z}$ de la famille Ξ_2 (v. 57 pour $p=0$ et 58 pour $p \geqq 1$) suffisamment fin pour que les conditions suivantes aient lieu: 1° pour chaque sommet Z de $\mathfrak{Z}$ l'inégalité $Z\Omega_1' \neq 0$ entraîne que $Z \subset \Omega_1$ ou bien $Z \subset \Omega_2$; 2° chaque sommet extérieur de $\mathfrak{Z}$ est un sous-ensemble de $\Omega_1' + \Omega_2'$; 3° pour chaque sommet Z de $\mathfrak{Z}$ l'inégalité $Z\Omega_2' \neq 0$ entraîne que $Z \subset \Omega_2$; 4° il n'existe aucune chaîne $D^{p+1}(\mathfrak{Z})$ dans $R - \Omega_2$ telle que $FD^{p+1}(\mathfrak{Z}) = C^p(\mathfrak{Z}) \bmod \bar{\Omega}_1$ (c'est possible, car $C^p \not\sim 0 \bmod A_1' \bar{\Omega}_1$ dans A_1 pour $A_1' = R - \Omega_2$); 5° aucun sommet de $\mathfrak{Z}$ ne rencontre simultanément A_1 et $\bar{\Omega}_2$; 6° si l'on a $ZA_2 \neq 0$ et $Z\Omega_1' \neq 0$ pour un sommet Z de $\mathfrak{Z}_1$ alors $Z \subset \Omega_2$. Choisissons $\mathfrak{T} \in \Delta(\mathfrak{Z})$ et supposons que $\mathfrak{U}$ parcoure la famille $\Phi(\mathfrak{Z}, \mathfrak{T})$ (v. 57 pour $p=0$ et 58 pour $p \geqq 1$).

D'après la propriété 2° du réseau $\mathfrak{Z}$, on a (v. 8) $\overline{R - R_0} \subset \Omega_1 + \Omega_2$. Il existe donc (v. la propriété 5° de $\mathfrak{Z}$) des nombres a_i tels que

$$C^p(\mathfrak{Z}) = \sum_{i=1}^{\alpha_p} a_i \sigma_i^p \bmod A_1 \bar{\Omega}_1 \text{ dans } A_1.$$

Posons

$$C^{n-p}(\mathfrak{U}) = \sum_{i=1}^{\alpha_p} b_i K^{n-p}(\sigma_i^p, \mathfrak{U})$$

les nombres $b_i \in \mathfrak{R}$ étant assujettis aux conditions suivantes: 1° $b_i = 0$ si σ_i^p possède un sommet σ_j^0 tel que $\sigma_j^0 \Omega_1' \neq 0$; 2° $\sum_{i=1}^{\alpha_p} \eta_{ji}^p b_i = 0$ pour chaque valeur de j $(1 \leqq j \leqq \alpha_{p+1})$ telle qu'aucun sommet de σ_j^{p+1} ne rencontre Ω_2'; 3° $\sum_{i=1}^{\alpha_p} a_i b_i = 1$. Supposons pour un moment qu'on puisse vérifier ces conditions. D'après 1°, la chaîne $C^{n-p}(\mathfrak{U})$ est située dans $R - \Omega_1' = A_2$. D'après 2°, 19.1, 3° et 19.3 la chaîne $FC^{n-p}(\mathfrak{U})$ est située dans $A_2 \overline{R - R_0} + \sum_j T(\sigma_j^{p+1})$, j parcourant seulement de telles valeurs que σ_j^{p+1} possède un sommet rencontrant Ω_2' et par suite (v. la propriété 3° de $\mathfrak{Z}$) contenu dans Ω_2; donc $FC^{n-p}(\mathfrak{U}) \subset \Omega_2$ (v. la propriété 6° de $\mathfrak{Z}$). Donc $C^{n-p}(\mathfrak{U})$ est un $(n-p, \mathfrak{U})$-cycle $\bmod A_2 \bar{\Omega}_2$ dans Ω_2. En faisant varier les $\mathfrak{U}$, on voit (v. 19.2) que les chaînes $C^{n-p}(\mathfrak{U})$ définissent un $(n-p, R)$-cycle $\bmod A_2 \bar{\Omega}_2$ dans A_2. D'après la propriété 3° des nombres b_i, on a $C^p C^{n-p} = 1$.

Reste à prouver que les conditions posées pour les nombres $b_i \in \mathfrak{R}$ sont réalisables. Il résulte de la théorie élémentaire des équations linéaires que, dans le cas contraire, il existerait des nombres $c_j \in \mathfrak{R}$ $(1 \leqq j \leqq \alpha_{p+1})$ tels que 1° $c_j = 0$ pour chaque valeur de j telle que σ_j^{p+1} possède un sommet

rencontrant Ω_2'; 2° pour chaque valeur de i telle qu'aucun sommet de σ_i^p ne rencontre Ω_1' on a

$$a_i = \sum_{j=1}^{\alpha_{p+1}} \eta_{ji}^p c_j.$$

Or des conditions imposées pour $\mathfrak{Z}$ on déduit sans peine que l'on aurait dans ce cas $C^p(\mathfrak{Z}) \sim 0 \bmod \bar{\Omega}_1$ dans $R - \Omega_2$ ce qui est une contradiction.

63. Soit A_2 un sous-ensemble bicompact de R tel que $A_2 S \subset S_2$. Soit C^{n-p} un $(n-p, R)$-cycle mod $A_2 S$ dans A_2. Supposons que C^{n-p}, considéré comme élément du module Γ_2, soit $\neq 0$. Il existe donc un entourage Ω_2 de S_2 tel que, A_2' étant un sous-ensemble bicompact de R assujetti à la condition $A_2' S \subset \Omega_2$, on n'ait jamais $C^{n-p} \sim 0 \bmod A_2' \bar{\Omega}_2$ dans A_2'. On doit prouver qu'il existe un élément C^p de Γ_1 tel que $C^p C^{n-p} = 1$.

Soit Ω_2' un entourage de S_2 tel que $\Omega_2' \Subset \Omega_2$. Posons $A_1 = R - \Omega_2'$ de sorte que A_1 est un sous-ensemble bicompact de R tel que $A_1 S \subset S_1$. Il suffit donc de prouver qu'il existe un (p, R)-cycle $C^p \bmod A_1 S$ dans A_1 tel que $C^p C^{n-p} = 1$. Or soit Θ la famille de tous les entourages Ω de S si petits que $A_2 \bar{\Omega} \subset \Omega_2'$. Pour chaque $\Omega \in \Theta$, désignons par $\Pi(\Omega)$ la famille de tous les (p, R)-cycles $C^p \bmod A_1 \bar{\Omega}$ dans A_1 et tels que $C^p C^{n-p} = 1$ (v. 59.2, où l'on remplace F par $A_1 \bar{\Omega}$). Lorsque $\Omega, \Omega^* \in \Theta$; $\Omega^* \subset \Omega$, on voit sans peine (v. 59.2) qu'à chaque $C^p \in \Pi(\Omega^*)$ il existe un ${}^*C^p \in \Pi(\Omega)$ tel que $C^p = {}^*C^p \bmod A_1 \bar{\Omega}$. Si $\Pi(\Omega) \neq 0$, c'est évidemment un système linéaire. D'après le théorème du n° 6.1 (v. aussi 59.2), où l'on remplace R par A_1 et S par $A_1 S$ (ce qui est permis), il suffit donc de montrer que $\Pi(\Omega) \neq 0$ pour chaque $\Omega \in \Theta$.

Soit donc $\Omega \in \Theta$ et choisissons un entourage Ω' de S si petit que $\Omega' \Subset \Omega$. Soit B un entourage de A_2 si petit que $BS \subset \Omega_2'$. Choisissons un réseau $\mathfrak{Z}$ de la famille Ξ_2 (v. 57 pour $p = 0$ et 58 pour $p \geqq 1$). Soit $\mathfrak{z} \in \Xi_2$ un affinement de $\mathfrak{Z}$ si fin que les conditions suivantes aient lieu: 1° pour chaque sommet extérieur z de $\mathfrak{z}$ on a $Bz \subset \Omega_2$; 2° $z \in \mathfrak{z}$, $z\,\Omega' \neq 0$ entraîne que $z \subset \Omega$; 3° $z \in \mathfrak{z}$, $z\,\Omega_2' \neq 0$ entraîne que $z \subset \Omega_2$; 4° $\mathfrak{z}$ est un affinement de $\mathfrak{Z}$ normal rel. aux cycles mod $A_1 \bar{\Omega}$ dans A_1.

Choisissons $\mathfrak{t} \in \Delta(\mathfrak{z})$ et supposons que $\mathfrak{U}$ parcoure la famille $\Phi(\mathfrak{z}, \mathfrak{t})$ (v. 57 pour $p = 0$ et 58 pour $p \geqq 1$). Choisissons une projection $\pi = Pr.(\mathfrak{z}, \mathfrak{Z})$. Construisons le réseau fermé $\mathfrak{T}$ correspondant à $\mathfrak{Z}$ d'après n° 24, en y faisant usage de $\mathfrak{z}$, $\mathfrak{t}$ et π. Employons les notations du n° 24. Evidemment (v. 58.5) on a $\mathfrak{T} \in \Delta(\mathfrak{Z})$, $\Phi(\mathfrak{Z}, \mathfrak{T}) \supset \Phi(\mathfrak{z}, \mathfrak{t})$.

D'après 31 et 24 (3), il existe des nombres $b_i \in \mathfrak{R}$[70] tels que

$$C^{n-p}(\mathfrak{U}) \sim \sum_{i=1}^{\alpha_p} b_i K^{n-p}(\sigma_i^p, \mathfrak{U}) \sim \sum_{\nu=1}^{\beta_p} b_\nu' k^{n-p}(\tau_\nu^p, \mathfrak{U}) \bmod \bar{\Omega}_2 \text{ dans } \bar{B}.$$

[70] On voit sans peine (v. 26) qu'on peut supposer que les nombres b_i soient indépendants du choix de $\mathfrak{U} \in \Phi(\mathfrak{z}, \mathfrak{t})$.

nous y avons posé: 1° $b'_\nu = 0$ pour $\gamma_p + 1 \leqq \nu \leqq \beta_p$ ainsi que pour $1 \leqq \nu \leqq \gamma_p$, $\pi \tau^p_\nu = 0$; 2° $b'_\nu = b_i$ pour $1 \leqq \nu \leqq \gamma_p$, $\pi \tau^p_\nu = \sigma^p_i$. Posons

$$C^p(\mathfrak{z}) = \sum_{\nu=1}^{\beta_p} a'_\nu \tau^p_\nu,$$

les nombres $a'_\nu \in \mathfrak{R}$ étant assujettis aux conditions suivantes: 1° $a'_\nu = 0$ si un sommet de τ^p_ν rencontre Ω'_2; 2° $\sum_{\nu=1}^{\beta_p} \zeta^{p-1}_{\nu\mu} a'_\nu = 0$ pour chaque valeur de μ $(1 \leqq \mu \leqq \beta_{p-1})$ telle qu'aucun sommet de τ^{p-1}_μ ne rencontre Ω' [71]; 3° $\sum_{\nu=1}^{\beta_p} a'_\nu b'_\nu = 1$. Supposons pour un moment qu'on puisse vérifier ces conditions. D'après 1°, la chaîne $C^p(\mathfrak{z})$ est située dans $R - \Omega'_2 = A_1$; d'après 2°, chaque simplexe de la chaîne $FC^p(\mathfrak{z})$ possède un sommet rencontrant Ω' de sorte que la chaîne $FC^p(\mathfrak{z})$ est située dans Ω en vertu de la propriété 2° du réseau $\mathfrak{z}$. Donc $C^p(\mathfrak{Z}) = \sum_{i=1}^{\alpha_p} a_i \sigma^p_i$ est un $(p, \mathfrak{Z})$-cycle essentiel (v. la propriété 4° de $\mathfrak{z}$) $\bmod A_1 \bar{\Omega}$ dans A_1. On voit sans peine que $\sum_{i=1}^{\alpha_p} a_i b_i = 1$. Il existe donc un (p, R)-cycle $C^p \bmod A_1 \bar{\Omega}$ dans A_1 et tel que $C^p C^{n-p} = 1$, c. q. f. d.

Reste à prouver que les conditions posées pour les nombres a'_ν sont réalisables. On voit sans peine que, dans le cas contraire, on pourrait déterminer des nombres $c_\mu \in \mathfrak{R}$ $(1 \leqq \mu \leqq \beta_{p-1})$ tels que: 1° $c_\mu = 0$ si le simplexe τ^{p-1}_μ possède un sommet rencontrant Ω'; 2° $b'_\nu = \sum_{\mu=1}^{\beta_{p-1}} \zeta^{p-1}_{\nu\mu} c_\mu$ pour chaque valeur de ν $(1 \leqq \nu \leqq \beta_p)$ telle qu'aucun sommet de τ^p_ν ne rencontre Ω'_2 [72]. Or d'après 1°, la chaîne

$$D^{n-p+1}(\mathfrak{U}) = \sum_{\mu=1}^{\beta_{p-1}} c_\mu k^{n-p+1}(\tau^{p-1}_\mu, \mathfrak{U})$$

serait située dans $R - \Omega'$ et d'après 2° on aurait, en tenant compte de la propriété 3° du réseau $\mathfrak{z}$,

$$FD^{n-p+1}(\mathfrak{U}) = \sum_{\nu=1}^{\beta_p} b'_\nu k^{n-p}(\tau^p_\nu, \mathfrak{U}) \mod \bar{\Omega}_2 \text{ [73]}.$$

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[71] Cette condition n'exige rien si $p = 0$.

[72] Pour $p = 0$: On aurait $b'_\nu = 0$ pour chaque valeur de ν $(1 \leqq \nu \leqq \beta_0)$ telle que $\tau^0_\nu \Omega'_2 = 0$.

[73] Pour $p = 0$: On aurait $\sum_{\nu=1}^{\beta_0} b'_\nu k^n(\tau^0_\nu, \mathfrak{U}) = 0 \bmod \bar{\Omega}_2$.

Donc on aurait $C^{n-p}(\mathfrak{U}) \sim 0 \bmod \overline{\Omega}_2$ dans $R - \Omega'$ et par suite, la famille $\Phi(\mathfrak{z}, t)$ étant complète, $C^{n-p} \sim 0 \bmod A_2' \overline{\Omega}_2$ dans A_2' pour $A_2' = R - \Omega'$, ce qui est une contradiction.

64. Pour $S = S_2$, Γ_2 est (v. 6.2) le $(n-p)^{\text{ème}}$ groupe de Betti de l'espace R mod S. En désignant par $\varkappa$ la famille de tous les sous-ensembles bicompacts de $R - S$, on voit sans peine que, dans le cas actuel $S = S_2$, Γ_1 est le $p^{\text{ème}}$ groupe de Betti *bicompact* (= d'espèce $\varkappa$ au sens de l'*Homologie*, V, 9) de l'espace $R - S$. Pareillement, pour $S_2 = 0$, Γ_2 est le $(n-p)^{\text{ème}}$ groupe de Betti bicompact de $R - S$ et Γ_1 est le $p^{\text{ème}}$ groupe de Betti de R mod S.

Donc, *pour $q = p$ ou $q = n - p$, le $q^{\text{ème}}$ groupe de Betti de l'espace R mod S est duel au $(n-q)^{\text{ème}}$ groupe de Betti bicompact de l'espace $R - S$.*

VIII.

65. *Soit $0 \leqq p \leqq \dfrac{n-1}{2}$. Supposons la validité de tous les axiomes du Chap. I et II ainsi que celle des axiomes G^k (v. 21) pour $n - p \leqq k \leqq n - 1$ et des axiomes H^k (v. 27) pour $n - p - 1 \leqq k \leqq n - 2$. Alors les axiomes G^k $(0 \leqq k \leqq p)$ sont aussi vérifiés.*

Démonstration. Il suffit de montrer la validité de G^p (pour G^k on remplace p par k). Soit donc $a \in R - S$. On doit prouver qu'à chaque entourage $U \subset R - S$ de A on peut attacher un entourage V tel que chaque (p, R)-cycle dans $\overline{V}$ est ~ 0 dans $\overline{U}$. Or il suffit de prouver que l'entourage V de a peut être choisi de telle sorte que chaque (p, R)-cycle dans $\overline{V}$ soit ~ 0 dans un sous-ensemble bicompact de $R - S$. En effet, le cas général s'en déduit en remplaçant S par $R - U$, ce qui est sans influence sur la validité de nos axiomes.

Soit donc $a \in R - S$; soit $U \Subset R - S$ un entourage de a; soit Ω un entourage de S si petit que $\overline{U}\,\overline{\Omega} = 0$. C^p étant un (p, R)-cycle dans $\overline{U}$ et C^{n-p} étant un $(n-p, R)$-cycle mod S, pour calculer le nombre $C^p\, C^{n-p}$ on peut (v. 59.2) remplacer C^{n-p} par un $(n-p, R)$-cycle C_1^{n-p} mod $\overline{\Omega}$ homologue à C^{n-p} mod $\overline{\Omega}$. Il existe évidemment un réseau $\mathfrak{Z}$ et un réseau fermé $\mathfrak{T}$ correspondant qui soient tels que: 1° l'énoncé du n° 31 est vrai pour $A = B = R$; 2° $R - R_0 \subset \Omega$; 3° le point a appartient à un sommet unique (nécessairement intérieur) de $\mathfrak{Z}$ de manière que $a \in R_0 - R_1$ (dans les notations de 8). Ceci étant, on peut (d'après 31) attacher à chaque C^{n-p} un $C_1^{n-p} \sim C^{n-p}$ mod $\overline{\Omega}$ de manière que (v. 19.3) C_1^{n-p} soit un $(n-p, R)$-cycle mod $\overline{\Omega}$ dans R_p. Or soit V un entourage de a si petit que $\overline{V} \subset R - \overline{\Omega} - R_1$; si C^p est un (p, R)-cycle dans $\overline{V}$, on a, en excluant d'abord le cas $p = 0$, $C^p\, C^{n-p} = 0$ d'après 59.1 et par suite aussi $C^p\, C^{n-p} = 0$ pour chaque choix du $(n-p, R)$-cycle C^{n-p} mod S. D'après 62 (où on pose $S_2 = S$;

v. 64) il en résulte que $C^p \sim 0$ dans un sous-ensemble bicompact de $R-S$, pour chaque choix du (p, R)-cycle C^p dans $\overline{V}$, c. q. f. d.

Dans le cas $p=0$ le raisonnement précédant ne s'applique plus. Or on peut alors (v. 14.1 et 17.5) déterminer un entourage $U \in R-S$ de a de manière qu'on puisse attacher à chaque (n, R)-cycle C^n mod S un nombre $r \in \mathfrak{R}$ de manière que $C^n - r G^n \sim 0 \bmod R-U$. Soit V un entourage de a tel que $V \in U$. Pour chaque 0-cycle C^0 dans $\overline{V}$ on a alors $C^0(C^n - r G^n) = 0$ d'après 59.1 (v. aussi 59.2). Il en résulte que si C^0 est tel que $C^0 G^n = 0$ on a $C^0 C^n = 0$ pour chaque (n, R)-cycle C^n mod S. Il suffit donc de montrer que $I(C^0) = 0$ (v. [38]) entraîne que $C^0 G^n = 0$. Or soit C_a^0 le $(0, R)$-cycle (dans $\overline{V}$) correspondant (v. *Homologie*, III, 17) au point a. Posons $C_a^0 G^n = s$. On a $s \neq 0$; en effet, dans le cas contraire, on aurait $C_a^0 C^n = 0$ pour chaque C^n et par suite, d'après 62 avec $S_2 = S$ (v. aussi 64) $C_a^0 \sim 0$ dans un sous-ensemble bicompact de R, ce qui entraînerait, comme on sait, que $I(C_a^0) = 0$, tandis que, évidemment, $I(C_a^0) = 1$. Puisque $s \neq 0$, on peut attacher à chaque 0-cycle C^0 dans $\overline{V}$ un nombre $t \in \mathfrak{R}$ tel que $(C^0 - t C_a^0) G^n = 0$, ce qui entraîne que $C^0 - t C_a^0 \sim 0$ dans un sous-ensemble bicompact de R, d'où $I(C^0 - t C_a^0) = 0$. Or $I(C^0 - t C_a^0) = I(C^0) - t I(C_a^0) = I(C^0) - t$, donc $t = I(C^0)$. Par suite $I(C^0) = 0$ entraîne que $t = 0$, d'où $C^0 G^n = 0$, c. q. f. d.

65.1 *Les axiomes des Chap. I et II entraînent que l'espace $R-S$ est localement connexe.*

Cela résulte de 65 (avec $p=0$), car l'axiome G^0 équivaut évidemment (v. *Homologie*, III, 14—18) à la connexité locale de $R-S$.

66. *Soit $0 \leqq p \leqq \frac{n}{2} - 1$. Supposons la validité de tous les axiomes des Chap. I et II ainsi que celle des axiomes G^k (v. 21) pour $n-p \leqq k \leqq n-1$ et des axiomes H^k (v. 27) pour $n-p-1 \leqq k \leqq n-2$. Alors les axiomes H^k $(0 \leqq k \leqq p)$ sont aussi vérifiés.*

La démonstration sera donnée au n° 66.2.

66.1. Lemme. Soit $0 \leqq p \leqq n-2$. Supposons que R soit un espace normal et que $S = \overline{S} \subset R$. Supposons la validité de l'axiome G^{n-p-1} (v. 21). Soit $a \in R-S$. Soit $U \in R-S$ un entourage de a. Il existe un entourage $V \in U$ de a jouissant de la propriété suivante: Soit C^{n-p} un $(n-p, R)$-cycle mod $(S+\overline{V})$; il existe un $(n-p, R)$-cycle C_1^{n-p} mod S et un $(n-p, R)$-cycle C_2^{n-p} mod $\overline{V}$ dans $\overline{U}$ tels que $C^{n-p} \sim C_1^{n-p} + C_2^{n-p}$ mod $S+\overline{V}$.

Démonstration. D'après l'axiome G^{n-p-1}, déterminons l'entourage $V \in U$ de manière que l'on ait $\Gamma^{n-p-1} \sim 0$ dans $\overline{U}$ pour chaque $(n-p-1, R)$-cycle Γ^{n-p-1} dans $\overline{V}$. Soit Φ la famille (complète) de tous les réseaux dont aucun sommet ne rencontre simultanément $\overline{V}$ et S. Pour chaque $\mathfrak{U} \in \Phi$, on a

$$FC^{n-p}(\mathfrak{U}) = \Gamma_1^{n-p-1}(\mathfrak{U}) + \Gamma_2^{n-p-1}(\mathfrak{U}),$$

où $I_1^{n-p-1}(\mathfrak{U})$ $[I_2^{n-p-1}(\mathfrak{U})]$ est un $(n-p-1, \mathfrak{U})$-cycle dans S [dans $\overline{V}$]. Lorsque $\mathfrak{V} \in \Phi$ est un affinement de $\mathfrak{U} \in \Phi$, on a, en posant $\pi = Pr.(\mathfrak{V}, \mathfrak{U})$, $\pi C^{n-p}(\mathfrak{V}) \sim C^{n-p}(\mathfrak{U}) \bmod (S + \overline{V})$. Il existe donc une $(n-p+1, \mathfrak{U})$-chaîne $E^{n-p+1}(\mathfrak{U})$ et deux $(n-p, \mathfrak{U})$-chaînes $E_1^{n-p}(\mathfrak{U})$, $E_2^{n-p}(\mathfrak{U})$ telles que $E_1^{n-p}(\mathfrak{U}) \subset S$, $E_2^{n-p}(\mathfrak{U}) \subset \overline{V}$ et

$$\pi C^{n-p}(\mathfrak{V}) - C^{n-p}(\mathfrak{U}) = FE^{n-p+1}(\mathfrak{U}) + E_1^{n-p}(\mathfrak{U}) + E_2^{n-p}(\mathfrak{U}),$$

d'où

$$\pi I_1^{n-p-1}(\mathfrak{V}) - I_1^{n-p-1}(\mathfrak{U}) - FE_1^{n-p}(\mathfrak{U})$$
$$= \pi I_2^{n-p-1}(\mathfrak{V}) - I_2^{n-p-1}(\mathfrak{U}) - FE_2^{n-p}(\mathfrak{U}).$$

Or la chaîne à gauche (à droite) est située dans S (dans $\overline{V}$); d'après la définition de la famille Φ, on en déduit que chacune de ces deux chaînes est égale à zéro, d'où $\pi I_2^{n-p-1}(\mathfrak{V}) \sim I_2^{n-p-1}(\mathfrak{U})$ dans $\overline{V}$. Il en résulte que $I_2^{n-p-1} = \{I_2^{n-p-1}(\mathfrak{U})\}$ est un $(n-p-1, R)$-cycle dans $\overline{V}$. D'après la définition de V, il existe donc pour chaque $\mathfrak{U} \in \Phi$ une $(n-p, \mathfrak{U})$-chaîne dans $\overline{U}$ dont la frontière est égale à $I_2^{n-p-1}(\mathfrak{U})$. Pour chaque $\mathfrak{U} \in \Phi$, soit $L_2^{n-p}(\mathfrak{U})$ la famille de tous les $(n-p, \mathfrak{U})$-cycles $C_2^{n-p}(\mathfrak{U}) \bmod \overline{V}$ dans $\overline{U}$ tels que $FC_2^{n-p}(\mathfrak{U}) \sim I_2^{n-p-1}(\mathfrak{U})$ dans $\overline{V}$. Nous venons de voir que $L_2^{n-p}(\mathfrak{U}) \neq 0$ pour chaque $\mathfrak{U} \in \Phi$. On voit sans peine que $L_2^{n-p}(\mathfrak{U})$ est un système linéaire et que $\pi L_2^{n-p}(\mathfrak{V}) \subset L_2^{n-p}(\mathfrak{U})$. D'après *Homologie*, II, 21, on peut donc choisir $C_2^{n-p}(\mathfrak{U}) \in L_2^{n-p}(\mathfrak{U})$ pour chaque $\mathfrak{U} \in \Phi$ de manière que $C_2^{n-p} = \{C_2^{n-p}(\mathfrak{U})\}$ soit un $(n-p, R)$-cycle mod $\overline{V}$ dans $\overline{U}$. Puisque $C_2^{n-p}(\mathfrak{U}) \in L_2^{n-p}(\mathfrak{U})$, il existe pour chaque $\mathfrak{U} \in \Phi$ une $(n-p, \mathfrak{U})$-chaîne $D_2^{n-p}(\mathfrak{U})$ dans $\overline{V}$ telle que

$$FC_2^{n-p}(\mathfrak{U}) = I_2^{n-p-1}(\mathfrak{U}) + FD_2^{n-p}(\mathfrak{U}).$$

Le cycle C_2^{n-p} étant déterminé, il s'agit encore de démontrer l'existence de C_1^{n-p}. Or désignons pour $\mathfrak{U} \in \Phi$ par $L_1^{n-p}(\mathfrak{U})$ la famille de tous les $(n-p, \mathfrak{U})$-cycles $C_1^{n-p}(\mathfrak{U}) \bmod S$ tels que :

$$C^{n-p}(\mathfrak{U}) \sim C_1^{n-p}(\mathfrak{U}) + C_2^{n-p}(\mathfrak{U}) \quad \bmod (S + \overline{V}).$$

En supposant pour un moment que $L_1^{n-p}(\mathfrak{U}) \neq 0$, on voit sans peine que c'est un système linéaire. Or on a évidemment $\pi L_1^{n-p}(\mathfrak{V}) \subset L_1^{n-p}(\mathfrak{U})$. Il résulte donc de l'*Homologie*, II, 21 que le cycle C_1^{n-p} existe. On doit encore démontrer que $L_1^{n-p}(\mathfrak{U}) \neq 0$; c'est évident, car on voit sans peine que

$$C^{n-p}(\mathfrak{U}) - C_2^{n-p}(\mathfrak{U}) + D_2^{n-p}(\mathfrak{U}) \in L_1^{n-p}(\mathfrak{U}).$$

66.2. Passons à la démonstration du théorème du n° 66. Il suffit de démontrer la validité de l'axiome H^p. Il suffit même de prouver l'énoncé E suivant : Soit $a \in R - S$; soit $P_1 \in R - S$ un entourage donné de a. A chaque

entourage $P_2 \subset P_1$ de a on peut attacher un entourage $P_3 \subset P_2$ de a de manière que : Si C^p est un (p, R)-cycle dans $\overline{P_1} - P_2$ tel que $C^p \sim 0$ dans un sous-ensemble bicompact de $R - S$, on ait aussi $C^p \sim 0$ dans un sous-ensemble bicompact de $R - (S + \overline{P_3})$. En effet, d'une part, l'axiome H^p, sous la forme énoncée au n° 27, est une conséquence de E si $P = R - S$; d'autre part, en remplaçant S par $R - P$ (et donc $R - S$ par P) on ne détruit pas la validité des axiomes supposés vrais au n° 66.

Soit donc $a \in R - S$ et soient P_1, P_2 deux entourages de a tels que $P_2 \subset P_1 \Subset R - S$. Choisissons un entourage $U \Subset P_2$ de a et déterminons l'entourage $V = P_3$ de a d'après 66.1. Soit C^p un (p, R)-cycle dans $\overline{P_1} - P_2$ tel que $C^p \sim 0$ dans un sous-ensemble bicompact de $R - S$. D'après 64, on a $C^p C^{n-p} = 0$ pour chaque $(n-p, R)$-cycle $C^{n-p} \bmod S$. On doit prouver que $C^p \sim 0$ dans un sous-ensemble bicompact de $R - (S + \overline{P_3})$. Or le théorème du n° 64 reste évidemment vrai en y remplaçant S par $S + \overline{P_3}$ (car les axiomes dont on a déduit ce théorème restent vrais). Or on a évidemment $\overline{P_1} - P_2 \subset R - (S + \overline{P_3})$ de sorte qu'il suffit de prouver que $C^p C^{n-p} = 0$ pour chaque $(n-p, R)$-cycle $C^{n-p} \bmod (S + \overline{P_3})$. D'après 66.1 il existe un $(n-p, R)$-cycle C_1^{n-p} et un $(n-p, R)$-cycle $C_2^{n-p} \bmod \overline{P_3}$ dans $\overline{U}$ tel que

$$C^{n-p} \sim C_1^{n-p} + C_2^{n-p} \mod (S + \overline{P_3}).$$

On a donc (v. 56, 3° et 4°)

$$C^p C^{n-p} = C^p C_1^{n-p} + C^p C_2^{n-p}.$$

Or nous savons que $C^p C_1^{n-p} = 0$; d'autre part, $C^p C_2^{n-p} = 0$ d'après 59.1, car $C^p \subset \overline{P_1} - P_2$, $C_2^{n-p} \subset \overline{U}$, $\overline{U}(\overline{P_1} - P_2) = 0$.

67. Faisons les hypothèses du n° 55. Il se peut qu'il existe un nombre infini de $(n-p, R)$-cycles $\bmod S$ lin. indépendants $\bmod S$ (c'est-à-dire tels qu'aucune combinaison linéaire de ces cycles ne soit $\sim 0 \bmod S$); or il résulte du théorème du n° 31 que, Ω étant un entourage donné de S, il n'existe qu'un nombre fini de $(n-p, R)$-cycles $\bmod S$ lin. indépendants $\bmod \overline{\Omega}$; autrement dit, Ω étant donné, on peut indiquer un nombre fini de $(n-p, R)$-cycles $\bmod S$ tels que chaque $(n-p, R)$-cycle $\bmod S$ soit homologue $\bmod S$ à une combinaison linéaire de ces cycles donnés, augmentée d'un $(n-p, R)$-cycle $\bmod S$ dans $\overline{\Omega}$.

Pareillement il peut exister un nombre infini de $(n-p, R)$-cycles absolus, dont chacun est situé dans un sous-ensemble bicompact de $R - S$, et tels qu'aucune combinaison linéaire de ces cycles ne soit homologue à zéro dans un sous-ensemble bicompact de $R - S$. Mais, A étant un sous-ensemble bicompact de $R - S$ donné, il existe un nombre fini de $(n-p, R)$-cycles dans A tels que chaque $(n-p, R)$-cycle situé dans A soit homologue

à une combinaison linéaire de ces cycles dans un sous-ensemble bicompact de $R-S$.

Du théorème du n° 64 on déduit (v. 59.1) que ces remarques restent vraies en remplaçant p par $n-p$.

Supposons en particulier que $S=0$. Alors l'ensemble $R-S=R$ lui même est bicompact, et par suite le $p^{\text{ème}}$, ainsi que le $(n-p)^{\text{ème}}$ nombre de Betti de R est fini. Ces deux nombres sont égaux l'un à l'autre d'après 64 *(théorème de dualité de Poincaré)*.

68. Soit $0 \leqq p \leqq n-1$. Supposons la validité des axiomes énumérés au n° 55, en y remplaçant S par 0.

Soit S un sous-ensemble bicompact de R. Soit Δ_1 l'ensemble de tous les (p, R)-cycles C^p dans A homologues à zéro dans R, où A parcourt la classe K de tous les sous-ensembles bicompacts de $R-S$; considérons deux éléments C_1^p, C_2^p de Δ_1 comme égaux si l'on peut déterminer $A \in K$ de manière que $C_1^p \sim C_2^p$ dans A. Soit Δ_2 l'ensemble de tous les $(n-p-1, R)$-cycles Γ^{n-p-1} dans S homologues à zéro dans R; deux éléments Γ_1^{n-p-1}, Γ_2^{n-p-1} de Δ_2 seront considérés comme égaux si $\Gamma_1^{n-p-1} \sim \Gamma_2^{n-p-1}$ dans S.

Nous allons définir une multiplication $C^p \times \Gamma^{n-p-1}$ rel. à laquelle les deux modules Γ_1 et Γ_2 sont duels (*théorème de dualité de M. Pontrjagin*[74]).

Soit $\Gamma^{n-p-1} \in \Delta_2$. Pour chaque réseau $\mathfrak{U}$, il existe un système linéaire $L^{n-p}(\mathfrak{U})$ de $(n-p, \mathfrak{U})$-cycles $C^{n-p}(\mathfrak{U})$ dans R tels que $FC^{n-p}(\mathfrak{U}) \sim \Gamma^{n-p-1}(\mathfrak{U})$ dans S. De l'*Homologie* II, 21 on déduit sans peine qu'on peut choisir $C^{n-p}(\mathfrak{U}) \in L^{n-p}(\mathfrak{U})$ de manière que $C^{n-p} = \{C^{n-p}(\mathfrak{U})\}$ soit un $(n-p, R)$-cycle dans S. Posons $C^{n-p} = \varphi(\Gamma^{n-p-1})$; la fonction φ n'est pas du reste univoque. Ceci étant, posons

$$C^p \times \Gamma^{n-p-1} = C^p \varphi(\Gamma^{n-p-1}).$$

Les propriétés 1°, 2°, 3° dans la définition de dualité de deux modules (n° 61) sons évidentes. Reste à montrer: A_1: le produit $C^p \times \Gamma^{n-p-1}$ est univoquement déterminé; A_2: si $C^p \in \Delta_1$ jouit de la propriété que $C^p \times \Gamma^{n-p-1} = 0$ pour chaque $\Gamma^{n-p-1} \in \Delta_2$, on a $C^p = 0$; A_3: si $\Gamma^{n-p-1} \in \Delta_2$ jouit de la propriété que $C^p \times \Gamma^{n-p-1} = 0$ pour chaque $C^p \in \Delta_1$, on a $\Gamma^{n-p-1} = 0$.

Démonstration de A_1. Soit

$$C_1^p = C_2^p, \quad \Gamma_1^{n-p-1} = \Gamma_2^{n-p-1},$$

$$C_1^{n-p} = \varphi(\Gamma_1^{n-p-1}), \quad C_2^{n-p} = \varphi(\Gamma_2^{n-p-1}).$$

On doit prouver que

$$C_1^p C_1^{n-p} = C_2^p C_2^{n-p}.$$

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[74] Göttinger Nachrichten, 1928, p. 448 (Satz I).

entourage $P_2 \subset P_1$ de a on peut attacher un entourage $P_3 \subset P_2$ de a de manière que : Si C^p est un (p, R)-cycle dans $\overline{P_1 - P_2}$ tel que $C^p \sim 0$ dans un sous-ensemble bicompact de $R - S$, on ait aussi $C^p \sim 0$ dans un sous-ensemble bicompact de $R - (S + \overline{P_3})$. En effet, d'une part, l'axiome H^p, sous la forme énoncée au n° 27, est une conséquence de E si $P = R - S$; d'autre part, en remplaçant S par $R - P$ (et donc $R - S$ par P) on ne détruit pas la validité des axiomes supposés vrais au n° 66.

Soit donc $a \in R - S$ et soient P_1, P_2 deux entourages de a tels que $P_2 \subset P_1 \Subset R - S$. Choisissons un entourage $U \Subset P_2$ de a et déterminons l'entourage $V = P_3$ de a d'après 66.1. Soit C^p un (p, R)-cycle dans $\overline{P_1 - P_2}$ tel que $C^p \sim 0$ dans un sous-ensemble bicompact de $R - S$. D'après 64, on a $C^p C^{n-p} = 0$ pour chaque $(n-p, R)$-cycle $C^{n-p} \bmod S$. On doit prouver que $C^p \sim 0$ dans un sous-ensemble bicompact de $R - (S + \overline{P_3})$. Or le théorème du n° 64 reste évidemment vrai en y remplaçant S par $S + \overline{P_3}$ (car les axiomes dont on a déduit ce théorème restent vrais). Or on a évidemment $\overline{P_1 - P_2} \subset R - (S + \overline{P_3})$ de sorte qu'il suffit de prouver que $C^p C^{n-p} = 0$ pour chaque $(n-p, R)$-cycle $C^{n-p} \bmod (S + \overline{P_3})$. D'après 66.1 il existe un $(n-p, R)$-cycle C_1^{n-p} et un $(n-p, R)$-cycle $C_2^{n-p} \bmod \overline{P_3}$ dans $\overline{U}$ tel que

$$C^{n-p} \sim C_1^{n-p} + C_2^{n-p} \mod (S + \overline{P_3}).$$

On a donc (v. 56, 3° et 4°)

$$C^p C^{n-p} = C^p C_1^{n-p} + C^p C_2^{n-p}.$$

Or nous savons que $C^p C_1^{n-p} = 0$; d'autre part, $C^p C_2^{n-p} = 0$ d'après 59.1, car $C^p \subset \overline{P_1 - P_2}$, $C_2^{n-p} \subset \overline{U}$, $\overline{U}(\overline{P_1 - P_2}) = 0$.

67. Faisons les hypothèses du n° 55. Il se peut qu'il existe un nombre infini de $(n-p, R)$-cycles $\bmod S$ lin. indépendants $\bmod S$ (c'est-à-dire tels qu'aucune combinaison linéaire de ces cycles ne soit $\sim 0 \bmod S$); or il résulte du théorème du n° 31 que, Ω étant un entourage donné de S, il n'existe qu'un nombre fini de $(n-p, R)$-cycles $\bmod S$ lin. indépendants $\bmod \overline{\Omega}$; autrement dit, Ω étant donné, on peut indiquer un nombre fini de $(n-p, R)$-cycles $\bmod S$ tels que chaque $(n-p, R)$-cycle $\bmod S$ soit homologue $\bmod S$ à une combinaison linéaire de ces cycles donnés, augmentée d'un $(n-p, R)$-cycle $\bmod S$ dans $\overline{\Omega}$.

Pareillement il peut exister un nombre infini de $(n-p, R)$-cycles absolus, dont chacun est situé dans un sous-ensemble bicompact de $R - S$, et tels qu'aucune combinaison linéaire de ces cycles ne soit homologue à zéro dans un sous-ensemble bicompact de $R - S$. Mais, A étant un sous-ensemble bicompact de $R - S$ donné, il existe un nombre fini de $(n-p, R)$-cycles dans A tels que chaque $(n-p, R)$-cycle situé dans A soit homologue

à une combinaison linéaire de ces cycles dans un sous-ensemble bicompact de $R-S$.

Du théorème du n° 64 on déduit (v. 59.1) que ces remarques restent vraies en remplaçant p par $n-p$.

Supposons en particulier que $S=0$. Alors l'ensemble $R-S=R$ lui même est bicompact, et par suite le $p^{\text{ème}}$, ainsi que le $(n-p)^{\text{ème}}$ nombre de Betti de R est fini. Ces deux nombres sont égaux l'un à l'autre d'après 64 *(théorème de dualité de Poincaré)*.

68. Soit $0 \leqq p \leqq n-1$. Supposons la validité des axiomes énumérés au n° 55, en y remplaçant S par 0.

Soit S un sous-ensemble bicompact de R. Soit Δ_1 l'ensemble de tous les (p, R)-cycles C^p dans A homologues à zéro dans R, où A parcourt la classe K de tous les sous-ensembles bicompacts de $R-S$; considérons deux éléments C_1^p, C_2^p de Δ_1 comme égaux si l'on peut déterminer $A \in K$ de manière que $C_1^p \sim C_2^p$ dans A. Soit Δ_2 l'ensemble de tous les $(n-p-1, R)$-cycles Γ^{n-p-1} dans S homologues à zéro dans R; deux éléments Γ_1^{n-p-1}, Γ_2^{n-p-1} de Δ_2 seront considérés comme égaux si $\Gamma_1^{n-p-1} \sim \Gamma_2^{n-p-1}$ dans S.

Nous allons définir une multiplication $C^p \times \Gamma^{n-p-1}$ rel. à laquelle les deux modules Γ_1 et Γ_2 sont duels *(théorème de dualité de M. Pontrjagin*[74]*)*.

Soit $\Gamma^{n-p-1} \in \Delta_2$. Pour chaque réseau $\mathfrak{U}$, il existe un système linéaire $L^{n-p}(\mathfrak{U})$ de $(n-p, \mathfrak{U})$-cycles $C^{n-p}(\mathfrak{U})$ dans R tels que $FC^{n-p}(\mathfrak{U}) \sim \Gamma^{n-p-1}(\mathfrak{U})$ dans S. De l'*Homologie* II, 21 on déduit sans peine qu'on peut choisir $C^{n-p}(\mathfrak{U}) \in L^{n-p}(\mathfrak{U})$ de manière que $C^{n-p} = \{C^{n-p}(\mathfrak{U})\}$ soit un $(n-p, R)$-cycle dans S. Posons $C^{n-p} = \varphi(\Gamma^{n-p-1})$; la fonction φ n'est pas du reste univoque. Ceci étant, posons

$$C^p \times \Gamma^{n-p-1} = C^p \varphi(\Gamma^{n-p-1}).$$

Les propriétés 1°, 2°, 3° dans la définition de dualité de deux modules (n° 61) sons évidentes. Reste à montrer: A_1: le produit $C^p \times \Gamma^{n-p-1}$ est univoquement déterminé; A_2: si $C^p \in \Delta_1$ jouit de la propriété que $C^p \times \Gamma^{n-p-1} = 0$ pour chaque $\Gamma^{n-p-1} \in \Delta_2$, on a $C^p = 0$; A_3: si $\Gamma^{n-p-1} \in \Delta_2$ jouit de la propriété que $C^p \times \Gamma^{n-p-1} = 0$ pour chaque $C^p \in \Delta_1$, on a $\Gamma^{n-p-1} = 0$.

Démonstration de A_1. Soit

$$C_1^p = C_2^p, \quad \Gamma_1^{n-p-1} = \Gamma_2^{n-p-1},$$

$$C_1^{n-p} = \varphi(\Gamma_1^{n-p-1}), \quad C_2^{n-p} = \varphi(\Gamma_2^{n-p-1}).$$

On doit prouver que

$$C_1^p C_1^{n-p} = C_2^p C_2^{n-p}.$$

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[74] Göttinger Nachrichten, 1928, p. 448 (Satz I).

L'égalité $C_1^p = C_2^p$ signifie qu'il existe un sous-ensemble bicompact A de $R-S$ tel que $C_1^p - C_2^p \sim 0$ dans A. Donc d'après 64, on a $(C_1^p - C_2^p) C_2^{n-p} = 0$, car C_2^{n-p} est un $(n-p, R)$-cycle mod S. Reste à prouver que $C_1^p (C_1^{n-p} - C_2^{n-p}) = 0$. Or l'égalité $\Gamma_1^{n-p-1} = \Gamma_2^{n-p-1}$ signifie que $\Gamma_1^{n-p-1} \sim \Gamma_2^{n-p-1}$ dans S. D'autre part, dans chaque réseau $\mathfrak{U}$ on a $FC_1^{n-p}(\mathfrak{U}) \sim \Gamma_1^{n-p-1}(\mathfrak{U})$ dans S, $FC_2^{n-p}(\mathfrak{U}) \sim \Gamma_2^{n-p-1}(\mathfrak{U})$ dans S d'où il résulte qu'on peut déterminer une $(n-p, \mathfrak{U})$-chaîne $D^{n-p}(\mathfrak{U})$ dans S telle que $C_1^{n-p}(\mathfrak{U}) - C_2^{n-p}(\mathfrak{U}) - D^{n-p}(\mathfrak{U}) \to 0$. Pour chaque réseau $\mathfrak{U}$ désignons par $L^{n-p}(\mathfrak{U})$ la famille de tous les $(n-p, \mathfrak{U})$-cycles absolus de la forme

$$C_1^{n-p}(\mathfrak{U}) - C_2^{n-p}(\mathfrak{U}) - D^{n-p}(\mathfrak{U}) + E^{n-p}(\mathfrak{U}) + FH^{n-p+1}(\mathfrak{U}),$$

où $E^{n-p}(\mathfrak{U})$ parcourt tous les $(n-p, \mathfrak{U})$-cycles dans S tandis que $H^{n-p+1}(\mathfrak{U})$ parcourt toutes les $(n-p+1, \mathfrak{U})$-chaînes. D'après *Homologie*, II, 21 on voit sans peine qu'il est possible de choisir $C^{n-p}(\mathfrak{U}) \in L^{n-p}(\mathfrak{U})$ de manière que $C^{n-p} = \{C^{n-p}(\mathfrak{U})\}$ soit un $(n-p, R)$-cycle (absolu). Evidemment $C^{n-p} \sim C_1^{n-p} - C_2^{n-p} \bmod S$, d'où $C_1^p (C_1^{n-p} - C_2^{n-p}) = C_1^p C^{n-p}$, de sorte qu'on doit seulement prouver que $C_1^p C^{n-p} = 0$, ce qui résulte de 64 (en y remplaçant S par 0), car $C_1^p \sim 0 \bmod R$.

Démonstration de A_2. Supposons que le cycle $C^p \in \Delta_1$ ne soit homologue à zéro dans aucun sous-ensemble bicompact de $R-S$. On doit prouver qu'il existe un $\Gamma^{n-p-1} \in \Delta_2$ tel que $C^p \times \Gamma^{n-p-1} = 1$. Or d'après 64 il existe un $(n-p, R)$-cycle $C^{n-p} \bmod S$ tel que $C^p C^{n-p} = 1$. Pour chaque réseau $\mathfrak{U}$ soit $\Gamma^{n-p-1}(\mathfrak{U}) = FC^{n-p}(\mathfrak{U})$. On voit sans peine que $\Gamma^{n-p-1} = \{\Gamma^{n-p-1}(\mathfrak{U})\}$ est un élément de Δ_2 tel que $C^{n-p} = \varphi(\Gamma^{n-p-1})$, d'où $C^p \times \Gamma^{n-p-1} = 1$.

Démonstration de A_3. Supposons que le cycle $\Gamma^{n-p-1} \in \Delta_2$ ne soit pas homologue à zéro dans S. Soit $C_0^{n-p} = \varphi(\Gamma^{n-p-1})$. On doit prouver qu'il existe un $C_0^p \in \Delta_1$ tel que $C^p C_0^{n-p} = 1$. On voit sans peine que le $(n-p, R)$-cycle $C^{n-p} \bmod S$ n'est homologue mod S à aucun $(n-p, R)$-cycle absolu. Supposons par impossible que, pour chaque réseau $\mathfrak{U}$, le $(n-p, \mathfrak{U})$-cycle mod S: $C^{n-p}(\mathfrak{U})$ soit homologue mod S à un $(n-p, \mathfrak{U})$-cycle absolu $B^{n-p}(\mathfrak{U})$; désignons par $L^{n-p}(\mathfrak{U})$ la famille de tous ces $B^{n-p}(\mathfrak{U})$; de l'*Homologie*, II, 21 on déduit sans peine qu'on peut choisir $B^{n-p}(\mathfrak{U}) \in L^{n-p}(\mathfrak{U})$ de manière que $B^{n-p} = \{B^{n-p}(\mathfrak{U})\}$ soit un $(n-p, R)$-cycle absolu, d'où il résulte la contradiction $C_0^{n-p} \sim B^{n-p} \bmod S$.

Soit donc $\mathfrak{U}_0$ un réseau tel que $C_0^{n-p}(\mathfrak{U})$ ne soit homologue mod S à aucun $(n-p, \mathfrak{U}_0)$-cycle absolu; on voit sans peine (v. la démonstration de 6.1) qu'il existe un entourage Ω de S tel que, dans le réseau $\mathfrak{U}_0$, les frontières, cycles, homologies mod S soient les mêmes comme mod $\bar{\Omega}$. Donc $C_0^{n-p}(\mathfrak{U})$ n'est homologue mod $\bar{\Omega}$ à aucun $(n-p, \mathfrak{U}_0)$-cycle absolu; par suite C_0^{n-p} n'est homologue mod $\bar{\Omega}$ à aucun $(n-p, R)$-cycle absolu.

D'après 67, il existe un ensemble fini τ de $(n-p, R)$-cycles mod S tel que chaque $(n-p, R)$-cycle mod S soit homologue mod S à une combinaison linéaire de ces cycles, augmentée d'un $(n-p, R)$-cycle mod S dans $\bar{\Omega}$, tandis qu'aucune combinaison linéaire de ces cycles ne soit $\sim 0 \bmod \bar{\Omega}$. On voit sans peine qu'on peut supposer que le cycle C_0^{n-p} appartienne à la famille τ; en outre, on peut supposer que la famille τ contienne des cycles C_1^{n-p}, $C_2^{n-p}, \cdots, C_h^{n-p}$ $(h = 0, 1, 2, \cdots)$ tels qu'un $(n-p, R)$-cycle mod S est $\sim \sum_{i=1}^{h} a_i C_i^{n-p} \bmod \Omega$ si et seulement s'il est homologue mod $\bar{\Omega}$ à un $(n-p, R)$-cycle absolu. Soit

$$C_0^{n-p}, C_1^{n-p}, \cdots, C_h^{n-p}, C_{h+1}^{n-p}, \cdots, C_k^{n-p}$$

$0 \leqq h \leqq k$) la famille τ. K_0 étant la famille de tous les sous-ensembles bicompacts de $R - \bar{\Omega}$, il résulte facilement de 64 (où l'on remplace S par $\bar{\Omega}$) qu'il existe des (p, R)-cycles absolus

$$C_0^p, C_1^p, \cdots, C_h^p, C_{h+1}^p, \cdots, C_k^p,$$

C_i^p $(0 \leqq i \leqq k)$ étant situé dans $A_i \in K_0$, tels que pour $0 \leqq i \leqq k$, $0 \leqq j \leqq k$: $C_i^p C_j^{n-p} = 1$ pour $i = j$, $= 0$ pour $i \neq j$.

Je dis que C_0^p est l'élément cherché du module Δ_1. Puisque $C_0^p C_0^{n-p} = 1$, il faut seulement démontrer que $C_0^p \sim 0$ dans R. D'après 64, il suffit de prouver que $C_0^p C^{n-p} = 0$ pour chaque $(n-p, R)$-cycle absolu C^{n-p}. Or d'après ce que nous venons de dire, il existe des nombres $a_i \in \mathfrak{R}$ et un $(n-p, R)$-cycle ${}^*C^{n-p} \bmod S$ dans $\bar{\Omega}$ tels que

$$C^{n-p} \sim \sum_{i=1}^{h} a_i C_i^{n-p} + {}^*C^{n-p} \mod S,$$

d'où

$$C_0^p C^{n-p} = \sum_{i=1}^{h} a_i C_0^p C_i^{n-p} + C_0^p {}^*C^{n-p}.$$

Or nous savons que $C_0^p C^{n-p} = 0$ pour $1 \leqq i \leqq h$; d'autre part, $C_0^p {}^*C^{n-p} = 0$, car $C_0^p \subset A_0$, ${}^*C^{n-p} \subset \bar{\Omega}$, $A_0 \bar{\Omega} = 0$.

Remarque. Soit $\delta_i = 1$ pour $i = 0$, $\delta_i = 0$ pour $i \geqq 1$. Supposons que le $p^{\text{ème}}$ et le $(n-p-1)^{\text{ème}}$ nombre de Betti de l'espace R soient resp. égaux à δ_p et à δ_{n-p-1}. Alors Γ_1 est le module de tous les (p, R)-cycles C^p bicompacts dans $R - S$ (tels que $I(C^p) = 0$ si $p = 0$) tandis que Γ_2 est le module de tous les $(n-p-1, R)$-cycles C^{n-p-1} dans S (tels que $I(C^{n-p-1}) = 0$ si $n-p-1 = 0$). Le fait que les deux modules Γ_1 et Γ_2 sont duels constitue *le théorème de dualité de M. Alexander.*

69. Faisons l'hypothèse du n° 55. Soit Γ_1 l'ensemble de tous les (p, R)-cycles absolu dans A, où A parcourt tous les sous-ensembles bicompacts de $R - S$; deux éléments C_1^p, C_2^p seront considérés comme égaux si $C_1^p \sim C_2^p$

mod S. Soit Γ_2 l'ensemble qui s'obtient de Γ_1 en remplaçant p par $n-p$. On voit sans peine (v. 56, 4°) que le produit $C^p C^{n-p}$ où $C^p \in \Gamma_1$, $C^{n-p} \in \Gamma_2$, est bien déterminé malgré les conventions faites sur l'égalité de deux éléments de Γ_1 ou de Γ_2. En procédant comme dans les n^{os} 62 et 63, on démontre sans peine que les deux modules Γ_1 et Γ_2 sont duels relativement à la multiplication $C^p C^{n-p}$. C'est *le second théorème de dualité de M. Lefschetz*[75].

70. Soient A_1 et A_2 deux sous-ensembles de $R-S$, fermés dans $R-S$ et tels que $\overline{A_1 A_2} \subset R-S$. Soit C^p un (p, R)-cycle mod $\overline{A_1} S$ dans A_1 ; soit C^{n-p} un $(n-p, R)$-cycle mod $\overline{A_2} S$ dans $\overline{A_2}$. On voit sans peine qu'on peut étendre la définition du produit $C^p C^{n-p}$ du Chap. VII (qui a été exposée seulement sous l'hypothèse $\overline{A_1}\,\overline{A_2} = 0$) au cas plus général ici envisagé, et que les propriétés 1°, 2°, 3°, 4° du n° 61 restent vraies.

Sous la forme ainsi généralisée, le produit $C^p C^{n-p}$ ne dépend évidemment (cf. 6.3) que de l'espace $R-S$ (et de son orientation).

On pourrait aussi généraliser le théorème général de dualité du n° 61 en lui donnant une forme qui ne dépend plus que de l'espace $R-S$[76]. Nous omettons de le faire, car on n'arrive qu'à un énoncé fort compliqué et peu susceptible d'applications ; d'ailleurs, il est évident que l'important théorème de dualité du n° 64 exprime une propriété topologique de l'espace $R-S$ (v. 6.3).

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[75] S. Lefschetz, Topology, p. 149, formule (20).

[76] Cf. Lefschetz, Topology, p. 314, formule (18).

UNTERSUCHUNGEN ÜBER LÜCKEN UND SINGULARITÄTEN VON POTENZREIHEN[1].

ZWEITE MITTEILUNG[2].

VON G. PÓLYA.

VIERTES KAPITEL.

Eine Ergänzung zu dem Hadamardschen Multiplikationssatz und einige damit zusammenhängende Lückensätze.

EINLEITUNG.

67. Der Hadamardsche Multiplikationssatz stellt einen Zusammenhang zwischen den Singularitäten der drei Potenzreihen

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \tag{1}$$

$$g(z) = b_0 + b_1 z + b_2 z^2 + \cdots, \tag{2}$$

$$h(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots \tag{3}$$

fest. Dieser Zusammenhang besteht im wesentlichen[3] darin, daß jeder singuläre Punkt von $h(z)$ die Form $\alpha\beta$ hat, wobei α ein singulärer Punkt von $f(z)$ und β ein singulärer Punkt von $g(z)$ ist. Dieser Satz gibt somit eine aus den Singularitäten von $f(z)$ und $g(z)$ konstruierte Menge an, worin die Menge der Singularitäten von $h(z)$ als Untermenge enthalten ist; er sagt uns nicht, wo die singulären Punkte von $h(z)$ liegen müssen, nur wo sie liegen können; er läßt die Frage, ob ein Punkt von der Form $\alpha\beta$ wirklich singulär für $h(z)$ ist oder nicht, vollständig offen. Gelingt es uns festzustellen, daß ein Punkt von der Form $\alpha\beta$ für $h(z)$ tatsächlich singulär ist, so sind wir über die Aussage des Hadamardschen Multiplikationssatzes wesentlich hinausgegangen.

Es ist zuerst E. Borel gelungen[4], über den Hadamardschen Multiplikationssatz in der besagten Richtung hinauszugehen. Der ausschlaggebende Spezialfall seines Resultates kann so formuliert werden:

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[1] Received May 26, 1933.

[2] Die erste Mitteilung (in Abkürzung: 1. Mtlg.) erschien in Math. Zeitschrift **29** (1929), S. 549—640. Die Numerierung der Paragraphen wird fortgesetzt, so daß Verweisungen auf Nr. 1—66 sich auf die 1. Mtlg. beziehen.

[3] Vgl. J. Hadamard, Acta mathematica **22** (1898), S. 55—63, ferner G. Faber, Jahresbericht der deutschen Math. Vereinigung **16** (1907), S. 285—298. Eine spezielle, aber dafür besonders durchsichtige Aussage findet sich bei L. Bieberbach, Lehrbuch der Funktionentheorie, Bd. II (1927). Vgl. S. 292—294.

[4] E. Borel, Bulletin de la Société Math. de France **26** (1898), S. 238—248.

Beide Reihen (1) *und* (2) *sollen die Eigenschaft haben, daß auf dem Konvergenzkreis nur ein einziger singulärer Punkt, und zwar ein Pol liegt. Heißt der auf dem Konvergenzkreis von* (1) *gelegene Pol* α, *der auf dem Konvergenzkreis von* (2) *gelegene* β, *so ist* $\alpha\beta$ *tatsächlich singulär für* $h(z)$, *nämlich der einzige auf dem Konvergenzkreis von* (3) *gelegene singuläre Punkt und ein Pol für* $h(z)$.

Ausschlaggebend nenne ich diesen Spezialfall, weil aus ihm das volle Resultat von Borel durch einfache Schlüsse herzuleiten ist; es ist dabei hauptsächlich zu beachten, daß die Funktionaloperation, die aus $f(z)$ und $g(z)$ die „Hadamardsche Komponierte" $h(z)$ erzeugt, sowohl in bezug auf $f(z)$ wie in bezug auf $g(z)$ linear (oder „distributiv") ist; vgl. Nr. 93—94. Eine kurze, und wohl auch genügend klare Fassung des ausgesprochenen Satzes ist die folgende:

Das Produkt von zwei Polen ist notwendigerweise ein singulärer Punkt, u. zw. wieder ein Pol.

Die Aufgabe, diesen Satz auf andere Arten von isolierten singulären Punkten auszudehnen, wurde durch die Untersuchung von Borel nahegelegt, aber erst nach einiger Zeit gelang es Faber[5], auf dem von Borel eingeschlagenen Weg weiterzuschreiten, und einen allgemeineren Satz zu beweisen, der in analoger Abkürzung so zu formulieren ist:

Das Produkt von zwei isolierten singulären Punkten, falls keiner von den beiden Verzweigungspunkt, ist notwendigerweise ein singulärer Punkt.

Ein isolierter singulärer Punkt, der kein Verzweigungspunkt ist, d. h. in dessen Umgebung die Funktion eindeutig bleibt, ist entweder ein Pol oder ein sogenannter „wesentlicher singulärer" Punkt. Die Einbeziehung der wesentlichen singulären Punkte ist das Neue an dem Faberschen Satz. Ferner gab Faber einen Beweisansatz für den folgenden, noch weiter gehenden Satz:

Das Produkt von zwei isolierten singulären Punkten, falls mindestens einer von den beiden kein Verzweigungspunkt, ist notwendigerweise ein singulärer Punkt.

Aber der Fabersche Beweisansatz ist insofern unvollständig, als er einen wesentlichen und gar nicht selbstverständlichen Hilfssatz über ganze Funktionen nur ausspricht und nicht beweist[6]. Diese Lücke ist allerdings auszufüllen (vgl. unter Nr. 77—79); aber selbst dann bleibt noch der folgende Satz unbewiesen:

Das Produkt von zwei beliebigen isolierten singulären Punkten ist notwendigerweise ein singulärer Punkt.

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[5] G. Faber, a. a. O. [3]; vgl. S. 294—296.

[6] Vgl. Faber, a. a. O. [3]; S. 297—298, insbesondere die Fußnote auf S. 297.

Einen schon vor einiger Zeit für diesen Satz angekündigten[7] Beweis auszuführen ist der Hauptgegenstand der gegenwärtigen Mitteilung.

68. Der darzustellende Beweis ist lang und nicht leicht. Ich hoffe, daß ich dem Leser die Bewahrung der Übersicht erleichtere, wenn ich hier einen Weg skizziere, auf welchem man in natürlicher Weise zu der zentralen Bemerkung des Beweises gelangt.

Die Voraussetzung des zu beweisenden Satzes lautet: Auf dem Konvergenzkreis von (1) liegt nur ein einziger singulärer Punkt; er heiße α. Auf dem Konvergenzkreis von (2) liegt auch nur ein einziger singulärer Punkt; er heiße β. Es ist α ein isolierter singulärer Punkt für $f(z)$, β für $g(z)$.

Zu beweisen ist: $\alpha\beta$ ist für $h(z)$ singulär.

Auf alle Fälle wissen wir aus dem Hadamardschen Multiplikationssatz, daß $h(z)$ in jedem Punkt der abgeschlossenen Kreisfläche $|z| \leqq |\alpha\beta|$ regulär ist, den einzigen Punkt $z = \alpha\beta$ möglicherweise ausgenommen. Daher ist die ursprüngliche Behauptung äquivalent mit der folgenden: *$h(z)$ ist in der abgeschlossenen Kreisfläche $|z| \leqq |\alpha\beta|$ nicht ausnahmslos regulär.* Wollen wir stärker betonen, daß $h(z)$ für $|z| < |\alpha\beta|$ sicher regulär ist, so können wir die Behauptung auch so aussprechen: *Der Konvergenzradius der Reihe* (3) *ist* $|\alpha\beta|$.

Es ist leicht zu sehen, daß der Konvergenzradius von (3) nicht kleiner als $|\alpha\beta|$ ist; jedoch warum nicht größer? Wir suchen vorerst, auf den ersten Anlauf, mindestens einen Teil, einen äußersten Fall der vorgelegten Frage zu beantworten: Warum ist der Konvergenzradius nicht unendlich groß? Oder noch extremer: Warum verschwindet $h(z)$ nicht identisch? So kommen wir dazu, zunächst eine schwächere Behauptung als Ziel aufzustellen, anstatt der ursprünglich vorgelegten: *Es können nicht alle Produkte $a_0 b_0, a_1 b_1, \cdots, a_n b_n, \cdots$ verschwinden.*

Die ausgesprochene Behauptung ist offenbar richtig, wenn die ursprüngliche richtig war. ($h(z)$ kann sicherlich nicht identisch verschwinden, wenn es einen singulären Punkt hat!) Durch ihre Aufstellung ist es uns gelungen, die Frage auf ein geläufigeres Gebiet zu verschieben: Es handelt sich um das Verschwinden gewisser Koeffizienten a_n und b_n, um die „Lücken" der Reihen (1) und (2). Hierüber ist uns einiges bekannt, insbesondere dies: Da nur ein singulärer Punkt auf der Konvergenzgrenze der Reihe (1) liegt, haben die nichtverschwindenden a_n die Maximaldichte 1 (vgl. Satz IVa in Nr. 58). Was sollten wir jetzt eigentlich über die Koeffizienten wissen, um schließen zu können, daß einige (womöglich unendlich viele) $a_n b_n$ nicht verschwinden?

[7] G. Pólya, Comptes Rendus, Paris **184** (1927), S. 579—581; vgl. auch die nachfolgende Bemerkung von Herrn Hadamard auf S. 581. Der Satz selbst wird auch im Referat von S. Mandelbrojt, Mémorial des Sciences Mathématiques, fasc. **54** (1932) auf S. 20 erwähnt.

Es handelt sich eigentlich um die Durchschnittsmenge, um die größte gemeinsame Untermenge von zwei Mengen von Indizes n:

A sei die Menge der Indizes, für welche $a_n \neq 0$;

B sei die Menge der Indizes, für welche $b_n \neq 0$.

Unser Ziel ist zu beweisen, daß das logische Produkt AB nicht leer ist. Wir wissen, nach Fabry, daß die Maximaldichte von A Eins ist — und was noch? Wir wissen, trivialerweise, daß die Vereinigungsmenge $A+B$ enthalten ist in der Menge aller Indizes $0, 1, 2, 3, \cdots$, deren Dichte Eins ist.

An dieser Stelle ist es natürlich, auf die Analogie zu Punktmengen hinüberzugreifen. A und B seien Punktmengen, z. B. in der Ebene; sie seien beide enthalten in einer Menge vom Maße 1, sagen wir in einem Quadrat von der Seitenlänge 1. Das äußere Maß von A ist 1. Was sollten wir jetzt eigentlich über die Menge B wissen, um schließen zu können, daß AB nicht leer ist?

Wir sollten wissen, daß das innere Maß von B nicht 0 ist. Gut! Also wir sollen versuchen zu beweisen, daß die *Minimaldichte der nichtverschwindenden* b_n *positiv* ist.

Indem wir diesen Beweisansatz nach Maßgabe der zur Verfügung stehenden Beweismittel und an Hand geeigneter Beispiele korrigieren, kommen wir schließlich dazu, die folgenden beiden Behauptungen als Beweisziel aufzustellen:

a) *Die obere Dichte der* a_n *ist Eins.*

b) *Die untere Dichte der* b_n *ist positiv.*

Beide Behauptungen haben dieselbe Voraussetzung: Auf der Konvergenzgrenze der betreffenden Potenzreihe befindet sich nur ein einziger singulärer Punkt, u. zw. ein isolierter. Aus beiden Behauptungen zusammen würde sich, wie man leicht sieht (vgl. Nr. 89), die ursprünglich vorgelegte ergeben: Wir dürfen an den Behauptungen a) und b) als Beweisziel festhalten.

Es würde zu weit führen, nun auseinanderzusetzen, welche heuristischen Gründe den Beweis der Behauptungen a) und b) nahelegen, und wie die Diskussion des gefundenen Beweises weitere Verschärfungen der gewonnenen Sätze veranlaßt. Wenn dem Leser die Motive, welche von der ursprünglichen Behauptung zu a) und b) hinüberleiten, klar geworden sind, so hat er den wesentlichsten Zusammenhang begriffen. Wir können jetzt zur Besprechung einiger Definitionen übergehen, die für die Formulierung der nachher aufzustellenden endgültigen Beweisziele wesentlich sind.

69. Die Beschäftigung mit den Behauptungen a) und b) drängt uns dazu, gewisse Typen von singulären Punkten, die sich durch ihre besondere Lagebeziehung zu den übrigen singulären Punkten der Funktion auszeichnen, hervorzuheben und zu benennen. Es soll jetzt erklärt werden, was im folgenden unter *fastisolierten, isolierten* und *nicht-kritischen* isolierten,

unter *isolierbaren* und unter *zugänglichen* singulären Punkten zu verstehen ist.

Die Definition dieser Singularitätstypen wird an die Angabe einer Potenzreihenentwicklung gebunden, an deren Konvergenzgrenze sie liegen. Wenn das Zentrum der Entwicklung z_0 ist, so hat die Potenzreihe die Form

$$(4) \qquad a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + a_n(z - z_0)^n + \cdots.$$

Der auf dem Konvergenzkreis der Potenzreihe (4) gelegene singuläre Punkt α heißt für diese Reihe *fastisoliert*, wenn die durch die Reihe (4) definierte Funktion in jeden Punkt einer genügend kleinen Umgebung von α hinein geradlinig von z_0 aus fortsetzbar ist, abgesehen natürlich von solchen Punkten der Umgebung, deren geradlinige Verbindungsstrecke mit z_0 den singulären Punkt α passiert.

Es wird also in dieser Definition die Existenz einer besonders beschaffenen (kleinen) offenen Kreisfläche $\mathfrak{K}$ vom Mittelpunkt α gefordert. Die Verbindungsgerade von z_0 und α schneidet die Peripherie von $\mathfrak{K}$ in zwei Punkten, von welchen derjenige, der nicht im Konvergenzkreis von (4) liegt, α' genannt werde. Alle Punkte der Kreisfläche $\mathfrak{K}$ mit Ausnahme derjenigen, die auf der abgeschlossenen Verbindungsstrecke von α und α' liegen, bilden die *aufgeschnittene Kreisfläche* $\mathfrak{K}^*$, ein einfach zusammenhängendes, eigentliches Teilgebiet von $\mathfrak{K}$. Die gegebene Definition nennt eine am Konvergenzrand der Reihe (4) gelegene singuläre Stelle α dann und nur dann fastisolierte singuläre Stelle der Entwicklung (4), wenn, bei passender Wahl der (kleinen) Kreisfläche $\mathfrak{K}$, die durch die Potenzreihe (4) in einem Teile der aufgeschnittenen Kreisfläche $\mathfrak{K}^*$ dargestellte Funktion im ganzen Gebiet $\mathfrak{K}^*$ regulär ist. Der Punkt α ist singulär für die Reihe; über andere Punkte der Verbindungsstrecke von α und α' wird in der Definition nichts ausgesagt, sie mögen sich beliebig verhalten: Sie sind als Randpunkte von $\mathfrak{K}^*$ doppelt zu zählen, sie sind nämlich vom Innern von $\mathfrak{K}^*$ aus von zwei Seiten her, auf zwei wesentlich verschiedenen Wegen anzunähern, und die Fortsetzung der durch (4) definierten Funktion kann auf beiden Wegen beliebig ausfallen, so daß der betreffende Punkt von beiden Seiten her singulär, von beiden regulär oder von der einen Seite her regulär, von der andern her singulär ausfallen kann.

Eine fastisolierte singuläre Stelle α der Entwicklung (4) heißt *isoliert*, wenn, bei passender Wahl der (genügend kleinen) Kreisfläche $\mathfrak{K}$, alle Punkte der Strecke α, α', abgesehen natürlich von α, für beide Fortsetzungsrichtungen regulär sind.

Eine isolierte singuläre Stelle α heißt eine *nichtkritische* isolierte Stelle, wenn die durch (4) definierte Funktion in der passend (genügend klein)

gewählten Kreisscheibe $\mathfrak{K}$, in deren Mittelpunkt sie singulär wird, eindeutig bleibt. Eine isolierte singuläre Stelle ist entweder kritisch oder nichtkritisch; kritische isolierte Stelle heißen (in deutscher Terminologie) üblicherweise Verzweigungsstellen; eine nichtkritische isolierte Stelle ist, wie wohl bekannt, entweder ein Pol oder ein sogenannter wesentlicher singulärer Punkt.

Die eben definierten Begriffe des fastisolierten und des isolierten singulären Punktes sind wesentlich an die Angabe einer Potenzreihe, eines Funktionselementes gebunden, an dessen Konvergenzgrenze sie liegen; ein fastisolierter singulärer Punkt α des Funktionselementes (4) ist nicht mehr fastisolierter Punkt für gewisse, passend gewählte unmittelbare Fortsetzungen davon. Ein nichtkritischer isolierter Punkt des Funktionselementes (4) hingegen behält diese Eigenschaft für alle Funktionselemente, an deren Konvergenzgrenze er liegt und die aus (4) durch Fortsetzung innerhalb $\mathfrak{K}$ entstehen.

Wir kommen zu einer weiteren Definition. Ein auf dem Konvergenzkreis der Reihenentwicklung (4) gelegener singulärer Punkt α heißt für diese Entwicklung *isolierbar,* wenn es in jedem (noch so kleinen) Kreis mit dem Mittelpunkt α eine doppelpunktlose geschlossene, den Punkt α umschließende Kurve gibt, entlang welcher die durch die Potenzreihe definierte Funktion fortsetzbar ist. Gemeint ist analytische Fortsetzung der Funktion entlang eines vollen Umlaufs der fraglichen Kurve, von einem bestimmten Ausgangspunkte aus, in einer bestimmten Richtung zum Ausgangspunkt zurück, wobei die in Betracht kommenden Funktionswerte entlang eines gewissen, in den Konvergenzkreis von (4) fallenden Bogens der Kurve durch (4) dargestellt sind.

Eine isolierte Stelle ist also, ob kritisch oder nicht kritisch, sicherlich auch isolierbar zu nennen. Eine fastisolierte Stelle ist dann und nur dann *nicht*-isolierbar, wenn die oben erwähnte Umgebung $\mathfrak{K}$ so klein gewählt werden kann, daß die Strecke α, α' von beiden Seiten her eine singuläre Linie wird.

Wir kommen zu einer letzten Definition, zu der eines zugänglichen singulären Punktes, die folgendermaßen lautet:

Es sei $f(z)$ eine analytische Funktion und α ein singulärer Punkt von $f(z)$. Man betrachte eine offene Kreisfläche mit dem Mittelpunkt α. Sie sei durch einen Durchmesser in zwei offene Halbkreise geteilt, der Durchmesser selbst sei durch α in zwei Radien ohne Endpunkte geteilt. Wenn es möglich ist, die Kreisfläche und den Durchmesser so zu wählen, daß derjenige Zweig von $f(z)$, der in α singulär wird, in einem der beiden Halbkreise und zugleich auf einem der beiden Radien regulär bleibt, so heiße α ein *zugänglicher* singulärer Punkt (des betreffenden Zweiges) von $f(z)$.

Wenn z. B. α fastisolierter singulärer Punkt eines Elementes der Funktion $f(z)$ ist, so ist α sicherlich ein zugänglicher singulärer Punkt von $f(z)$. Ist hingegen α ein von den beiden Endpunkten verschiedener Punkt einer geraden Strecke, die eine singuläre Linie für $f(z)$ bildet, so ist α nicht zugänglich.

Diese Definitionen sind nicht müßig: Sie beschreiben einfache geometrische Eigenschaften der Singularitäten, die in verhältnismäßig einfachen Eigenschaften der Koeffizientenfolge des angrenzenden Funktionselementes sich wiederspiegeln oder beim Verhalten gegenüber gewissen Funktionaloperationen maßgebend sind, wie es sich später zeigen wird (vgl. Hilfssatz III in Nr. 83). Sie spielen bei dem bevorstehenden Beweis, ja schon bei der scharfen, zweckmäßigen Formulierung der ins Auge gefaßten Ergänzung zum Hadamardschen Multiplikationssatz eine wesentliche Rolle.

70. Nach geeigneter Verschärfung und Verallgemeinerung gehen die in Nr. 68 erwähnten Behauptungen a) und b) in die folgenden beiden Sätze über:

SATZ A. *Wenn auf dem Konvergenzkreise einer Potenzreihe nur ein einziger singulärer Punkt liegt, und dieser singuläre Punkt für die Potenzreihe fastisoliert ist, so ist die obere Dichte der Koeffizienten* 1.

SATZ B. *Wenn die untere Dichte der Koeffizienten einer Potenzreihe Null ist, so ist die dargestellte Funktion eindeutig, und ihr Existenzgebiet ist einfach zusammenhängend*[8].

Aus diesen beiden Sätzen wird mehr folgen, als in Nr. 67 behauptet wurde: Nicht bloß das Produkt von zwei beliebigen isolierten singulären Punkten ist notwendigerweise singulär, sondern auch das Produkt eines fastisolierten und eines isolierbaren singulären Punktes. Die ausführliche Fassung lautet so:

SATZ C. *Wenn auf dem Konvergenzkreis der Potenzreihe*

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

nur ein einziger singulärer Punkt α liegt, und α fastisoliert ist, wenn ferner auf dem Konvergenzkreis der Potenzreihe

$$b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots$$

ebenfalls nur ein einziger singulärer Punkt β liegt, und β isolierbar ist, so ist der Punkt $\alpha\beta$ für die Potenzreihe

[8] Unter Existenzgebiet ist die Gesamtheit der Regularitätspunkte zu verstehen. (Pole und algebraische Verzweigungspunkte gehören also *nicht* zum Existenzgebiet.) „Einfach zusammenhängend" heißt ausführlicher: Einfach zusammenhängendes Teilgebiet der *z-Ebene*. (Nicht der *z-Kugel*!) Insofern nicht ausdrücklich das Gegenteil erwähnt wird, gilt auch im folgenden, insbesondere in den Nrn. 83–87, daß der Punkt $z = \infty$ nicht als Punkt der Gebiete gerechnet wird, deren Zusammenhangsverhältnisse untersucht werden.

$$a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots + a_n b_n z^n + \cdots$$

notwendigerweise ein singulärer Punkt, und zwar der einzige auf deren Konvergenzkreis gelegene singuläre Punkt.

Die Genauigkeit dieses Satzes ist daran zu ermessen, daß das Produkt von zwei fastisolierten singulären Punkten nicht singulär zu sein braucht, wie es später (in Nr. 91) durch ein Beispiel belegt werden soll.

Der Satz C beantwortet nur eine Einzelfrage, allerdings eine besonders einschneidende Einzelfrage aus einer allgemeineren Fragestellung, die den Hadamardschen Multiplikationssatz betrifft: Unter welchen Bedingungen kann behauptet werden, daß das Produkt $\alpha\beta$ aus einem singulären Punkt α der Funktion (1) und aus einem singulären Punkt β der Funktion (2) tatsächlich singulär für die Funktion (3) ist? Einige Andeutungen über diese allgemeine Fragestellung sind in den Nr. 92—93 enthalten.

71. Das Material für den Beweis der Sätze A, B, C ist aus vielen, weit auseinanderliegenden Gebieten zusammenzutragen. Um die Darstellung nicht über Gebühr auszudehnen, habe ich darauf verzichtet, sie in ähnlichem Maße „voraussetzungslos" zu gestalten, wie die Darstellung der ersten Mitteilung über diese Untersuchungen gestaltet werden konnte. Mehrere wesentliche Hilfsmittel sind in dieser ersten Mitteilung zu finden (in den Kapiteln I und II; Kapitel III kommt weniger in Frage); eine ganz wesentliche Stütze des Satzes B ist einer meiner vorangehenden Publikationen zu entnehmen[9], usw. Das Benutzte wird unter Angabe der Herkunftstelle in geeigneter Form ausführlich formuliert, meistens in Form von Hilfssätzen[10].

Zerlegung der Singularitäten.

72. Eine Stütze des Beweises für Satz A ist die folgende Tatsache:

Satz I. *Wenn die Potenzreihe*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

auf ihrem Konvergenzkreise nur einen einzigen singulären Punkt hat, nämlich den Punkt $z = 1$, und dieser Punkt ein fastisolierter singulärer Punkt für $f(z)$ ist, so läßt $f(z)$ die Zerlegung

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[9] G. Pólya, Nachrichten der Gesellschaft der Wissenschaften zu Göttingen 1927, S. 187—195.

[10] An dieser Stelle erwähne ich zwei eigene Arbeiten zur Ergänzung der in der Fußnote [2] der 1. Mtlg. gegebenen Liste: k) Atti del Congresso Internazionale dei Matematici, Bologna 1928, Bd. III, S. 243—247; l) Göttinger Nachrichten (1930), Fachgruppe I, S. 19—27. Auch zwei der Eidg. Technischen Hochschule in Zürich eingereichte Dissertationen seien hier erwähnt: R. Jungen, *Sur les séries de Taylor n'ayant que des singularités algébrico-logarithmiques sur leur cercle de convergence* (Commentarii Mathematici Helvetici **3** (1931), S. 266—306) und E. Boller, *Über ganze Funktionen vom Exponentialtypus und ihre Indikatordiagramme*, 1932.

$$f(z) = \Phi(z) + f^*(z) \tag{5}$$

zu, mit folgenden Eigenschaften:

$f^*(z)$ ist regulär im Innern eines Kreises, dessen Radius größer als 1 ist;

$\Phi(z)$ ist regulär und eindeutig außerhalb einer endlichen, auf der reellen Achse gelegenen Strecke. Der linke Endpunkt dieser Strecke ist $z = 1$;

$\Phi(z)$ bleibt auch für $z = \infty$ regulär, und es ist $\Phi(\infty) = 0$.

Das außerhalb der besagten Strecke gelegene Gebiet ist einfach zusammenhängend zu nennen, *wenn* der Punkt $z = \infty$ dazu gerechnet wird; man stelle sich das Gebiet auf der Kugel vor. Die Einfachheit des Zusammenhangs macht die behauptete Eindeutigkeit selbstverständlich.

Dieser Satz ist im wesentlichen bekannt[11]; er soll aber der Vollständigkeit halber bewiesen werden, u. zw. soll er, um den später wieder zu gebrauchenden Gedankengang klar herauszuarbeiten, in zwei Schritten bewiesen werden.

Nach Voraussetzung gibt es eine Zahl ϱ, $\varrho > 1$, so beschaffen, daß $f(z)$ im Kreise $|z| \leqq \varrho$ regulär ist, abgesehen von denjenigen Punkten der reellen Achse, die auf der Strecke zwischen $z = 1$ und $z = \varrho$ liegen; $z = 1$ muß, die übrigen Punkte der Strecke können singulär sein. Versetzen wir uns zunächst in den Spezialfall, in welchem der Punkt $z = \varrho$ regulär bleibt, sowohl bei Annäherung von der oberen, wie bei Annäherung von der unteren Seite der reellen Achse her. Wir können dann $f(z)$, für genügend kleines $|z|$, folgendermaßen durch das Cauchysche Integral darstellen: Es ist

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)\, dw}{w - z}, \tag{6}$$

wobei die geschlossene, im positiven Sinne durchlaufene Kurve C in zwei Teile zerfällt:

$$C = \Gamma + C^*.$$

C^* ist die Kreislinie $|z| = \varrho$, im positiven Sinne durchlaufen.

Γ ist eine geschlossene Kurve ohne Doppelpunkt, die durch den Punkt $z = \varrho$ hindurchgeht, das Geradenstück zwischen $z = 1$ und $z = \varrho$, abgesehen vom rechten Endpunkt, im Innern enthält, den Punkt $z = 0$ außerhalb läßt, und selbst im Innern von C^* enthalten ist, wieder abgesehen vom Punkte $z = \varrho$; Γ ist im negativen Sinne durchlaufen.

$C = \Gamma + C^*$ ist also eine geschlossene Kurve mit einem Doppelpunkt an der Stelle $z = \varrho$, die im positiven Sinne durchlaufene Umrandung eines

[11] Vgl. A. Haar, Nachrichten der Gesellschaft der Wiss. Göttingen (1914), S. 114—123. Vgl. ferner die auf eine verwandte aber etwas verschiedene Aufgabe gerichtete Arbeit von G. Faber, Math. Annalen **60** (1905), S. 379—397, insbesondere die Ausführungen S. 391—395, die auch für den Beweis des Satzes I verwendet werden konnten. Für ein allgemeineres Resultat vgl. die Ankündigung von N. Aronszajn, Comptes Rendus, Paris **194** (1932), S. 155—156.

den Punkt $z = 0$ enthaltenden Gebietes. Gemäß (6) ist in einer Umgebung des Punktes $z = 0$

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(w)\,dw}{w-z} + \frac{1}{2\pi i}\int_{C^*}\frac{f(w)\,dw}{w-z}$$
$$= \Phi(z) + f^*(z) ;$$

wir haben $\Phi(z)$ mit dem Integral über Γ, $f^*(z)$ mit dem über C^* identifiziert. Aus dieser Integraldarstellung sind die in Satz I ausgesagten Regularitätseigenschaften von $\Phi(z)$ und $f^*(z)$ ohne weiteres ersichtlich (durch geeignete Deformation der Kurve Γ). Insbesondere gilt für $|z| > \varrho$

$$\Phi(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(w)}{w-z}\,dw = -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\int_{\Gamma} f(w)\,w^n\,dw .$$

Diese Reihenentwicklung zeigt, daß $\Phi(z)$ im Punkt $z = \infty$ regulär ist und den Wert 0 annimmt.

73. Die Einschränkung, unter welcher der Beweis von Satz I erbracht wurde, läßt sich abmildern: Der Punkt $z = \varrho$ muß nicht regulär sein, es genügt, wenn von beiden Seiten der reellen Achse Wege von endlicher Länge in $z = \varrho$ hineinführen, entlang welcher $f(z)$ beschränkt bleibt. Unter dieser erweiterten Annahme bleibt der gegebene Beweis im wesentlichen bestehen, nur die Integrationslinien Γ und C^* sind insofern zu modifizieren, daß wir sie in der Nähe des Punktes $z = \varrho$ mit den erwähnten Wegen zusammenfallen lassen müssen.

Diese Bemerkung führt uns auf die folgende, von Poincaré herrührende Beweismethode[12]. Man kann[13] eine solche ganze Funktion $G(z)$ angeben, die keine Nullstellen hat und entlang der positiven reellen Achse eine vorgegebene (beliebig stark anwachsende!) Funktion dem Betrage nach übertrifft. Man kann also die nie verschwindende ganze Funktion $G(z)$ bei vorgegebenem $f(z)$ so wählen, daß

$$\frac{f(z)}{G\left(-\frac{1}{(z-\varrho)^2}\right)} = g(z)$$

gegen 0 strebt, wenn z entlang einer Parallelen zur imaginären Achse in den Punkt $z = \varrho$ hineinrückt, u. zw. für beide Annäherungen, sowohl von

[12] H. Poincaré, American Journal of Math. **14** (1892), S. 201—221; vgl. S. 213—221. Dieselbe Methode wendet auch Haar an, vgl. a. a. O. [11], S. 117—122.

[13] Poincaré, a. a. O. [12], S. 213—215. Vgl. z. B. auch G. Pólya und G. Szegő, *Aufgaben und Lehrsätze aus der Analysis* (1925), Bd. II, Nr. II 180, S. 32 und S. 211. Man könnte hier auch etwas anders vorgehen, indem man einen (allerdings tiefer liegenden) Satz von T. Carleman, Arkiv för Matematik, Astronomi och Fysik, **20 B** (1927), Nr. 4, benutzt.

der unteren wie von der oberen Halbebene her. (Wir haben hier stillschweigend Regularität in einem etwas größeren Gebiet vorausgesetzt als ursprünglich. Dies schadet aber nicht, da wir ja ϱ durch ϱ' ersetzen könnten, wo $1<\varrho'<\varrho$.) Die Methode der vorangehenden Nr. 72 ist also, mit der am Anfang der vorliegenden Nummer angegebenen Modifikation, auf $g(z)$ anwendbar: Es ist

$$g(z) = \Psi(z) + g^*(z),$$

wobei $g^*(z)$ im Kreise $|z|<\varrho$, und $\Psi(z)$ außerhalb der Strecke zwischen $z=1$ und $z=\varrho$ regulär ist; ferner bleibt $\Psi(z)$ auch für $z=\infty$ regulär, und es ist $\Psi(\infty)=0$. Wird nun

$$\Psi(z)\, G\left(\frac{1}{-(z-\varrho)^2}\right) = \Phi(z), \quad g^*(z) G\left(-\frac{1}{(z-\varrho)^2}\right) = f^*(z)$$

gesetzt, so besitzen $\Phi(z)$ und $f^*(z)$ alle von ihnen in Satz I ausgesagten Eigenschaften.

Interpolation der Koeffizienten.

74. Man sagt, daß eine ganze Funktion $F(z)$ die Koeffizienten einer vorgelegten Potenzreihe

$$\Phi(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots \tag{7}$$

interpoliert, wenn für $n = 0, 1, 2, 3, \cdots$ die Gleichung

$$F(n) = c_n \tag{8}$$

besteht. Die Koeffizientenfolge einer beliebigen Potenzreihe kann durch eine ganze Funktion interpoliert werden, sogar — dieser Punkt geht uns besonders an — durch eine ganze Funktion *vom Exponentialtypus,* falls die Potenzreihe nicht stets divergiert. In der Tat[14], es kann der Koeffizient c_n der als nicht stets divergent vorausgesetzten Reihe (7) in der Form

$$c_n = \frac{1}{2\pi i} \oint \Phi(w)\, w^{-n-1}\, dw \tag{9}$$

dargestellt werden, wobei das Integral im positiven Sinne entlang eines genügend kleinen Kreises vom Mittelpunkt $w=0$ erstreckt ist. Man betrachte das Integral

$$F(Z) = \frac{1}{2\pi i} \oint \Phi(w)\, w^{-Z-1}\, dw, \tag{10}$$

erstreckt über denselben Kreis, der in (9) auftritt; die in w mehrdeutige Funktion w^{-Z} ist als $e^{-Z \log w}$ zu interpretieren, wobei der Imaginärteil von

[14] Die darzustellende Methode rührt wohl von C. W. Oseen her; vgl. Arkiv för Matematik, Astronomi och Fysik, **2** (1906), Nr. 21.

log w zwischen $-\pi$ und π enthalten sei (wir zerschneiden den Integrationskreis durch die reelle negative Halbachse). Daß $F(z)$ eine ganze Funktion ist und der Gleichung (8) für $n = 0, 1, 2, \cdots$ genügt, ist klar; ich behaupte, daß $F(z)$ eine ganze Funktion *vom Exponentialtypus* ist. In der Tat, es geht das Integral (10) durch die Variablentransformation $w = e^{-z}$ in

$$F(Z) = \frac{1}{2\pi i} \int \Phi(e^{-z})\, e^{zZ}\, dz \tag{11}$$

über, wobei das Integral in bezug auf z entlang einer geradlinigen Strecke von der Länge 2π zu erstrecken ist, die durch die reelle Achse der z-Ebene halbiert wird und darauf senkrecht steht. Das Integral (11) ist mit dem in Kapitel II behandelten Integral (27) zu vergleichen; die Überlegung von Nr. 24 a), angewandt auf (11) (anstelle des dort betrachteten (27)) zeigt, daß $F(z)$ eine ganze Funktion vom Exponentialtypus ist, deren konjugiertes Diagramm in der beschriebenen Integrationsstrecke von der Länge 2π enthalten ist. (Es ist mit dieser Strecke identisch, wie aus Nr. 43 hervorgeht, aber dies wird in der Folge nicht benutzt.)

75. Die Integrationslinie in (9) kann auf mannigfache Art abgeändert werden, ohne den Wert des Integrals zu ändern; dieselbe Änderung der Integrationslinie wird aber das durch (10) definierte $F(z)$ durch eine Funktion ersetzen, die zwar auch die Koeffizientenfolge c_n interpoliert und ebenfalls vom Exponentialtypus ist, aber sonst von $F(z)$ ganz verschieden sein kann. Wird z. B. der ursprünglich gewählte Kreis durch einen andern, konzentrischen ersetzt, so geht das ursprüngliche $F(z)$ in ein verschiedenes über, sogar das konjugierte Diagramm erfährt eine Horizontalverschiebung. (Bei Beibehaltung des Integrationskreises würde eine abgeänderte Wahl des Zweiges von $\log w$, vermittels einer andern Zerschneidung des Integrationskreises, eine Vertikalverschiebung des konjugierten Diagramms bewirken.) Eine unter besonderen Umständen zulässige spezielle Verlagerung des Integrationsweges ergibt, unter Hinzuziehung von früher Bewiesenem, den

Satz II. *Damit die durch die Potenzreihe* (7) *definierte Funktion* $\Phi(z)$ *entlang eines vom Punkt* $z = 0$ *auslaufenden Halbstrahls bis in den Punkt* $z = \infty$ *hinein fortsetzbar sei und bei dieser Fortsetzung den Funktionswert* $\Phi(\infty) = 0$ *erhalte, ist notwendig und hinreichend die Existenz einer ganzen Funktion* $F(z)$ *vom Exponentialtypus, die die Koeffizienten von* $\Phi(z)$ *im Sinne von* (8) *interpoliert, und von dessen Indikatordiagramm die Breite senkrecht zur reellen Achse kleiner als* 2π *ist*[15].

[15] Vgl. F. Carlson, Sur une classe de séries de Taylor, Thèse, Upsala 1914. Vgl. die weiteren Zitate der 1. Mtlg. S. 553, Fußnote [8].

Daß $\Phi(z)$ „bis in den Punkt $z = \infty$ hinein" fortsetzbar ist, ist natürlich so zu verstehen, daß der durch die besagte Fortsetzung zustandegekommene Zweig sich für große Werte von z in eine Reihe von der Form

$$\Phi(z) = d_0 + \frac{d_1}{z} + \frac{d_2}{z^2} + \cdots \tag{12}$$

entwickeln läßt; ferner ist $\Phi(\infty) = d_0 = 0$ gefordert. Somit verschwindet $\Phi(w)w^{-1}$ für $w = \infty$ wie w^{-2}, und wir können, indem wir die Voraussetzung betreffend die Fortsetzbarkeit von $\Phi(z)$ benutzen, die ursprüngliche kreisförmige Integrationslinie in (9) für $n = 0, 1, 2, \cdots$ durch eine neue, sagen wir $\mathfrak{L}$, ersetzen, welche aus folgenden vier Stücken besteht: Aus einem Kreisbogen mit Zentrum $w = 0$, der im Konvergenzgebiet von (7) verläuft; aus einem zum erstgenannten konzentrischen und in bezug auf das gemeinsame Zentrum ähnlich gelegenen Kreisbogen, der im Konvergenzgebiet von (12) verläuft; schließlich aus zwei, die entsprechenden Endpunkte der genannten Bögen verbindenden Geradenstücken, deren Verlängerung durch das gemeinsame Zentrum $w = 0$ geht. Es sei jetzt das Integral (10) auch entlang dieser Linie $\mathfrak{L}$ erstreckt. Bei der Abbildung $w = e^{-z}$ geht die Linie $\mathfrak{L}$ in ein in der z-Ebene gelegenes Rechteck $\mathfrak{R}$ über, dessen vertikale Seiten die *Länge* 2π *nicht erreichen*; entlang dieses Rechteckes $\mathfrak{R}$ ist nun das Integral (11) zu erstrecken. Die Anwendung der Überlegungen von Nr. 24 a) auf (11) ergibt, daß $F(z)$ eine ganze Funktion vom Exponentialtypus ist, dessen konjugiertes Diagramm in der durch $\mathfrak{R}$ begrenzten Rechtecksfläche liegt. Damit ist bewiesen, daß die in Satz II genannte Bedingung notwendig ist; daß sie auch hinreicht, haben wir schon in Nr. 44 besprochen.

76. Zwischen dem Indikatordiagramm $\mathfrak{J}$ der ganzen Funktion $F(z)$ und den Singularitäten der Potenzreihe $\Phi(z)$ besteht eine sehr enge, in Nrn. 43—44 ausführlich besprochene Beziehung. Wir brauchen im folgenden diese Beziehung nur in einem sehr speziellen Falle: Wenn $\Phi(z)$ außerhalb einer auf der positiven reellen Achse gelegenen geradlinigen Strecke regulär ist, dann ist das Indikatordiagramm von $F(z)$ entweder eine auf der reellen Achse gelegene geradlinige Strecke oder eine dazu parallele Strecke im Abstand $2\pi m$ von der reellen Achse, wobei m ganz ist; dies geht aus Nrn. 43—44 hervor. Nun ist aber $F(z)$ durch $\Phi(z)$ nicht eindeutig bestimmt: Genau dieselben in Satz II beschriebenen Beziehungen, welche zwischen $\Phi(z)$ und $F(z)$ bestehen, bestehen auch zwischen $\Phi(z)$ und $F(z)\,e^{2\pi i m z}$ [16]. Die Multiplikation von $F(z)$ mit $e^{2\pi i m z}$ bedeutet aber ersichtlicherweise eine Vertikalverschiebung des Indikatordiagrammes um den

[16] Aber abgesehen von diesem Faktor ist $F(z)$ vollständig bestimmt. Vgl. Carlson, a. a. O. [15], ferner Nr. 43b (1. Mtlg.).

Vektor $-2\pi mi$ (vgl. Nrn. 31, 32). Indem wir diese Vertikalverschiebung passend wählen und uns auf die Sätze I, II, ferner auf Nrn. 43—44 stützen, erhalten wir schließlich das folgende Resultat, das wir von den bisher getroffenen Vorbereitungen zum Beweise des Satzes A allein weiter brauchen werden:

SATZ III. *Die notwendige und hinreichende Bedingung dafür, daß die Potenzreihe*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

auf ihrem Konvergenzkreis nur einen singulären Punkt habe, nämlich den Punkt $z = 1$, und daß dieser Punkt ein fastisolierter singulärer Punkt für $f(z)$ sei, ist die Möglichkeit einer Zerlegung

$$a_n = F(n) + a_n^*,$$

wobei

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n^*|} < 1$$

und $F(z)$ eine ganze Funktion vom Exponentialtypus ist, deren Indikatordiagramm ein auf der reellen Achse gelegenes Geradenstück ist, dessen rechter Endpunkt im Nullpunkt liegt.

Gemäß der Definition des Indikatordiagrammes und des Indikators gilt für die im Satz erwähnte Funktion $F(z)$

$$(13) \qquad h(\varphi) = \overline{\lim_{r\to\infty}} r^{-1} \log |F(re^{i\varphi})| = 0 \quad \text{für} \quad -\frac{\pi}{2} \leqq \varphi \leqq \frac{\pi}{2}.$$

Vom Satz III wird im folgenden eigentlich nur ein Teil gebraucht: Nur die Notwendigkeit der besprochenen Bedingung, und von den Eigenschaften von $F(z)$ bloß soviel, daß $F(z)$ für $\Re z \geqq 0$ regulär ist und (13) genügt.

Auf dem zum Beweis von Satz III führenden Wege erhält man auch den wohlbekannten und noch auf vielen anderen Wegen erreichbaren Satz von Wigert[17], den man auch so aussprechen kann:

ZUSATZ ZU SATZ III. *Soll der Punkt $z = 1$ für $f(z)$ nicht bloß fastisoliert, sondern isoliert, und zwar kein Verzweigungspunkt (also ein Pol oder eine wesentliche singuläre Stelle) sein, so besteht das Indikatordiagramm von $F(z)$ aus dem Nullpunkt.*

Man kann die hier behauptete Eigenschaft von $F(z)$ in der üblichen Terminologie auch so aussprechen, daß $F(z)$ entweder einer Ordnung unterhalb 1 oder (höchstens) dem Minimaltypus der Ordnung 1 angehört. Die durch Formel (13) ausgedrückte (weniger besagende) Eigenschaft von $F(z)$

[17] Öfversigt af K. Vet. Ak. Förhandlingar **57** (1900), S. 1001—1011. Vgl. ferner 1. Mtlg. S. 553, Fußnote [8] und etwa L. Bieberbach, a. a. O. [3], S. 288—292.

kann so ausgesprochen werden, daß $F(z)$ *in der rechten Halbebene* höchstens vom Minimaltypus der Ordnung 1 ist.

Beweis des Satzes A.

77. Der zu beweisende Satz A ist nahe verwandt mit dem folgenden, von Faber aufgestellten Satz[18]:

Satz IV. *Wenn auf dem Konvergenzkreise einer Potenzreihe nur ein einziger singulärer Punkt liegt, und dieser singuläre Punkt isoliert und nicht-kritisch, also ein Pol oder eine wesentliche singuläre Stelle ist, so ist die Dichte der Koeffizienten* 1.

Der neue Satz A geht aus dem Faberschen Satz IV durch eine doppelte Änderung hervor: Einerseits wird die Voraussetzung erweitert; man geht von speziellen, nicht-kritischen isolierten Singularitäten zu allgemeineren fastisolierten Singularitäten über. Andererseits wird die Behauptung abgeschwächt, anstatt von Dichte nur von der oberen Dichte gesprochen. Zur Erläuterung der Methode soll dem Beweis des Satzes A der Beweis des analogen leichteren Satzes IV voraufgeschickt werden.

Sowohl in Satz A, wie in Satz IV können wir ohne wirklichen Verlust an Allgemeinheit annehmen, daß der Konvergenzradius 1 und daß der einzige auf dem Konvergenzkreis liegende singuläre Punkt $z = 1$ ist. Dem Beweise von Satz A wurde durch Satz III wesentlich vorgearbeitet, dem Beweise von Satz IV durch den Zusatz zu Satz III. In Kenntnis dieses Zusatzes können wir Satz IV auf den folgenden[19] reduzieren:

Satz V. *Es sei $F(z)$ eine ganze Funktion, deren Anwachsen den Minimaltypus der Ordnung* 1 *nicht übertrifft, und α eine gegebene (kleine) positive Zahl. Dann haben diejenigen positiven ganzen Zahlen n, für welche die Ungleichung*

$$|F(n)| > e^{-\alpha n}$$

erfüllt ist, die Dichte 1.

In Kenntnis des Satzes III können wir Satz A auf den folgenden reduzieren:

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[18] G. Faber, a. a. O. [3] S. 297. In wesentlich schwächerer Form wurde der Satz von N. Obreschkoff wiedergefunden; vgl. Comptes Rendus, Paris, **182** (1926), S. 307—309, insbesondere S. 309. Vgl. ferner die Bemerkungen von G. Valiron und G. Pólya, Comptes Rendus, Paris, **185** (1927), S. 831—833 bzw. S. 1107—1108.

[19] G. Faber a. a. O. [3] S. 297; auch die Fußnote ist zu beachten. Ich gebe hier den in meiner Note von 1927 a. a. O. [7] angekündigten Beweis. Vgl. für verwandte aber verschiedene Sätze das Kapitel IX des in Druck befindlichen Buches von V. Bernstein, Leçons sur les progrès récents de la théorie des séries de Dirichlet (Paris, 1933). [Vgl. noch die Nrn. 95—99, insbesondere die Fußnote [38].]

SATZ VI. *Die Funktion $F(z)$ soll in der Halbebene $\Re z \geqq 0$ regulär sein, ihr Anwachsen daselbst soll den Minimaltypus der Ordnung* 1 *nicht übertreffen, und es sei für reell wachsendes z*

$$\overline{\lim_{z \to +\infty}} \; z^{-1} \log |F(z)| = 0. \tag{14}$$

Es sei α eine gegebene (kleine) positive Zahl. Dann haben diejenigen positiven ganzen Zahlen n, für welche die Ungleichung

$$|F(n)| > e^{-\alpha n}$$

erfüllt ist, die obere Dichte 1.

Satz VI ist also das eigentliche Beweisziel; er wird mit Hülfe der Lagrangeschen Interpolationsformel bewiesen[20]. Zur methodischen Vorbereitung, und auch zur Ergänzung einer öfters erwähnten Faberschen Untersuchung, wird der Beweis von Satz V voraufgeschickt.

78. Der Beweis des Satzes V kann auf folgende Bemerkung gestützt werden:

HILFSSATZ I. *Es sei $P(z) = z^n + \cdots$ ein Polynom n-ten Grades vom höchsten Koeffizienten* 1. *Dann gibt es unter den $n+1$ Werten*

$$P(x_0), P(x_1), P(x_2), \cdots, P(x_n),$$

die das Polynom an $n+1$ verschiedenen ganzzahligen Stellen $x_0, x_1, x_2, \cdots, x_n$ annimmt, mindestens einen, dessen Betrag $\left(\frac{n}{2e}\right)^n$ übertrifft.

Zum Beweis[21] interpoliere man das Polynom $P(z)$ an den $n+1$ besagten ganzzahligen Stellen. Es ist

$$P(z) = \sum_{\nu=0}^{n} P(x_\nu) \frac{(z - x_0) \cdots (z - x_{\nu-1})(z - x_{\nu+1}) \cdots (z - x_n)}{(x_\nu - x_0) \cdots (x_\nu - x_{\nu-1})(x_\nu - x_{\nu+1}) \cdots (x_\nu - x_n)}.$$

Indem man den Koeffizienten von z^n links und rechts vergleicht, und hierbei eine wesentliche Voraussetzung benutzt, erhält man

$$1 = \sum_{\nu=0}^{n} \frac{P(x_\nu)}{(x_\nu - x_0) \cdots (x_\nu - x_{\nu-1})(x_\nu - x_{\nu+1}) \cdots (x_\nu - x_n)}. \tag{15}$$

Man kann annehmen, daß die betrachteten ganzzahligen Stellen aufsteigend numeriert sind, das heißt, daß

$$x_0 < x_1 < x_2 < \cdots < x_n.$$

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[20] Ich teile hier den in meiner Note von 1927 a. a. O. [7] erwähnten Beweis mit. Herr Rolf Nevanlinna fand im Wintersemester 1928/29 einen zweiten, gänzlich verschiedenen, sehr interessanten Beweis, worauf bei anderer Gelegenheit eingegangen werden soll.

[21] Vgl. G. Pólya, Jahresbericht der deutschen Math. Vereinigung **28** (1919), S. 31—40, insbesondere S. 32.

Die Beachtung der Abstände zwischen den sukzessiven Punkten ergibt offenbar

$$x_\nu - x_{\nu-1} \geqq 1,\ x_\nu - x_{\nu-2} \geqq 2,\ \cdots,\ x_\nu - x_0 \geqq \nu,$$
$$x_{\nu+1} - x_\nu \geqq 1,\ x_{\nu+2} - x_\nu \geqq 2,\ \cdots,\ x_n - x_\nu \geqq n - \nu,$$

und somit folgt aus (15), wenn wir das Maximum der $n+1$ Beträge $|P(x_0)|$, $|P(x_1)|, \cdots, |P(x_n)|$ mit M bezeichnen

$$1 \leqq M \sum_{\nu=0}^{n} \frac{1}{(x_\nu - x_0)\cdots(x_\nu - x_{\nu-1})(x_{\nu+1} - x_\nu)\cdots(x_n - x_\nu)}$$

$$\leqq M \sum_{\nu=0}^{n} \frac{1}{\nu \cdots 2 \cdot 1 \cdot 1 \cdot 2 \cdots (n-\nu)} \tag{16}$$

$$= \frac{M}{n!} \sum_{\nu=0}^{n} \frac{n!}{\nu!\,(n-\nu)!} = \frac{M}{n!}\, 2^n < M \frac{2^n e^n}{n^n},$$

also

$$M > \left(\frac{n}{2e}\right)^n,$$

w. z. b. w.

79. Jetzt kommen wir zum Beweise des Satzes V.

Ich bezeichne mit $n(r)$ die Anzahl derjenigen Nullstellen der ganzen Funktion $F(z)$, deren Betrag $\leqq r$ ist. Die Voraussetzung des Satzes V hat bekanntlich[22] zur Folge, daß

$$\lim_{r\to\infty} r^{-1} n(r) = 0 \tag{17}$$

ist, ferner, daß $F(z)$ höchstens vom Geschlecht 1, also von der Form

$$F(z) = e^{az} \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{a_\nu}\right) e^{\frac{z}{a_\nu}} \tag{18}$$

ist. Die beim Hinschreiben der Formel (18) gemachten Beschränkungen ($F(0) = 1$ und $n(r) \to \infty$ für $r \to \infty$) sind unwesentlich.

Ich bezeichne mit $N(r)$ die Anzahl derjenigen positiven ganzen Zahlen x, für welche die beiden Ungleichungen

$$|F(x)| \leqq e^{-ax},$$
$$x \leqq r$$

gelten. Die Behauptung des Satzes betrifft $N(r)$, die Anzahl der „kleinen" ganzzahligen Stellen bis zur Grenze r, und besagt, daß

$$\lim_{r\to\infty} r^{-1} N(r) = 0 \tag{19}$$

ist. Es ist also (19) zu beweisen.

[22] Vgl. z. B. Bieberbach, a. a. O. [3], S. 235–243.

Es sei β gegeben, $0<\beta<1$. Ich betrachte die ganzzahligen Werte x im Intervall

$$\beta R < x \leqq R. \tag{20}$$

Ich setze abkürzungsweise

$$n(2R) = n, \tag{21}$$
$$|a_\nu| = c_\nu \qquad (\nu = 0, 1, 2, 3, \cdots)$$

(so daß $c_{n+1} > 2R$), und betrachte

$$\begin{aligned} |F(x)F(-x)| &= \prod_{\nu=1}^{\infty}\left|1-\frac{x^2}{a_\nu^2}\right| \geqq \prod_{\nu=1}^{\infty}\left|1-\frac{x^2}{c_\nu^2}\right| \\ &= \prod_{\nu=1}^{n}\frac{c_\nu+x}{c_\nu}\,\frac{|x-c_\nu|}{c_\nu}\prod_{\nu=n+1}^{\infty}\frac{1}{1+\dfrac{x^2}{c_\nu-x^2}} \\ &\geqq \frac{1}{(2R)^n}\left|\prod_{\nu=1}^{n}(x-c_\nu)\right|\prod_{\nu=n+1}^{\infty}\frac{1}{1+\dfrac{x^2}{c_\nu^2-\left(\dfrac{c_\nu}{2}\right)^2}} \\ &\geqq \frac{1}{(2R)^n}|P(x)|\left\{\prod_{\nu=1}^{\infty}\left(1+\frac{4x^2}{3c_\nu^2}\right)\right\}^{-1}. \end{aligned} \tag{22}$$

Unter $P(z) = z^n + \cdots$ ist hierbei das Polynom mit dem höchsten Koeffizienten 1 verstanden, dessen Nullstellen $c_1, c_2, \cdots, c_n$ sind. Nach Hilfssatz I gilt, *mit eventueller Ausnahme von höchstens n Stellen*, daß an einer ganzzahligen Stelle

$$|P(x)| > \left(\frac{n}{2e}\right)^n$$

ist, und hieraus folgt weiter, gemäß (22), wenn die betreffende ganzzahlige Stelle x im Intervall (20) liegt, daß

$$\begin{aligned} |F(x)| &> \frac{1}{(2R)^n}\left(\frac{n}{2e}\right)^n\left\{|F(-x)|\prod_{\nu=1}^{\infty}\left(1+\frac{4x^2}{3c_\nu^2}\right)\right\}^{-1} \\ &> (4e)^{-\frac{x}{\beta}\frac{n}{R}}\, e^{-\frac{x}{\beta}\frac{n}{R}\log\frac{R}{n}}\left\{|F(-x)|\prod_{\nu=1}^{\infty}\left(1+\frac{4x^2}{3c_\nu^2}\right)\right\}^{-1} \end{aligned} \tag{23}$$

ist; zum Schluß wurde die erste Ungleichung von (20) berücksichtigt.

Da $F(-x)$, ebenso wie $F(x)$ den Minimaltypus der Ordnung 1 nicht übersteigt, und dasselbe von der ganzen Funktion von x gilt, die rechts in (23) als zweiter Faktor in der geschweiften Klammer steht, folgt auf Grund von (21) und (17), daß für genügend großes R für alle in (20)

liegenden ganzzahligen x, mit eventueller Ausnahme von höchstens n Werten x,

$$|F(x)| > e^{-\alpha x}$$

gilt. Für die fraglichen R ist also

$$N(R) \leqq \beta R + n(2R).$$

Es folgt, wieder mit Rücksicht auf (17),

$$\overline{\lim_{R\to\infty}} R^{-1} N(R) \leqq \beta.$$

Da hierin β beliebig klein genommen werden kann, ist (19) bewiesen.

80. Der wesentlichste Inhalt von Satz VI wird ausführlicher, aber auch schwerfälliger formuliert in dem folgenden Satz VII, der vor VI bewiesen und nachher zur Ableitung von VI benutzt werden soll.

SATZ VII. *Es sei $F(z)$ regulär in der Halbebene $\Re z \geqq 0$. Das Maximum von $F(z)$ in dem Halbkreis $\Re z \geqq 0$, $|z| \leqq r$ sei mit $M(r)$ bezeichnet, und es sei*

$$\lim_{r\to\infty} r^{-1} \log M(r) = 0. \tag{24}$$

Es seien gegeben drei positive Zahlen α, γ, Δ, alle $\leqq \frac{1}{2}$.

Es gibt dann drei positive Zahlen z_0 (groß), η (klein) und β,

$$0 < \beta < \Delta \leqq \frac{1}{2}, \tag{25}$$

folgendermaßen beschaffen: Wenn $z > z_0$ und

$$\Delta \leqq \delta \leqq \frac{1}{2} \tag{26}$$

ist, und im Intervall

$$\frac{z(1-\beta)}{\delta} \leqq x < \frac{z}{\delta} \tag{27}$$

die Dichte derjenigen ganzzahligen Werte x, für welche

$$|F(x)| \leqq e^{-\alpha x} \tag{28}$$

besteht, größer als γ ist, so gilt notwendigerweise

$$|F(z)| < e^{-\eta z}. \tag{29}$$

Es liegt z links vom Intervall (27), wegen (25) und (26).

Ungenauer, aber anschaulicher gefaßt, besagt der Satz VII dies: Das Vorhandensein von ganzzahligen Stellen x in genügender γ-Dichte, mit einem niedrigen α-Niveau gestattet den Rückschluß, daß auch die vorangehende Stelle z ein ziemlich niedriges η-Niveau hatte. Noch kürzer:

Wenn viele Nachfolger auf einem niedrigen Niveau stehen, so stand sicherlich auch kein Vorgänger auf einem hohen Niveau.

Der interessante Fall des Satzes VII ist, wenn alle drei fest vorgegebene Größen α, γ, Δ *klein* sind: Ein kleines α entspricht einem verhältnismäßig weniger tiefen Niveau, ein kleines γ einer kleinen Dichte, ein kleines Δ einer großen Entfernung der Nachfolger vom Vorgänger z. Die aus α, γ, Δ zu bestimmende, der Länge des Intervalles (27) proportionale Zahl β wird sich zum Schluß auch als klein herausstellen.

Der Beweisansatz erinnert an den des Hilfssatzes I; er beruht auf der Lagrangeschen Interpolationsformel mit Restglied[23]:

$$F(z) = \sum_{\nu=0}^{n} F(x_\nu) \frac{(z-x_0)\cdots(z-x_{\nu-1})(z-x_{\nu+1})\cdots(z-x_n)}{(x_\nu-x_0)\cdots(x_\nu-x_{\nu-1})(x_\nu-x_{\nu+1})\cdots(x_\nu-x_n)} + \frac{1}{2\pi i}\oint F(u)\frac{(z-x_0)(z-x_1)\cdots(z-x_n)}{(u-x_0)(u-x_1)\cdots(u-x_n)}\frac{du}{u-z}. \tag{30}$$

Die Interpolation erfolgt an den ganzzahligen Stellen $x_0, x_1, x_2, \cdots, x_n$ mit dem niedrigen α-Niveau, wodurch der Wert von $F(z)$ heruntergedrückt wird, während das Restglied, bei passender Wahl der Integrationslinie, wegen (24) nicht zu groß werden kann.

81. Ich gehe zu genauen Einzelheiten über. Es seien β, δ den Ungleichungen (25), (26) unterworfen und fest gewählt; die nähere Bestimmung des Wertes von β behalte ich mir vor. Ganzzahlige x-Stellen, die der Bedingung (28) genügen, seien im Intervalle (27) insgesamt $n+1$ vorhanden, nämlich die Stellen $x_0, x_1, x_2, \cdots, x_n$,

$$\frac{z(1-\beta)}{\delta} \leqq x_0 < x_1 < x_2 < \cdots < x_n < \frac{z}{\delta}. \tag{31}$$

Die Stellen $x_0, x_1, x_2, \cdots, x_n$ sind also so beschaffen, daß

$$|F(x_\nu)| \leqq e^{-\alpha x_\nu} \leqq e^{-\frac{\alpha z(1-\beta)}{\delta}}. \tag{32}$$

Ferner ist offenbar

$$x_\nu - x_\mu \geqq \nu - \mu, \quad \text{wenn } \nu > \mu, \tag{33}$$

$$n \leqq x_n - x_0 < \frac{z\beta}{\delta}, \tag{34}$$

$$0 < x_\nu - z < \frac{z}{\delta} - z = z\frac{1-\delta}{\delta}. \tag{35}$$

Nach Voraussetzung ist das Verhältnis der Anzahl $n+1$ zu der Länge des Intervalles (27) nicht kleiner als γ, d. h.

[23] Vgl. z. B. Pólya und Szegő, a. a. O.[13] Bd. I, Nr. III 163, S. 116 und S. 289.

$$n+1 \geqq \gamma \frac{z\beta}{\delta}. \tag{36}$$

Das Integral rechts in (30) sei entlang des im positiven Sinne durchzulaufenden Randes von dem Halbkreis geführt, der durch die Ungleichungen

$$\Re u \geqq 0, \qquad |u| \leqq \frac{2z}{\delta}$$

abgegrenzt ist.

Auf der ganzen Integrationslinie ist ersichtlicherweise

$$|u-z| \geqq z; \tag{37}$$

auf dem geradlinigen Teil gilt

$$|u-x_\nu| \geqq x_\nu \geqq \frac{z(1-\beta)}{\delta},$$

auf dem krummlinigen Teil

$$|u-x_\nu| \geqq |u|-x_\nu = \frac{2z}{\delta}-x_\nu > \frac{2z}{\delta}-\frac{z}{\delta},$$

und somit auf der ganzen Integrationslinie

$$|u-x_\nu| \geqq \frac{z(1-\beta)}{\delta}. \tag{38}$$

Auf der ganzen Integrationslinie gilt ferner für jedes feste $\varepsilon > 0$, sobald nur z und damit $z\delta^{-1} \geqq 2z$ genügend groß ist,

$$|F(u)| < e^{\frac{\varepsilon 2z}{\delta}}. \tag{39}$$

Unter sukzessiver Beachtung von (32), (35), (33), (39), (35), (38), (37) ergibt (30) für beliebig kleines positives ε bei genügend großem z

$$|F(z)| \leqq e^{-\frac{z\alpha(1-\beta)}{\delta}} \left(\frac{z(1-\delta)}{\delta}\right)^n \sum_{\nu=0}^{n} \frac{1}{\nu!\,(n-\nu)!} + \\ + \frac{1}{2\pi} e^{\frac{\varepsilon 2z}{\delta}} \left(\frac{z(1-\delta)}{\delta} \frac{\delta}{z(1-\beta)}\right)^{n+1} \frac{1}{z}\left(2\frac{2z}{\delta}+\pi\frac{2z}{\delta}\right).$$

Zieht man eine auch beim Beweise von Hilfssatz I benutzte Rechnung (vgl. (16)) heran, so ergibt sich weiter

$$|F(z)| < e^{-\frac{z\alpha(1-\beta)}{\delta}} \left(\frac{2e(1-\delta)z}{n\delta}\right)^n + \frac{2}{\delta} e^{\frac{2\varepsilon z}{\delta}} \left(\frac{1-\delta}{1-\beta}\right)^{n+1}$$

$$= \exp\left(\frac{z}{\delta}\left[-\alpha(1-\beta)+\frac{n\delta}{z}\log 2e(1-\delta)\frac{z}{n\delta}\right]\right) \tag{40}$$

$$+ \frac{2}{\delta}\exp\left(\frac{z}{\delta}\left[2\varepsilon-\frac{(n+1)\delta}{z}\log\frac{1-\beta}{1-\delta}\right]\right).$$

Man beachte, daß die Funktion der positiven Variablen t

$$t \log 2e(1-\delta)\, t^{-1} \tag{41}$$

monoton wächst, solang $t \leqq 2(1-\delta)$, also (vgl. (26)) sicherlich, wenn $t \leqq 1$. Somit ergibt sich aus (40) wegen (34), (25), (26) und unter Beachtung der wesentlichen Voraussetzung (36) weiter

$$|F(z)| < \exp\left(\frac{z}{\delta}[-\alpha(1-\beta)+\beta\log 2e(1-\delta)\,\beta^{-1}]\right) + \frac{2}{\delta}\exp\left(\frac{z}{\delta}\left[2\varepsilon-\gamma\beta\log\frac{1-\beta}{1-\delta}\right]\right),$$

und schließlich mit Berufung auf (26)

$$|F(z)| < \exp\left(\frac{z}{\delta}[-\alpha(1-\beta)+\beta\log 2e(1-\Delta)\,\beta^{-1}]\right) + \frac{2}{\Delta}\exp\left(\frac{z}{\delta}\left[2\varepsilon-\gamma\beta\log\frac{1-\beta}{1-\Delta}\right]\right). \tag{42}$$

Die Funktion (41) strebt für $t \to 0$ gegen 0. Daher ist es möglich, zuerst β so klein zu wählen, daß die erste Exponentialgröße rechts unter (42) einen *negativen* Exponenten erhält. Dann kann man ε so klein (und dementsprechend z oberhalb einer so großen Grenze z_0) wählen, daß auch die zweite Exponentialgröße einen *negativen* Exponenten erhält. *Nach* dieser Wahl erhält man, unter Beachtung von (26),

$$\begin{aligned}|F(z)| &\leqq \exp\left(2z[-\alpha(1-\beta)+\beta\log 2e(1-\Delta)\,\beta^{-1}]\right) \\ &\quad + 2\Delta^{-1}\exp\left(2z\left[2\varepsilon-\gamma\beta\log\frac{1-\beta}{1-\Delta}\right]\right) \\ &= e^{-\eta_1 z} + 2\Delta^{-1}e^{-\eta_2 z},\end{aligned}$$

wo η_1, η_2, nach der vorgenommenen Wahl von β und ε, durch α, γ und Δ völlig bestimmte positive Größen sind, w. z. b. w.

82. Satz VI entsteht ganz leicht aus Satz VII, unter Heranziehung der maßgebenden Bedingung (14). Aus der in Nr. 80 besprochenen vagen Formulierung des Satzes VII schließt man unmittelbar: Wenn also ein Vorgänger von Niveau da war, so wird auch das durchschnittliche Niveau seiner Nachfolger eine gewisse Zeit lang gehoben. Genau wird der Schluß so durchgeführt: Die Anzahl derjenigen im Intervall $0 < x \leqq r$ liegenden positiven ganzzahligen Stellen x, für welche (28) gilt, sei mit $N(r)$ bezeichnet. Wir haben nachzuweisen — die Fassung ist mit der ursprünglichen äquivalent — daß die untere Dichte der (28) genügenden Stellen, also daß

$$\varlimsup_{r\to\infty} r^{-1} N(r)$$

verschwindet.

Es seien die positiven Zahlen α, γ, Δ, wie in Satz VII, beliebig klein, aber fest gegeben, und es sei, gemäß der ausschlaggebenden Bedingung (14) des Satzes VI, eine positive Zahl z' (ein „großer Vorgänger") vorhanden, für welche

$$F(z') \geqq e^{-\eta z'}, \qquad z' > z_0 \tag{43}$$

gilt. Es sei β dem Satz VII gemäß und die positive ganze Zahl m durch die Ungleichung

$$\frac{\Delta}{(1-\beta)^{m-1}} \leqq \frac{1}{2} < \frac{\Delta}{(1-\beta)^m} \tag{44}$$

bestimmt. Wir wenden Satz VII m Mal, mit

$$\delta = \frac{\Delta}{(1-\beta)^{m-1}}, \cdots, \frac{\Delta}{1-\beta}, \Delta$$

an; da die Behauptung (29) für $z = z'$ gemäß (43) nicht zutrifft, kann auch die Voraussetzung, betreffend die Dichte der ganzzahligen x-Stellen, die (28) genügen, nicht zutreffen: Diese Dichte wird in keinem der Intervalle

$$\left(\frac{z'(1-\beta)^m}{\Delta}, \frac{z'(1-\beta)^{m-1}}{\Delta}\right), \cdots, \left(\frac{z'(1-\beta)^2}{\Delta}, \frac{z'(1-\beta)}{\Delta}\right), \left(\frac{z'(1-\beta)}{\Delta}, \frac{z'}{\Delta}\right)$$

den Wert γ übertreffen, woraus folgt, daß die Anzahl der fraglichen, (28) genügenden x-Stellen im Gesamtintervall

$$\frac{z'(1-\beta)^m}{\Delta} \leqq x < \frac{z'}{\Delta}$$

höchstens $\gamma z' \Delta^{-1}[1-(1-\beta)^m]$ ist. Im Intervall

$$0 < x < \frac{z'(1-\beta)^m}{\Delta}$$

befinden sich, wegen (44), weniger als $2z'$ ganzzahlige Stellen, also ist sicher

$$N\left(\frac{z'}{\Delta}\right) \leqq 2z' + \gamma \frac{z'}{\Delta}[1-(1-\beta)^m],$$

$$\frac{\Delta}{z'} N\left(\frac{z'}{\Delta}\right) < 2\Delta + \gamma.$$

Somit ist, da gemäß (14) beliebig große, (43) erfüllende z'-Stellen vorhanden sind,

$$\varlimsup_{r\to\infty} r^{-1} N(r) \leqq 2\Delta + \gamma.$$

Hier muß aber, da γ und Δ beliebig klein gewählt werden können, die linke Seite Null sein, w. z. b. w.

Beweis des Satzes B.

83. Den Beweis des Satzes B werde ich weniger vollständig darstellen als den des Satzes A[24]. Ich werde mich darauf beschränken, anzugeben, wieso Satz B aus der Kombination von zwei anderen Sätzen zu gewinnen ist, ohne diese selbst ab ovo zu analysieren.

Die eine Quelle des Satzes B ist ein Resultat von Ostrowski[25], das ich folgendermaßen formuliere:

Hilfssatz II. *Wenn die Potenzreihe*

$$C_1 z^{\mu_1} + C_2 z^{\mu_2} + \cdots + C_n z^{\mu_n} + \cdots$$

der Bedingung

$$\overline{\lim_{n\to\infty}} \frac{\mu_{n+1}}{\mu_n} = \infty \tag{45}$$

genügt, so ist die dargestellte Funktion eindeutig und ihr Existenzgebiet ist einfach zusammenhängend[26].

Dieser bemerkenswerte Ostrowskische Satz war der Ausgangspunkt und ist ein wesentlicher Spezialfall des zu beweisenden Satzes B, in ähnlichem Sinne, wie der Hadamardsche Lückensatz der Ausgangspunkt des Fabryschen Lückensatzes war und ein wesentlicher Spezialfall davon ist. Satz B entsteht aus Hilfssatz II durch eine Erweiterung der Voraussetzung: Der Satz B stellt dieselbe Behauptung auf, stützt sich aber, anstatt auf (45), nur auf die weniger einschränkende Bedingung

$$\overline{\lim_{n\to\infty}} \frac{n}{\mu_n} = 0.$$

(Man beachte, daß, wegen $\mu_{n-1} \geqq n-2$,

$$\frac{n}{\mu_n} \leqq \frac{\mu_{n-1}}{\mu_n} \frac{n}{n-2}$$

ist.) Der Übergang von dem in Hilfssatz II formulierten Ostrowskischen Resultat zum Satz B ist analog dem bekannten Faberschen Übergang vom

[24] Satz B habe ich zuerst in den Comptes Rendus, Paris **185** (1927), S. 1107—1108 ausgesprochen. Der Beweis ist in meiner Arbeit a. a. O. [9] zum Teil ausgeführt, zum Teil angedeutet. Eine Darstellung findet sich im Buche von V. Bernstein, a. a. O. [19], vgl. Kapitel VII.

[25] A. Ostrowski, Journal of the London Math. Society **1** (1926), S. 251—263; vgl. insbesondere S. 256—258, auch die Fußnote auf S. 257 und die dort zitierten Arbeiten von J. F. Ritt und E. Hille.

[26] Vgl. Fußnote [8].

Hadamardschen zum Fabryschen Lückensatz; man kann aber, scheint mir, im jetzigen Fall nicht durchkommen, ohne aus einer wesentlich neuen Quelle zu schöpfen.

Die andere Quelle des Satzes B ist ein Resultat, das ich in einer früheren Untersuchung ausführlich bewiesen habe[27] und hier folgendermaßen formuliere:

HILFSSATZ III. *Es sei $\Psi(z)$ eine analytische Funktion, und die Potenzreihe*

$$l_0 + \frac{l_1 z}{1!} + \frac{l_2 z^2}{2!} + \cdots + \frac{l_n z^n}{n!} + \cdots = F(z)$$

soll eine nicht identisch verschwindende ganze Funktion vom Exponentialtypus darstellen, deren Indikatordiagramm der Nullpunkt ist. Dann ist

$$\Psi^*(z) = F(-D)\Psi(z) = l_0 \Psi(z) - \frac{l_1}{1!}\Psi'(z) + \frac{l_2}{2!}\Psi''(z) - \cdots$$

eine analytische Funktion, die im ganzen Existenzgebiet von $\Psi(z)$ regulär ist, für welche jedoch jeder ***zugängliche*** *singuläre Punkt von $\Psi(z)$ sicherlich singulär ist.*

Die Voraussetzung betreffend $F(z)$ läßt sich auch so fassen, daß das Anwachsen dieser nicht identisch verschwindenden ganzen Funktion den Minimaltypus der Ordnung 1 nicht übersteigt. Die von $\Psi(z)$ zu $\Psi^*(z)$ führende regularitätserhaltende lineare Funktionaloperation haben wir schon früher, in Nr. 36—40 betrachtet, vgl. insbesondere Nr. 40; zu dem dort Gesagten tritt jetzt als wesentlich neu hinzu, daß diese Funktionaloperation die *zugänglichen* singulären Punkte nicht zerstört.

84. Um den Satz B zu beweisen, schreibe man die darin erwähnte Potenzreihe in der Form

$$B_1 z^{\lambda_1} + B_2 z^{\lambda_2} + \cdots + B_n z^{\lambda_n} + \cdots = g(z). \tag{46}$$

Durch diese Schreibweise werden die von Null verschiedenen Koeffizienten hervorgehoben.

Die Voraussetzung des Satzes B besagt, daß die untere Dichte der ganzzahligen wachsenden Folge $\lambda_1, \lambda_2, \lambda_3, \cdots$ verschwindet. Daher kann man, gemäß dem im I. Kapitel unter Nr. 12 bewiesenen Satz X, die Folge in zwei elementfremde, zueinander komplementäre Teilfolgen $\lambda_1', \lambda_2', \lambda_3', \cdots$ und $\lambda_1'', \lambda_2'', \lambda_3'', \cdots$ zerlegen, so daß

$\lambda_1', \lambda_2', \lambda_3', \cdots$ die Dichte Null hat, und

$\lambda_1'', \lambda_2'', \lambda_3'', \cdots$ der Bedingung

[27] A. a. O. [9]; vgl. Satz 3, S. 191. Vgl. auch die Darstellung von V. Bernstein, a. a. O. [19], Kapitel III. Vgl. auch Boller, a. a. O. [10], S. 16—23.

$$\overline{\lim_{n\to\infty}} \frac{\lambda''_{n+1}}{\lambda''_n} = \infty \tag{47}$$

genügt.

Man konstruiere nun mit den Elementen der Teilfolge $\lambda'_1, \lambda'_2, \lambda'_3, \cdots$, deren Dichte verschwindet, die Funktion

$$F(z) = \prod_{\mu=1}^{\infty} \left(1 - \frac{z^2}{\lambda'^2_\mu}\right) = l_0 + \frac{l_1 z}{1!} + \frac{l_2 z^2}{2!} + \cdots.$$

Diese Funktion $F(z)$ ist ganz, vom Exponentialtypus und hat den Nullpunkt zum Indikatordiagramm, wie man es entweder direkt oder durch Anwendung von Nr. 14 leicht einsieht.

Man wende nun die in Hilfssatz III erwähnte Funktionaloperation auf die Funktion $\Psi(z) = g(e^{-z})$ an:

$$\begin{aligned} F(-D)\Psi(z) &= F(-D)g(e^{-z}) = F(-D)\sum_{n=1}^{\infty} B_n e^{-\lambda_n z} \\ &= \sum_{n=1}^{\infty} B_n F(\lambda_n) e^{-\lambda_n z} \\ &= \sum_{\mu=1}^{\infty} B_{\mu'} F(\lambda'_\mu) e^{-\lambda'_\mu z} + \sum_{\nu=1}^{\infty} B_{\nu''} F(\lambda''_\nu) e^{-\lambda''_\nu z} \\ &= \sum_{\nu=1}^{\infty} B_{\nu''} F(\lambda''_\nu) e^{-\lambda''_\nu z} = \Psi^*(z). \end{aligned} \tag{48}$$

Ein Schritt dieser Rechnung ist schon in Nr. 41 unter c) betrachtet worden. Mit μ' bezeichne ich denjenigen Index, zu welchem λ'_μ in der ursprünglichen. Folge $\lambda_1, \lambda_2, \lambda_3, \cdots$ gehörte, so daß $\lambda'_\mu = \lambda_{\mu'}$ ist; ähnlicherweise ist ν'' definiert, so daß $\lambda''_\nu = \lambda_{\nu''}$ ist.

Hilfssatz III besagt, daß auf allen Wegen, auf welchen $\Psi(z) = g(e^{-z})$ fortsetzbar ist, auch $\Psi^*(z)$ fortgesetzt werden kann, und daß jeder *zugängliche* singuläre Punkt von $\Psi(z)$ auch für $\Psi^*(z)$ singulär ist.

Hilfssatz II besagt, vgl. (47), daß

$$\Psi^*(-\log z) = \sum_{\nu=1}^{\infty} B_{\nu''} F(\lambda''_\nu) z^{\lambda''_\nu} = g^*(z)$$

eine eindeutige Funktion ist, deren Existenzgebiet ein *einfach zusammenhängender* Teil der z-Ebene ist.

Aus dieser Sachlage sollen wir schließen, daß auch $g(z)$ eindeutig und daß auch sein Existenzgebiet ein einfach zusammenhängender Teil der z-Ebene ist.

85. Wir haben also die Kette der folgenden vier Funktionen:

$$g^*(z), \quad \Psi^*(z), \quad \Psi(z), \quad g(z) \tag{49}$$

zu betrachten. Sie sind miteinander durch die Beziehungen

$$g^*(e^{-z}) = \Psi^*(z), \quad \Psi^*(z) = F(-D)\Psi(z), \quad \Psi(z) = g(e^{-z})$$

verbunden. Von $g^*(z)$ wissen wir und von $g(z)$ sollen wir beweisen, daß das Existenzgebiet ein einfach zusammenhängender Teil der z-Ebene ist; wir wollen nun dasselbe auch von $\Psi^*(z)$ und $\Psi(z)$ zeigen.

Der Übergang von $g^*(z)$ zu $\Psi^*(z)$ und auch der von $\Psi(z)$ zu $g(z)$ ist eigentlich trivial; wir verschieben beide auf Nr. 87. Der Übergang von $\Psi^*(z)$ zu $\Psi(z)$ ist die Hauptsache. Diesen Übergang gründen wir auf den folgenden

HILFSSATZ IV. *Es sei die analytische Funktion $\Psi(z)$ regulär im Punkte $z = z_0$, und es sei $\mathfrak{k}$ eine in der z-Ebene verlaufende doppelpunktlose Kurve, die durch den Punkt z_0 hindurchgeht. Es sei $\Psi(z)$ von z_0 ausgehend, entlang eines vollen Umlaufs der Kurve $\mathfrak{k}$ in einer bestimmten Richtung bis zum Punkte z_0 zurück analytisch fortsetzbar, jedoch sei die Kurve $\mathfrak{k}$ nicht auf einen Punkt stetig zusammenziehbar ohne Singularitäten von $\Psi(z)$ zu überschreiten. Dann liegt innerhalb $\mathfrak{k}$ ein zugänglicher singulärer Punkt von $\Psi(z)$.*

Der Beweis von Hilfssatz IV folgt in Nr. 86. Nehmen wir ihn vorderhand als bewiesen und das Existenzgebiet von $\Psi^*(z)$ als einfach zusammenhängend an. Dann kann, behaupte ich, weder $\Psi(z)$ mehrdeutig, noch das Existenzgebiet von $\Psi(z)$ mehrfach zusammenhängend sein. Im gegenteiligen Falle existierte nämlich eine Kurve $\mathfrak{k}$, die zu $\Psi(z)$ in der im Hilfssatz IV beschriebenen Beziehung steht. Nach Hilfssatz III wäre $\Psi^*(z) = F(-D)\Psi(z)$, wie $\Psi(z)$ selbst, in z_0 regulär und in der besagten Richtung, entlang eines vollen Umlaufs der Kurve $\mathfrak{k}$ bis zum Punkte z_0 zurück fortsetzbar. Die Kurve $\mathfrak{k}$ würde, nach Hilfssatz IV, einen *zugänglichen* singulären Punkt von $\Psi(z)$ umschließen. Schließlich wäre, nach der wichtigsten Aussage des Hilfssatzes IV, der durch $\mathfrak{k}$ umschlossene zugängliche singuläre Punkt von $\Psi(z)$ auch für $\Psi^*(z)$ singulär. Jedoch kann die ganz im einfach zusammenhängenden Existenzgebiete von $\Psi^*(z)$ verlaufende Kurve $\mathfrak{k}$ keinen singulären Punkt von $\Psi^*(z)$ umschließen. Der Widerspruch löst sich nur, wenn man zugibt, daß $\Psi(z)$ eine eindeutige Funktion und ihr Existenzgebiet einfach zusammenhängend ist.

Hiermit ist Satz B im wesentlichen bewiesen. Es bleiben nur einige Schlüsse von geometrischer Natur nachzutragen: der Beweis des Hilfssatzes IV, die Übergänge von $g^*(z)$ zu $\Psi^*(z)$ und von $\Psi(z)$ zu $g(z)$. Diese Schlüsse sind nahezu selbstverständlich, immerhin sollen sie mit Vorsicht durchgeführt werden.

86. Zunächst ist der Beweis des Hilfssatzes IV nachzutragen.

Wir dürfen annehmen, daß die doppelpunktlose geschlossene Kurve $\mathfrak{k}$ ein Polygon und daß z_0 eine Ecke dieses Polygons ist. Wir wollen zunächst den Fall betrachten, in dem $\mathfrak{k}$ ein Dreieck ist.

Es sei also $\mathfrak{k}$ ein Dreieck mit den Ecken a, b, c. Es sei $\Psi(z)$ in a regulär und entlang der drei Seiten des Dreiecks $\mathfrak{k}$ von a nach b, von b nach c und von c nach a zurück analytisch fortsetzbar. Man setze

$$b^* = b + (c - b)\,t,$$

man lasse die reelle Variable t von 0 bis 1 wachsen und betrachte das variable Dreieck $\mathfrak{k}^*$ mit den Ecken a, b^*, c. Für $t = 0$ und für genügend kleines positives t läßt sich $\Psi(z)$ entlang der Seiten von $\mathfrak{k}^*$ von a aus über b^* und c nach a zurück analytisch fortsetzen. Diese Fortsetzbarkeit kann jedoch nicht für jedes t im Intervalle $0 \leqq t \leqq 1$ stattfinden, sonst wäre eben das Dreieck $\mathfrak{k}$ auf die Strecke von a bis c, und nachher auf den Punkt a ohne Überschreitung von Singularitäten der Funktion $\Psi(z)$ stetig zusammenziehbar. Es gibt daher einen kleinsten Wert von t, sagen wir t_0, so daß die geradlinige Strecke von a bis $b_0^* = b + (c - b)\,t_0$ nicht von Singularitäten von $\Psi(z)$ frei ist, und auf dieser Strecke a, b_0^* gibt es einen zu a nächstgelegenen singulären Punkt: dieser ist ersichtlicherweise zugänglich im Sinne der in Nr. 69 auseinandergesetzten Definition. (Es ist in diesem Schluß zweimal benutzt worden, daß die regulären Punkte eine offene Menge bilden.)

Nun sei $\mathfrak{k}$ ein Polygon mit n Ecken, $n > 3$, und es sei angenommen, daß Hilfssatz IV für Polygone mit weniger als n Ecken schon bewiesen ist. Da die Summe der n Innenwinkel $(n - 2)\,\pi$ ist, gibt es mindestens drei ausspringende Ecken, d. h. Ecken mit einem Innenwinkel $< \pi$. Es sei b eine von z_0 verschiedene ausspringende Ecke, a die vorangehende und c die nachfolgende Ecke. Man betrachte, wie vorher einen beweglichen Punkt b^*, der stetig von b nach c wandert, und das variable Polygon $\mathfrak{k}^*$, das aus $\mathfrak{k}$ dadurch hervorgeht, daß die Seite a, b von $\mathfrak{k}$ sich in a, b^* neigt und die Seite b, c sich zu b^*, c verkürzt, während die $n - 2$ übrigen Seiten unverändert bleiben. Die Deformation soll aufhören, wenn $\Psi(z)$ aufhört, entlang $\mathfrak{k}^*$ fortsetzbar, oder wenn $\mathfrak{k}^*$ aufhört ein Polygon mit n Seiten zu sein. Beim Aufhören seiner Bewegung sei der Punkt b^* in b_0^* angelangt. Es sind nur drei Fälle denkbar:

1. Auf der geradlinigen Strecke a, b_0^* liegt ein singulärer Punkt von $\Psi(z)$.
2. Es ist $b_0^* = c$.
3. Es liegt auf der geradlinigen Strecke a, b_0^* eine von c verschiedene Ecke von $\mathfrak{k}$.

Im ersten Falle finden wir (wie vorher, bei Betrachtung des Dreiecks) einen in $\mathfrak{k}$ liegenden zugänglichen singulären Punkt von $\Psi(z)$, im zweiten und im dritten Falle wird die Frage auf ein Polygon mit höchstens $n - 1$ Ecken zurückgeführt.

87. Es bleibt noch übrig, in der Kette der vier Funktionen (49) den Übergang von $g^*(z)$ zu $\Psi^*(z)$ und den von $\Psi(z)$ zu $g(z)$ durchzuführen. Der erste Übergang ist aber dem Rückweg von $g(z)$ zu $\Psi(z)$ analog, so daß wir uns auf die Betrachtung dieser beiden Funktionen beschränken dürfen. Somit verbleibt uns die Aufgabe, die folgende Behauptung über die Existenzbereiche der Funktionen $g(z)$ und $\Psi(s) = g(e^{-s})$ zu beweisen: *Entweder sind sie beide einfach zusammenhängend, oder es ist keiner von den beiden einfach zusammenhängend.*

Den Existenzbereich von $g(z)$ bezeichne ich mit $\mathfrak{Z}$, den von $\Psi(s)$ mit $\mathfrak{S}$; es ist $\mathfrak{Z}$ in der z-Ebene, $\mathfrak{S}$ in der s-Ebene gelegen. Die Ebenen sind hierbei ohne den Punkt ∞ (also wie punktierte Kugeln) zu nehmen. Es wird $\mathfrak{Z}$ auf $\mathfrak{S}$ mittels

$$-\log z = s \tag{50}$$

abgebildet, abgesehen jedoch vom Punkte $z = 0$, der zu $\mathfrak{Z}$ gehört, dem aber kein Bildpunkt in $\mathfrak{S}$ entspricht.

a) Es sei $\mathfrak{Z}$ einfach zusammenhängend. Wir entfernen aus $\mathfrak{Z}$ diejenige geradlinige Strecke $\mathfrak{l}$, die in $z = 0$ beginnt und entlang der negativen reellen Achse bis zum ersten Randpunkt von $\mathfrak{Z}$ (eventuell bis $-\infty$) läuft; das übrigbleibende offene Gebiet sei mit $\mathfrak{Z}^*$ bezeichnet. $\mathfrak{Z}^*$ ist einfach zusammenhängend, der Punkt $z = 0$ ist ein Randpunkt, aber kein Punkt von $\mathfrak{Z}^*$, daher ist jeder Zweig der Funktion (50) eindeutig in $\mathfrak{Z}^*$, und es wird $\mathfrak{Z}^*$ mittels eines bestimmten Zweiges dieser Funktion auf ein einfach zusammenhängendes offenes Gebiet $\mathfrak{S}_0^*$ der s-Ebene abgebildet. Durch die Gesamtheit aller Zweige von (50) wird $\mathfrak{Z}^*$ in eine Reihe von kongruenten und gleichgelegenen Gebieten $\cdots \mathfrak{S}_{-2}^*, \mathfrak{S}_{-1}^*, \mathfrak{S}_0^*, \mathfrak{S}_1^*, \mathfrak{S}_2^*, \cdots$ abgebildet, die aus $\mathfrak{S}_0^*$ mittels der Translationen $\cdots -4\pi i, -2\pi i, 0, 2\pi i, 4\pi i, \cdots$ hervorgehen und zu je zweien punktfremd sind. Wenn man diese Gebiete $\mathfrak{S}_n$ durch die in der s-Ebene gelegenen (äquidistanten, geradlinigen) Bilder der Strecke $\mathfrak{l}$ (gleichsam „wie mit Mörtel“) verbindet, so entsteht schließlich $\mathfrak{S}$. Werden dieselben Operationen (Verschiebungen um $2n\pi i$, Verbindung „mit Mörtel“) mit einem Rechteck von der horizontalen Basis b und der Höhe 2π vorgenommen, so entsteht ein Vertikalstreifen von der Breite b, worauf $\mathfrak{S}$ ersichtlicherweise eindeutig und stetig abgebildet werden kann. Daher ist $\mathfrak{S}$, wie der Vertikalstreifen, von einfachem Zusammenhang.

b) Es sei $\mathfrak{S}$ einfach zusammenhängend. Man zeichne einen Punkt z_0 von $\mathfrak{Z}$ aus, der im Innern einer zu $\mathfrak{Z}$ gehörigen Umgebung des Nullpunktes, der Bestimmtheit halber im Innern des Konvergenzgebietes der Reihe (46), gelegen ist; jedoch sei $z_0 \neq 0$. Es sei $\mathfrak{z}$ eine von z_0 ausgehende, ganz im Gebiet $\mathfrak{Z}$ verlaufende, den Punkt $z = 0$ vermeidende, nach z_0 zurückkehrende, geschlossene Kurve; ich habe zu zeigen, daß $\mathfrak{z}$ durch stetige Deformation innerhalb $\mathfrak{Z}$ auf einen Punkt zusammenziehbar ist.

Es entspricht dem Punkt z_0 mittels eines bestimmten Zweiges von (50) ein Punkt s_0 der s-Ebene, und der Kurve $\mathfrak{z}$, durch stetige Fortsetzung des besagten Zweiges von (50), eine Kurve $\mathfrak{f}$, die aber nicht geschlossen zu sein braucht, sondern einen Endpunkt s_0' hat, welcher, von s_0 im allgemeinen verschieden, aber, ebenso wie s_0, ein durch (50) entworfenes Bild von z_0 ist. Die geradlinige, vertikale Verbindungsstrecke von s_0 und s_0' liegt in $\mathfrak{S}$, sie liegt nämlich im Bild des Konvergenzkreises von (46). Da $\mathfrak{S}$, nach Voraussetzung, einfach zusammenhängend ist, kann man $\mathfrak{f}$ durch stetige Deformation innerhalb $\mathfrak{S}$ in die geradlinige Verbindungsstrecke von s_0 und s_0' überführen. Simultan wird, durch Umkehrung der Abbildung (50), $\mathfrak{z}$ durch stetige Deformation in einen, im allgemeinen mehrfach durchlaufenen Kreis mit dem Mittelpunkt $z = 0$, der innerhalb des Konvergenzgebietes von (46) liegt, übergeführt. Diesen Kreis kann man aber offenbar auf seinen Mittelpunkt innerhalb $\mathfrak{Z}$ zusammenziehen. Hierdurch ist gezeigt, daß $\mathfrak{Z}$ einfach zusammenhängt.

Beweis des Satzes C.

88. Der Satz C, das Hauptziel der vorliegenden Mitteilung, ergibt sich durch eine einfache Kombination der im Vorangehenden bewiesenen Sätze A und B. Wir müssen den Gedankengang, dessen heuristisches Auffinden in Nr. 68 skizziert wurde, nun genau ausführen.

Wir wollen also die Betrachtung der drei Potenzreihen

$$(1) \qquad a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots = f(z),$$

$$(2) \qquad b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots = g(z),$$

$$(3) \qquad a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots + a_n b_n z^n + \cdots = h(z)$$

wieder aufnehmen. Wir wollen die Voraussetzung des Satzes C zuerst nicht vollständig heranziehen, sondern bloß den Umstand benutzen, daß α der einzige singuläre Punkt auf dem Konvergenzkreise von (1) und β der einzige auf dem von (2) ist. Schon aus diesem Umstande allein folgt, auf Grund des Hadamardschen Multiplikationssatzes[28], daß die Potenzreihe (3) in der abgeschlossenen Kreisfläche $|z| \leqq |\alpha\beta|$ keinen andern singulären Punkt haben kann, als den Punkt $z = \alpha\beta$. Je nach dem der Punkt $z = \alpha\beta$ für sie singulär oder regulär ist, wird die Potenzreihe (3) den Konvergenzradius $|\alpha\beta|$ oder einen größeren Konvergenzradius haben. Wir sehen, daß die Behauptung von C äquivalent ist mit der Behauptung, daß (3) für $|z| > |\alpha\beta|$ divergiert. Wenn wir noch, was wir ohne wirklichen Verlust an Allgemeinheit tun können, $\alpha = 1$, $\beta = 1$ annehmen, können wir den Satz C in der folgenden Form aussprechen:

[28] Es genügt hier die von Bieberbach a. a. O. [3] gegebene Fassung.

SATZ C'. **Voraussetzung:**

(α) *Es ist* 1 *der einzige singuläre Punkt auf dem Konvergenzkreis der Reihe* (1), *und zwar ist* 1 *für die Reihe* (1) *ein fastisolierter singulärer Punkt.*

(β) *Es ist* 1 *der einzige singuläre Punkt auf dem Konvergenzkreis der Reihe* (2), *und zwar ist* 1 *für die Reihe* (2) *ein isolierbarer singulärer Punkt.*

(γ) *Es ist* $\varrho > 1$.

Behauptung: *Die Reihe*

$$a_0 b_0 + a_1 b_1 \varrho + a_2 b_2 \varrho^2 + \cdots + a_n b_n \varrho^n + \cdots \tag{51}$$

divergiert.

Um die Äquivalenz der Sätze C und C' völlig einzusehen, beachte man noch, daß ϱ beliebig nahe an 1 gewählt werden kann.

89. Wir setzen

$$\varrho = \frac{1}{(1-\varepsilon)^2}; \tag{52}$$

es ist, $\sqrt{\varrho}$ positiv genommen,

$$\varepsilon = 1 - \frac{1}{\sqrt{\varrho}} > 0,$$

womit wir die Voraussetzung (γ) wesentlich benutzt haben. Setzen wir weiter

$$a_n^* = \begin{cases} a_n & \text{wenn } |a_n| \geqq (1-\varepsilon)^n, \\ 0 & \text{wenn } |a_n| < (1-\varepsilon)^n; \end{cases} \tag{53}$$

$$b_n^* = \begin{cases} b_n & \text{wenn } |b_n| \geqq (1-\varepsilon)^n, \\ 0 & \text{wenn } |b_n| < (1-\varepsilon)^n; \end{cases} \tag{54}$$

$$a_0^* + a_1^* z + a_2^* z^2 + \cdots + a_n^* z^n + \cdots = f^*(z), \tag{1*}$$

$$b_0^* + b_1^* z + b_2^* z^2 + \cdots + b_n^* z^n + \cdots = g^*(z). \tag{2*}$$

Ferner bedeute:

$N_a(r)$ die Anzahl der von 0 verschiedenen unter den $[r]+1$ Zahlen $a_0^*, a_1^*, a_2^*, \cdots, a_{[r]}^*$;

$N_b(r)$ die Anzahl der von 0 verschiedenen unter den $[r]+1$ Zahlen $b_0^*, b_1^*, b_2^*, \cdots, b_{[r]}^*$;

$N_{ab}(r)$ die Anzahl der von 0 verschiedenen unter den $[r]+1$ Produkten $a_0^* b_0^*, a_1^* b_1^*, a_2^* b_2^*, \cdots, a_{[r]}^* b_{[r]}^*$.

Die Reihe (1*) hat, gemäß (53), in der offenen Kreisfläche $|z| < (1-\varepsilon)^{-1}$ dieselben Singularitäten wie die Reihe (1). Daher ist, nach Voraussetzung (α), $z = 1$ der einzige singuläre Punkt auf dem Konvergenzkreis von (1*), und zwar ein fastisolierter singulärer Punkt für diese Reihe. Die Anwendung des Satzes A auf die Reihe (1*) ergibt

$$\overline{\lim_{r \to \infty}}\, r^{-1} N_a(r) = 1. \tag{55}$$

Die Reihe (2*) hat, gemäß (54), in der offenen Kreisscheibe $|z|<(1-\varepsilon)^{-1}$ dieselben Singularitäten wie die Reihe (2). Daher ist, nach Voraussetzung (β), $z=1$ ein isolierbarer singulärer Punkt der Reihe (2*). Es folgt jedoch aus der Definition eines isolierbaren singulären Punktes (vgl. Nr. 69), daß eine Funktion, die einen solchen singulären Punkt hat, entweder nicht eindeutig ist oder mindestens keinen einfach zusammenhängenden Existenzbereich besitzt. Die Anwendung des Satzes B auf die Reihe (2*) ergibt (da die Behauptung nicht zutrifft, kann die Voraussetzung nicht erfüllt sein)

$$\overline{\lim_{r\to\infty}}\, r^{-1} N_b(r) > 0. \tag{56}$$

Nun ist folgendes zu beachten: Wenn für ein bestimmtes n sowohl a_n^* in der Anzahl $N_a(r)$ wie b_n^* in der Anzahl $N_b(r)$ mitgezählt wurde, so wurde $a_n^* b_n^*$ in $N_{ab}(r)$ mitgezählt. Folglich ist

$$N_a(r)+N_b(r) \leqq [r]+1+N_{ab}(r).$$

Hieraus folgt (die in Betracht kommende elementare Rechenregel wurde in Nr. 5 unter (9) explizite aufgestellt), daß

$$\overline{\lim_{r\to\infty}}\, r^{-1} N_a(r) + \overline{\lim_{r\to\infty}}\, r^{-1} N_b(r) \leqq 1 + \overline{\lim_{r\to\infty}}\, N_{ab}(r). \tag{57}$$

Die Kombination von (55), (56) und (57) ergibt

$$\overline{\lim_{r\to\infty}}\, r^{-1} N_{ab}(r) > 0. \tag{58}$$

Es gibt also (dies drückt nur einen Teil von (58) aus) unendlich viele n', so daß

$$a_{n'}^* b_{n'}^* \neq 0$$

ist. Dann gilt aber, gemäß den Definitionen (53) und (54), sofort

$$|a_{n'}^* b_{n'}^*| = |a_{n'} b_{n'}| \geqq (1-\varepsilon)^{n'} (1-\varepsilon)^{n'}.$$

Es gibt also — jetzt greife man auf (52) zurück — unendlich viele n', so daß

$$|a_{n'} b_{n'}| \varrho^{n'} = |a_{n'} b_{n'}| \frac{1}{(1-\varepsilon)^{2n'}} \geqq 1$$

ist. Dies setzt die Divergenz der Reihe (51), also die Behauptung des Satzes C', und somit den Satz C in Evidenz.

Bemerkungen und Zusätze.

90. Zunächst sei ein Beispiel angegeben, daß zur Erläuterung aller drei Sätze A, B und C geeignet ist[29].

[29] Das hier zu besprechende Beispiel habe ich schon in meiner unter [7] zitierten Arbeit angegeben; ein analoges, aber anders konstruiertes Beispiel behandelt Boller a. a. O.[10], S. 36 – 38.

Ich behaupte zunächst, daß die aus der analytischen Fortsetzung der beiden Potenzreihen

$$\sum_{n=2}^{\infty} z^n e^{n(-1+\cos\log\log n)\log\log n} = f(z), \tag{59}$$

$$\sum_{n=2}^{\infty} z^n e^{n(-1+\sin\log\log n)\log\log n} = g(z) \tag{60}$$

entspringenden Funktionen $f(z)$ *und* $g(z)$ *in der von* 1 *bis* $+\infty$ *entlang der positiven reellen Achse aufgeschlitzten Ebene regulär sind.* D. h., daß sie in allen Punkten der Ebene regulär sind, mit eventueller Ausnahme der Punkte z der reellen Achse, für welche $z \geqq 1$ gilt; es sind alle Logarithmen mit der reellen Bestimmung zu nehmen.

Es genügt wohl, die ausgesprochene Behauptung nur für die Reihe (59) darzutun, und hierzu genügt es sicher, auf Grund eines bekannten Satzes von E. Lindelöf[30], darzutun, daß die Funktion

$$F(z) = e^{z[-1+\cos(\log\log z)]\log\log z} \tag{61}$$

in der rechten Halbebene vom Minimaltypus der Ordnung 1 ist. Wir dürfen und werden uns darauf beschränken, die Funktion (61) nur in dem durch die beiden Ungleichungen

$$\Re z > 0, \quad |z| > 1 \tag{62}$$

abgegrenzten Teilgebiete der rechten Halbebene zu untersuchen. Wir betrachten dabei einen innerhalb (62) regulären Zweig von $\log\log z$; er ist eindeutigerweise dadurch definiert, daß für positives, 1 übersteigendes z der Wert von $\log z$ positiv und der von $\log\log z$ reell zu nehmen ist.

Die zu beweisende Behauptung wird ausführlicher durch die Formel (13) ausgedrückt. Zum Beweise setze man mit reellen x, y, φ, U, V

$$\begin{aligned} z &= x+iy = |z|e^{i\varphi}, \\ \log\log z &= \log(\log|z|+i\varphi) \\ &= \frac{1}{2}\log[(\log|z|)^2+\varphi^2]+i\operatorname{arctg}\frac{\varphi}{\log|z|} \\ &= U + iV ; \end{aligned}$$

es ist innerhalb (62)

$$x>0, \quad -\frac{\pi}{2}<\varphi<\frac{\pi}{2}, \quad -\frac{\pi}{2}<V<\frac{\pi}{2}$$

und für $|z|\to\infty$

$$U \sim \log\log|z|, \quad V = O\left(\frac{1}{\log|z|}\right).$$

Man schreibe

$$\begin{aligned} F(z) &= e^{\psi(z)}, \\ \psi(z) &= z[-1+\cos(\log\log z)]\log\log z. \end{aligned}$$

[30] E. Lindelöf, *Le calcul des résidus* (Paris, 1905); vgl. S. 109.

Es ist

$$\Re\psi(z) = (xU - yV)(-1 + \cos U \operatorname{ch} V) + (xV + yU)\sin U \operatorname{sh} V$$
$$= xU(-1 + \cos U) + O\left(\frac{|z| \log\log|z|}{\log|z|}\right).$$

Hieraus folgt, da das erste Glied für genügend großes $|z|$ nie positiv ist, die durch die Formel (13) dargestellte Behauptung, und somit die Fortsetzbarkeit von (59) in dem behaupteten Umfang.

Es ist hinzuzufügen, daß $z = 1$ *ein singulärer Punkt von* (59) ist. Denn der Konvergenzradius von (59) ist 1, wie man leicht durch die Betrachtung einer großen ganzen Zahl k und desjenigen entsprechenden ganzen n findet, für welches

$$\log\log n \leqq 2k\pi < \log\log(n+1)$$

gilt: Die n-te Wurzel des Koeffizienten von z^n in der Reihe (59) ist nämlich nahezu 1 für ein derartiges n.

Schließlich bemerke man, daß diejenigen Koeffizienten, für welche

$$\cos(\log\log n) < 0 \tag{63}$$

ist, kleiner sind, als die entsprechenden Koeffizienten der Reihe

$$\sum_{n=2}^{\infty}\left(\frac{z}{\log n}\right)^n,$$

welche ersichtlicherweise eine ganze Funktion darstellt. Wenn die der Bedingung (63) genügenden Glieder der Reihe (59) herausgeworfen (durch Nullen ersetzt) werden, so entsteht aus (59) eine neue Reihe, welche wir die „gereinigte" Reihe (59) nennen wollen. Durch diese „Reinigung" wird insofern nichts geändert, als die im Endlichen gelegenen Singularitäten dieselben bleiben.

91. Von dem Verlauf der Funktionen, die durch die Reihen (59), (60) und durch die „gereinigte" Reihe (59) definiert sind, wissen wir bisher dies: Für alle ist der Punkt $z = 1$ singulär, die Punkte auf der reellen Achse rechts von $z = 1$ sind vielleicht singulär, vielleicht regulär, in allen übrigen Punkten der z-Ebene bleiben die Funktionen bei geradliniger Fortsetzung vom Anfangspunkt $z = 0$ aus sicher regulär. Alle drei betrachteten Reihen haben nur einen singulären Punkt auf dem Konvergenzkreis, den Punkt $z = 1$, und dieser Punkt ist fastisoliert.

Gemäß Satz A muß die obere Dichte der Koeffizienten der „gereinigten" Reihe (59) Eins sein. Dies ist leicht direkt einzusehen, nämlich folgendermaßen: Es sei n so beschaffen, daß n der Bedingung (63) genügt, $n+1$ hingegen nicht mehr genügt, und es sei n' die kleinste auf n folgende ganze Zahl, welche (63) wieder genügt. Dann ist ersichtlicherweise

$$\log\log n' - \log\log n > \pi, \qquad n' > n^a,$$

wenn abkürzungsweise $e^\pi = a$ gesetzt wird; ferner ist die Anzahl der von Null verschiedenen Koeffizienten bis und mit demjenigen von $z^{n'}$ nicht kleiner als $n' - n - 1$; und schließlich ist

$$\frac{n' - n - 1}{n'} > 1 - \frac{n-1}{n^a},$$

welche Zahl, in Übereinstimmung mit Satz A, für $n \to \infty$ gegen 1 strebt. Dieselbe Rechnung zeigt weiter, daß man unter der Voraussetzung des Satzes A nicht behaupten kann, daß die *Dichte* der Koeffizienten 1 ist: Sie zeigt nämlich, daß die *untere Dichte* der nicht verschwindenden Koeffizienten der „gereinigten" Reihe (59) *Null* ist.

Gemäß Satz B folgt aus dem letzterwähnten Umstande, daß die durch die „gereinigte" Reihe (59) definierte Funktion eindeutig und ihr Existenzbereich einfach zusammenhängend ist. Wir schließen hieraus, daß für diese Funktion die von $z = 1$ aus entlang der reellen Achse ins $+\infty$ laufende Halbgerade von beiden Seiten her eine *singuläre Linie* sein muß. (Das gleiche Verhalten von *beiden* Seiten her ist auch aus der Realität der Koeffizienten ersichtlich.) Das Gleiche gilt offenbar für die ursprüngliche, nicht-gereinigte Reihe (59) und für (60). So hat Satz B zur Feststellung der analytischen Natur der durch diese Reihen definierten Funktionen wesentlich beigetragen. Übrigens leistet im gegenwärtigen Falle der speziellere Ostrowskische Satz (Hilfssatz II) ebensoviel wie Satz B.

Zum Satz C ist dies zu bemerken: Die Hadamardsche Komposition, die aus den Reihen (1) und (2) die Reihe (3) erzeugt, läßt aus (59) und (60) die Reihe

$$\sum_{n=2}^{\infty} z^n \, e^{n(-2+\cos\log\log n + \sin\log\log n)\log\log n}$$

entstehen, die ersichtlicherweise stets konvergiert, eine ganze Funktion darstellt, überhaupt keine Singularitäten hat, auch das Produkt $1 \cdot 1 = 1$ aus den beiden fastisolierten Punkten der Reihen (59) und (60) nicht zum singulären Punkte hat. Dieses Beispiel erweist die schon in Nr. 70 aufgestellte Behauptung, die die Genauigkeit des Satzes C in das richtige Licht setzt: *Das Produkt von zwei fastisolierten singulären Punkten braucht nicht singulär zu sein*[31].

[31] In der Neuauflage des klassischen Werkes von J. Hadamard, *La série de Taylor* (2. éd. unter Mitarbeit von S. Mandelbrojt, Paris, 1926), wird S. 101, Zeilen 11—13, eine Frage aufgeworfen (sie wird in dem unter [7] zitierten Werk von Mandelbrojt, S. 39, Zeilen 17—19 wiederholt), die auf die Frage hinausläuft, ob der folgende Satz wahr sei: *Wenn eine Reihe nur einen singulären Punkt auf ihrem Konvergenzkreise besitzt, so ist für jede Reihe mit der komplementären Exponentenfolge der Konvergenzkreis natürliche*

92. Der Satz C steht nicht vereinzelt da, man kann ihm mehrere analoge Sätze zur Seite stellen. Um den Satz C in seiner richtigen Umgebung zu zeigen, werde ich ihn (in etwas verallgemeinerter Form) als die letzte in eine Reihe von vier parallelen Aussagen einordnen, die zu dem folgenden Satz zusammengefaßt werden:

SATZ VIII. *Es sei α der einzige auf dem Konvergenzkreis der Reihe*

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots = f(z) \tag{1}$$

gelegene singuläre Punkt, und es sei β einer derjenigen singulären Punkte, die auf dem Konvergenzkreis der Reihe

$$b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots = g(z) \tag{2}$$

liegen. Man darf behaupten, daß der Punkt $\alpha\beta$ ein singulärer Punkt, u. zw. ein auf dem Konvergenzkreise gelegener singulärer Punkt der Reihe

$$a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots + a_n b_n z^n + \cdots = h(z) \tag{3}$$

ist, falls eine der folgenden vier Bedingungen zutrifft:

(I) *α ist ein Pol.* (β kann ein singulärer Punkt von beliebiger Natur sein.)

(II) *α ist eine nichtkritische isolierte singuläre Stelle von $f(z)$ und β ist auf dem Konvergenzkreis von* (2) *halbisoliert.*

(III) *α ist eine algebraisch-logarithmische Stelle von $f(z)$ und β ist auf dem Konvergenzkreis von* (2) *isoliert.*

(IV) *α ist ein fastisolierter singulärer Punkt der Reihenentwicklung* (1) *und β ist ein isolierbarer, auf dem Konvergenzkreis isolierter singulärer Punkt von* (2).

Ergänzend sei hinzugefügt: Ein auf dem Konvergenzkreis gelegener singulärer Punkt β der Reihe (2) heißt *auf dem Konvergenzkreis isoliert*, wenn es auf diesem Kreise einen Bogen mit dem *Mittelpunkte* β gibt, dessen sämtliche Punkte, mit Ausnahme von β, für (2) regulär sind; β heißt *auf dem Konvergenzkreise halbisoliert*, wenn es auf diesem Kreise einen Bogen mit dem *Endpunkt* β gibt, dessen sämtliche Punkte, mit Ausnahme von β, für (2) regulär sind. Es heißt α eine *algebraisch-logarithmische Stelle* von $f(z)$, wenn $f(z)$ in einer Umgebung von α sich als eine Summe von endlich vielen Gliedern von der Form

$$(z-\alpha)^a [\log(z-\alpha)]^p \varphi(z)$$

Grenze. Die Frage ist zu *verneinen*. Gegenbeispiel: Man „reinige" beide Reihen (59), (60), indem man aus (59) die Glieder mit $\cos \log\log n < 0.8$ und aus (60) die mit $\sin \log\log n < 0.8$ hinauswirft. Beide „gereinigte" Reihen haben nur einen singulären Punkt auf der Konvergenzgrenze, nämlich den Punkt 1 (sie sind in der Tat regulär außerhalb der singulären Halbgeraden von 1 bis $+\infty$), und sie sind exponentenfremd, d. h. ihre Hadamardsche Komponierte verschwindet identisch.

darstellen läßt, wobei a eine beliebige komplexe, p eine nichtnegative ganze Zahl ist, und die Funktion $\varphi(z)$ in der besagten Umgebung von α regulär bleibt.

In allen sukzessiven Übergängen von (I) zu (II), (III) und (IV) ändern sich die Bedingungen in systematischer Weise ab: die Bedingung für α wird weiter, die für β enger (mit einer Ausnahme in bezug auf α: der Übergang von (II) zu (III)). Die Zusammenstellung eröffnet einen Ausblick auf eine allgemeinere Aufgabe: Zu einem gegebenen Singularitätstyp von α den „komplementären" Singularitätstyp für β zu finden, womit ein möglichst weiter Typ für β gemeint ist, welcher die Behauptung „$\alpha\beta$ singulär für $h(z)$" zuläßt. Ein Rückblick auf den Beweis von C lehrt, daß die Untersuchung der den betreffenden Singularitätstypen zukommenden „Lückenbedingungen" für die gestellte Aufgabe nützlich sein kann. Schon die bisher bekannten Methoden gestatten übrigens, die Aufgabe etwas über das in Satz VIII Formulierte hinaus zu fördern.

Die Beweise für (I) und (III) darf ich hier übergehen[32], die für (II) und (IV) werde ich skizzieren.

Betrachten wir die Bedingung (II) und nehmen wir $\alpha = 1$ an. Dann läßt sich a_n auf die in Satz III angegebene Weise in zwei Summanden zerlegen, u. zw. hat die dabei auftretende ganze Funktion $F(z)$, gemäß dem Zusatz zu Satz III, den Nullpunkt zum Indikatordiagramm. Nehmen wir, ohne wirklichen Verlust an Allgemeinheit, $a_n^* = 0$ an. Dann ist

$$\begin{aligned} h(e^{-z}) &= \sum_0^\infty F(n)\, b_n\, e^{-nz} = F(-D) \sum_0^\infty b_n\, e^{-nz} \\ &= F(-D)\, g(e^{-z}). \end{aligned} \tag{64}$$

Aus dem singulären Punkt β, der auf dem Konvergenzkreis von (2) halbisoliert ist, entsteht ein zugänglicher singulärer Punkt von $g(e^{-z})$, und dieser wird, gemäß Hilfssatz III, durch die Funktionaloperation $F(-D)$ nicht zerstört.

Der in Bedingung (IV) erwähnte Fall ist zwar etwas allgemeiner als der in Satz C behandelte, er läßt sich aber darauf leicht zurückführen. Man zerlegt, mit der Methode der Nr. 72 (die von Nr. 73 kommt nicht in Frage) $g(z)$ in zwei Summanden

$$g(z) = \Psi(z) + g^*(z),$$

[32] (I) ist in meiner Note, Comptes Rendus, Paris **184** (1927), S. 502—504 bewiesen (vgl. das Lemma); es handelt sich dabei um denjenigen Spezialfall der bald aufzustellenden Formel (64), in welchem $F(z)$ ein Polynom ist. Für (III) vgl. R. Jungen, a. a. O. [10], S. 293, Théorème 5. Der Zusammenhang mit dem vorangehenden Théorème 4 auf S. 292 erläutert ebenfalls die eben gemachte Bemerkung über Lückenbedingungen. Vgl. hierzu noch Faber, a. a. O. [3], S. 297, und meine in Fußnote [10] unter k) angegebene Arbeit, S. 246—247.

so daß $g^*(z)$ im Punkt $z = \beta$ regulär ist und $\Psi(z)$ den Punkt $z = \beta$ als einzigen singulären Punkt auf ihrem Konvergenzkreise hat[33]. Die Anwendung des Hadamardschen Multiplikationssatzes auf $f(z)$ und $g^*(z)$ und die des Satzes C auf $f(z)$ und $\Psi(z)$ ergibt das Gewünschte.

93. Wir haben bisher den Fall betrachtet, in welchem α auf dem Konvergenzkreis von (1) und β auf dem von (2) gelegen ist. Der allgemeinere Fall, in welchem die singulären Punkte α und β erst im weiteren Verlaufe der analytischen Fortsetzung angetroffen werden, läßt sich, im wesentlichen, auf den bisher untersuchten speziellen zurückführen. Da das Prinzip dieser Zurückführung wohl bekannt ist[34], will ich mich bei seiner Erläuterung auf einen besonders durchsichtigen, einfach formulierbaren Spezialfall beschränken.

Die Formulierung macht Gebrauch von dem geläufigen Begriff des Mittag Lefflerschen Hauptsternes[35]. Als Hauptstern der Reihe (1) wird, wie üblich, die Gesamtheit derjenigen Punkte der z-Ebene bezeichnet, in welche die durch (1) definierte Funktion von dem Punkte $z = 0$ aus geradlinig fortgesetzt werden kann. Der Hauptstern ist ein offenes Gebiet. Als *Ecke* des Hauptsternes wird ein solcher Randpunkt α bezeichnet, der mit dem Punkt $z = 0$ durch eine gerade Strecke verbunden werden kann, welche, abgesehen natürlich vom Endpunkte α, ganz im Hauptstern verläuft. Eckpunkte müssen, andere Randpunkte des Hauptsterns können singuläre Punkte des im Hauptstern definierten Zweiges von $f(z)$ sein.

SATZ IX. *Es sei von den beiden Potenzreihen* (1) *und* (2) *vorausgesetzt, daß auf dem Rande des Hauptsternes nur abzählbar viele singuläre Punkte liegen, die keinen Häufungspunkt im Endlichen besitzen, und zwar sollen alle singulären Punkte in den Ecken des Hauptsternes liegen, in den übrigen Randpunkten soll sich die im Hauptstern definierte Fortsetzung regulär verhalten. Die Ecken des Hauptsternes von* (1) *seien mit* $\alpha_1, \alpha_2, \alpha_3, \cdots$ *bezeichnet, die von* (2) *mit* $\beta_1, \beta_2, \beta_3, \cdots$.

Dann haben alle am Rande des Hauptsternes der Hadamardschen Komponierten (3) *gelegenen singulären Punkte die Form* $\gamma = \alpha_k \beta_l$, *und zwar ist* γ *sicherlich ein singulärer Punkt, wenn* **nur ein** *Paar von Indizes* k, l *existiert, so daß* $\gamma = \alpha_k \beta_l$ *ist.*

Zur Erläuterung dieser Aussage sei bemerkt: Der Fall, daß nur endlich viele $\alpha_\varkappa$ oder β_λ vorhanden sind, wird zugelassen. In den von den Ecken verschiedenen Randpunkten des Hauptsterns ist bei *beiderseitiger* Annähe-

[33] Dieselbe Zerlegung für denselben Zweck wird ausführlicher dargestellt durch Jungen, a. a. O. [10], S. 293.

[34] Vgl. E. Borel, a. a. O. [4], ferner, insbesondere für die Behandlung der Verzweigungspunkte, G. Faber, a. a. O. [3] und [11].

[35] Vgl. z. B. L. Bieberbach, a. a. O. [3], S. 292.

rung Regularität vorausgesetzt. Der Punkt $z = \infty$ wurde weder für die Reihen (1) und (2) noch für (3) als singulärer Punkt oder als Randpunkt des Hauptsternes mitgerechnet.

Beim Beweis des Satzes IX dürfen wir annehmen, daß

$$(65) \qquad |\alpha_1| \leqq |\alpha_2| \leqq |\alpha_3| \leqq \cdots, \qquad |\beta_1| \leqq |\beta_2| \leqq |\beta_3| \leqq \cdots.$$

Wir wollen unter „Funktion $f(z)$" stets nur den im Hauptstern der Reihe (1) definierten Zweig verstehen, und ähnlich für $g(z)$. Wir zeichnen ein bestimmtes Paar $\alpha_k \beta_l$ aus, und wir wollen nach der Methode der Nr. 72 (die verfeinerte Methode in Nr. 73 kommt nicht in Frage) die Singularitäten von $f(z)$ und $g(z)$ passend zerlegen.

Man wähle zu diesem Zwecke zwei positive Zahlen a und b, so daß

$$(66) \qquad a\,|\beta_1| > |\alpha_k \beta_l|, \qquad |\alpha_1|\,b > |\alpha_k \beta_l|,$$

ferner so, daß auf dem Kreis $|z| = a$ keine Singularität von $f(z)$ und auf $|z| = b$ keine von $g(z)$ liegt. Es seien K, L durch

$$(67) \qquad |\alpha_K| < a < |\alpha_{K+1}|, \qquad |\beta_L| < b < |\beta_{L+1}|$$

bestimmt, wenn es unendlich viele $\alpha_\varkappa$ und β_λ gibt. (Wenn es nur endlich viele $\alpha_\varkappa$ gibt, sei α_K das letzte $\alpha_\varkappa$, die von $z = 0$ entfernteste Ecke des Hauptsterns, und es sei a von vornherein so gewählt, daß $|\alpha_K| < a$; ähnlich für die β_λ.)

Man schneide den Halbstrahl, der vom Punkte $z = 0$ aus durch $\alpha_\varkappa$ gezogen ist, mit dem Kreis $|z| = a$ und bezeichne den Schnittpunkt mit $\alpha'_\varkappa$; anders gesagt, es ist

$$(68) \qquad |\alpha'_1| = |\alpha'_2| = \cdots = |\alpha'_K| = a,$$

und die Quotienten

$$\frac{\alpha'_1}{\alpha_1}, \frac{\alpha'_2}{\alpha_2}, \cdots, \frac{\alpha'_K}{\alpha_K}$$

sind positive Zahlen größer als 1. Man entferne aus der offenen Kreisfläche $|z| < a$ zuerst diejenigen Punkte, die auf den K geradlinigen Strecken von α_1 bis α'_1, von α_2 bis α'_2, $\cdots$, von α_K bis α'_K liegen, ferner diejenigen, die in den K abgeschlossenen Kreisflächen vom Radius ε liegen, deren Mittelpunkte $\alpha_1, \alpha_2, \cdots, \alpha_K$ sind. Bei passender Wahl von einem genügend kleinen ε bleibt von der Kreisfläche $|z| < a$ ein zusammenhängendes Gebiet $\mathfrak{G}$ übrig. Wenn z in $\mathfrak{G}$ liegt, so ist

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)\,dw}{w - z};$$

das Integral ist entlang der Umrandung von $\mathfrak{G}$ im positiven Sinne erstreckt. Die Integrationslinie wird durch die Punkte $\alpha_1', \alpha_2', \cdots, \alpha_K'$ in $2K$ Teile zerlegt. Es soll die Gesamtheit der K auf dem Kreise $|z| = a$ liegenden Bögen mit $\mathfrak{S}^*$ bezeichnet werden; es bleiben noch K „Schlingen" $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_K$ übrig: es geht $\mathfrak{S}_\varkappa$ von $\alpha_\varkappa'$ aus geradlinig auf $\alpha_\varkappa$ zu, umfährt $\alpha_\varkappa$ im negativen Sinne entlang eines Kreises vom Radius ε und kehrt geradlinig nach $\alpha_\varkappa'$ zurück. Diese Zerlegung der Integrationslinie ergibt

$$(69)\qquad \begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\mathfrak{S}^*}\frac{f(w)\,dw}{w-z}+\sum_{\varkappa=1}^{K}\frac{1}{2\pi i}\int_{\mathfrak{S}_\varkappa}\frac{f(w)\,dw}{w-z} \\ &= f^*(z) \qquad +\sum_{\varkappa=1}^{K} f_\varkappa(z) \qquad ; \end{aligned}$$

es wurde $f^*(z)$ mit dem Integral über $\mathfrak{S}^*$, $f_\varkappa(z)$ mit dem über $\mathfrak{S}_\varkappa$ identifiziert, $\varkappa = 1, 2, \cdots, K$. Ersichtlicherweise ist $f^*(z)$ im Kreise $|z| < a$ regulär, und $f_\varkappa(z)$ ist, wie die Deformation der Schlinge $\mathfrak{S}_\varkappa$ zeigt, regulär (übrigens auch eindeutig) außerhalb der geradlinigen Strecke $\alpha_\varkappa, \alpha_\varkappa'$; es bleibt $f_\varkappa(z)$ sogar in allen inwendigen Punkten dieser Strecke regulär, u. zw. beim Heranrücken von beiden Seiten her, aber der Endpunkt $\alpha_\varkappa$ muß und der Endpunkt $\alpha_\varkappa'$ kann für $f_\varkappa(z)$ singulär sein.

Ähnlich wie $f(z)$ in (69), sei auch $g(z)$ zerlegt:

$$(70)\qquad g(z) = g^*(z)+\sum_{\lambda=1}^{L} g_\lambda(z).$$

$g^*(z)$ ist innerhalb des Kreises $|z| = b$, $g_\lambda(z)$ außerhalb einer geraden Strecke regulär, die in $z = \beta_\lambda$ beginnt, radial von $z = 0$ wegläuft, die Länge $b - |\beta_\lambda|$ hat und in einem Punkte endet, den wir mit β_λ' bezeichnen wollen; es ist $|\beta_\lambda'| = b$.

94. Man bezeichne die Hadamardsche Komponierte (3) der beiden Reihen $f(z)$ und $g(z)$ mit $[f \cdot g]$. Mit dieser Abkürzung erhalten wir aus (69) und (70)

$$(71)\qquad [f\cdot g] = \sum_{\varkappa=1}^{K}\sum_{\lambda=1}^{L}[f_\varkappa\cdot g_\lambda]+\sum_{\lambda=1}^{L}[f^*\cdot g_\lambda]+\sum_{\varkappa=1}^{K}[f_\varkappa\cdot g^*]+[f^*\cdot g^*].$$

Der Hadamardsche Multiplikationssatz (man denke den Beweis in dem vorliegenden Falle nochmals durch) zeigt, daß $[f_\varkappa \cdot g_\lambda]$ regulär bleibt außerhalb des Halbstrahles, der von $\alpha_\varkappa\beta_\lambda$ ausgeht und über $\alpha_\varkappa'\beta_\lambda'$ nach ∞ läuft; ferner bleibt $[f_\varkappa \cdot g_\lambda]$ sogar in allen Punkten dieses Halbstrahls regulär, u. zw. bei beiderseitiger Annäherung, abgesehen (vielleicht) von den folgenden 4 Punkten: $\alpha_\varkappa\beta_\lambda$, $\alpha_\varkappa'\beta_\lambda$, $\alpha_\varkappa\beta_\lambda'$, $\alpha_\varkappa'\beta_\lambda'$[36].

[36] Es ist, nebenbei bemerkt, $[f_\varkappa \cdot g_\lambda]$ außerhalb der geraden Strecke $\alpha_\varkappa\beta_\lambda, \alpha_\varkappa'\beta_\lambda'$ eindeutig, wie es am bequemsten aus Satz II folgt.

Der Satz C zeigt, daß $\alpha_\varkappa \beta_\lambda$ für $[f_\varkappa \cdot g_\lambda]$ *singulär* ist. In der Tat, $\alpha_\varkappa$ liegt als einziger singulärer Punkt auf dem Konvergenzkreis der Maclaurinschen Reihe von $f_\varkappa(z)$ und β_λ liegt ähnlicherweise auf dem von $g_\lambda(z)$. Es ist $\alpha_\varkappa$ für $f_\varkappa(z)$ und β_λ für $g_\lambda(z)$ ein isolierter Punkt, im Sinne der in Nr. 69 gegebenen Definition, also sowohl fastisoliert wie isolierbar, sodaß die Voraussetzungen des Satzes C reichlich erfüllt sind.

Man bezeichne die kleinere (nicht größere) der beiden positiven Zahlen $a|\beta_1|$ und $|\alpha_1|b$ mit c. Gemäß dieser Definition folgt aus (68) und (65)

$$|\alpha'_\varkappa \beta_\lambda| = a|\beta_\lambda| \geqq a|\beta_1| \geqq c, \quad |\alpha_\varkappa \beta'_\lambda| = |\alpha_\varkappa| b \geqq |\alpha_1| b \geqq c.$$

Wir können zusammenfassend dies sagen: Der Hauptstern von $[f_\varkappa \cdot g_\lambda]$ hat eine einzige Ecke $\alpha_\varkappa \beta_\lambda$, und diese Ecke ist der einzige auf dem Rande des Hauptsternes von $[f_\varkappa \cdot g_\lambda]$ gelegene singuläre Punkt, der im Kreise

$$|z| < c \tag{72}$$

enthalten sein kann.

Nun ist wegen (65) und (66)

$$a|\beta_\lambda| \geqq a|\beta_1| \geqq c, \quad |\alpha_\varkappa| b \geqq |\alpha_1| b \geqq c, \quad ab > |\alpha_1| b \geqq c,$$

so daß kraft (eines trivialen Falls) des Hadamardschen Multiplikationssatzes alle Funktionen

$$[f^* \cdot g_\lambda], \quad [f_\varkappa \cdot g^*], \quad [f^* \cdot g^*]$$

innerhalb des Kreises (72) regulär sind.

Gemäß Formel (71) kann also im Kreise (72) der Hauptstern von $[f \cdot g]$ keine anderen Ecken und $[f \cdot g]$ selbst keine anderen singulären Stellen haben als die Punkte

$$\alpha_\varkappa \beta_\lambda \quad (\varkappa = 1, 2, \cdots, K; \ \lambda = 1, 2, \cdots, L). \tag{73}$$

Nicht alle Punkte (73) müssen im Kreis (72) liegen, wohl aber der Punkt $\alpha_k \beta_l$, wegen (66). Jeder Punkt (73) ist für einen Summanden der Doppelsumme rechts unter (71) wirklich singulär, wie wir auf Grund des Satzes C eingesehen haben; liegt er in (72) und ist er *nur für einen* der besagten Summanden singulär, so ist er singulär für $[f \cdot g]$, w. z. b. w.

Nachtrag zum Beweis des Satzes A[37].

95. Eine Mitteilung von Herrn Vladimir Bernstein[38] führte mich zu einer wesentlich kürzeren und zugleich weiter führenden Beweisanordnung des Satzes A, die ich im folgenden mitteilen möchte.

[37] Hinzugefügt während der Korrektur, den 10. August 1933.

[38] Herr Bernstein machte mich darauf aufmerksam, daß die im vorangehenden bewiesenen, von mir in 1927 gefundenen und in meiner unter [7] zitierten Note implizite mitangekündigten Sätze V und VI zu den Theoremen II und I einer neuerdings erschienenen Note von Miss Mary L. Cartwright in Beziehung stehen. Meine Sätze können zwar in diesen Theoremen

Ich beginne mit einer neuen Definition, die sich den in Nr. 69 gegebenen Definitionen anschließt. Ich betrachte wieder die dort erwähnte Potenzreihe (4) und einen singulären Punkt α auf dem Konvergenzkreis von (4). Die Verbindungsgerade von α und z_0 wird durch den Punkt α in zwei Halbstrahlen zerschnitten; denjenigen Halbstrahl, der nicht durch z_0 hindurchgeht, nenne ich $\mathfrak{h}$. Mit diesen Bezeichnungen lautet nun meine Definition so: Der singuläre Punkt α heißt für die Reihe (4) *gut zugänglich*, wenn es einen abgeschlossenen Winkelraum $\mathfrak{W}$ gibt, dessen Spitze α, dessen Winkelhalbierende $\mathfrak{h}$ und dessen Öffnung $< \pi$ ist, so beschaffen, daß die durch die Reihe definierte Funktion sich von z_0 aus geradlinig fortsetzen läßt in alle solche Punkte einer genügend kleinen Umgebung von α, die nicht in $\mathfrak{W}$ liegen.

Vergleicht man diese Definition mit den in Nr. 69 gegebenen, so sieht man, daß der Begriff „gut zugänglich" enger ist, als „zugänglich", jedoch umfassender als „fastisoliert". Im Besitze des neuen Begriffes kann man den Satz A folgendermaßen erweitern:

Satz A*. *Wenn auf dem Konvergenzkreise einer Potenzreihe nur ein einziger singulärer Punkt liegt, und dieser singuläre Punkt für die Potenzreihe gut zugänglich ist, so ist die obere Dichte der Koeffizienten 1.*

Dieser Satz ist sehr genau; ersetzt man darin „gut zugänglich" durch „zugänglich" (im Sinne der Nr. 69), so entsteht eine falsche Aussage[39]. Ich will Satz A* in den folgenden Nrn. 96—98 beweisen. Aus Satz A* und dem unveränderten Satz B folgt durch die Schlüsse der vorangehenden Nrn. 88—89 eine Erweiterung des Satzes C (und auch des Satzes VIII), die ich nicht ausführlich formulieren, nur in abgekürzter Form aussprechen will: *Das Produkt eines gut zugänglichen Punktes und eines isolierbaren Punktes ist notwendigerweise singulär.*

96. Der Beweis des allgemeineren Satzes A* ist zunächst darin einfacher als der Beweis von A, daß die in Nr. 72—73 vorgenommene Zerlegung der Singularitäten dazu nicht nötig ist. Die in Nr. 74—76 besprochene Interpolation der Koeffizienten wird hingegen mit einer gewissen Abänderung

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dem Wortlaut nach nicht enthalten sein, denn die in den letzteren erwähnte Ausnahmemenge der r-Werte braucht nur ein kleines Maß zu haben, konnte also wohl alle ganzzahligen Werte von r enthalten. Meine Sätze können aber tatsächlich leicht aus derjenigen Modifikation der Theoreme von Miss Cartwright hergeleitet werden, die Herr Bernstein in einer demnächst in den Annali di Matematica erscheinenden Note beweisen wird.

[39] Die Koeffizienten der Entwicklung der Funktion

$$\sum_{n=1}^{\infty} \left(\frac{4z+4z^2+z^6}{9}\right)^{n!}$$

im Einheitskreis haben die obere Dichte 5/6, und der einzige auf dem Einheitskreis gelegene singuläre Punkt, $z=1$, ist zugänglich.

wieder gebraucht. Man setze voraus, daß die Reihe (7) auf ihrem Konvergenzkreise nur einen singulären Punkt besitzt, nämlich den Punkt $z=1$, und daß dieser singuläre Punkt für (7) gut zugänglich ist. Man wähle als Integrationsweg in (9) und auch in (10) eine einfache geschlossene Kurve, die den Punkt $w=0$ im Innern enthält, durch den Punkt $-e^{\delta}$ hindurchgeht ($\delta>0$, δ genügend klein, die Wahl von δ wird noch präzisiert) und nur solche Punkte im Innern enthält und nur durch solche hindurchgeht, in denen die durch (7) definierte Funktion $\Phi(w)$ regulär bleibt. Durch die Variablenvertauschung $w=e^{-z}$ geht (10) in (11) über; das Integral (11) ist entlang eines Kurvenbogens erstreckt, dessen beide Endpunkte $-\delta-i\pi$ und $-\delta+i\pi$ sind. Bei der Abbildung $w=e^{-z}$ gehen die beiden Schenkel des Winkelraumes $\mathfrak{W}$ in zwei Kurven über, die im Punkte $z=0$ miteinander einen durch die negative reelle Achse halbierten Winkel bilden, dessen Öffnung kleiner als π ist. Man kann also — hier wird benutzt, daß $z=1$ für (7) gut zugänglich und der einzige singuläre Punkt auf dem Konvergenzkreise ist — das positive δ so klein wählen, daß sowohl auf den beiden abgeschlossenen geradlinigen Strecken von 0 zu $-\delta-i\pi$ und von 0 zu $-\delta+i\pi$, wie auch auf dem von diesen beiden Strecken rechts gelegenen Teil des Streifens $-\pi\leqq\Im z\leqq\pi$ die Funktion $\Phi(e^{-z})$ regulär bleibt, abgesehen natürlich von dem einzigen, offenbar singulären Punkt $z=0$. Bei dieser Wahl von δ kann man die Integrationslinie von (11) den beiden gleichen Seiten des gleichschenkligen Dreiecks $(-\delta-i\pi,\ 0,\ -\delta+i\pi)$ so nahe bringen, wie man will, und die Überlegung der Nr. 24 a), angewandt auf (11), ergibt, daß $F(Z)$ eine ganze Funktion vom Exponentialtypus ist, deren konjugiertes Diagramm in dem Dreieck $(-\delta-i\pi,\ 0,\ -\delta+i\pi)$ enthalten ist. Andererseits ist, auf Grund der Nr. 43—44, die in $z=0$ liegende Spitze dieses Dreiecks, die in der Abbildung $w=e^{-z}$ dem singulären Punkt $w=1$ von $\Phi(w)$ entspricht, sicherlich ein Punkt (u. zw. ein extremer Punkt) des konjugierten Diagramms von $F(Z)$.

Das Gesagte beweist die eine Hälfte des folgenden Satzes III, nämlich die Notwendigkeit der darin auszusprechenden Bedingung; es empfiehlt sich die Bezeichnungen etwas zu ändern, c_n durch a_n und δ/π durch $\operatorname{tg}\alpha$ zu ersetzen.

SATZ III*. *Die notwendige und hinreichende Bedingung dafür, daß die Potenzreihe*

$$f(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n+\cdots$$

auf ihrem Konvergenzkreise nur einen singulären Punkt habe, nämlich den Punkt $z=1$, und daß dieser Punkt ein gut zugänglicher Punkt für $f(z)$ sei, ist die Möglichkeit eine ganze Funktion $F(z)$ vom Exponentialtypus und eine positive Zahl α $\left(0<\alpha<\frac{\pi}{2}\right)$ zu finden, so daß

$$a_n = F(n) \text{ für } n = 0, 1, 2, \cdots$$

und

$$h(\varphi) = \overline{\lim_{r\to\infty}}\, r^{-1} \log |F(re^{i\varphi})| = 0 \text{ für } -\alpha \leqq \varphi \leqq \alpha \tag{74}$$

sei.

Daß die ausgesprochene Bedingung hinreicht, ergibt sich ohne weiteres aus Nr. 43—44, wird aber im folgenden gar nicht gebraucht. Man vergleiche den Satz III* mit dem komplizierteren III und insbesondere (74) mit (13).

97. Zum Beweis des Satzes A* haben wir die durch Satz III* eingeführte Funktion $F(z)$ zu untersuchen. Um diese Untersuchung übersichtlich zu gestalten, empfiehlt sich eine geläufige funktionentheoretische Schlußweise[40] auf die folgende allgemeine Fassung zu bringen:

HILFSSATZ V. *Es sei $\mathfrak{M}$ ein in der z-Ebene gelegenes zusammenhängendes offenes Gebiet, $\mathfrak{N}$ und $\mathfrak{O}$ abgeschlossene Teile von $\mathfrak{M}$.*

Es sei die analytische Funktion $F(z)$ regulär und eindeutig in $\mathfrak{M}$ und erfülle daselbst die Ungleichung

$$|F(z)| \leqq M. \tag{75}$$

Ferner sei N die Anzahl der mit Multiplizität gerechneten Nullstellen von $F(z)$ in $\mathfrak{N}$.

Dann gilt in $\mathfrak{O}$ die obere Abschätzung

$$|F(z)| \leqq Me^{-\chi N}$$

wobei die positive Zahl χ von $F(z)$, M und N unabhängig, nur von $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ abhängig ist, $\chi = \chi(\mathfrak{M}, \mathfrak{N}, \mathfrak{O})$. Es hängt χ sogar nur von der Form und der Konfiguration der Punktmengen $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ ab, ist aber von deren Lage und Größe in folgendem Sinne unabhängig: Wenn eine Ähnlichkeitstransformation der komplexen Zahlenebene $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ bzw. in $\mathfrak{M}^*$, $\mathfrak{N}^*$, $\mathfrak{O}^*$ überführt, so ist $\chi(\mathfrak{M}, \mathfrak{N}, \mathfrak{O}) = \chi(\mathfrak{M}^*, \mathfrak{N}^*, \mathfrak{O}^*)$.

Ich gebe hier der Vollständigkeit halber einen Beweis; er wird in mehreren Schritten erbracht.

Erster Spezialfall. Es seien

$$\mathfrak{M}, \qquad \mathfrak{N}, \qquad \mathfrak{O}$$

konzentrische Kreisflächen, nämlich

$$|z| < 1, \qquad |z| \leqq \varrho, \qquad |z| \leqq \sigma$$

[40] Sie wird z. B. auch von Miss Cartwright a. a. O. [38] gebraucht; sie kommt in dem bekannten Ansatz von F. und R. Nevanlinna zum genauesten Ausdruck, vgl. Acta Societatis Scientiarum Fennicae 50 (1922) Nr. 5.

bzw., wobei $0<\varrho<1$, $0<\sigma<1$. Es seien $a_1, a_2, \cdots, a_N$ die Nullstellen von $F(z)$ in $\mathfrak{N}$, mehrfache mit der richtigen Multiplizität gesetzt. Dann ist die Funktion

$$F(z)\prod_{n=1}^{N}\frac{1-\bar{a}_n z}{z-a_n}$$

in $\mathfrak{M}$ (im Einheitskreise) regulär und ihr Betrag hat daselbst wieder die obere Schranke M. Daher ist

$$|F(z)| \leqq M\prod_{n=1}^{N}\left|\frac{z-a_n}{1-\bar{a}_n z}\right|. \tag{76}$$

Man beachte, daß (wie geläufig und leicht ersichtlich)

$$\left|\frac{z-a}{1-\bar{a}\,z}\right| \leqq \frac{|a|+|z|}{1+|a||z|} \quad \text{wenn} \quad |a|<1, \quad |z|<1,$$

man berücksichtige ferner, daß a_n in $\mathfrak{N}$ liegt, und nehme schließlich an, daß z in $\mathfrak{O}$ liegt; dann ergibt (76), daß

$$|F(z)| \leqq M\left(\frac{\varrho+\sigma}{1+\varrho\sigma}\right)^N = Me^{-\chi N}$$

in $\mathfrak{O}$, wobei

$$\chi = \log\frac{1+\varrho\sigma}{\varrho+\sigma} \tag{77}$$

gesetzt wurde. In diesem Spezialfall ist also Hilfssatz V bewiesen.

Der Beweis ist im Falle $\varrho=\sigma=0$ überflüssig, er bleibt in den beiden Fällen $0<\varrho<1$, $\sigma=0$ und $\varrho=0$, $0<\sigma<1$ bestehen und liefert im ersteren Falle eine wichtige Folgerung der Jensenschen Ungleichung und im letzteren Falle, für $N=1$, das Schwarzsche Lemma. Hilfssatz V ist also als eine Verallgemeinerung des Schwarzschen Lemma anzusehen.

Zweiter Spezialfall. Es sei $\mathfrak{M}$ (wie vorher) der offene Einheitskreis, $\mathfrak{N}$ und $\mathfrak{O}$ beliebige abgeschlossene Teilmengen davon. Der größte Abstand vom Nullpunkt heiße ϱ für $\mathfrak{N}$ und σ für $\mathfrak{O}$; es ist $\varrho<1$, $\sigma<1$. Die abgeschlossenen Kreisflächen $|z|\leqq\varrho$, $|z|\leqq\sigma$ sollen bzw. $\mathfrak{N}'$ und $\mathfrak{O}'$ heißen, und die Anzahl der Nullstellen von $F(z)$ in $\mathfrak{N}'$ heiße N'; offenbar ist $\mathfrak{N}$ in $\mathfrak{N}'$, $\mathfrak{O}$ in $\mathfrak{O}'$ enthalten und es ist $N\leqq N'$. Durch Anwendung des im vorangehenden Spezialfalle gewonnenen Resultates findet man, mit dem Wert (77) für χ, daß

$$|F(z)| \leqq Me^{-\chi N'} \leqq Me^{-\chi N}$$

in $\mathfrak{O}'$ und um so mehr in $\mathfrak{O}$.

Ein allgemeiner Fall. Es sei $\mathfrak{M}$ einfach zusammenhängend, $\mathfrak{N}$, $\mathfrak{O}$ beliebige abgeschlossene Teilmengen von $\mathfrak{M}$. Man bilde das in der z-Ebene gelegene

$\mathfrak{M}$ eineindeutig und konform auf den offenen Einheitskreis der w-Ebene, der im folgenden $\mathfrak{M}^*$ genannt wird, ab. Diese Abbildung sei durch die Funktion $z = z(w)$ vermittelt und führe $\mathfrak{N}$ in $\mathfrak{N}^*$, $\mathfrak{O}$ in $\mathfrak{O}^*$ über. Setzt man

$$F(z(w)) = F^*(w),$$

so ist ersichtlicherweise M eine obere Schranke von $|F^*(w)|$ in $\mathfrak{M}^*$, N die Anzahl der Nullstellen von $F^*(w)$ in $\mathfrak{N}^*$, und die in $\mathfrak{O}^*$ nach dem vorangehenden Spezialfall bestehende Abschätzung

$$Me^{-\chi N} \geqq |F^*(w)| = |F(z(w))|$$

ergibt, wenn z in $\mathfrak{O}$ liegt, das gewünschte Resultat.

Der allgemeinste Fall, in welchem $\mathfrak{M}$ einen beliebigen Zusammenhang hat, ist auf den Fall des einfachen Zusammenhanges leicht zurückzuführen (Bedeckung von $\mathfrak{N}$ und $\mathfrak{O}$ durch endlich viele Kreise, Zerschneidung von $\mathfrak{M}$); ich behandle ihn hier nicht, da er im folgenden nicht benötigt wird. Man sieht aus dem Vorangehenden, daß χ nicht nur bei Abbildungen der Form $az+b = w$ invariant bleibt, wie im Hilfssatz V behauptet wurde, sondern bei beliebigen eineindeutigen konformen Abbildungen.

98. Ich wende den Hilfssatz V auf die im Satz III* erwähnte ganze Funktion $F(z)$ an. Es sei β eine feste positive Zahl und

$\mathfrak{M} = \mathfrak{M}_r$ der Kreissektor, der durch die Ungleichungen $-\alpha < \arc z < \alpha$, $0 < |z| < r(1+\beta)$ begrenzt wird;

$\mathfrak{N} = \mathfrak{N}_r$ die geradlinige Strecke auf der reellen Achse zwischen $z = r\beta$ und $z = r$;

$\mathfrak{O} = \mathfrak{O}_r$ der Punkt $z = r$;

$M(r(1+\beta))$ das Maximum von $|F(z)|$ in $\mathfrak{M}_r$;

$n(r)$ die Anzahl der Nullstellen von $F(z)$ in $\mathfrak{N}_r$.

Zwar hängen $\mathfrak{M}_r$, $\mathfrak{N}_r$, $\mathfrak{O}_r$ von r ab, erfahren aber bei wachsendem r nur eine Dehnung, die ihre Konfiguration unverändert läßt; daher bleibt auch das in Hilfssatz erwähnte χ unverändert, und es ist

$$|F(r)| \leqq M(r(1+\beta))\, e^{-\chi n(r)}, \tag{78}$$

χ unabhängig von r. Man bezeichne mit $N(r)$ die Anzahl der verschwindenden unter den $[r]$ Funktionswerten

$$F(1),\ F(2),\ \cdots,\ F([r]);$$

es handelt sich um $N(r)$ in Satz A*. Es ist offenbar

$$N(r) \leqq \beta r + n(r). \tag{79}$$

Die Kombination von (78) und (79) ergibt (auf Grund der Regel (9) in Nr. 5),

$$\overline{\lim_{r\to\infty}}\, r^{-1}\log|F(r)| + \chi \varlimsup_{r\to\infty} r^{-1}N(r)$$
$$\leqq (1+\beta)\overline{\lim_{r\to\infty}}\,(r(1+\beta))^{-1}\log M(r(1+\beta)) + \chi\beta.$$

Beachtet man (74) und die Bedeutung von $M(r)$ (und auch das in Nr. 26 Gesagte), so erhält man

(80) $$\chi \varlimsup_{r\to\infty} r^{-1}N(r) \leqq \chi\beta.$$

Da χ positiv ist und β beliebig klein gewählt werden kann, muß der links in (80) stehende Grenzwert verschwinden. Die obere Dichte der Koeffizienten $a_n = F(n)$ ist aber

$$1 - \varlimsup_{r\to\infty} r^{-1}N(r).$$

Damit ist Satz A* bewiesen.

99. Die im Beweise des Satzes A* nicht benutzten Teile des Beweises vom Satz A sind insofern nicht ganz überflüssig gemacht, daß sie bei analogen Untersuchungen verwendet werden können, z. B. bei der Begründung vom folgenden[41]

SATZ X. *Wenn auf der Konvergenzgrenze einer Potenzreihe nur ein einziger singulärer Punkt liegt, und dieser singuläre Punkt ein fastisolierter Punkt* ***endlicher Ordnung*** *ist, so ist die Dichte der Koeffizienten 1.*

Ich nenne eine fastisolierte Stelle α von *endlicher Ordnung*, wenn, die Bezeichnungen der Nr. 69 beibehalten, die aufgeschnittene Kreisfläche $\mathfrak{K}^*$ so klein gewählt werden kann, daß darin

$$|f(z)|\, e^{-\left(\frac{1}{\delta}\right)^a}$$

beschränkt bleibt; hierin ist $f(z)$ die durch (4) definierte Funktion, δ der Abstand des Punktes z von der Strecke α, α', und a eine positive Konstante, deren Existenz die Hauptsache an der Definition ist. Auf den Beweis von Satz X gehe ich hier nicht ein. Aus Satz X folgt leicht eine weitgehende Verallgemeinerung des Falles (III) von Satz VIII, nämlich daß das *Produkt eines fastisolierten Punktes endlicher Ordnung und eines auf dem Konvergenzkreise isolierten Punktes sicher singulär ist.*

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[41] Ich habe diesen Satz in 1929 im Anschluß an die in [20] erwähnte Mitteilung von Herrn Rolf Nevanlinna gefunden; der Beweis wird durch die in [38] erwähnte Untersuchung von Herrn V. Bernstein etwas vereinfacht.

PRINCETON UNIVERSITY.

ON A METHOD OF SUMMATION OF FOURIER SERIES.*

(SECOND PAPER.)

By A. F. Moursund.

1. **Introduction.** This paper is a continuation of an earlier paper in which we give theorems concerning the effectiveness of the N_q method for the summation of Fourier series, conjugate Fourier series, and the first derived series of Fourier and conjugate Fourier series.[1] We are concerned here primarily with the question of summation of the pth derived series of Fourier series.

In a paper published in 1926 M. Jacob shows that the pth derived series of the Fourier series of a Lebesgue integrable function is summable by the Cesàro method, $(C, p+\delta)$, $\delta>0$, to the value of the pth generalized derivative (in the sense of de la Vallée-Poussin[2]) of the function wherever that derivative exists.[3] He shows by an example that the restriction $\delta>0$ is necessary.

We set up here based upon the N_q method a summation method, the N_{q_p} method, which contains as a special case the Riesz equivalent of the Cesàro method, $(C, p+\delta)$, $0<\delta\leqq 1$; and obtain for the N_{q_p} method a result analogous to the result of M. Jacob, given above, for the Cesàro method. We obtain an analogous result by superimposing upon the Riesz method, (R, p), the N_{q_0} method.

We give, also, theorems concerning the N_{q_1} summability of the Fourier series and the N_{q_0} summability of the conjugate Fourier series.

2. **The N_{q_p} and the N_{q_0}—(R, p) methods.** Let $F(n)$ be defined for all values of $n>0$, and let it be desired to consider a "generalized limit" of $F(n)$ as $n\to\infty$.

The N_{q_p} method consists in forming the N_{q_p} transform or mean

$$N_{q_p}F(n)\equiv\int_0^1 q_p(t)\,F(nt)\,dt$$

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* Received December 24, 1932. — This paper is a part of a dissertation presented for the degree of Doctor of Philosophy at Brown University.

[1] Annals of Mathematics (2), vol. 33 (1932), pp. 773–784.

[2] Ch. J. de la Vallée-Poussin, *Sur l'approximation des fonctions d'une variable réelle*, Acad. Bruxelles, Bulletin 1908.

[3] M. Jacob, *Über die Cesàro'sche Summierbarkeit des Fourier'schen Integrales*, Bulletin International, Classe des Sciences Mathématiques et Naturelles, Cracow Academy of Science (A), 1926, pp. 41–74.

and then considering the limit

$$\lim_{n\to\infty} N_{q_p} F(n),$$

where

(1) $q_p(t) \geqq 0$ for $0 \leqq t \leqq 1$;

(2) $\int_0^1 q_p(t)\,dt = 1$;

(3) $\frac{d^\beta}{dt^\beta} q_p(t)$ exists and is absolutely continuous for $0 \leqq t \leqq 1$, $(\beta = 1, 2, \cdots, p-1)$;

(4) $\frac{d^\beta}{dt^\beta} q_p(t) = 0$ for $t = 1$, $(\beta = 0, 1, 2, \cdots, p-1)$;

(5) $\frac{d^p}{dt^p} q_p(t)$ exists for $0 < t < 1$;

(6) $(-1)^p \frac{d^p}{dt^p} q_p(t) \geqq 0$ and monotone increasing;

(7) $\lim_{\delta\to 0} \int_0^{1-\delta} \frac{d^p}{dt^p} q_p(t) \log \frac{1}{1-t}\,dt$ exists.

Similarly, the $N_{q_0}-(R, p)$ method consists in forming the $N_{q_0}-(R, p)$ transform

$$N_{q_0}-(R, p)\,F(n) \equiv \int_0^1 q_0(v)\,dv \int_0^1 p(1-t)^{p-1} F(ntv)\,dt$$

and then considering the limit

$$\lim_{n\to\infty} N_{q_0}-(R, p)\,F(n),$$

where the function $q_0(v)$ is an N_{q_p} kernel of order zero, i. e. satisfies conditions (6), (2), and (7), for $p = 0$.

It is essential for the application of the N_{q_p} and $N_{q_0}-(R, p)$ methods that $F(n)$ be defined for almost all values of $n > 0$; hence if this function is defined only for integral values of n, for example the nth partial sum of a series, it must be defined for the intermediary values by some method such as linear interpolation.

In the following lemmas we state, without proof, some general properties of the N_{q_p} method.

LEMMA 2.1. *The N_{q_p} method is regular.*[4]

LEMMA 2.2. *The kernel $q_p(t)$ is monotone decreasing, its derivatives of odd order less than p are negative and monotone increasing, its derivatives of*

[4] F. Nevanlinna, *Über die Summation der Fourier'schen Reihen und Integrale*, Översikt av Finska Vetenskaps-Societetens Förhandlingar, 64, A, No. 3, 14 pp. (1921–22).

even order less than p are positive and monotone decreasing, and there exists a constant A_p such that

$$\left|\frac{d^\beta}{dt^\beta} q_p(t)\right| < A_p \quad (\beta = 0, 1, 2, \cdots, p-1), \tag{1}$$

$$\int_0^1 \left|\frac{d^p}{dt^p} q_p(t)\right| dt < A_p. \tag{2}$$

LEMMA 2.3. *For $p = 0$ the N_{q_p} method is identical with the N_q method.*[5]

LEMMA 2.4. *If the kernel*

$$q_p(t) \equiv (p+\delta)(1-t)^{p+\delta-1} \qquad (0 < \delta \leqq 1),$$

then the N_{q_p} method reduces to the Riesz method, $(R, p+\delta)$, and is consequently equivalent to the Cesàro method, $(C, p+\delta)$.

In the two remaining lemmas we give properties of the $N_{q_0}-(R, p)$ method.

LEMMA 2.5. *The $N_{q_0}-(R, p)$ method is regular.*

LEMMA 2.6. *If $F(n)$ is such that the double integral*

$$\int_0^1 q_0(v)\, dv \int_0^1 r(1-t)^{r-1} |F(nt v)|\, dt$$

exists, and also

$$N_{q_0}-(R, r)\, F(n) \to L \quad as \quad n \to \infty;$$

then for $p > r$

$$N_{q_0}-(R, p)\, F(n) \to L \quad as \quad n \to \infty.$$

3. Notation and general preliminary lemmas. The following notation will be used throughout the paper. We consider only functions $f(x)$ which are periodic of period 2π and Lebesgue integrable $[f(x) \subset L]$. We set

$$a_n \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\cos ns\, ds,$$

$$b_n \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\sin ns\, ds.$$

Then $f(x)$ generates the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{3.1}$$

whose conjugate series is

$$\sum_{n=1}^{\infty}(-b_n \cos nx + a_n \sin nx). \tag{3.2}$$

[5] Loc.cit. [1], p. 773.

We designate by $S_n(x)$ and $\sigma_n(x)$ the nth partial sums of the Fourier and conjugate Fourier series, respectively, and by $S_n^{(p)}(x)$ the nth partial sum of the pth derived series of the Fourier series.

We set

$$\varphi(s) \equiv f(x+s)+f(x-s)-2f(x),$$

$$\varphi_1(s) = \int_0^s \varphi(t)\,dt;^{6} \tag{3.3}$$

$$\psi^*(s) \equiv f(x+s)-f(x-s). \tag{3.4}$$

In connection with the conjugate Fourier series we use the function

$$\tilde{f}(x) \equiv -\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi}\psi^*(s)\,\cot\frac{s}{2}\,ds, \tag{3.5}$$

which, for $f(x)\subset L$, exists almost everywhere;[7] and in connection with the pth derived series of the Fourier series the notion of a generalized derivative given by the definitions below.

DEFINITION 3.1. *If at a point x the function $f(x)$ satisfies an equation of the form*

$$\frac{f(x+s)+f(x-s)}{2} = \sum_{i=0}^{k} f^{(2i)}(x)\,\frac{s^{2i}}{(2i)!}+\omega(x,s)\,\frac{s^{2k}}{(2k)!},$$

where $\omega(x,s)\to 0$ as $s\to 0$, then $f^{(2k)}(x)$ is the $2k$th generalized derivative of $f(x)$.

DEFINITION 3.2. *If at a point x the function $f(x)$ satisfies an equation of the form*

$$\frac{f(x+s)-f(x-s)}{2} = \sum_{i=0}^{k} f^{(2i+1)}(x)\,\frac{s^{2i+1}}{(2i+1)!}+\omega(x,s)\,\frac{s^{2k+1}}{(2k+1)!},$$

where $\omega(x,s)\to 0$ as $s\to 0$, then $f^{(2k+1)}(x)$ is the $(2k+1)$th generalized derivative of $f(x)$.

It follows from the existence of the pth generalized derivative that the $(p-2)$th, $(p-4)$th, $\cdots$, also exist. If the ordinary derivative $\frac{d^p}{dx^p}f(x)$ exists it is identical with the generalized derivative $f^{(p)}(x)$.[8]

We say that:

DEFINITION 3.3. *A point at which the generalized derivative $f^{(p)}(x)$ exists is G_p regular with respect to $f(x)$.*

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[6] In our earlier paper, loc. cit. [1], we set $\Phi^*(s) = \int_0^s \varphi(t)\,dt$. Our present notation is in general use.

[7] A. Plessner, *Zur Theorie der konjugierten trigonometrischen Reihen*, Mitteilungen des Mathematischen Seminars der Universität Gießen, Heft 10 (1923).

[8] de la Vallée-Poussin, loc. cit. [2].

DEFINITION 3.4. *A summation method is G_p effective if it sums the pth derivedseries of the Fourier series of $f(x) \subset L$ to $f^{(p)}(x)$ at the G_p regular points.*

In order to simplify the expressions for $S_n^{(2k)}(x)$ and $S_n^{(2k+1)}(x)$, which we give in the next section, we set

$$u(k, \alpha, n) \equiv \left[\sum_{\beta=0}^{2k-2\alpha} \left\{ \frac{(2k-2\alpha)!}{\beta!\,(2k-2\alpha-\beta)!} \frac{d^\beta}{ds^\beta} \cot\frac{s}{2} \frac{d^{2k-2\alpha-\beta}}{ds^{2k-2\alpha-\beta}} \sin ns \right\} + \frac{d^{2k-2\alpha}}{ds^{2k-2\alpha}} \cot ns \right]_{s=\pi}; \tag{3.6}$$

and then

$$\begin{aligned} U_{2i}(2k, n) &\equiv -\sum_{\alpha=1}^{i} \frac{\pi^{2i-2\alpha}}{(2i-2\alpha+1)!}\, u(k, \alpha, n), \\ U_{2i+1}(2k+1, n) &\equiv -\sum_{\alpha=0}^{i} \frac{\pi^{2i-2\alpha}}{(2i-2\alpha+1)!}\, u(k, \alpha, n); \end{aligned} \tag{3.7}$$

and state the following lemmas.

LEMMA 3.1. *For n a positive integer*

$$\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}} = 1+2[\cos s+\cos 2s+\cdots+\cos ns];$$

$$\frac{d^{2k+1}}{ds^{2k+1}} \frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}} = 0 \qquad (s=0, \pi);$$

$$\frac{1}{\pi}\int_0^\pi \frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\, ds = 1.$$

LEMMA 3.2.

$$\frac{1}{\pi}\int_0^\pi \frac{s^{2i}}{(2i)!} \frac{d^{2k}}{ds^{2k}} \frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\, ds = \begin{cases} U_{2i}(2k, n) & ((i<k), \\ 1+U_{2k}(2k, n) & i=k). \end{cases}$$

LEMMA 3.3.

$$\frac{1}{\pi}\int_0^\pi \frac{s^{2i+1}}{(2i+1)!} \frac{d^{2k+1}}{ds^{2k+1}} \frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\, ds = \begin{cases} -U_{2i+1}(2k+1, n) & (i<k), \\ -1-U_{2k+1}(2k+1, n) & (i=k). \end{cases}$$

4. **The N_{q_p} and $N_{q_0}-(R, p)$ transforms.** From our earlier paper we have for $f(x) \subset L$ and periodic of period 2π:[9]

$$S_n(x) = \frac{a_0}{2} + \sum_1^n (a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s) \frac{\sin\left(n+\frac{1}{2}\right)(x-s)}{\sin\frac{x-s}{2}}\,ds \tag{4.1}$$

$$= f(x) + \frac{1}{2\pi}\int_0^{\pi} \varphi(s) \frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds,$$

and

$$\sigma_n(x) = \sum_1^n (-b_n \cos nx + a_n \sin nx)$$

$$= -\frac{1}{2\pi}\int_0^{\pi} \psi^*(s) \cot\frac{s}{2}(1-\cos ns)\,ds - \frac{1}{2\pi}\int_0^{\pi} \psi^*(s) \sin ns\,ds. \tag{4.2}$$

At a point where $\varphi_1(s) = o(s)$, as $s \to 0$, we obtain from (4.1) by integrating by parts the equation,

$$S_n(x) = f(x) + \frac{\varphi_1(\pi)}{2\pi}\cos \pi n$$

$$-\frac{1}{2\pi}\int_0^{\pi} \varphi_1(s) \frac{d}{ds} \frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds; \tag{4.3}$$

and then

$$N_{q_1} S_n(x) = f(x) + \frac{\varphi_1(\pi)}{2\pi}\int_0^1 q_1(t)\cos \pi n t\,dt$$

$$-\frac{1}{2\pi}\int_0^1 q_1(t)\,dt \int_0^{\pi} \varphi_1(s) \frac{d}{ds} \frac{\sin\left(nt+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds \tag{4.4}$$

$$\equiv f(x) + C_1 + C_2.$$

From (4.2) we have

$$N_{q_0} \sigma_n(x) = -\frac{1}{2\pi}\int_0^1 q_0(t)\,dt \int_0^{\pi} \psi^*(s) \cot\frac{s}{2}(1-\cos nts)\,ds$$

$$-\frac{1}{2\pi}\int_0^1 q_0(t)\,dt \int_0^{\pi} \psi^*(s) \sin nts\,ds \tag{4.5 a}$$

[9] Loc. cit. [1], pp. 776–777.

$$= -\frac{1}{2\pi}\int_{\frac{\delta}{n}}^{\pi} \psi^*(s)\cot\frac{s}{2}\,ds$$

$$-\frac{1}{2\pi}\int_0^1 q_0(t)\,dt\int_0^{\frac{\delta}{n}} \psi^*(s)\cot\frac{s}{2}(1-\cos nts)\,ds$$

$$+\frac{1}{2\pi}\int_0^1 q_0(t)\,dt\int_{\frac{\delta}{n}}^{\eta} \psi^*(s)\left(\cot\frac{s}{2}-\frac{2}{s}\right)\cos nts\,ds$$

(4.5 b)

$$+\frac{1}{\pi}\left[\int_0^1\int_{\frac{A}{n}}^{\eta} + \int_0^1 q_0(t)\,dt\int_{\frac{\delta}{n}}^{\frac{A}{n}} \frac{\psi^*(s)}{s}\cos nts\,ds\right]$$

$$+\frac{1}{2\pi}\int_0^1 q_0(t)\,dt\int_{\eta}^{\pi} \psi^*(s)\cot\frac{s}{2}\cos nts\,ds$$

$$-\frac{1}{2\pi}\int_0^1 q_0(t)\,dt\int_0^{\pi} \psi^*(s)\sin nts\,ds$$

$$\equiv -\frac{1}{2\pi}\int_{\frac{\delta}{n}}^{\pi} \psi^*(s)\cot\frac{s}{2}\,ds + D_1 + D_2 + D_3 + D_4 + D_5 + D_6.$$

It follows from (4.1) that

$$S_n^{(p)}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\frac{\partial^p}{\partial x^p}\frac{\sin\left(n+\frac{1}{2}\right)(x-s)}{\sin\frac{x-s}{2}}\,ds$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\frac{d^p}{d(x-s)^p}\frac{\sin\left(n+\frac{1}{2}\right)(x-s)}{\sin\frac{x-s}{2}}\,ds$$

(4.6 a)

$$= -\frac{1}{2\pi}\int_{x+\pi}^{x-\pi} f(x-s)\frac{d^p}{ds^p}\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-s)\frac{d^p}{ds^p}\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds$$

$$= \frac{1}{2\pi}\int_0^{\pi} f(x-s)\frac{d^p}{ds^p}\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds$$

(4.6 b)
$$+(-1)^{p+1}\int_{\pi}^{0} f(x+s)\frac{d^p}{ds^p}\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{(-1)^p f(x+s)+f(x-s)}{2}\,\frac{d^p}{ds^p}\,\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds.$$

At the G_{2k}, G_{2k+1} regular points we have then, respectively,

$$S_n^{(2k)}(x) = \frac{1}{\pi}\int_0^{\pi}\left[\sum_{i=0}^{k} f^{(2i)}(x)\frac{s^{2i}}{(2i)!}+\omega(x,s)\frac{s^{2k}}{(2k)!}\right]$$

(4.7)
$$\times\frac{d^{2k}}{ds^{2k}}\,\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds,$$

and

$$S_n^{(2k+1)}(x) = -\frac{1}{\pi}\int_0^{\pi}\left[\sum_{i=0}^{k} f^{(2i+1)}(x)\frac{s^{2i+1}}{(2i+1)!}+\omega(x,s)\frac{s^{2k+1}}{(2k+1)!}\right]$$

(4.8)
$$\times\frac{d^{2k+1}}{ds^{2k+1}}\,\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds;$$

which become, respectively, by means of Lemmas 3.1, 3.2, and 3.3:

$$S_n^{(2k)}(x) = f^{(2k)}(x)+\sum_{i=1}^{k} f^{(2i)}(x)\,U_{2i}(2k,n)$$

(4.9)
$$+\frac{1}{\pi}\int_0^{\pi}\omega(x,s)\frac{s^{2k}}{(2k)!}\,\frac{d^{2k}}{ds^{2k}}\,\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds,$$

and

$$S_n^{(2k+1)}(x) = f^{(2k+1)}(x)+\sum_{i=0}^{k} f^{(2i+1)}(x)\,U_{2i+1}(2k+1,n)$$

(4.10)
$$-\frac{1}{\pi}\int_0^{\pi}\omega(x,s)\frac{s^{2k+1}}{(2k+1)!}\,\frac{d^{2k+1}}{ds^{2k+1}}\,\frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds.$$

Hence

$$N_{q_p} S_n^{(2k)}(x) = f^{(2k)}(x) + \sum_{i=1}^{k} f^{(2i)}(x) \int_0^1 q_p(t)\, U_{2i}(2k, nt)\, dt$$

$$+ \frac{1}{\pi} \int_0^1 q_p(t)\, dt \int_0^\pi \omega(x, s) \frac{s^{2k}}{(2k)!} \frac{d^{2k}}{ds^{2k}} \frac{\sin\left(nt + \frac{1}{2}\right) s}{\sin \frac{s}{2}}\, ds \tag{4.11}$$

$$\equiv f^{(2k)}(x) + \sum_{i=1}^{k} f^{(2i)}(x) V_{2i} + W_{2k},$$

and

$$N_{q_p} S_n^{(2k+1)}(x)$$

$$= f^{(2k+1)}(x) + \sum_{i=0}^{k} f^{(2i+1)}(x) \int_0^1 q_p(t)\, U_{2i+1}(2k+1, nt)\, dt$$

$$- \frac{1}{\pi} \int_0^1 q_p(t)\, dt \int_0^\pi \omega(x, s) \frac{s^{2k+1}}{(2k+1)!} \frac{d^{2k+1}}{ds^{2k+1}} \frac{\sin\left(nt + \frac{1}{2}\right) s}{\sin \frac{s}{2}}\, ds \tag{4.12}$$

$$\equiv f^{(2k+1)}(x) + \sum_{i=1}^{k} f^{(2i+1)}(x)\, V_{2i+1} - W_{2k+1}.$$

Also

$$N_{q_0} - (R, 2k)\, S_n^{(2k)}(x)$$

$$= f^{(2k)}(x) + \sum_{i=1}^{k} f^{(2i)}(x) \int_0^1 q_0(v)\, dv \int_0^1 2k(1-t)^{2k-1}\, U_{2i}(2k, ntv)\, dt$$

$$+ \frac{1}{\pi} \int_0^1 q_0(v)\, dv \int_0^1 2k(1-t)^{2k-1}\, dt \int_0^\pi \omega(x, s)$$

$$\times \frac{s^{2k}}{(2k)!} \frac{d^{2k}}{ds^{2k}} \frac{\sin\left(ntv + \frac{1}{2}\right) s}{\sin \frac{s}{2}}\, ds \tag{4.13}$$

$$\equiv f^{(2k)}(x) + \sum_{i=1}^{k} f^{(2i)}(x)\, X_{2i} + Y_{2k},$$

and

$$N_{q_0} - (R, 2k+1)\, S_n^{(2k+1)}(x)$$

$$= f^{(2k+1)}(x) + \sum_{i=0}^{k} f^{(2i+1)}(x) \int_0^1 q_0(v)\, dv$$

$$\times \int_0^1 (2k+1)(1-t)^{2k}\, U_{2i+1}(2k+1, ntv)\, dt$$

$$- \frac{1}{\pi} \int_0^1 q_0(v)\, dv \int_0^1 (2k+1)(1-t)^{2k}\, dt \int_0^\pi \omega(x, s) \tag{4.14}$$

$$\times \frac{s^{2k+1}}{(2k+1)!} \frac{d^{2k+1}}{ds^{2k+1}} \frac{\sin\left(ntv + \frac{1}{2}\right) s}{\sin \frac{s}{2}}\, ds$$

$$\equiv f^{(2k+1)}(x) + \sum_{i=0}^{k} f^{(2i+1)}(x)\, X_{2i+1} - Y_{2k+1}.$$

5. **Preliminary lemmas for the N_{q_p} method theorems.** Throughout the proofs of our N_{q_p} method Theorems 9.1, 9.2, and 9.3 we use the following lemmas. For convenience we set

$$Q_p(h) = \int_{1-h}^{1} \left| t^p \frac{d^p}{dt^p} q_p(t) \right| dt, \quad h = \frac{\pi}{ns} \leqq 1.$$

We have then:

LEMMA 5.1. $$\left| \int_0^1 \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} nts \frac{d^p}{dt^p} q_p(t)\, dt \right| \leqq Q_p(h).$$

LEMMA 5.2. *$Q_p(h)$ is a continuous monotone increasing function of h, $Q_p(0) = 0$, and $Q_p(h) < A_p$.*

LEMMA 5.3. $$\int_0^1 \frac{Q_p(s)}{s}\, ds = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{Q_p(s)}{s}\, ds \text{ exists.}$$

Proofs of Lemmas 5.1, 5.2, and 5.3. As a consequence of condition (6) on $q_p(t)$ [Sec. 2] we have,

$$Q_p(h) = \int_{1-h}^{1} \left| t^p \frac{d^p}{dt^p} q_p(t) \right| dt = \int_{1-h}^{1} (-1)^p t^p \frac{d^p}{dt^p} q_p(t)\, dt.$$

To complete the proofs we have only to replace in the proofs of Lemmas 5.1, 5.2, and 5.3 of our earlier paper:[10]

$$q(t) \text{ by } (-1)^p t^p q_p^{(p)}(t),$$
$$Q(h) \text{ by } Q_p(h),$$

and

condition (3) by condition (7).

6. **Preliminary lemmas for Theorems 9.1 and 9.3.** In this section we give lemmas which we use, in addition to the lemmas of Sec. 5, in proving our Theorem 9.1 on the N_{q_1} summability of the Fourier series and our Theorem 9.3 on the N_{q_p} summability of the pth derived series of the Fourier series. Throughout the proofs of the lemmas we use the facts that:

$$\left| s^\beta \frac{d^\beta}{ds^\beta} \cot \frac{s}{2} \right| = O\left(\frac{1}{s}\right) \qquad (0 \leqq s \leqq \pi); \tag{6.1}$$

and

$$s^\beta \frac{d^\beta}{ds^\beta} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} nts = t^\beta \frac{d^\beta}{dt^\beta} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} nts. \tag{6.2}$$

In the proofs of Lemmas 6.2 and 6.3 sin may be replaced by cos.

LEMMA 6.1. *The functions*

$$\frac{d^\beta}{dt^\beta} [t^\beta q_p(t)], \quad \beta < p,$$

[10] Loc. cit. [1], pp. 778–779.

and

$$\frac{d^\beta}{dt^\beta} q_p(t) \frac{d^{p-\beta}}{dt^{p-\beta}} t^p \quad \beta < p,$$

are of bounded variation on (0,1).

LEMMA 6.2. *For* $\beta < p$,

$$\left| s^\beta \int_0^1 q_p(t) \frac{d^\beta}{ds^\beta} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} nts \right\} dt \right| = O\left(\frac{1}{ns}\right).$$

Proof. We have, using (6.2), integrating β times by parts, and then using condition (4) on $q_p(t)$ [Sec. 2]:

$$\left| s^\beta \int_0^1 q_p(t) \frac{d^\beta}{ds^\beta} \sin nts\, dt \right|$$

$$= \left| \int_0^1 \frac{d^\beta}{dt^\beta} [t^\beta q_p(t)] \sin nts\, dt \right| = O\left(\frac{1}{ns}\right)$$

as a result of Lemma 6.1.

LEMMA 6.3. *If* $\frac{\pi}{n} \leqq s \leqq \pi$, *then*

$$\left| s^p \int_0^1 q_p(t) \frac{d^p}{ds^p} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} nts \right\} dt \right| = O\left[\frac{1}{ns} + Q_p\left(\frac{\pi}{ns}\right)\right].$$

Proof. Proceeding as in the proof of Lemma 6.2, we have, by Lemmas 6.1 and 5.1,

$$\left| s^p \int_0^1 q_p(t) \frac{d^p}{ds^p} \sin nts\, dt \right|$$

$$= \left| \int_0^1 \frac{d^p}{dt^p} [t^p q_p(t)] \sin nts\, dt \right|$$

$$\leqq \sum_{\beta=0}^{p-1} \frac{p!}{\beta!\,(p-\beta)!} \left| \int_0^1 \frac{d^\beta}{dt^\beta} q_p(t) \frac{d^{p-\beta}}{dt^{p-\beta}} t^p \sin nts\, dt \right|$$

$$+ \left| \int_0^1 t^p \frac{d^p}{dt^p} q_p(t) \sin nts\, dt \right|$$

$$= O\left[\frac{1}{ns} + Q_p\left(\frac{\pi}{ns}\right)\right].$$

LEMMA 6.4. *For* $s = \pi$ *and* $\beta \leqq p$,

$$\int_0^1 q_p(t) \frac{d^\beta}{ds^\beta} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} nts \right\} dt = o(1).$$

Proof. The lemma follows from Lemmas 6.2, 6.3, and 5.2.

LEMMA 6.5. *If* $0 \leqq s \leqq \dfrac{\pi}{n}$ *and* $0 \leqq t \leqq 1$, *then*

$$\left| s^p \frac{d^p}{ds^p} \frac{\sin\left(nt+\frac{1}{2}\right)s}{\sin\frac{s}{2}} \right| = O(n).$$

Proof. We have, using (6.1),

$$\begin{aligned}
&\left| s^p \frac{d^p}{ds^p} \frac{\sin\left(nt+\frac{1}{2}\right)s}{\sin\frac{s}{2}} \right| \\
&\quad \leqq \sum_{\beta=0}^{p} \frac{p!}{\beta!\,(p-\beta)!} s^p \left| \frac{d^\beta}{ds^\beta} \cot\frac{s}{2} \right| \left| \frac{d^{p-\beta}}{ds^{p-\beta}} \sin nts \right| \\
&\qquad + s^p \left| \frac{d^p}{ds^p} \cos nts \right| \\
&\quad \leqq \sum_{\beta=0}^{p} \frac{p!}{\beta!\,(p-\beta)!} O(s^{p-\beta-1}) \cdot n^{p-\beta} + (ns)^p = O(n).
\end{aligned}$$

LEMMA 6.6. *For* $\dfrac{\pi}{n} \leqq s \leqq \pi$ *and* $0 \leqq t \leqq 1$,

$$\left| s^p \int_0^1 q_p(t) \frac{d^p}{ds^p} \frac{\sin\left(nt+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\, dt \right| = O\left[\frac{1}{ns^2} + \frac{1}{s} Q_p\left(\frac{\pi}{ns}\right)\right].$$

Proof. Proceeding as in the proof of Lemma 6.5, we have by (6.1) and Lemmas 6.2 and 6.3

$$\begin{aligned}
&\left| s^p \int_0^1 q_p(t) \frac{d^p}{ds^p} \frac{\sin\left(nt+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\, dt \right| \\
&\quad \leqq \sum_{\beta=1}^{p} \frac{p!}{\beta!\,(p-\beta)!} s^\beta \left| \frac{d^\beta}{ds^\beta} \cot\frac{s}{2} \right| \left| s^{p-\beta} \int_0^1 q_p(t) \frac{d^{p-\beta}}{ds^{p-\beta}} \sin nts\, dt \right| \\
&\qquad + \left| \cot\frac{s}{2} \right| \left| s^p \int_0^1 q_p(t) \frac{d^p}{ds^p} \sin nts\, dt \right| \\
&\qquad + \left| s^p \int_0^1 q_p(t) \frac{d^p}{ds^p} \cos nts\, dt \right| \\
&\quad = \sum_{\beta=1}^{p} \left[O\left(\frac{1}{s}\right) \cdot O\left(\frac{1}{ns}\right) \right] + O\left(\frac{1}{s}\right) \cdot O\left[\frac{1}{ns} + Q_p\left(\frac{\pi}{ns}\right)\right] \\
&\qquad + O\left[\frac{1}{ns} + Q_p\left(\frac{\pi}{ns}\right)\right] = O\left[\frac{1}{ns^2} + \frac{1}{s} Q_p\left(\frac{\pi}{ns}\right)\right].
\end{aligned}$$

Lemma 6.6 enables us to write

$$(6.3)\qquad \left| s^p \int_0^1 q_p(t) \frac{d^p}{ds^p} \frac{\sin\left(nt+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\, dt \right| \leqq K_p\left[\frac{1}{ns^2}+\frac{1}{s}Q_p\left(\frac{\pi}{ns}\right)\right], \qquad (K_p = \text{const.}).$$

LEMMA 6.7. *If* $\frac{\pi}{n} \leqq s \leqq \pi$, $0 \leqq t \leqq 1$, *and* $\beta < p$; *then*

$$\left| s^\beta \int_0^1 q_p(t) \frac{d^\beta}{ds^\beta} \frac{\sin\left(nt+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\, dt \right| = O\left(\frac{1}{ns^2}\right).$$

LEMMA 6.8. *If* $F(s) = o(1)$, *as* $s \to 0$, *and* $\beta \leqq p$, *then*

$$\int_0^\pi F(s)\, s^\beta\, ds \int_0^1 q_p(t) \frac{d^\beta}{ds^\beta} \frac{\sin\left(nt+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\, dt = o(1).$$

Proof. For the case $\beta = p$, we have

$$\left| \int_0^\pi F(s)\, s^p\, ds \int_0^1 q_p(t) \frac{d^p}{ds^p} \frac{\sin\left(nt+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\, dt \right|$$

$$\leqq \int_0^\pi |F(s)| \left| s^p \int_0^1 q_p(t) \frac{d^p}{ds^p} \frac{\sin\left(nt+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\, dt \right| ds$$

$$= \int_0^{\frac{\pi}{n}} \int_0^1 + \int_{\frac{\pi}{n}}^{\delta} \int_0^1 + \int_\delta^\pi \int_0^1 \equiv E_1 + E_2 + E_3;$$

then, by Lemma 6.5

$$|E_1| = o(1) \int_0^{\frac{\pi}{n}} O(n)\, ds = o(1);$$

by (6.3) and Lemma 5.3

$$|E_2| \leqq [o(1) \text{ as } \delta \to 0]\, K_p \int_{\frac{\pi}{n}}^{\delta} \left[\frac{1}{ns^2}+\frac{1}{s}Q_p\left(\frac{\pi}{ns}\right)\right] ds$$

$$\leqq [o(1) \text{ as } \delta \to 0]\, K_p \left\{\left[-\frac{1}{ns}\right]_{\frac{\pi}{n}}^{\delta} + \int_0^1 \frac{Q_p(s)}{s}\, ds\right\}$$

$$= o(1), \text{ as } \delta \to 0;$$

and by (6.3) and Lemma 5.2

$$|E_3| \leqq \int_\delta^\pi |F(s)|\, K_p\left[\frac{1}{ns^2} + \frac{1}{s}\, Q_p\left(\frac{\pi}{ns}\right)\right] ds$$

$$\leqq K_p\left[\frac{1}{n\delta^2} + \frac{1}{\delta}\, Q_p\left(\frac{\pi}{n\delta}\right)\right] = o(1) \text{ as } n \to \infty, \text{ for a fixed } \delta.$$

In proving the lemma for the case $\beta < p$ we proceed as in the proof for the case $\beta = p$, but use Lemma 6.7 instead of (6.3).

7. **Preliminary lemmas for Theorem 9.2.** In proving our Theorem 9.2 concerning the N_{q_0} summability of the conjugate Fourier series we will use, in addition to the lemmas of Section 5, the lemmas given in this section.

LEMMA 7.1. *The limit*

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_\varepsilon^\pi \frac{\psi^*(s)}{s}\, ds$$

exists wherever $\tilde{f}(x)$ *exists.*

Proof. Since the function, $\frac{2}{s} - \cot\frac{s}{2}$, is summable on $(0, \pi)$ we have wherever $\tilde{f}(x)$ exists,

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_\varepsilon^\pi \frac{\psi^*(s)}{s}\, ds$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi}\left[\int_\varepsilon^\pi \psi^*(s)\left(\frac{2}{s} - \cot\frac{s}{2}\right) ds + \int_\varepsilon^\pi \psi^*(s) \cot\frac{s}{2}\, ds\right]$$

$$= \frac{1}{2\pi} \int_0^\pi \psi^*(s)\left(\frac{2}{s} - \cot\frac{s}{2}\right) ds + \tilde{f}(x)$$

exists.

LEMMA 7.2. *If* $\beta > \alpha > 0$, *then*

$$\int_\alpha^\beta \frac{\psi^*(s)}{s}\, ds = o(1), \text{ as } \alpha, \beta \to 0.$$

Proof. The lemma follows from Lemma 7.1.

LEMMA 7.3. *If* $F(t) \subset L$ *on* $(0, 1)$, *then*

$$\frac{d}{ds} \int_0^1 F(t) \cos st\, dt = -\int_0^1 tF(t) \sin st\, dt.$$

LEMMA 7.4. *If* $F(t) \subset L$ *on* $(0,1)$, *then the function*

$$G(s) \equiv \int_0^1 F(t) \cos st\, dt = P_1(s) - P_2(s),$$

where $P_1(s)$ *and* $P_2(s)$ *are monotone non-increasing bounded functions for* $0 \leqq s \leqq S$.

Proof. The total variation of $G(s)$ on $(0, S)$ is, by Lemma 7.3,

$$\int_0^S \left| \frac{d}{ds} \int_0^1 F(t) \cos st \, dt \right| ds = \int_0^S \left| \int_0^1 tF(t) \sin st \, dt \right| ds$$
$$\leqq \int_0^S \int_0^1 |F(t)| \, dt \, ds \leqq S \int_0^1 |F(t)| \, dt,$$

hence is finite.

LEMMA 7.5. *For $\beta > \alpha > 0$, the integral*

$$\int_\alpha^\beta \frac{Q_0(s)}{s} ds = o(1), \text{ as } \alpha, \beta \to 0.$$

Proof. The lemma is an immediate consequence of Lemma 5.3.

8. **Preliminary lemmas for Theorem 9.4.** In proving our Theorem 9.4 on the $N_{q_0} - (R, p)$ summability of the pth derived series of the Fourier series we will use, in addition to lemmas previously given, the lemmas of this section. We set, for convenience,

$$\left| s^{\beta+1} \frac{d^\beta}{ds^\beta} \cot \frac{s}{2} \right| \leqq R_\beta \qquad (0 \leqq s \leqq \pi), \tag{8.1}$$

and

$$\left| \frac{d^\beta}{dt^\beta} [p(1-t)^{p-1} t^\beta] \right| \leqq F_\beta, \tag{8.2}$$

and use in the proofs of our lemmas, without further reference to it, the fact that $\omega(x, s) \to 0$, as $s \to 0$.

LEMMA 8.1. *For $s = \pi$ and $\beta \leqq p$,*

$$\int_0^1 q_0(v) \, dv \int_0^1 p(1-t)^{p-1} \frac{d^\beta}{dt^\beta} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} nsv \, dt = o(1).$$

Proof. For $\beta < p$, we have, using (6.2) and integrating β times by parts,

$$\int_0^1 q_0(v) \, dv \int_0^1 p(1-t)^{p-1} \frac{d^\beta}{ds^\beta} \sin nsv \, dt$$
$$= \frac{(-1)^\beta}{s^\beta} \int_0^1 q_0(v) \, dv \int_0^1 \frac{d^\beta}{dt^\beta} [p(1-t)^{p-1} t^\beta] \sin nsv \, dt$$
$$= o(1) \text{ as } ns \to \infty,$$

by the Riemann-Lebesgue theorem and Lemma 2.1. Similarly, we have

$$\int_0^1 q_0(v) \, dv \int_0^1 p(1-t)^{p-1} \frac{d^p}{ds^p} \sin nsv \, dt$$
$$= \frac{p!}{s^p} \int_0^1 q_0(v) \sin nsv \, dv$$
$$+ \frac{(-1)^p}{s^p} \int_0^1 q_0(v) \, dv \int_0^1 \frac{d^p}{dt^p} [p(1-t)^{p-1} t^p] \sin nsv \, dt$$
$$= o(1) \text{ as } ns \to \infty.$$

LEMMA 8.2.

$$\int_0^\pi \omega(x,s)\cot\frac{s}{2}\,ds\int_0^1 q_0(v)\sin nsv\,dv = o(1) \quad \text{as} \quad n\to\infty.$$

Proof. We set

$$\int_0^\pi\int_0^1 = \int_0^{\frac{\pi}{n}}\int_0^1 + \int_{\frac{\pi}{n}}^{\delta}\int_0^1 + \int_\delta^\pi\int_0^1 \equiv H_1+H_2+H_3.$$

Then

$$|H_1| \leqq R_0\int_0^{\frac{\pi}{n}} |\omega(x,s)|\,ds\int_0^1 q_0(v)\left|\frac{\sin nsv}{s}\right| dv = o(1);$$

by Lemmas 5.1 and 5.3

$$|H_2| \leqq R_0\int_{\frac{\pi}{n}}^{\delta} |\omega(x,s)|\,\frac{Q_0\left(\frac{\pi}{ns}\right)}{s}\,ds$$

$$\leqq R_0[o(1) \text{ as } \delta\to 0]\int_0^1 \frac{Q_0(s)}{s}\,ds = o(1), \text{ as } \delta\to 0;$$

and, interchanging the order of integration, by the Riemann-Lebesgue theorem and Lemma 2.1

$$|H_3| = \left|\int_0^1 q_0(v)\,dv\int_\delta^\pi \omega(x,s)\cot\frac{s}{2}\sin nsv\,ds\right|$$

$$= o(1) \text{ as } n\to\infty, \text{ for a fixed } \delta.$$

LEMMA 8.3. *For* $\beta \leqq p$,

$$\int_0^\pi \omega(x,s)\,s^{p-\beta}\,\frac{d^{p-\beta}}{ds^{p-\beta}}\cot\frac{s}{2}\,ds$$

$$\times\int_0^1 q_0(v)\,dv\int_0^1 \frac{d^\beta}{dt^\beta}\,[p(1-t)^{p-1}t^p]\sin ntsv\,dt = o(1).$$

Proof. Changing the order of integration, we have using (8.1) and (8.2):

$$\left|\int_0^\pi(\cdots)\,ds\int_0^1(\cdots)\,dv\int_0^1(\cdots)\,dt\right| = \left|\int_0^\pi(\cdots)\,ds\int_0^1(\cdots)\,dt\int_0^1(\cdots)\,dv\right|$$

$$\leqq \int_0^\pi |\omega(x,s)|\,R_{p-\beta}\,\frac{1}{s}\,ds\int_0^1 F_\beta\,dt\left|\int_0^1 q_0(v)\sin ntsv\,dv\right|$$

$$= R_{p-\beta}F_\beta\left[\int_0^{\frac{\pi}{n}}\int_0^1\int_0^1 + \int_{\frac{\pi}{n}}^{\delta}\int_0^{\frac{\pi}{ns}}\int_0^1 + \int_{\frac{\pi}{n}}^{\delta}\int_{\frac{\pi}{ns}}^1\int_0^1 + \int_\delta^\pi\int_0^\eta\int_0^1 + \int_\delta^\pi\int_\eta^1\int_0^1\right]$$

$$\equiv R_{p-\beta}F_\beta[L_1+L_2+L_3+L_4+L_5].$$

We have immediately

$$|L_1| \leqq o(1)\int_0^{\frac{\pi}{n}} ds\int_0^1 dt\int_0^1 q_0(v)\left|\frac{\sin nstv}{s}\right| dv = o(1);$$

$$|L_2| \leqq [o(1) \text{ as } \delta\to 0]\int_{\frac{\pi}{n}}^{\delta}\frac{ds}{s}\int_0^{\frac{\pi}{ns}} dt\int_0^1 q_0(v)\,dv$$

$$\leqq [o(1) \text{ as } \delta\to 0]\frac{\pi}{n}\left[-\frac{1}{s}\right]_{\frac{\pi}{n}}^{\delta} = o(1), \text{ as } \delta\to 0;$$

and by Lemma 5.1

$$|L_3| \leqq [o(1) \text{ as } \delta\to 0]\int_{\frac{\pi}{n}}^{\delta}\frac{1}{s}\,ds\int_{\frac{\pi}{ns}}^1 dt\int_{1-\frac{\pi}{nts}}^1 q_0(v)\,dv$$

$$= [o(1) \text{ as } \delta\to 0]\left\{\int_{\frac{\pi}{n}}^{\delta}\int_{\frac{\pi}{ns}}^1\int_{1-\frac{\pi}{nts}}^{1-\frac{\pi}{ns}} + \int_{\frac{\pi}{n}}^{\delta}\int_{\frac{\pi}{ns}}^1\int_{1-\frac{\pi}{ns}}^1 q_0(v)\,dv\right\}$$

$$= [o(1) \text{ as } \delta\to 0]\{M_1+M_2\} = o(1) \text{ as } \delta\to 0,$$

provided M_1 and M_2 are bounded. We have at once

$$|M_2| \leqq \int_{\frac{\pi}{n}}^{\delta}\frac{1}{s}\,Q_0\left(\frac{\pi}{ns}\right)ds \leqq \int_0^1\frac{Q_0(s)}{s}\,ds$$

and hence, by Lemma 5.3, M_2 is bounded; also,

$$M_1 = \int_{\frac{\pi}{n}}^{\delta}\frac{1}{s}\,ds\int_0^{1-\frac{\pi}{ns}} q_0(v)\,dv\int_{\frac{\pi}{ns}}^{\frac{\pi}{ns}\left(\frac{1}{1-v}\right)} dt$$

$$\leqq \int_{\frac{\pi}{n}}^{\delta}\frac{1}{s}\,ds\int_0^{1-\frac{\pi}{ns}} q_0(v)\,dv = \int_0^{1-\frac{\pi}{n\delta}} q_0(v)\log\frac{1}{1-v}\,dv$$

and hence, by condition (7) on $q_p(v)$ [Sec. 2], M_1 is bounded. Finally we have

$$L_4 \leqq \frac{\eta}{\delta}\int_{\delta}^{\pi}|\omega(x,s)|\,ds = o(1) \text{ as } \eta\to 0, \text{ for a fixed } \delta;$$

and by Lemmas 5.1 and 5.2

$$L_5 \leqq \frac{1}{\delta}\int_{\delta}^{\pi}|\omega(x,s)|\,ds\int_{\eta}^1 Q_0\left(\frac{\pi}{nts}\right)dt$$

$$\leqq Q_0\left(\frac{\pi}{n\delta\eta}\right)\left(\frac{1-\eta}{\delta}\right)\int_{\delta}^{\pi}|\omega(x,s)|\,ds = o(1) \text{ as } n\to\infty, \text{ for fixed } \delta, \eta.$$

The lemma follows. In an analogous, and even simpler, fashion we can prove

LEMMA 8.4.

$$\int_0^1 q_0(v)\,dv \int_0^\pi \omega(x,s)\,ds \int_0^1 \frac{d^p}{dt^p}\left[p(1-t)^{p-1}t^p\right]\cos ntsv\,dt = o(1).$$

In conformity with the notation of (4.13) and (4.14) we set

$$Y_p \equiv \frac{1}{2\pi}\int_0^1 q_0(v)\,dv \int_0^1 p(1-t)^{p-1}\,dt \times \int_0^\pi \omega(x,s)\frac{s^p}{p!}\frac{d^p}{ds^p}\frac{\sin\left(ntv+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,ds; \tag{8.3}$$

and have as our final lemma:

LEMMA 8.5. $Y_p = o(1)$.

Proof. By changing the order of integration, using Leibniz's formula for the pth derivative of a product, (6.2), and repeated integration by parts we have, the computations being similar to those in the proof of Lemma 8.1,

$$\begin{aligned}
Y_p &= \int_0^\pi (\cdots)\,ds \int_0^1 (\cdots)\,dv \int_0^1 (\cdots)\,dt \\
&= \frac{1}{2\pi}\sum_{\beta=0}^{p-1}\frac{(-1)^\beta}{\beta!\,(p-\beta)!}\int_0^\pi \omega(x,s)\,s^{p-\beta}\frac{d^{p-\beta}}{ds^{p-\beta}}\cot\frac{s}{2}\,ds \\
&\qquad\times\int_0^1 q_0(v)\,dv\int_0^1\frac{d^\beta}{dt^\beta}\left[p(1-t)^{p-1}t^p\right]\sin ntsv\,dt \\
&\quad+\frac{1}{2\pi}\int_0^\pi \omega(x,s)\cot\frac{s}{2}\,ds\int_0^1 q_0(v)\sin nsv\,dv \\
&\quad+\frac{(-1)^p}{2\pi\,p!}\int_0^\pi \omega(x,s)\cot\frac{s}{2}\,ds \\
&\qquad\times\int_0^1 q_0(v)\,dv\int_0^1\frac{d^p}{dt^p}\left[p(1-t)^{p-1}t^p\right]\sin ntsv\,dt \\
&\quad+\frac{1}{2\pi}\int_0^\pi \omega(x,s)\,ds\int_0^1 q_0(v)\cos nsv\,dv \\
&\quad+\frac{(-1)^p}{2\pi}\int_0^\pi \omega(x,s)\,ds\int_0^1 q_0(v)\,dv\int_0^1\frac{d^p}{dt^p}\left[p(1-t)^{p-1}t^p\right]\cos ntsv\,dt \\
&\equiv P_1+P_2+P_3+P_4+P_5.
\end{aligned}$$

Then, by Lemma 8.3, P_1 and $P_3 = o(1)$; by Lemma 8.2, $P_2 = o(1)$; also, changing the order of integration, by the Riemann-Lebesgue theorem and Lemma 2.1, $P_4 = o(1)$; and finally, by Lemma 8.4, $P_5 = o(1)$. Hence we get the desired result.

9. Theorems. In this section we state and prove our principal theorems.

THEOREM 9.1. *If the function $f(x) \subset L$ and is periodic of period 2π, then the N_{q_1} sum of its Fourier series is equal to $f(x)$ at each point x for which $\varphi_1(s) = o(s)$, as $s \to 0$.*

Proof. At a point x for which $\varphi_1(s) = o(s)$, as $s \to 0$, we have from 4.4, $C_1 = o(1)$ by the Riemann-Lebesgue theorem; and, interchanging the order of integration and using Lemma 6.8,

$$C_2 = -\frac{1}{2\pi}\int_0^\pi \frac{\varphi_1(s)}{s}\,s\,ds \int_0^1 q_1(t)\,\frac{d}{ds}\,\frac{\sin\left(nt+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\,dt = o(1).$$

We may choose $q_1(t) \equiv (1+\varepsilon)(1-t)^\varepsilon$ and thus obtain as a special case of our theorem a theorem equivalent to the usual theorem on summability $C, 1+\varepsilon)$.

THEOREM 9.2. *If $f(x)$ is bounded and measurable, and is periodic of period 2π, then the N_{q_0} sum of its conjugate Fourier series is $\tilde{f}(x)$ wherever $\tilde{f}(x)$ exists. The N_{q_0} sum is definitely infinite wherever $\tilde{f}(x)$ is definitely infinite.*

Proof. Since $f(x)$ is bounded, $\psi^*(s)$ is bounded; we set throughout this proof

$$|\psi^*(s)| \leqq M.$$

At a point x where $\tilde{f}(x)$ exists we have, from (4.5):

$$|D_1| = O\left[\frac{Mn}{\pi}\int_0^1 q_0(t)\,dt\int_0^{\frac{\delta}{n}} ds\right] = o(1) \text{ as } \delta \to 0;$$

$$|D_2| = O\left[\frac{M}{2\pi}\int_0^1 q_0(t)\,dt\int_{\frac{\delta}{n}}^{\eta} ds\right] = o(1) \text{ as } \eta \to 0;$$

by an interchange of the order of integration, and Lemmas 5.1 and 7.5

$$|D_3| = \frac{1}{\pi}\left|\int_{\frac{A}{n}}^{\eta}\frac{\psi^*(s)}{s}\,ds\int_0^1 q_0(t)\cos nts\,dt\right|$$

$$\leqq \frac{M}{\pi}\int_{\frac{A}{n}}^{\eta}\frac{1}{s}\,Q_0\left(\frac{\pi}{ns}\right)ds \leqq \frac{M}{\pi}\int_{\frac{\pi}{n\eta}}^{\frac{\pi}{A}}\frac{Q_0(s)}{s}\,ds = o(1)$$

as $A \to \infty$, for a fixed η; also, by an interchange of the order of integration, Lemma 7.4, the second mean value theorem, and Lemma 7.2

$$D_4 = \frac{1}{\pi}\int_{\frac{\delta}{n}}^{\frac{A}{n}} \frac{\psi^*(s)}{s}\,ds \int_0^1 q_0(t)\cos nts\,dt$$

$$= \frac{1}{\pi}\int_{\delta}^{A} \frac{\psi^*\left(\frac{s}{n}\right)}{s}\,ds \int_0^1 q_0(t)\cos st\,dt$$

$$= \frac{1}{\pi}\int_{\delta}^{A} \frac{\psi^*\left(\frac{s}{n}\right)}{s} P_1(s)\,ds - \frac{1}{\pi}\int_{\delta}^{A} \frac{\psi^*\left(\frac{s}{n}\right)}{s} P_2(s)\,ds$$

$$= \frac{P_1(\delta)}{\pi}\int_{\delta}^{\xi_1} \frac{\psi^*\left(\frac{s}{n}\right)}{s}\,ds - \frac{P_2(\delta)}{\pi}\int_{\delta}^{\xi_2} \frac{\psi^*(s)}{s}\,ds \quad \left(\delta \leqq \begin{matrix}\xi_1\\ \xi_2\end{matrix} \leqq A\right)$$

$$= \frac{P_1(\delta)}{\pi}\int_{\frac{\delta}{n}}^{\frac{\xi_1}{n}} \frac{\psi^*(s)}{s}\,ds - \frac{P_2(\delta)}{\pi}\int_{\frac{\delta}{n}}^{\frac{\xi_2}{n}} \frac{\psi^*(s)}{s}\,ds$$

$= o(1)$ as $n \to \infty$, for a fixed δ and A;

and finally by the Riemann-Lebesgue theorem and Lemma 2.1, $D_5 = o(1)$ as $n \to \infty$, for a fixed η, and $D_6 = o(1)$.

Hence

$$N_{q_0}\sigma_n(x) = -\frac{1}{2\pi}\int_{\frac{\delta}{n}}^{\pi} \psi^*(s)\cot\frac{s}{2}\,ds + o(1) \to \tilde{f}(x) \text{ as } n \to \infty.$$

From the above proof of the first part of the theorem it follows that, for a bounded $f(x)$, $|D_1|$, $|D_2|$, $|D_3|$, $|D_5|$, and $|D_6|$ are bounded for fixed δ, η, and A; we have also, for a fixed δ,

$$|D_4| \leqq \frac{P_1(\delta)}{\pi}\cdot M\log\frac{\xi_1}{\delta} + \frac{P_2(\delta)}{\pi}\cdot M\log\frac{\xi_2}{\delta} = O(1).$$

Hence, wherever $\tilde{f}(x)$ is definitely infinite, $N_{q_0}\sigma_n(x)$ becomes definitely infinite as $n \to \infty$.

By choosing $q_0(t) = \alpha(1-t)^{\alpha-1}$, $\alpha > 0$, we obtain, on account of the equivalence of (R, α) and (C, α), the theorem of B. N. Prasad on summability (C, α) as a special case of our theorem.[11]

THEOREM 9.3. *The N_{q_p} method is G_r effective, $(r = 0, 1, 2, \cdots, p)$.*

Proof. We consider two cases, $r = 2k$, and $r = 2k+1$. At the G_{2k} regular points we have for $p \geqq 2k$, from (4.11),

[11] B. N. Prasad, *A Theorem on the Cesàro Summability of the Allied Series of a Fourier Series*, Journal London Mathematical Society, 6 (1931), pp. 274-278.

$$\sum_{i=1}^{k} f^{(2i)}(x)\, V_{2i} = o(1)$$

by Lemma 6.4, since V_{2i}, $(0 \leqq i \leqq k)$, is a finite sum of terms of the type treated in that lemma multiplied by constants; and by Lemma 6.8

$$W_{2k} = o(1).$$

Hence

$$N_{q_p}\, s_n^{(2k)}(x) \to f^{(2k)}(x) \text{ as } n \to \infty.$$

The result for $r = 2k+1$ is obtained, from (4.12), in a similar fashion.

We obtain the Riesz equivalent of the theorem of M. Jacob on summability $(C, p+\delta)$, mentioned in the introduction, by choosing

$$q_p(t) = (p+\delta)(1-t)^{p+\delta-1}.$$

THEOREM 9.4. *The $N_{q_0}-(R, p)$ method is G_r effective, $(r = 0, 1, \cdots, p)$.*

Proof. The proof is analogous to that of Theorem 9.3. We have only to start from (4.13), (4.14) respectively, and to use Lemmas 8.1, 8.5, and 2.6.

By choosing $q_0(t) = \delta(1-t)^{\delta-1}$ we obtain as a special case of our theorem M. Jacob's theorem on summability $(C, p+\delta)$.[12]

[12] Loc. cit. [3].

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ORTHOGONAL TRIGONOMETRIC SUMS.[1]

By Dunham Jackson.

1. **Introduction.** The purpose of this paper is to develop the beginnings of a theory analogous to that of systems of polynomials orthogonal with respect to a given weight function, for the corresponding case of trigonometric sums.[2]

It is well known that if a set of functions $\psi_1(x)$, $\psi_2(x)$, $\cdots$ is given in an interval (a, b), each of the ψ's being summable over the interval together with its square, and any finite number of them being properly independent, in the sense that every linear combination with coefficients that do not all vanish is different from zero over a set of positive measure, then it is possible to construct a normalized orthogonal system $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$, in which the general function $\varphi_n(x)$ is a linear combination of $\psi_1(x), \cdots, \psi_n(x)$, so that

$$\int_a^b \varphi_m(x)\,\varphi_n(x)\,dx = 0 \qquad (m \neq n),$$

$$\int_a^b [\varphi_n(x)]^2\,dx = 1.$$

Let $\varrho(x)$ be a non-negative summable function of period 2π, different from zero over a set of positive measure in any interval of length 2π, and let $\psi_1(x)$, $\psi_2(x)$, $\cdots$ be the set of functions

$$[\varrho(x)]^{1/2},\ [\varrho(x)]^{1/2}\cos x,\ [\varrho(x)]^{1/2}\sin x,\ [\varrho(x)]^{1/2}\cos 2x,\ [\varrho(x)]^{1/2}\sin 2x,\ \cdots.$$

The corresponding φ's for a period interval will be a sequence[3]

$$[\varrho(x)]^{1/2}u_0(x),\ [\varrho(x)]^{1/2}u_1(x),\ [\varrho(x)]^{1/2}v_1(x),\ [\varrho(x)]^{1/2}u_2(x),\ [\varrho(x)]^{1/2}v_2(x),\ \cdots,$$

in which $u_n(x)$ and $v_n(x)$ are trigonometric sums of the nth order, the former containing no term in $\sin nx$, and the characteristic properties of the sequence are expressed by the relations

[1] Received February 21, 1933.—Presented to the American Mathematical Society, August 30, 1932, and April 15, 1933.

[2] The treatment is to a considerable extent parallel to that in the writer's expository paper entitled *Series of orthogonal polynomials*, Annals of Mathematics, vol. 34 (1933), pp. 527–545.

[3] See e. g. the writer's *Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. XI, New York, 1930, pp. 89–91.

$$\int_{-\pi}^{\pi} \varrho(x)\, u_m(x)\, u_n(x)\, dx = 0, \quad \int_{-\pi}^{\pi} \varrho(x)\, v_m(x)\, v_n(x)\, dx = 0, \quad (m \neq n),$$

$$\int_{-\pi}^{\pi} \varrho(x)\, u_m(x)\, v_n(x)\, dx = 0 \text{ for all } m \text{ and } n, \tag{1}$$

$$\int_{-\pi}^{\pi} \varrho(x)\, [u_n(x)]^2\, dx = 1, \quad \int_{-\pi}^{\pi} \varrho(x)\, [v_n(x)]^2\, dx = 1.$$

The content of the equations with right-hand member zero can be expressed by saying that any two functions of the sequence of u's and v's are orthogonal to each other with respect to $\varrho(x)$ as weight function. The following discussion is concerned with further properties of the u's and v's belonging to a given weight function $\varrho(x)$, and in particular with the convergence of series expansions in terms of them.

It is apparent from the manner of construction of the orthogonal system that, in the general formulation, $\varphi_n(x)$ as written in terms of the ψ's actually contains a term in $\psi_n(x)$, with non-vanishing coefficient, for each value of n, and hence that each $\psi_n(x)$ is linearly expressible in terms of $\varphi_1(x), \cdots, \varphi_n(x)$. Consequently any linear combination of $\psi_1, \cdots, \psi_{n-1}$ is a linear combination of $\varphi_1, \cdots, \varphi_{n-1}$, and $\varphi_n(x)$ is orthogonal to any such combination. In the special case under consideration, $u_n(x)$ actually has a term in $\cos nx$, and $v_n(x)$ actually has a term in $\sin nx$. Any trigonometric sum of the nth order containing no term in $\sin nx$ is linearly expressible in terms of $u_0(x)$, $u_1(x)$, $v_1(x)$, $\cdots$, $u_{n-1}(x)$, $v_{n-1}(x)$, $u_n(x)$, and any trigonometric sum of the nth order is expressible in terms of $u_0(x)$, $\cdots$, $u_n(x)$, $v_n(x)$. With respect to the weight function $\varrho(x)$, $u_n(x)$ and $v_n(x)$ are orthogonal to any trigonometric sum of order lower than the nth, and $v_n(x)$ is orthogonal also to any sum of the nth order lacking the term in $\sin nx$.

2. **Recursion formulas.** The u's and v's are connected by recursion formulas analogous to that of Darboux[4] for orthogonal polynomials, though somewhat less simple in form. Let μ be any constant. (The essence of the method would be sufficiently illustrated by taking $\mu = 0$, but it is desirable for the applications to have the degree of generality conferred by the presence of the arbitrary constant.) The product $\cos(x-\mu)\, u_n(x)$ is a trigonometric sum of order $n+1$, and as such can be expressed in terms of $u_0(x), \cdots, u_{n+1}(x), v_{n+1}(x)$:

$$\begin{aligned}\cos(x-\mu)\, u_n(x) = {} & \alpha_n u_{n+1}(x) + A_n u_n(x) + A'_n u_{n-1}(x) + A''_n u_{n-2}(x) + \cdots \\ & + \beta_n v_{n+1}(x) + B_n v_n(x) + B'_n v_{n-1}(x) + B''_n v_{n-2}(x) + \cdots.\end{aligned} \tag{2}$$

[4] G. Darboux, *Sur l'approximation des fonctions de très-grands nombres ...*, Journal de mathématiques pures et appliquées (3), vol. 4 (1878), pp. 377-416, see pp. 411-416.

The coefficients of course depend on μ, but it is not necessary to indicate that in the notation. For any value of k from 0 to n, multiplication by $\varrho(x)\, u_{n-k}(x)$ and integration from $-\pi$ to π, with attention to the relations (1), gives

$$\int_{-\pi}^{\pi} \varrho(x) \cos(x-\mu)\, u_n(x)\, u_{n-k}(x)\, dx = A_n^{(k)}. \tag{3}$$

If the trigonometric factors under the integral sign are regarded as forming the product of $u_n(x)$ and $\cos(x-\mu)\, u_{n-k}(x)$, the latter is a trigonometric sum of order lower than n if k has any value greater than 1, and the integral consequently vanishes: $A_n^{(k)} = 0$ for $k \geqq 2$. Similarly

$$\int_{-\pi}^{\pi} \varrho(x) \cos(x-\mu)\, u_n(x)\, v_{n-k}(x)\, dx = B_n^{(k)} \tag{4}$$

for $k = 0, 1, \cdots, n-1$ (there is no $v_0(x)$), and the integral vanishes for $k \geqq 2$. Replacement of n by $n-1$ in (2) gives an identity of the form

$$\cos(x-\mu)\, u_{n-1}(x) = \alpha_{n-1}\, u_n(x) + \beta_{n-1}\, v_n(x) + \cdots. \tag{5}$$

If this expression for $\cos(x-\mu)\, u_{n-1}(x)$ is substituted in (3), for $k = 1$, the value of the integral defining A_n' is seen to be α_{n-1}. So (2) may be rewritten in the form

$$\begin{aligned} \cos(x-\mu)\, u_n(x) = {} & \alpha_n\, u_{n+1}(x) + A_n\, u_n(x) + \alpha_{n-1}\, u_{n-1}(x) \\ & + \beta_n\, v_{n+1}(x) + B_n\, v_n(x) + B_n'\, v_{n-1}(x). \end{aligned} \tag{6}$$

Further specifications with regard to the coefficients will be brought out later.

After the manner of (2) there may be written down a corresponding identity

$$\begin{aligned} \cos(x-\mu)\, v_n(x) = {} & \gamma_n\, u_{n+1}(x) + C_n\, u_n(x) + C_n'\, u_{n-1}(x) + C_n''\, u_{n-2}(x) + \cdots \\ & + \delta_n\, v_{n+1}(x) + D_n\, v_n(x) + D_n'\, v_{n-1}(x) + D_n''\, v_{n-2}(x) + \cdots. \end{aligned} \tag{7}$$

The coefficients are given by the integrals

$$\begin{aligned} \int_{-\pi}^{\pi} \varrho(x) \cos(x-\mu)\, v_n(x)\, u_{n-k}(x)\, dx &= C_n^{(k)}, \\ \int_{-\pi}^{\pi} \varrho(x) \cos(x-\mu)\, v_n(x)\, v_{n-k}(x)\, dx &= D_n^{(k)}, \end{aligned} \tag{8}$$

and it is seen that $C_n^{(k)} = D_n^{(k)} = 0$ for $k \geqq 2$. Substitution from the identity

$$\cos(x-\mu)\, v_{n-1}(x) = \gamma_{n-1}\, u_n(x) + \delta_{n-1}\, v_n(x) + \cdots \tag{9}$$

gives $D_n' = \delta_{n-1}$. Also, (9) may be used in (4) and (5) in (8) to show that $B_n' = \gamma_{n-1}$ and $C_n' = \beta_{n-1}$. And it appears from comparison of (4) and (8) for $k = 0$ that $B_n = C_n$. So the identities (6) and (7) take the form

$$\begin{aligned}(10)\qquad \cos(x-\mu)\,u_n(x) &= \alpha_n u_{n+1}(x) + A_n u_n(x) + \alpha_{n-1} u_{n-1}(x) \\ &\quad + \beta_n v_{n+1}(x) + B_n v_n(x) + \gamma_{n-1} v_{n-1}(x),\end{aligned}$$

$$\begin{aligned}(11)\qquad \cos(x-\mu)\,v_n(x) &= \gamma_n u_{n+1}(x) + B_n u_n(x) + \beta_{n-1} u_{n-1}(x) \\ &\quad + \delta_n v_{n+1}(x) + D_n v_n(x) + \delta_{n-1} v_{n-1}(x).\end{aligned}$$

These hold for $n \geqq 1$, provided that when $n = 1$ it is understood that $v_0(x)$ is to be replaced by 0. For $n = 0$ no calculation is required to justify the formula

$$\cos(x-\mu)\,u_0(x) = \alpha_0 u_1(x) + \beta_0 v_1(x) + A_0 u_0(x).$$

3. **Christoffel-Darboux formula.** From the recursion formulas it is possible to obtain an analogue of the Christoffel-Darboux formula[5] in the theory of orthogonal polynomials. It is a question of an identity for the sum

$$\begin{aligned}(12)\qquad K_n(x,t) \equiv u_0(x)\,u_0(t) + u_1(x)\,u_1(t) + \cdots + u_n(x)\,u_n(t) \\ + v_1(x)\,v_1(t) + \cdots + v_n(x)\,v_n(t).\end{aligned}$$

Let (10) and (11) be multiplied by $u_n(t)$ and $v_n(t)$ respectively, and let the corresponding formulas for $\cos(t-\mu)\,u_n(x)\,u_n(t)$ and $\cos(t-\mu)\,v_n(x)\,v_n(t)$ be written down by interchange of t and x. Combination of the resulting identities by the indicated additions and subtractions yields

$$\begin{aligned}(13)\qquad &[\cos(t-\mu) - \cos(x-\mu)]\,[u_n(x)\,u_n(t) + v_n(x)\,v_n(t)] \\ &= \alpha_n[u_n(x)\,u_{n+1}(t) - u_{n+1}(x)\,u_n(t)] - \alpha_{n-1}[u_{n-1}(x)\,u_n(t) - u_n(x)\,u_{n-1}(t)] \\ &+ \beta_n[u_n(x)\,v_{n+1}(t) - v_{n+1}(x)\,u_n(t)] - \beta_{n-1}[u_{n-1}(x)\,v_n(t) - v_n(x)\,u_{n-1}(t)] \\ &+ \gamma_n[v_n(x)\,u_{n+1}(t) - u_{n+1}(x)\,v_n(t)] - \gamma_{n-1}[v_{n-1}(x)\,u_n(t) - u_n(x)\,v_{n-1}(t)] \\ &+ \delta_n[v_n(x)\,v_{n+1}(t) - v_{n+1}(x)\,v_n(t)] - \delta_{n-1}[v_{n-1}(x)\,v_n(t) - v_n(x)\,v_{n-1}(t)].\end{aligned}$$

These relations hold for $n \geqq 1$, with the previous understanding as to v_0. For $n = 0$,

$$\begin{aligned}(14)\qquad &[\cos(t-\mu) - \cos(x-\mu)]\,u_0(x)\,u_0(t) \\ &= \alpha_0[u_0(x)\,u_1(t) - u_1(x)\,u_0(t)] + \beta_0[u_0(x)\,v_1(t) - v_1(x)\,u_0(t)].\end{aligned}$$

The desired identity for $K_n(x,t)$ is derived by writing (13) with n replaced by k for the values $k = 1, 2, \cdots, n$ successively, summation over these values of the index, and addition of the terms obtained from (14):

[5] Darboux, loc. cit.

$$[\cos(t-\mu)-\cos(x-\mu)]\,K_n(x,t)$$
$$=\alpha_n[u_n(x)\,u_{n+1}(t)-u_{n+1}(x)\,u_n(t)]+\beta_n[u_n(x)\,v_{n+1}(t)-v_{n+1}(x)\,u_n(t)]$$
$$+\gamma_n[v_n(x)\,u_{n+1}(t)-u_{n+1}(x)\,v_n(t)]+\delta_n[v_n(x)\,v_{n+1}(t)-v_{n+1}(x)\,v_n(t)],$$

or, if the long right-hand member is denoted by $N_n(x,t)$,

$$K_n(x,t)=\frac{N_n(x,t)}{[\cos(t-\mu)-\cos(x-\mu)]}. \tag{15}$$

The constants α_n, β_n, γ_n, δ_n can be evaluated in terms of the leading coefficients in the expressions of $u_n(x)$ and $v_n(x)$ as combinations of sines and cosines. More important for the subsequent applications, however, is the fact that *their absolute values are less than unity.* Multiplication of (10) by $\varrho(x)\,u_{n+1}(x)$ and integration from $-\pi$ to π gives

$$\alpha_n=\int_{-\pi}^{\pi}\varrho(x)\cos(x-\mu)\,u_n(x)\,u_{n+1}(x)\,dx.$$

Hence

$$|\alpha_n|\leqq\int_{-\pi}^{\pi}\varrho(x)\,|\cos(x-\mu)\,u_n(x)\,u_{n+1}(x)|\,dx.$$

Let Schwarz's inequality be applied to this integral, the integrand being regarded as the product of the factors $[\varrho(x)]^{1/2}\,|\cos(x-\mu)\,u_n(x)|$ and $[\varrho(x)]^{1/2}\,|u_{n+1}(x)|$:

$$\alpha_n^2\leqq\int_{-\pi}^{\pi}\varrho(x)\cos^2(x-\mu)\,[u_n(x)]^2\,dx\int_{-\pi}^{\pi}\varrho(x)\,[u_{n+1}(x)]^2\,dx$$
$$<\int_{-\pi}^{\pi}\varrho(x)\,[u_n(x)]^2\,dx\int_{-\pi}^{\pi}\varrho(x)\,[u_{n+1}(x)]^2\,dx=1;$$

the replacement of $\cos^2(x-\mu)$ by unity actually increases the value of the integrand over a set of positive measure and increases the value of the integral, the alternative of equality being ruled out. A corresponding calculation gives a similar result for β_n, γ_n, and δ_n (and incidentally for each of the coefficients denoted by A_n, B_n, D_n in the preceding section; in the case of A_n and D_n the fact can also be recognized directly from the form of the integrals defining them, without the use of Schwarz's inequality).

If the terms of the nth order in $u_n(x)$ and $v_n(x)$ are denoted by $a_n\cos nx$, $c_n\cos nx$, and $d_n\sin nx$ respectively, the explicit formulas for α_n, β_n, γ_n, δ_n are found to be as follows:

$$\alpha_n=\frac{1}{2}\frac{a_n}{a_{n+1}}\left(\cos\mu-\frac{c_{n+1}}{d_{n+1}}\sin\mu\right),$$

$$\beta_n=\frac{1}{2}\frac{a_n}{d_{n+1}}\sin\mu,$$

$$\gamma_n = \frac{1}{2\,a_{n+1}}\left[(c_n \cos\mu - d_n \sin\mu) - \frac{c_{n+1}}{d_{n+1}}(c_n \sin\mu + d_n \cos\mu)\right],$$

$$\delta_n = \frac{1}{2\,d_{n+1}}(c_n \sin\mu + d_n \cos\mu),$$

for $n \geqq 1$, while

$$\alpha_0 = \frac{a_0}{a_1}\left(\cos\mu - \frac{c_1}{d_1}\sin\mu\right), \qquad \beta_0 = \frac{a_0}{d_1}\sin\mu.$$

These are set down merely as a matter of record, and will not be used further.

4. **Formal expansion of an arbitrary function.** An arbitrary function $f(x)$ of period 2π, subject to the condition that $\varrho(x) f(x)$ be summable over a period, can be formally expanded in a series

$$\begin{aligned} &a_0 u_0(x) + a_1 u_1(x) + a_2 u_2(x) + \cdots \\ &\quad + b_1 v_1(x) + b_2 v_2(x) + \cdots \end{aligned} \tag{16}$$

by the procedure that is usual in connection with series of orthogonal functions; the meaning of a_n here of course has no connection with the use of the same symbol in the preceding paragraph. The general coefficients are

$$a_k = \int_{-\pi}^{\pi} \varrho(x) f(x)\, u_k(x)\, dx, \qquad b_k = \int_{-\pi}^{\pi} \varrho(x) f(x)\, v_k(x)\, dx.$$

The partial sum of the series through terms of the nth order is

$$s_n(x) = \int_{-\pi}^{\pi} \varrho(t) f(t)\, K_n(x, t)\, dt, \tag{17}$$

where $K_n(x, t)$ is defined as above. In particular,

$$1 \equiv \int_{-\pi}^{\pi} \varrho(t)\, K_n(x, t)\, dt;$$

multiplication of this identity by $f(x)$, which is independent of the variable of integration, and subtraction from (17) gives

$$s_n(x) - f(x) = \int_{-\pi}^{\pi} \varrho(t)\,[f(t) - f(x)]\, K_n(x, t)\, dt. \tag{18}$$

5. **Analogue of the Riemann-Lebesgue theorem.** If $\varrho(x)\,[f(x)]^2$ as well as $\varrho(x) f(x)$ is summable, further calculation which is well known as applied to orthogonal functions in general yields the relation

$$\int_{-\pi}^{\pi} \varrho(x)\,[f(x) - s_n(x)]^2\, dx = \int_{-\pi}^{\pi} \varrho(x)\,[f(x)]^2\, dx - a_0^2 - \sum_{k=1}^{n}(a_k^2 + b_k^2).$$

As the magnitude of the sum on the right can never exceed that of the integral on the right as n increases, by reason of the non-negative character

of the left-hand member, the corresponding infinite series is convergent, and the general term approaches zero:

$$\lim_{k\to\infty} a_k = 0, \qquad \lim_{k\to\infty} b_k = 0.$$

If the sums $u_k(x)$, $v_k(x)$ are uniformly bounded, as will be true at least in particular cases, this conclusion can be made more general by omission of the requirement that ϱf^2 be summable, the hypothesis of summability being imposed merely on ϱ and ϱf. Suppose $|u_k(x)| \leqq H$, $|v_k(x)| \leqq H$, for all k and all x. For arbitrary positive N, let $f_N(x) = f(x)$ when $|f(x)| \leqq N$, $f_N(x) = 0$ when $|f(x)| > N$. If ε is any positive number,

$$\int_{-\pi}^{\pi} \varrho(x)\,|f(x) - f_N(x)|\,dx < \frac{\varepsilon}{2H}$$

when N is sufficiently large, and hence

$$\left|\int_{-\pi}^{\pi} \varrho(x) f(x)\, u_k(x)\,dx - \int_{-\pi}^{\pi} \varrho(x) f_N(x)\, u_k(x)\,dx\right| < \frac{1}{2}\varepsilon,$$

and similarly with $u_k(x)$ replaced by $v_k(x)$, for all values of k. Let N be given a fixed value so large that these inequalities hold. Since ϱf_N^2 is summable,

$$\left|\int_{-\pi}^{\pi} \varrho(x) f_N(x)\, u_k(x)\,dx\right| < \frac{1}{2}\varepsilon, \qquad \left|\int_{-\pi}^{\pi} \varrho(x) f_N(x)\, v_k(x)\,dx\right| < \frac{1}{2}\varepsilon,$$

for all values of k from a certain point on, and when this is true

$$|a_k| < \varepsilon, \qquad |b_k| < \varepsilon,$$

if a_k and b_k are the coefficients defined for the original function $f(x)$.

It will be desirable subsequently to have the conclusion formulated for a function other than the one whose series expansion is under investigation, in the form of a

LEMMA. *If $\varrho(t)\,\varphi(t)$ and $\varrho(t)\,[\varphi(t)]^2$ are summable, or if $\varrho(t)\,\varphi(t)$ is summable and the functions $u_k(t)$, $v_k(t)$ are uniformly bounded, the integrals*

$$\int_{-\pi}^{\pi} \varrho(t)\,\varphi(t)\,u_k(t)\,dt, \qquad \int_{-\pi}^{\pi} \varrho(t)\,\varphi(t)\,v_k(t)\,dt$$

approach zero as k becomes infinite.

(It is understood throughout the paper that the weight function itself is summable.)

6. **Special uniformly bounded systems.** As there has been occasion above to suppose the functions $u_k(x)$, $v_k(x)$ uniformly bounded, and as this hypothesis will be fundamental in a large part of what follows, it is worth while, pending investigation of general conditions under which

the property in question is realized, to point out that in a particular class of cases its validity can be recognized immediately. In other cases it can be seen that the u's and v's are bounded for specified values of x, or uniformly bounded over a part of a period, and this also is significant for the theory of convergence.

Let $\varrho(x)$ be any weight function for which the u's and v's are bounded for a particular value of x as k becomes infinite, or uniformly bounded over a designated range of values of x:

$$|u_k(x)| \leqq H, \qquad |v_k(x)| \leqq H, \tag{19}$$

for values of x belonging to a point set E. Certainly this condition is fulfilled everywhere in the special case $\varrho(x) \equiv 1$, the functions of the normalized system for $k \geqq 1$ being $(1/\pi^{1/2}) \cos kx$ and $(1/\pi^{1/2}) \sin kx$. Let $U_k(x)$, $V_k(x)$ be the sums of the kth order in the normalized orthogonal system corresponding to the weight function $\varrho(x)\,\tau(x)$, where $\tau(x)$ is any non-negative trigonometric sum. It will appear that $U_k(x)$, $V_k(x)$ are bounded at any point of E where $\tau(x) \neq 0$, and uniformly bounded over any part of E where $\tau(x)$ has a positive lower bound.

Let p be the order of the sum $\tau(x)$. Since $\tau(x)\,U_n(x)$ is a trigonometric sum of order $n+p$ it can be expressed in the form

$$\begin{aligned}\tau(x)\,U_n(x) \equiv\ & A_0\,u_0(x) + A_1\,u_1(x) + \cdots + A_{n+p}\,u_{n+p}(x) \\ & + B_1\,v_1(x) + \cdots + B_{n+p}\,v_{n+p}(x),\end{aligned}$$

the coefficients depending on n, to be sure, as well as on the index which appears explicitly. (The present A's and B's have no relation to those used previously). By the property of orthogonality of the sums $u_k(x)$, $v_k(x)$,

$$A_k = \int_{-\pi}^{\pi} \varrho(x)\,\tau(x)\,U_n(x)\,u_k(x)\,dx.$$

But $U_n(x)$ is orthogonal to any trigonometric sum of lower order with respect to the weight function $\varrho(x)\,\tau(x)$ for which the U's and V's are constructed, and hence $A_k = 0$ for $k < n$. Similarly, $B_k = 0$ for $k < n$. So

$$U_n(x) \equiv \frac{1}{\tau(x)}\,[A_n u_n(x) + \cdots + A_{n+p} u_{n+p}(x) + B_n v_n(x) + \cdots + B_{n+p} v_{n+p}(x)],$$

the number of terms in the bracket being $2p+2$, and remaining finite as n increases. In the same way $V_n(x)$ has the form

$$V_n(x) \equiv \frac{1}{\tau(x)}\,[C_n u_n(x) + \cdots + C_{n+p} u_{n+p}(x) + D_n v_n(x) + \cdots + D_{n+p} v_{n+p}(x)].$$

For x in E, if $\tau(x) \neq 0$,

$$|U_n(x)| \leqq \frac{H}{\tau(x)} (|A_n| + \cdots + |A_{n+p}| + |B_n| + \cdots + |B_{n+p}|),$$

whence, as any one of the absolute values in the parenthesis is certainly not greater than the square root of the sum of the squares of all of them,

$$(20) \quad |U_n(x)| \leqq \frac{H}{\tau(x)} (2p+2)(A_n^2 + \cdots + A_{n+p}^2 + B_n^2 + \cdots + B_{n+p}^2)^{1/2}.$$

(By Schwarz's inequality the last relation holds in fact with $(2p+2)$ replaced by $(2p+2)^{1/2}$). If M_0 is the maximum of $\tau(x)$,

$$A_n^2 + \cdots + A_{n+p}^2 + B_n^2 + \cdots + B_{n+p}^2 = \int_{-\pi}^{\pi} \varrho(x) [\tau(x)\, U_n(x)]^2 \, dx$$
$$\leqq M_0 \int_{-\pi}^{\pi} \varrho(x)\, \tau(x)\, [U_n(x)]^2 \, dx = M_0,$$

the last equality resulting from the normalization of the U's with respect to $\varrho(x)\, \tau(x)$ as weight function. So the whole right-hand member of (20) has an upper bound independent of n. Similar reasoning applies to $V_n(x)$. So the assertion with regard to the boundedness of the U's and V's is justified. In particular, if (19) holds for all values of x and if $\tau(x)$ has a positive minimum *the U's and V's are uniformly bounded everywhere.*

7. **First convergence theorem.** Let $\varrho(x)$ be a non-negative summable weight function of period 2π for which the sums $u_k(x)$, $v_k(x)$ are everywhere uniformly bounded. Let $f(x)$ be a function of period 2π such that $\varrho(x) f(x)$ is summable. Let $s_n(x)$, as above, be the partial sum of the series (16) through terms of the nth order. This section is concerned with the setting up of conditions for the convergence of $s_n(x)$ toward $f(x)$ for an arbitrarily specified value of x, to be held fast throughout the discussion. The question of uniformity of convergence will be reserved for treatment by a different method in the next section.

The value of x being given, let $\psi_1(t)$ and $\psi_2(t)$ be functions of period 2π, defined in the interval $x-(\pi/2) \leqq x < x+(3\pi/2)$ as follows:

$$\psi_1(t) = f(t)-f(x) \text{ for } |t-x| \leqq \frac{\pi}{2}, \quad \psi_1(t) = 0 \qquad \text{for } |t-x| > \frac{\pi}{2},$$

$$\psi_2(t) = 0 \qquad \text{for } |t-x| \leqq \frac{\pi}{2}, \quad \psi_2(t) = f(t)-f(x) \text{ for } |t-x| > \frac{\pi}{2}.$$

Then $\psi_1(t) + \psi_2(t) = f(t)-f(x)$ everywhere, and by (18)

$$s_n(x)-f(x) = \int_{-\pi}^{\pi} \varrho(t)\, \psi_1(t)\, K_n(x,t)\, dt + \int_{-\pi}^{\pi} \varrho(t)\, \psi_2(t)\, K_n(x,t)\, dt$$
$$= I_1 + I_2.$$

It will be recalled that in the expression (15) for $K_n(x, t)$ the constant μ is arbitrary, the coefficients α_n, β_n, γ_n, δ_n which enter into the numerator depending on μ, but being always less than 1 in absolute value. For substitution of (15) in the integral I_1, let $\mu = x+(\pi/2)$. Then

$$\cos(x-\mu) = \cos\left(-\frac{\pi}{2}\right) = 0, \quad \cos(t-\mu) = \cos\left(t-x-\frac{\pi}{2}\right) = \sin(t-x),$$

$$\cos(t-\mu) - \cos(x-\mu) = \sin(t-x).$$

So I_1 is the sum of

$$(21) \qquad \alpha_n \int_{-\pi}^{\pi} \varrho(t) \frac{\psi_1(t)}{\sin(t-x)} [u_n(x)\, u_{n+1}(t) - u_{n+1}(x)\, u_n(t)]\, dt$$

and three other integrals of similar form. Let

$$\Phi(t) \equiv \frac{f(t)-f(x)}{t-x},$$

and let it be supposed that the function $f(t)$ is such that there is an interval of values of t about the point x over which $\varrho(t)\,\Phi(t)$ is summable. Then $\varrho(t)\,\Phi(t)$ is summable over any finite interval, being the product of the summable function $\varrho(t)\,[f(t)-f(x)]$ by the continuous function $1/(t-x)$ over any interval which does not approach the point $t = x$. If

$$\varphi_1(t) \equiv \frac{\psi_1(t)}{\sin(t-x)},$$

$\varrho(t)\,\varphi_1(t)$ is summable over the interval $x-(\pi/2) \leqq t \leqq x+(\pi/2)$, being the product of $\varrho(t)\,\Phi(t)$ by the continuous function $(t-x)/\sin(t-x)$; it is summable from $x+(\pi/2)$ to $x+(3\pi/2)$, being identically zero except at the point $t = x+\pi$ where it is not defined; and therefore it is summable over the entire period from $x-(\pi/2)$ to $x+(3\pi/2)$, or equally well over any other period interval, in particular that from $-\pi$ to π. The expression (21) is equal to

$$\alpha_n \left[u_n(x) \int_{-\pi}^{\pi} \varrho(t)\, \varphi_1(t)\, u_{n+1}(t)\, dt - u_{n+1}(x) \int_{-\pi}^{\pi} \varrho(t)\, \varphi_1(t)\, u_n(t)\, dt\right].$$

Here each integral approaches zero as n becomes infinite, by the Lemma of Section 5; $u_n(x)$ and $u_{n+1}(x)$ are bounded by hypothesis; and $|\alpha_n| < 1$. So the whole expression approaches 0. Similar reasoning, with a similar conclusion, is applicable to each of the other three parts of I_1. Hence $\lim_{n\to\infty} I_1 = 0$.

For the application of (15) in I_2, let $\mu = x$,

$$\cos(t-\mu) - \cos(x-\mu) = \cos(t-x) - 1,$$

and let $\varphi_2(t) \equiv \psi_2(t)/[\cos(t-x)-1]$. With new values of $\alpha_n, \cdots, \delta_n$, the integral I_2 is the sum of

$$\alpha_n\left[u_n(x)\int_{-\pi}^{\pi}\varrho(t)\,\varphi_2(t)\,u_{n+1}(t)\,dt - u_{n+1}(x)\int_{-\pi}^{\pi}\varrho(t)\,\varphi_2(t)\,u_n(t)\,dt\right]$$

and three other expressions of like character. The product $\varrho(t)\,\varphi_2(t)$ is summable from $x-(\pi/2)$ to $x+(\pi/2)$, where it is identically zero except for $t=x$, and from $x+(\pi/2)$ to $x+(3\pi/2)$, where $|\cos(t-x)-1| \geqq 1$, and so is summable over any period interval. By use of the Lemma again it is seen that $\lim_{n\to\infty} I_2 = 0$. The fact that I_1 and I_2 both approach zero means that

$$\lim_{n\to\infty} s_n(x) = f(x).$$

The proof of convergence goes equally well without the requirement that the u's and v's be uniformly bounded, if it is assumed that they are bounded at the point x and that $\varrho(t)\,[f(t)]^2$ and $\varrho(t)\,[\Phi(t)]^2$ are summable.

These results are expressed in

THEOREM I. *The series* (16) *will converge to the value* $f(x)$ *for a specified value of* x *if the functions* $u_k(t)$ *and* $v_k(t)$ *are uniformly bounded, if* $\varrho(t)\,f(t)$ *is summable, and if there is an interval about the point* x *over which* $\varrho(t)\,\Phi(t)$ *is summable, where* $\Phi(t)$ *denotes the difference quotient* $[f(t)-f(x)]/(t-x)$; *or if* $u_k(x)$ *and* $v_k(x)$ *are bounded for the particular value of* x *in question, if* $\varrho(t)\,[f(t)]^2$ *is summable, and if* $\varrho(t)\,[\Phi(t)]^2$ *is summable over an interval about the point* x.

The conditions on $\Phi(t)$ can of course be interpreted in more specific terms[6] with reference to properties of $f(t)$. They will be more than satisfied, for example, if $f(t)$ satisfies a Lipschitz condition in the neighborhood of $t=x$.

More particularly still, the condition on $\Phi(t)$ will be satisfied if $f(t)$ vanishes identically throughout an interval about the point x. If $f_1(t)$ and $f_2(t)$ are two functions which are equal throughout such an interval, the series for their difference, which is the difference of their respective series, will converge toward zero at the point x; if the series for f_1 converges, the series for f_2 will converge likewise, regardless of any difference between the functions outside the interval designated, provided ϱf_1 and ϱf_2 are summable, and, in case of need, ϱf_1^2 and ϱf_2^2 also. *The convergence of the series for a function* $f(x)$ *at a specified point depends only on the behavior of the function in the neighborhood of the point,* if the u's and v's are uniformly bounded and ϱf is summable, or if the u's and v's are bounded at the point in question and ϱf^2 is summable.

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[6] Cf. e. g. H. Lebesgue, *Leçons sur les séries trigonométriques*, Paris, 1906, Chapter III.

8. **Second convergence theorem.** Let $T_n(x)$ be any trigonometric sum of the nth order. It can be expressed as a linear combination of $u_0(x)$, $u_1(x)$, $v_1(x)$, $\cdots$, $u_n(x)$, $v_n(x)$, and this linear expression is at the same time the series development of $T_n(x)$ in terms of the u's and v's. The partial sum of the series through terms of order n is identical with $T_n(x$ itself. This means that

$$T_n(x) \equiv \int_{-\pi}^{\pi} \varrho(t)\, T_n(t)\, K_n(x, t)\, dt.$$

Let $f(x) - T_n(x) \equiv r_n(x)$. Then $f(x) = T_n(x) + r_n(x)$, and if $s_n(x)$ is the partial sum of the series for $f(x)$,

$$\begin{aligned} s_n(x) &= \int_{-\pi}^{\pi} \varrho(t) f(t)\, K_n(x, t)\, dt \\ &= \int_{-\pi}^{\pi} \varrho(t)\, T_n(t)\, K_n(x, t)\, dt + \int_{-\pi}^{\pi} \varrho(t)\, r_n(t)\, K_n(x, t)\, dt \\ &= T_n(x) + \int_{-\pi}^{\pi} \varrho(t)\, r_n(t)\, K_n(x, t)\, dt, \end{aligned}$$

$$f(x) - s_n(x) = r_n(x) - \int_{-\pi}^{\pi} \varrho(t)\, r_n(t)\, K_n(x, t)\, dt.$$

If $|r_n(x)| \leqq \varepsilon_n$,

$$\left| \int_{-\pi}^{\pi} \varrho(t)\, r_n(t)\, K_n(x, t)\, dt \right| \leqq \varepsilon_n \int_{-\pi}^{\pi} \varrho(t)\, |K_n(x, t)|\, dt,$$

and consequently

$$|f(x) - s_n(x)| \leqq \varepsilon_n [1 + \lambda_n(x)],$$

if

$$\lambda_n(x) = \int_{-\pi}^{\pi} \varrho(t)\, |K_n(x, t)|\, dt.$$

So the convergence of $s_n(x)$ toward $f(x)$ can be discussed in terms of the order of magnitude of ε_n and the order of magnitude of $\lambda_n(x)$. Information with regard to the former, when the sums $T_n(x)$ are appropriately chosen, is given by known theorems on degree of approximation. The latter will be examined here.[7]

Let it be supposed that $\varrho(x)$ *is such that* $u_k(x)$, $v_k(x)$ *are uniformly bounded, and furthermore,* for simplicity at least, *that* $\varrho(x)$ *itself is bounded.* Let $\varrho(x) \leqq G$, $|u_k(x)| \leqq H$, $|v_k(x)| \leqq H$. In view of the periodicity of the functions involved, the integral defining $\lambda_n(x)$, instead of being taken from $-\pi$ to π, may be extended over any other interval of length 2π,

[7] For a corresponding treatment of the problem of polynomial approximation see J. Shohat, *On the development of continuous functions in series of Tchebycheff polynomials,* Transactions of the American Mathematical Society, vol. 27 (1925), pp. 537–550.

say that from $x-(\pi/2)$ to $x+(3\pi/2)$. Let the integrals from $x-(\pi/2)$ to $x+(\pi/2)$ and from $x+(\pi/2)$ to $x+(3\pi/2)$ be represented by J_1 and J_2 respectively, and let J_1 be further subdivided into the integrals from $x-(\pi/2)$ to $x-(1/n)$, from $x-(1/n)$ to $x+(1/n)$, and from $x+(1/n)$ to $x+(\pi/2)$, denoted respectively by J_1', J_1'', and J_1''', so that

$$\lambda_n(x) = J_1+J_2 = J_1'+J_1''+J_1'''+J_2,$$

the integrand each time being $\varrho(t)\,|K_n(x,t)|$.

In J_1' and J_1''' let $K_n(x,t)$ be represented by (15) with $\mu = x+(\pi/2)$, whereby the denominator takes the form

$$\cos(t-\mu)-\cos(x-\mu) = \sin(t-x).$$

Since $|t-x| \leqq \pi/2$ over the range in question, $|\sin(t-x)| \geqq (2/\pi)\,|t-x|$. In the first of the four parts of the numerator, the relations

$$|a_n|<1, \quad |u_n(x)| \leqq H, \quad |u_{n+1}(t)| \leqq H, \quad |u_{n+1}(x)| \leqq H, \quad |u_n(t)| \leqq H$$

give $|a_n\,[u_n(x)\,u_{n+1}(t)-u_{n+1}(x)\,u_n(t)]| \leqq 2H^2$. As corresponding inequalities hold for the other three parts, and as $\varrho(t) \leqq G$,

$$\varrho(t)\,|K_n(x,t)| \leqq \frac{8\pi G H^2}{2\,|t-x|}.$$

By explicit integration,

$$\int_{x-(\pi/2)}^{x-(1/n)} \frac{dt}{|t-x|} = \int_{x+(1/n)}^{x+(\pi/2)} \frac{dt}{|t-x|} = \log\left(\frac{\pi}{2}\right)+\log n,$$

and J_1' and J_1''' have each an upper bound of the order of $\log n$. In J_1'' it suffices to read off from the definition of $K_n(x,t)$ in (12), together with the hypothesis on the u's and v's, that $|K_n(x,t)| \leqq (2n+1)H^2$; the length of the interval being $2/n$, it is seen that $J_1'' \leqq 2[2+(1/n)]GH^2 \leqq 6GH^2$. Consequently the whole of J_1 does not exceed a constant multiple of $\log n$.

In J_2 the formula (15) is to be used with $\mu = x$. As $t-x$ ranges from $\pi/2$ to $3\pi/2$,

$$|\cos(t-\mu)-\cos(x-\mu)| = |\cos(t-x)-1| \geqq 1,$$

and $\varrho(t)\,|K_n(x,t)| \leqq 8GH^2$, $J_2 \leqq 8\pi GH^2$.

By combination of these results it is recognized that

$$\lambda_n(x) \leqq C \log n$$

for $n>1$, where C is independent of n and x.

The conclusions with regard to convergence and degree of convergence are expressed in

THEOREM II. *If $\varrho(x)$ is bounded, if $u_k(x)$ and $v_k(x)$ are uniformly bounded, and if trigonometric sums $T_n(x)$ of the nth order exist*[8] *so that*

$$|f(x)-T_n(x)| \leqq \varepsilon_n,$$

then

$$|f(x)-s_n(x)| \leqq C_1 \varepsilon_n \log n$$

for $n>1$, where C_1 is independent of n and x; in particular,[9] *the series converges uniformly to the value $f(x)$ for all values of x if $f(x)$ has a modulus of continuity $\omega(\delta)$ such that $\lim_{\delta\to 0} \omega(\delta) \log \delta = 0$.*

9. **General weight functions.** Some information as to the order of magnitude of the u's and v's, when they are not known to remain bounded, is readily accessible in the case of weight functions of considerable generality.

Suppose that $\varrho(x)$ is summable and has a positive lower bound for all values of x:

$$\varrho(x) \geqq v>0. \tag{22}$$

Let $T_n(x)$ be any trigonometric sum of the nth order (not necessarily one of those in the orthogonal sequence) satisfying the condition that

$$\int_{-\pi}^{\pi} \varrho(x)\,[T_n(x)]^2\,dx = 1. \tag{23}$$

In consequence of (22),

$$\int_{-\pi}^{\pi} [T_n(x)]^2\,dx \leqq \frac{1}{v}. \tag{24}$$

Let c_0 be the constant term in the trigonometric sum $[T_n(x)]^2$, and let $S_n(x) = [T_n(x)]^2 - c_0$, so that $S_n(x)$ is a trigonometric sum of order $2n$ without constant term. The integral (24) has the value $2\pi c_0$, and hence $|c_0| \leqq 1/(2\pi v)$. For any value of x between $-\pi$ and π,

$$\int_{-\pi}^{x} [T_n(x)]^2\,dx \leqq \int_{-\pi}^{\pi} [T_n(x)]^2\,dx \leqq \frac{1}{v},$$

and

$$\left|\int_{-\pi}^{x} S_n(x)\,dx\right| \leqq \int_{-\pi}^{x} [T_n(x)]^2\,dx + \int_{-\pi}^{x} |c_0|\,dx \leqq \frac{2}{v}.$$

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[8] For the requisite theorems on degree of approximation see e. g. the writer's Colloquium, cited above, Chapter I.

[9] For the case of Fourier series, and for the special theorem on degree of approximation needed here, see also Lebesgue, *Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz*, Bulletin de la Société Mathématique de France, vol. 38 (1910), pp. 184–210; pp. 201–202, 209.

The integral between bars in the left-hand member is itself a trigonometric sum of order $2n$, having $S_n(x)$ for its derivative. Hence, by Bernstein's theorem,

$$|S_n(x)| \leqq \frac{4n}{v}.$$

It follows that $[T_n(x)]^2 \equiv c_0 + S_n(x)$ has an upper bound of the order of n, and $|T_n(x)|$ has an upper bound of the order of $n^{1/2}$. In particular, $|u_n(x)|$ and $|v_n(x)|$ can not exceed a constant multiple of $n^{1/2}$.

Now let the requirement that $\varrho(x)$ have a positive lower bound be replaced by a less restrictive one: let it be assumed merely that $[\varrho(x)]^{-1}$ as well as $\varrho(x)$ is summable over a period. Let $T_n(x)$ again be a trigonometric sum of the nth order satisfying (23). Let Schwarz's inequality be applied to the integral of the product of the factors $[\varrho(x)]^{-1/2}$ and $[\varrho(x)]^{1/2}\,|T_n(x)|$:

$$\left[\int_{-\pi}^{\pi} |T_n(x)|\,dx\right]^2 \leqq \int_{-\pi}^{\pi} [\varrho(x)]^{-1}\,dx \int_{-\pi}^{\pi} \varrho(x)\,[T_n(x)]^2\,dx = \int_{-\pi}^{\pi} [\varrho(x)]^{-1}\,dx.$$

The square root of the value of the last integral being denoted by I,

$$\int_{-\pi}^{\pi} |T_n(x)|\,dx \leqq I.$$

Let c_0 this time be the constant term in $T_n(x)$, and $S_n(x)$ the sum of the remaining terms. It is seen that

$$2\pi\,|c_0| = \left|\int_{-\pi}^{\pi} T_n(x)\,dx\right| \leqq I, \qquad |c_0| \leqq \frac{I}{2\pi}.$$

Furthermore,

$$\int_{-\pi}^{\pi} |S_n(x)|\,dx \leqq \int_{-\pi}^{\pi} |T_n(x)|\,dx + \int_{-\pi}^{\pi} |c_0|\,dx \leqq 2I,$$

$$\left|\int_{-\pi}^{x} S_n(x)\,dx\right| \leqq \int_{-\pi}^{x} |S_n(x)|\,dx \leqq \int_{-\pi}^{\pi} |S_n(x)|\,dx \leqq 2I.$$

Since $\int S_n(x)\,dx$ is a trigonometric sum of the nth order, Bernstein's theorem gives for $|S_n(x)|$, and consequently for $|T_n(x)|$, an upper bound of the order of n. Specifically, $|u_n(x)|$ and $|v_n(x)|$ can not exceed a constant multiple of n.

These bounds for the u's and v's, however, do not seem to lead immediately to any better theorems on convergence than can be obtained by direct use of the least-square property of the series.[10] Consider for example

[10] See e. g. the writer's Colloquium, already cited, Chapter III; D. Jackson, *A generalized problem in weighted approximation*, Transactions of the American Mathematical Society, vol. 26 (1924), pp. 133–154; J. Shohat, *On the polynomial and trigonometric approximation of measurable bounded functions on a finite interval*, Mathematische Annalen, vol. 102 (1930), pp. 157–175.

the case of the last paragraph, under which $\varrho(x)$ and $[\varrho(x)]^{-1}$ are summable. It follows from well-known properties of orthogonal functions in general that the integral

$$\gamma_n = \int_{-\pi}^{\pi} \varrho(x)\,[f(x) - s_n(x)]^2\,dx$$

has a smaller value than the corresponding integral with $s_n(x)$ replaced by any other trigonometric sum $t_n(x)$ of the nth order. Suppose sums $t_n(x)$ exist so that

$$|f(x) - t_n(x)| \leqq \varepsilon_n$$

for all values of x. Then it is certain that γ_n does not exceed a constant multiple of ε_n^2, say $\gamma_n \leqq h^2\,\varepsilon_n^2$. Let Schwarz's inequality be applied with $[\varrho(x)]^{-1/2}$ and $[\varrho(x)]^{1/2}\,|f(x) - s_n(x)|$ as factors under the integral sign:

$$\left[\int_{-\pi}^{\pi} |f(x) - s_n(x)|\,dx\right]^2 \leqq I^2\gamma_n,$$

I having the same meaning as before. It follows that

$$\int_{-\pi}^{\pi} |f(x) - s_n(x)|\,dx \leqq Ih\varepsilon_n. \tag{25}$$

If this integral is denoted by g_n it is easily shown[11] that

$$|f(x) - s_n(x)| \leqq 4\,ng_n + 5\,\varepsilon_n,$$

which means, in consequence of (25), that $s_n(x)$ converges uniformly toward $f(x)$ if $\lim_{n\to\infty} n\,\varepsilon_n = 0$. And it will be possible to construct sums $t_n(x)$ so that this condition is satisfied, if $f(x)$ has a continuous derivative.[12] The proof of convergence thus obtained applies even at points where $\varrho(x)$ vanishes.

[11] See Colloquium, pp. 87–88. It is to be observed that the function denoted by $\varrho(x)$ in the passage cited is not the $\varrho(x)$ of the present problem, but is to be taken as 1, while the sum $T_n(x)$ there is to be taken as the present $s_n(x)$, and $m = 1$.

[12] See Colloquium, p. 12, Theorem IV, Corollary.

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INTEGRALS OVER SURFACES IN PARAMETRIC FORM.[1]

By E. J. McShane.[2]

Introduction. In a previous paper[3] I have investigated certain properties of the class of all rectifiable surfaces; i. e., of surfaces having representations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ in which the functions x, y, z satisfy a Lipschitz condition. The principal aim of that study was to obtain conditions sufficient to insure the semi-continuity of integrals $\int\int f(x, y, z, X, Y, Z)\, du\, dv$, where X, Y, Z are the three jacobians of x, y, z with respect to u, v.

In the present paper a similar investigation is conducted for a wider class of surfaces, including for example surfaces for which the derivatives x_u, etc., are summable together with their squares, and also surfaces $z = z(x, y)$ which are absolutely continuous in the sense of Tonelli. Along with theorems concerning semi-continuity, we find that for our class of surfaces the Lebesgue area is given by the classical double integral. The consideration of this wider class of surfaces is desirable not merely for the sake of generalization, but also because of the fact that the important class of saddle surfaces bounded by a Jordan curve and of finite area is not included in the class of rectifiable surfaces, but is included (as will be shown in a later paper) in the class of surfaces for which x_u^2, etc., are summable.

Since this paper is intended as a sequel to the preceding one[4] on double integrals, we retain the notation and definitions of that paper, and request the reader to read the first portion of it as an introduction. The first part of the present paper is concerned with the definition of the class of surfaces considered and with the statement of the problem. In the second part we obtain some lemmas on approximation by polyhedra which enable us to reduce the problem to a simpler one. The third part contains the statement of the principal theorems of the paper. The fourth part is devoted in part to the presentation of certain sets of conditions on the functions x, y, z which are sufficient to assure us that the surface $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ belongs to our class of surfaces, and in part to the indication of certain fragmentary extensions of our theorems.

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[1] Received August 7, 1932.

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[3] *On the semi-continuity of double integrals of the calculus of variations*, Annals of Mathematics, vol. 33 (1932), pp. 460–484.

[4] Loc. cit. [3].

Concerning notation, we sometimes find it convenient to utilize the notation $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ for surfaces, with X, Y and Z denoting the jacobians; this avoids excessive use of subscripts. On the other hand, it is often desirable for compactness to replace the symbols x, y, z, X, Y, Z, u, v by $x^1, x^2, x^3, X^1, X^2, X^3, u^1, u^2$ respectively, and use the single letter x to denote the triple (x^1, x^2, x^3), and similarly for X and u. These notations will be regarded as interchangeable; we will use whichever seems more convenient in each section.

PART I. Surfaces of type C.

1. Let $x(u, v)$, $y(u, v)$, $z(u, v)$ be a system of three functions defined and continuous on a point set B consisting of a Jordan curve and its interior. We say then that the equations

$$(1.1) \qquad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \quad \text{on} \quad B$$

form a representation of a continuous surface. The letters X, Y, Z will designate the three jacobians

$$X = \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, \qquad Y = \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix}, \qquad Z = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}.$$

Suppose now that the six partial derivatives $x_u, \cdots, z_v$ are all defined almost everywhere in B, and are summable[5] over B. We cannot conclude that X, Y, and Z are summable over B, since the product of two summable functions is not necessarily summable.

However, let us consider a rectangle r: $a \leqq u \leqq b$, $c \leqq v \leqq d$ interior to B. If we choose $\varepsilon > 0$ small enough, the rectangle $a - \varepsilon \leqq u \leqq b + \varepsilon$, $c - \varepsilon \leqq u \leqq d + \varepsilon$ will also lie in B. It is then known[6] that the function

$$(1.2) \qquad Z^{(x)}(u, v; h, k) = x_u(u+h, v+k)\, y_v(u, v) - x_v(u+h, v+k)\, y_u(u, v)$$

is summable over r for almost all numbers (h, k) such that $|h| < \varepsilon$, $|k| < \varepsilon$, and moreover the quadruple integral

$$(1.3) \qquad \int_\alpha^\beta \int_\gamma^\delta \int_a^b \int_c^d Z^{(x)}(u, v; h, k)\, dv\, du\, dk\, dh$$

---

[5] That is, summable over B when assigned arbitrary values over the null-set at which they are undefined. We shall throughout this paper continue to say that a function f, defined almost everywhere in a set E, is summable over E, provided that it is summable over E when set equal to zero where undefined.

[6] W. H. Young, *A new set of conditions for a formula for an area*, Proc. London Math. Soc., vol. 21 (1922), p. 91. The proof in Hobson, *Theory of Functions of a Real Variable*, vol. II, § 394, is for one variable, but can easily be extended to the general case.

exists, provided that $-\varepsilon \leqq \alpha < \beta \leqq \varepsilon$, $-\varepsilon \leqq \gamma < \delta \leqq \varepsilon$. A like statement holds true for

$$(1.4)\quad Z^{(y)}(u, v; h, k) = x_u(u, v)\, y_v(u+h, v+k) - x_v(u, v)\, y_u(u+h, v+k),$$

and also for the functions $X^{(y)}$, $X^{(z)}$ defined by replacing x, y by y, z respectively in the right members of (1.2), (1.4), and also for $Y^{(z)}$, $Y^{(x)}$, defined by replacing x, y by z, x respectively in the right members of (1.2), (1.4).

It follows immediately that if the partial derivatives $x_u, \cdots, z_v$ and the jacobians X, Y, Z are summable over B, then the integral

$$(1.5)\quad \int_\alpha^\beta \int_\gamma^\delta \int_a^b \int_c^d |\, Z^{(x)}(u, v; h, k) - Z(u, v)\,|\, dv\, du\, dk\, dh$$

and its five analogues for $Z^{(y)}$, $X^{(y)}$, etc., all exist. This fact being established, we are able to proceed to the definition of the class of surfaces with which most of the remainder of this paper will deal.

2. The class of surfaces with which we shall be chiefly concerned are those possessing representations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ satisfying the following conditions.

(2.1 a) *For almost all fixed values u_0 of u the functions $x(u_0, v)$, $y(u_0, v)$, $z(u_0, v)$ are absolutely continuous functions of v on each of the intervals of the line $u = u_0$ interior to B.*

(2.1 b) *For almost all fixed values v_0 of v the functions $x(u, v_0)$, $y(u, v_0)$, $z(u, v_0)$ are absolutely continuous functions of u on each of the intervals of the line $v = v_0$ interior to B.*

(2.1 c) *The derivatives*[7] *x_u, x_v, y_u, y_v, z_u, z_v and the jacobians X, Y, Z are summable over B.*

(2.1 d) *For every rectangle r: $a \leqq u \leqq b$, $c \leqq v \leqq d$ interior to B we have*

$$(2.2)\quad \lim \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} \int_a^b \int_c^d |\, Z^{(x)}(u, v; h, k) - Z(u, v)\,|\, dv\, du\, dk\, dh = 0,$$

together with a like statement[8] *for $Z^{(y)}$, $X^{(y)}$, $X^{(z)}$, $Y^{(z)}$ and $Y^{(x)}$.*

A surface S possessing a representation satisfying conditions (2.1) will be said to be *of type C*, and each representation of S which satisfies conditions (2.1) will be called a *typical representation.*

Since all of our theorems will concern surfaces of type C, it behooves us to examine the generality of this class. However, we do not wish to

[7] If (2.1a) and (2.1b) are satisfied, these derivatives and jacobians are defined almost everywhere in B.

[8] It can be shown that if (2.2) holds for $Z^{(x)}$, it necessarily holds for $Z^{(y)}$, and conversely; likewise for the other two pairs ($X^{(y)}$, $X^{(z)}$) and ($Y^{(z)}$, $Y^{(x)}$). But we shall not stop to prove this.

interrupt our progress at this point, and therefore the discussion of conditions (2.1) will be postponed to § 16. The reader may if he wishes turn immediately to that section, since it can be read independently of the rest of the paper.

3. Suppose[9] now that $f(x, X)$ is defined for all points (x) of a closed set A of $(x^1x^2x^3)$-space and for all numbers X^1, X^2, X^3, and satisfies the usual conditions[10] of continuity and homogeneity. Then if $x = x(u)$, u on B is a typical representation of a surface S of type C lying in the set A, we can show that the integral

$$\int_B f(x(u), X(u))\, du \tag{3.1}$$

exists. For let M be the maximum value of $f(x, X)$ on the bounded closed set[11] [x on S, X such that $\|X\| = 1$]; then

$$|f(x(u), X(u))| \leqq M \cdot \|X\| \leqq M(|X^1| + |X^2| + |X^3|).$$

By (2.1c) the right member of the inequality is summable over B, and therefore the integral (3.1) exists.

We now turn our attentions to sequences of surfaces of type C approaching a surface of type C as limit. For definiteness of notation we consider surfaces S_n $(n = 1, 2, \cdots)$ and S_0, having representations

$$S_0: \quad x = x_0(u), \qquad u \text{ on } B_0, \tag{3.2}$$

$$S_n: \quad x = x_n(u), \qquad u \text{ on } B_n, \tag{3.3a}$$

in which the functions $x_0(u)$ and $x_n(u)$ satisfy conditions (2.1), and in which each region B_n can be mapped on B_0 by a topological mapping T_n of positive index such that for all points u, $\overline{u}$ (of B_0, B_n respectively) corresponding under T_n we have

$$\|x_0(u), x_n(\overline{u})\| \leqq \eta_n, \quad \text{where } \lim \eta_n = 0. \tag{3.3b}$$

All of our theorems depend on our proving that if the function $f(x, X)$ satisfies certain conditions, then

$$\liminf \int_{B_n} f(x_n, X_n)\, du \geqq \int_{B_0} f(x_0, X_0)\, du. \tag{3.4}$$

It is convenient for us to prove that if

$$f(x, X) \geqq 0 \tag{3.5}$$

[9] We here replace the symbols x, y, z, X, Y, Z, u, v used heretofore by $x^1, x^2, x^3, X^1, X^2, X^3, u^1, u^2$ respectively.

[10] Loc. cit. [3], conditions (2.1). This paper (loc. cit. [3]) will henceforth be referred to as DI.

[11] As in DI, $\|X\| = [(X^1)^2 + (X^2)^2 + (X^3)^2]^{1/2}$.

for all x in a neighborhood of the points of S_0 and all X, then in order to prove that inequality (3.4) holds for all systems (3.3) we have only to show that (3.4) holds for systems of a simpler type than (3.3).

A first simplification is easily obtained, and can be thus stated: Given a surface S_0 of type C on which (3.5) is satisfied; if there exists a sequence $\{S_n\}$ satisfying (3.3) for which

$$\liminf \int_{B_n} f(x_n, X_n)\, du < \int_{B_0} f(x_0, X_0)\, du, \tag{3.6}$$

then there exists a square Q contained in B_0 and a sequence $\{\bar{S}_n\}$ of surfaces $x = \bar{x}_n(u)$, u on $\bar{B}_n$ satisfying (3.3) and such that

$$\liminf \int_{\bar{B}_n} f(\bar{x}_n, \bar{X}_n)\, du < \int_{Q} f(x_0, X_0)\, du. \tag{3.7}$$

The proof is identical with that of the first simplification in DI, p. 466, lines 3 to 12.

Further simplifications depend on our being able to approximate surfaces of type C by polyhedra, and will be deferred to Part II.

PART II. Approximation by Polyhedra.

4. The principal objects of this part of the paper are first to express the integral of the jacobian $\partial(x, y)/\partial(u, v)$ over a square in terms of a line integral around the boundary of the square, and second to show that the sequence $\{S_n\}$ in inequality (3.6) can be considered to be a sequence of polyhedra. To attain these objects we establish a sequence of lemmas. The first lemma enables us to turn our attention to a simpler type of region B.

LEMMA 1. *If $x^i = x^i(u)$, u on B, is a typical representation of a surface S of type C, then for every $\varepsilon > 0$ we can find a simply connected region $\bar{B}$, contained in B and consisting of a finite number of rectangles, such that*

$$\int_{B-\bar{B}} \|X\|\, du\, dv < \varepsilon, \tag{4.1}$$

$$\|\bar{S}, S\| < \varepsilon, \tag{4.2}$$

where S is the surface $x^i = x^i(u)$, u on $\bar{B}$.

Let the functions $\bar{u}^i = \bar{u}^i(u)$, u on B, map the region B conformally on the unit circle K_1: $\|\bar{u}\| \leqq 1$. The functions $\bar{x}^i(\bar{u}) \equiv x^i(u(\bar{u}))$ are uniformly continuous on K_1, hence there exists a $\delta > 0$ such that if $\|\bar{u}_1, \bar{u}_2\| < \delta$, then

$$\|\bar{x}(\bar{u}_1), \bar{x}(\bar{u}_2)\| < \frac{\varepsilon}{2}. \tag{4.3}$$

Also, since $\|X\|$ is summable over B, there exists a $\beta>0$ such that for every set E contained in B and of measure $m(E)<\beta$ we have

$$\int_E \|X\|\, du < \frac{\varepsilon}{2}. \tag{4.4}$$

Corresponding to the circle K_r: $\|\bar{u}\| \leqq r$ there is a subregion B_r of B, bounded by an analytic Jordan curve. Let r be a number such that $1-\delta<r<1$, and large enough so that $m(B-B_r)<\beta$. Then by (4.4) we have

$$\int_{B-B_r} \|X\|\, du < \frac{\varepsilon}{2}; \tag{4.5}$$

and if we map each point (ϱ, θ) of K_1 on the point $(r\varrho, \theta)$ of K_r, we find by (4.3) that

$$\|S_r, S\| < \frac{\varepsilon}{2}, \tag{4.6}$$

S_r being the surface $x^i = x^i(u)$, u on B_r.

The functions $x^i(u)$ are uniformly continuous on B, hence there exists a $\gamma>0$ such that if $\|u_1, u_2\|<\gamma$, then

$$\|x(u_1), x(u_2)\| < \frac{\varepsilon}{2}. \tag{4.7}$$

Since the boundary curve of B_r is analytic, we can construct a Jordan polygon each of whose sides is parallel to one or the other of the u^i-axes, and with alternate vertices on the boundary of B_r; and we may furthermore require that the sides of this polygon be small enough so that the region $\bar{B}$ consisting of polygon and interior lies interior to B, and can be mapped on B_r by a topological mapping under which corresponding points of B_r and $\bar{B}$ have distance $<\gamma$. Moreover, if the sides are small enough we have $m(B_r-\bar{B})<\beta$. This construction is quite simple, and will not be discussed in detail. The region $\bar{B}$ satisfies the conditions required; for by (4.7) we have $\|S_r, \bar{S}\|<\varepsilon/2$, where $\bar{S}$ is defined as in the statement of the lemma; hence by (4.6) this implies $\|S, \bar{S}\|<\varepsilon$. And (4.1) is satisfied, as follows from (4.5), (4.4) and the inequality $m(B_r-\bar{B})<\beta$.

5. For the next few sections we shall revert to the (x, y, z) notation, as mentioned in the last paragraph of the introduction.

We shall several times have occasion to use integrals taken around the boundary of a square Q. In order to have a uniform notation for such integrals, we map the boundary C of the square Q upon the interval $0 \leqq t \leqq 4$, beginning at the origin and traversing C counter-clockwise, in such a way that the four sides are mapped linearly on the intervals (0, 1),

(1, 2), (2, 3), (3, 4) respectively. If we are given a pair of functions $x(u, v)$, $y(u, v)$, defined on C, we define $x(t) \equiv x(u, v)$ and $y(t) \equiv y(u, v)$, where (u, v) is the point on C corresponding to t.

With this notation, we proceed to prove

LEMMA 2. *If the functions $x(t)$, $y(t)$ are absolutely continuous on C, then*

$$\iint \mu(x, y)\, dx\, dy = \int_0^4 x(t)\, y'(t)\, dt = -\int_0^4 y(t)\, x'(t)\, dt = \frac{1}{2}\int_0^4 (xy' - yx')\, dt, \tag{5.1}$$

where $\mu(x, y)$ is the index[12] *of the point (x, y) with respect to the curve $x = x(t)$, $y = y(t)$.*

We shall prove the first equality; the second follows from the first by an integration by parts, and the third is a consequence of the others. First[13] we suppose that $x(t)$ and $y(t)$ satisfy a Lipschitz condition. Then it is easy to construct functions $x(u, v)$, $y(u, v)$ which satisfy a Lipschitz condition on Q and coincide with $x(t)$, $y(t)$ on C. On the one hand, Young[14] has proved

$$\iint_Q \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv = \int_0^4 x(t)\, dy(t) = \int_0^4 x\, y'\, dt; \tag{5.2}$$

on the other hand, Schauder[15] has shown that

$$\iint_Q \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv = \iint \mu(x, y)\, dx\, dy; \tag{5.3}$$

whence equation (5.1) follows.

In case x and y are not Lipschitzian, we introduce the new variable

$$\tau \equiv \frac{4\left(t + \int_0^t [x'^2 + y'^2]^{1/2}\, dt\right)}{4 + \int_0^4 [x'^2 + y'^2]^{1/2}\, dt}.$$

The inverse function $t = t(\tau)$ is monotonic and Lipschitzian, and for each point (x, y) the index $\mu_1(x, y)$ with respect to the curve $x = x(t(\tau))$, $y = y(t(\tau))$ is equal to the index $\mu(x, y)$ with respect to the curve $x = x(t)$, $y = y(t)$; hence by what we have just proved

[12] DI, footnote [12].

[13] A direct proof is fairly easy to construct, but in order to save space we adopt the proof here given.

[14] W. H. Young, *On a formula for an area,* Proc. London Math. Soc., 18 (1919), p. 339; in particular, Theorem 5.

[15] J. Schauder, *Über stetige Abbildungen,* Fund. Math. 12 (1928), p. 47; in particular, Satz VIII.

$$\text{(5.4)}\qquad \begin{aligned}\iint \mu(x, y)\, dx\, dy &= \int_0^4 x\,(t(\tau))\,\frac{d}{d\tau}\,y\,(t(\tau))\,d\tau \\ &= \int_0^4 x\,(t(\tau))\,\frac{d}{dt}\,y(t)\,\frac{dt}{d\tau}\,d\tau = \int_0^4 x(t)\,y'(t)\,dt,\end{aligned}$$

where in the last integral we have merely changed variables[16]. The lemma is thus established.

6. Let us now suppose that the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ satisfy conditions (2.1) on a region $\bar{B}$, and that B is a region interior to $\bar{B}$ and consisting of a finite number of rectangles $r_1, \cdots, r_m$. Then for every point (u, v) of B the point $(u+h, v+k)$ is in $\bar{B}$, provided that $|h|$ and $|k|$ are less than a certain $\delta > 0$.

Now for every number ξ such that $0 < \xi < \delta$ we define

$$\text{(6.1a)}\qquad \bar{x}(u, v) \equiv \frac{1}{4\xi^2}\int_{u-\xi}^{u+\xi}\int_{v-\xi}^{v+\xi} x(u, v)\, du\, dv;$$

likewise we define

$$\text{(6.1b)}\qquad \bar{y}(u, v) = \frac{1}{4\eta^2}\int_{v-\eta}^{v+\eta}\int_{u-\eta}^{u+\eta} y(u, v)\, du\, dv,$$

$$\text{(6.1c)}\qquad \bar{z}(u, v) = \frac{1}{4\zeta^2}\int_{v-\zeta}^{v+\zeta}\int_{u-\zeta}^{u+\zeta} z(u, v)\, du\, dv,$$

where $0 < \eta < \delta$, $0 < \zeta < \delta$.

Since $x(u, v)$ is uniformly continuous on $\bar{B}$, the integral

$$\text{(6.2)}\qquad \int_{v-\xi}^{v+\xi} x(u, v)\, du\, dv$$

is a continuous function of u and v; so by (2.1 b) we have

$$\text{(6.3)}\qquad \begin{aligned}\frac{\partial \bar{x}}{\partial u} &= \frac{1}{4\xi^2}\int_{v-\xi}^{v+\xi} [x(u+\xi, v) - x(u-\xi, v)]\, dv \\ &= \frac{1}{4\xi^2}\int_{v-\xi}^{v+\xi}\int_{u-\xi}^{u+\xi} \frac{\partial x}{\partial u}\, du\, dv,\end{aligned}$$

with analogous expressions for x_v, y_u, etc. Equations (6.3) make it evident that for fixed values of ξ, η, ζ the functions $\bar{x}$, $\bar{y}$, $\bar{z}$ have continuous partial derivatives.

From the uniform continuity of $x(u, v)$ we have immediately

$$\text{(6.4a)}\qquad \lim_{\xi \to 0} \bar{x}(u, v) = x(u, v),$$

[16] Carathéodory, *Vorlesungen über Reelle Funktionen*, § 499, Satz 4; § 495, Satz 12.

the convergence being uniform over B. Likewise

$$\lim_{\eta\to 0}\bar{y}(u,v)=y(u,v),\qquad \lim_{\zeta\to 0}\bar{z}(u,v)=z(u,v), \tag{6.4b}$$

the convergence in each case being uniform over B.

7. We shall now utilize the approximating functions of § 6 to prove

LEMMA 3. *Let the functions $x(u,v)$, $y(u,v)$, $z(u,v)$ satisfy conditions (2.1) on a region B, and let S be the surface represented by the equations*

$$x=x(u,v),\quad y=y(u,v),\quad z=z(u,v),\quad (u,v)\text{ on }B,$$

where B is a region interior to $\bar{B}$ and consisting of a finite number of rectangles $r_1,\cdots,r_m$. Then for every $\varepsilon>0$ we can find a surface

$$\bar{S}\colon\ x=\bar{x}(u,v),\quad y=\bar{y}(u,v),\quad z=\bar{z}(u,v),\quad (u,v)\text{ on }B \tag{7.1}$$

satisfying the conditions:

$$\text{all six partial derivatives }\frac{\partial\bar{x}}{\partial u},\text{ etc., are continuous on }B; \tag{7.2}$$

$$[(\bar{x}(u)-x(u))^2+(\bar{y}(u)-y(u))^2+(\bar{z}(u)-z(u))^2]^{1/2}<\varepsilon\text{ for every }u\text{ in }B; \tag{7.3}$$

$$\iint_B(|\bar{X}-X|+|\bar{Y}-Y|+|\bar{Z}-Z|)\,du\,dv<\varepsilon. \tag{7.4}$$

We prove this by successive modifications of x, y and z. Defining $\bar{x}$ by equation (6.1a) we have by (6.3) and its analogue for $\bar{x}_v$

$$\begin{aligned}&\iint_B\left|\frac{\partial(\bar{x},y)}{\partial(u,v)}-\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv\\&=\sum_{j=1}^{m}\frac{1}{4\xi^2}\iint_B\int_{-\xi}^{\xi}\int_{-\xi}^{\xi}|x_u(u+h,v+k)y_v(u,v)-x_v(u+h,v+k)y_u(u,v)\\&\qquad -x_u(u,v)\,y_v(u,v)+x_v(u,v)\,y_u(u,v)|\,dh\,dk\,du\,dv\\&=\sum_{j=1}^{m}\frac{1}{4\xi^2}\iint_{r_j}\int_{-\xi}^{\xi}\int_{-\xi}^{\xi}|Z^{(x)}(u,v;h,k)-Z(u,v)|\,dh\,dk\,du\,dv.\end{aligned} \tag{7.5}$$

Hence by (2.1d)

$$\lim_{\xi\to 0}\iint_B\left|\frac{\partial(\bar{x},y)}{\partial(u,v)}-\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv=0. \tag{7.6}$$

Therefore by (7.6) and (6.4) we see that we can choose ξ small enough so that

$$\iint_B\left|\frac{\partial(\bar{x},y)}{\partial(u,v)}-\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv<\frac{\varepsilon}{6}, \tag{7.7}$$

$$|\bar{x}(u,v)-x(u,v)|<\frac{\varepsilon}{3}\text{ for all }(u,v)\text{ on }B. \tag{7.8}$$

Repeating the argument with z in place of y, we obtain

$$\lim_{\xi \to 0} \int\int_B \left| \frac{\partial(z, \overline{x})}{\partial(u, v)} - \frac{\partial(z, x)}{\partial(u, v)} \right| du\, dv = 0, \tag{7.9}$$

so that by (7.9) and (6.4) we can choose ξ small enough so that

$$\int\int_B \left| \frac{\partial(z, \overline{x})}{\partial(u, v)} - \frac{\partial(z, x)}{\partial(u, v)} \right| du\, dv < \frac{\varepsilon}{6}, \tag{7.10}$$

inequalities (7.7) and (7.8) remaining valid.

Now, having fixed ξ, we can by the same argument as above show

$$\lim_{\eta \to 0} \int\int_B \left| \frac{\partial(\overline{y}, z)}{\partial(u, v)} - \frac{\partial(y, z)}{\partial(u, v)} \right| du\, dv = 0. \tag{7.11}$$

To show that

$$\lim_{\eta \to 0} \int\int_B \left| \frac{\partial(\overline{x}, \overline{y})}{\partial(u, v)} - \frac{\partial(\overline{x}, y)}{\partial(u, v)} \right| du\, dv = 0 \tag{7.12}$$

we need only observe that x_u and x_v are continuous, hence bounded; whence it follows[17] that the system $(\overline{x}, y, z)$ satisfies conditions (2.1), and (7.12) follows exactly as (7.10) did. Hence, by choosing η small enough, we can through (7.11), (7.12) and (6.4) obtain

$$\int\int_B \left| \frac{\partial(\overline{y}, z)}{\partial(u, v)} - \frac{\partial(y, z)}{\partial(u, v)} \right| du\, dv < \frac{\varepsilon}{6}, \tag{7.13}$$

$$\int\int_B \left| \frac{\partial(\overline{x}, \overline{y})}{\partial(u, v)} - \frac{\partial(\overline{x}, y)}{\partial(u, v)} \right| du\, dv < \frac{\varepsilon}{6}, \tag{7.14}$$

$$|\overline{y}(u, v) - y(u, v)| < \frac{\varepsilon}{3} \text{ for all } (u, v) \text{ in } B. \tag{7.15}$$

Finally, by another repetition of the argument, the numbers ξ and η being fixed, we obtain a ζ such that

$$\int\int_B \left| \frac{\partial(\overline{z}, \overline{x})}{\partial(u, v)} - \frac{\partial(z, \overline{x})}{\partial(u, v)} \right| du\, dv < \frac{\varepsilon}{6}, \tag{7.16}$$

$$\int\int_B \left| \frac{\partial(\overline{y}, \overline{z})}{\partial(u, v)} - \frac{\partial(\overline{y}, z)}{\partial(u, v)} \right| du\, dv < \frac{\varepsilon}{6}, \tag{7.17}$$

$$|\overline{z}(u, v) - z(u, v)| < \frac{\varepsilon}{3} \text{ for all } (u, v) \text{ in } B. \tag{7.18}$$

This proves our lemma; for (7.3) follows from (7.8), (7.15) and (7.18), while (7.4) follows from (7.7), (7.10), (7.13), (7.14), (7.16) and (7.17).

[17] The proof of this is contained in § 16.

8. The devices of the preceding section enable us also to prove

LEMMA 4. *If the functions $x(u, v)$ and $y(u, v)$ satisfy conditions*[18] (2.1) *on a region B, and q is a square interior to B, and x and y are absolutely continuous on the boundary of q, then*

$$\int\int_q \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv = \int_0^4 xy'\, dt, \tag{8.1}$$

the latter integral being taken around the boundary of q as described in the second paragraph of § 5.

We assume that (8.1) is known to be true whenever the partial derivatives x_u, x_v, y_u, y_v are all continuous;[19] thus defining x, y by equations (6.1), we have

$$\int\int_q \frac{\partial(\bar{x}, \bar{y})}{\partial(u, v)}\, du\, dv = \int_0^4 \bar{x}\bar{y}'\, dt = -\int_0^4 \bar{y}\bar{x}'\, dt. \tag{8.2}$$

Using the square q as the region B of § 7, we have from (7.12) and (7.6)

$$\begin{aligned} \lim_{\xi\to 0}\left\{\lim_{\eta\to 0}\int\int_q \frac{\partial(\bar{x}, \bar{y})}{\partial(u, v)}\, du\, dv\right\} &= \lim_{\xi\to 0}\int\int_q \frac{\partial(\bar{x}, y)}{\partial(u, v)}\, du\, dv \\ &= \int\int_q \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv. \end{aligned} \tag{8.3}$$

On the other hand, by (6.4) we have

$$\begin{aligned} \lim_{\xi\to 0}\left\{\lim_{\eta\to 0}\left(-\int_0^4 \bar{y}(t)\,\bar{x}'(t)\, dt\right)\right\} &= \lim_{\xi\to 0}\left\{-\int_0^4 y(t)\,\bar{x}'(t)\, dt\right\} \\ &= \lim_{\xi\to 0}\left\{\int_0^4 \bar{x}(t)\, y'(t)\, dt\right\} \\ &= \int_0^4 x(t)\, y'(t)\, dt, \end{aligned} \tag{8.4}$$

which with (8.2) and (8.3) proves the lemma.

9. Reverting again to the more compact (x^1, x^2, x^3) notation, we collect several preceding lemmas together with certain results of DI in

LEMMA 5. *Let the function $f(x, X)$ be defined and satisfy the usual conditions of continuity and homogeneity on a bounded closed point set A in (x^1, x^2, x^3) space. If the surface S of type C lies interior to A and is typically represented in the form $x^i = x^i(u)$, u on B, then for every $\varepsilon > 0$ there exists a polyhedron Π typically represented in the form*[20] *$x^i = \bar{x}^i(\bar{u})$, $\bar{u}$ on Q, such that*

[18] I. e., conditions (2.1) with all references to z omitted.

[19] W. H. Young, *On a formula for an area*, Proc. London Math. Soc., vol. 18 (1919), p. 339; or Schauder, loc. cit. [15].

[20] The letter Q is used here as elsewhere to connote a square.

$$\|\Pi, S\| < \varepsilon, \tag{9.1}$$

$$\left| \int_B f(x, X)\, du - \int_Q f(\bar{x}, \bar{X})\, d\bar{u} \right| < \varepsilon. \tag{9.2}$$

Let M be the upper bound of $f(x, X)$ for all x in A and all X such that $\|X\| = 1$. By Lemma 1 we can find a region $\bar{B}$ interior to B and consisting of a finite number of rectangles, such that

$$\int_{B-\bar{B}} \|X\|\, du < \frac{\varepsilon}{4M}, \tag{9.3}$$

$$\|S_1, S\| < \frac{\varepsilon}{4}, \tag{9.4}$$

where S_1 is the surface represented by the equations $x^i = x^i(u)$, u on $\bar{B}$. By (9.3) we have

$$\left| \int_B f(x, X)\, du - \int_{\bar{B}} f(x, X)\, du \right| \leqq \int_{B-\bar{B}} M\|X\|\, du < \frac{\varepsilon}{4}. \tag{9.5}$$

Now let N be an upper bound for $|f_{X^i}(x, X)|$ for all x in A and all X such that $\|X\| = 1$; since f_{X^i} is positively homogeneous of degree 0 in X, this implies

$$|f_{X^i}(x, X)| \leqq N \tag{9.6}$$

for all x and all $X \neq 0$. According to Lemma 4 there exists a surface $\bar{S}$: $x^i = \bar{x}^i(u)$, u on $\bar{B}$, for which all six partial derivatives $\partial \bar{x}^i / \partial u^j$ are continuous, and for which the inequalities

$$\|\bar{S}, S_1\| < \frac{\varepsilon}{4}, \tag{9.7}$$

$$\int_{\bar{B}} (|\bar{X}^1 - X^1| + |\bar{X}^2 - X^2| + |\bar{X}^3 - X^3|)\, du < \frac{\varepsilon}{8N} \tag{9.8}$$

are satisfied. In addition, $f(x, X)$ is uniformly continuous for all x in A and all X such that $\|X\| = 1$; we can therefore find a $\gamma > 0$ such that if $\|x_1, x_2\| < \gamma$, then for every unit vector X the inequality

$$|f(x_1, X) - f(x_2, X)| < \frac{\varepsilon}{8\left(\int_{\bar{B}} \|X\|\, du + 1\right)}$$

holds; and it follows that for all X we have

$$|f(x_1, X) - f(x_2, X)| < \frac{\varepsilon}{8\left(\int_{\bar{B}} \|X\|\, du + 1\right)} \cdot \|X\|.$$

By Lemma 3 we can consider that the inequality $\|\bar{x}(u), x(u)\| < \gamma$ holds for every value of u; hence

$$|f(\bar{x}, \bar{X}) - f(x, X)| \leqq |f(\bar{x}, \bar{X}) - f(\bar{x}, X)| + |f(\bar{x}, X) - f(x, X)|$$

$$\leqq N \cdot \sum_i |\bar{X}^i - X^i| + \|X\| \cdot \frac{\varepsilon}{8\left(\int_{\bar{B}} \|X\|\, du + 1\right)}.$$

Integrating over $\bar{B}$, we find

$$\left|\int_{\bar{B}} f(\bar{x}, \bar{X})\, du - \int_{\bar{B}} f(x, X)\, du\right|$$

$$\leqq N \frac{\varepsilon}{8N} + \int_{\bar{B}} \|X\|\, du \cdot \frac{\varepsilon}{8\left(\int_{\bar{B}} \|X\|\, du + 1\right)} \tag{9.9}$$

$$\leqq \frac{\varepsilon}{4}.$$

The surface S being rectifiable, we can apply the second simplification in DI (p. 466) to obtain[21] a rectifiable surface S_2, represented in the form $x^i = x_2^i(u)$, u on Q, such that

$$\|\bar{S}, S_2\| < \frac{\varepsilon}{4}, \tag{9.10}$$

$$\left|\int_Q f(x_2, X_2)\, du - \int_{\bar{B}} f(\bar{x}, \bar{X})\, du\right| < \frac{\varepsilon}{4}. \tag{9.11}$$

Finally, the proof of the third simplification of DI yields a polyhedron Π, typically represented in the form $x^i = x_3^i(u)$, u on Q, such that

$$\|\Pi, S_2\| < \frac{\varepsilon}{4}, \tag{9.12}$$

$$\left|\int_Q f(x_3, X_3)\, du - \int_Q f(x_2, X_2)\, du\right| < \frac{\varepsilon}{4}. \tag{9.13}$$

By inequalities (9.4) to (9.13), this polyhedron satisfies inequalities (9.1) and (9.2).

Remark 1. From inequalities (9.3) and (9.8) and the portions of DI referred to, we easily see that the polyhedron Π may be required to satisfy the further inequality

$$\int_Q \|\bar{X}\|\, d\bar{u} < \int_B \|X\|\, du + \varepsilon. \tag{9.10}$$

10. Lemma 5 yields the following simplification of our problem:

If the surface S_0: $x^i = x_0^i(u)$, u *on* Q *and the sequence* $\{S_n\}$ *of surfaces* S_n: $x^i = x_n^i(u)$, u *on* B_n *satisfy conditions* (3.2), (3.3) *and* (3.5), *and*

[21] We call the reader's attention to Remark 2, p. 467, DI.

$$\liminf \int_{B_n} f(x_n, X_n)\, du < \int_Q f(x_0, X_0)\, du, \tag{10.1}$$

then there exists a sequence $\{\Pi_n\}$ *of polyhedra, each* Π_n *being typically represented in the form* $x^i = \bar{x}^i_n(u)$, u *on* Q, *satisfying conditions* (3.3), *and also satisfying inequality* (10.1). For we need only choose a sequence $\{\varepsilon_n\}$ of positive numbers tending to zero, and for each n choose a polyhedron Π_n: $x^i = x^i_n(u)$, u on Q (of whose existence Lemma 5 assures us) such that

$$\|\Pi_n, S_n\| < \varepsilon_n, \tag{10.2}$$

$$\left| \int_{B_n} f(x_n, X_n)\, du - \int_Q f(\bar{x}_n, \bar{X}_n)\, du \right| < \varepsilon_n; \tag{10.3}$$

then $\lim \Pi_n = \lim S_n = S_0$, and

$$\liminf \int_Q f(\bar{x}_n, \bar{X}_n)\, du = \liminf \int_{B_n} f(x_n, X_n)\, du < \int_Q f(x_0, X_0)\, du.$$

The fourth simplification of DI can now be utilized without change; we thus obtain another representation of each Π_n, the integrals being unchanged and the new functions defining Π_n tending uniformly to $x^i_0(u)$ on Q.

11. Combining § 10 with the first simplification (end of § 3) we arrive at the following result:

Let the function $f(x, X)$ *be defined and satisfy the usual conditions of continuity and homogeneity on a closed point set* A *of* (x^1, x^2, x^3) *space. Let* S_0 *be a surface of type C interior to* A *and typically represented in the form* $x^i = x^i_0(u)$, u *on* B_0, *and suppose further that*

(3.5) $f(x, X) \geqq 0$ for all x in a neighborhood of the points of S_0 and for all X.

Then in order to prove that inequality (3.4) *holds for all systems satisfying conditions* (3.2) *and* (3.3), *we need only show that for every square* Q *in* B_0 *and every sequence* $x_n(u)$ *of functions defined and quasi-linear on* Q *and converging uniformly on* Q *to* $x_0(u)$ *we have*

$$\liminf \int_Q f(x_n, X_n)\, du \geqq \int_Q f(x_0, X_0)\, du.$$

PART III. The principal theorems.

12. We have in the preceding parts developed the analytic machinery which we are to use in our generalizations of the lemmas and theorems in DI. We begin with a generalization of Lemma 1 of DI.

LEMMA 7. *If*

(a) *the functions* $x_0^i(u)$ *are defined and continuous on a square* q, *and are absolutely continuous on the boundary of* q;

(b) *the derivatives* $\partial x^i / \partial u^j$ *are well-defined almost everywhere in* q, *and satisfy the relations*

$$\begin{aligned}
\int_q X_0^1 \, du &= \frac{1}{2} \int_0^4 (x_0^2 x_0^{3\prime} - x_0^3 x_0^{2\prime}) \, dt, \\
\int_q X_0^2 \, du &= \frac{1}{2} \int_0^4 (x_0^3 x_0^{1\prime} - x_0^1 x_0^{3\prime}) \, dt, \\
\int_q X_0^3 \, du &= \frac{1}{2} \int_0^4 (x_0^1 x_0^{2\prime} - x_0^2 x_0^{1\prime}) \, dt;
\end{aligned} \tag{12.1}$$

(c) *the functions* $x_n^i(u)$ $(n = 1, 2, 3, \cdots)$ *are quasi-linear on* q;

(d) $\lim x_n^i(u) = x_0^i(u)$ *uniformly on* q;

then for every set of constants a_1, a_2, a_3 *we can find a sequence* $\{V_n\}$ *of measurable subsets of* q *such that*

$$\lim \int_{V_n} a_\alpha X_n^\alpha \, du = \int_q a_\alpha X_0^\alpha \, du.$$

There is clearly no loss of generality in assuming $a_\alpha a_\alpha = 1$. Let b_i, c_i be six numbers such that the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

is orthogonal, and apply the transformation

$$\xi^1 = a_\alpha x^\alpha, \quad \xi^2 = b_\alpha x^\alpha, \quad \xi^3 = c_\alpha x^\alpha; \tag{12.2}$$

the inverse transformation is

$$x^i = a_i \xi^1 + b_i \xi^2 + c_i \xi^3 \qquad (i = 1, 2, 3). \tag{12.3}$$

Applying this transformation to each of the sets of functions $x_j(u)$ $(j = 0, 1, 2, \cdots)$, we find that at every point at which all six partial derivatives $\partial x_j^i / \partial u^k$ exist (i.e., almost everywhere in q) we have $\Xi_j^1 = a_\alpha X_j^\alpha$; hence we need only find sets V_n such that

$$\lim \int_{V_n} \Xi_n^1 \, du = \int_q \Xi_0^1 \, du.$$

Now in (12.1) we multiply the three equations by a_1, a_2 and a_3 respectively and add. The left member is, as we have seen,

$$\int_q \Xi_0^1 \, du;$$

and if in the right member we make the substitution (12.3) and collect terms, we find

$$\int_q \Xi_0^1 \, du = \frac{1}{2} \int_0^4 (\xi_0^2 \xi_0^{3\prime} - \xi_0^3 \xi_0^{2\prime}) \, dt. \tag{12.4}$$

In the rest of this proof we shall replace ξ^i, Ξ^i by x^i, X^i respectively;[22] then by Lemma 3 we find

$$\int_q X_0^1 \, du = \int\int \mu_0(x^2, x^3) \, dx^2 \, dx^3, \tag{12.5}$$

where $\mu_0(x^2, x^3)$ is the order of the point (x^2, x^3) of the (x^2, x^3)-plane with respect to the curve Γ_0: $x^2 = x_0^2(t)$, $x^3 = x_0^3(t)$, $0 \leqq t \leqq 4$, which is the image in the (x^2, x^3)-plane of the boundary of q under the transformation $x^i = x_0^i(u)$ $(i = 2, 3)$. We likewise define $\mu_n(x^2, x^3)$ as the index of the point (x^2, x^3) with respect to the curve Γ_n: $x^2 = x_n^2(t)$, $x^3 = x_n^3(t)$, $0 \leqq t \leqq 4$.

Having established equation (12.5), we are now able to apply the proof of Lemma 1 of DI without change, beginning with the last paragraph on page 469, and thus conclude the proof of our Lemma 7.

13. From this point on we are able to follow the development of DI from § 5 to the end, with only trivial changes. We shall content ourselves with stating the generalizations of our theorems and indicating such changes in the proofs as may be required.

LEMMA 8. *Let the function $f(x, X)$ be defined and satisfy the usual conditions of continuity and homogeneity* (DI, 2.1) *on a closed point set A of (x^1, x^2, x^3)-space. Let $x^i = x_0^i(u)$, u on B_0, be a typical representation of a surface S_0 of type C interior to A. Let $f(x, X)$ satisfy the conditions*

(13.1 a) *$f(x, X) \geqq 0$ for every x in some neighborhood of the points of S_0, and for every X;*

(13.1 b) *$E(x_0(u), X_0(u), \overline{X}) \geqq 0$ for almost all points u in B_0 at which $X_0(u)$ is defined and $\| X_0(u) \| \neq 0$, and for all $\overline{X}$.*

Then for every sequence $\{S_n\}$ of surfaces of type C typically represented in the form S_n: $x^i = x_n^i(u)$, u on B_n, for which

$$\lim S_n = S_0, \qquad \int_{B_n} \| X_n \| \, du \leqq N$$

(N a constant independent of n) it is true that

$$\liminf \int_{B_n} f(x_n, X_n) \, du \geqq \int_{B_0} f(x_0, X_0) \, du. \tag{13.2}$$

[22] This is a mere change of notation; we do *not* mean a substitution of the expressions (12·2).

By § 11 it is sufficient to show that (13.2) holds for all systems

$$S_0\colon\ x^i = x_0^i(u),\quad u \text{ on } Q, \tag{13.3}$$

$$S_n\colon\ x^i = x_n^i(u),\quad u \text{ on } Q, \tag{13.4}$$

in which Q is an arbitrary square in B_0, the functions $x_n^i(u)$ are quasi-linear, and

$$\lim x_n^i(u) = x_0^i(u) \text{ uniformly on } Q. \tag{13.5}$$

We can also assume $m(Q) = 1$, for Q can be mapped linearly on the unit square $0 \leqq u^i \leqq 1$, the hypotheses remaining valid. We now proceed with the proof as given, requiring however that the squares $q_1, \cdots, q_m$ be so chosen that the $x_0^i(u)$ are absolutely continuous on their boundaries. The reference to Lemma 1 is of course replaced by a reference to Lemma 7 of this paper. To establish inequalities (5.11) and (5.12) of DI, we have only to observe that the functions of sets

$$\int_E X_0^i(u)\,du, \qquad \int_E f(x_0(u),\ X_0(u))\,du$$

are absolutely continuous, hence for all points u_0 of a set E_1 of measure 1 they possess derivatives equal respectively to $X_0^i(u_0)$, $f(x_0(u_0),\ X_0(u_0))$; from this the inequalities follow immediately.

Now the proof of Theorems I—IV of DI can be used without change to establish the following theorems:

THEOREM I. *Let $f(x, X)$ be defined and satisfy conditions* (DI 2.1) *on a closed point set A of (x^1, x^2, x^3)-space. Let the surface S of type C be interior to A, and have the two typical representations*

$$S\colon\ x^i = x^i(u),\quad u \text{ on } B, \tag{13.6}$$

$$S\colon\ x^i = \bar{x}^i(\bar{u}),\quad \bar{u} \text{ on } \bar{B}. \tag{13.7}$$

Then

$$\int_{\bar{B}} f(\bar{x}, \bar{X})\,du = \int_B f(x, X)\,du. \tag{13.8}$$

This theorem permits us to use the symbol $F(S)$ to denote the integral $\int_B f(x, X)\,du$, which is now seen to depend only on S and not on the particular typical representation used.

THEOREM II. *If the surface S be of type C and typically represented in the form S: $x^i = x^i(u)$, u on B, then*

$$L(S) = \int_B \|X\|\,du, \tag{13.9}$$

where $L(S)$ denotes the area (in the sense of Lebesgue) of the surface S.

THEOREM III. *Let the function $f(x, X)$ be defined and satisfy conditions* (DI 2.1) *on a closed point set A of (x^1, x^2, x^3)-space. Let $x^i = x_0^i(u)$, u on B_0, be a typical representation of a surface S_0 of type C interior to A. Let $f(x, X)$ satisfy the conditions* (13.1). *Then for every positive number N the functional $F(S) = \int f(x, X)\, du$ is lower semi-continuous at S_0 with respect to the class of all surfaces of type C having area $\leqq N$.*

14. In generalizing Theorems V and VI of DI we encounter no difficulty. Introducing as in DI the notation ${}_uX$ for unit vectors, so that for every vector $\bar{X}$ for which $\|\bar{X}\| \neq 0$ we have ${}_u\bar{X} = \bar{X}/\|\bar{X}\|$, we obtain

THEOREM V. *If in the hypotheses of Theorem* III *we replace* (13.1 a) *by the condition*

(14.1) $f(x_0, {}_uX) > 0$ *for every point x_0 on S_0 and every ${}_uX$,*

then $F(S)$ is lower semi-continuous at S_0 with respect to the class of all surfaces of type C interior to A.

As a corollary we have

THEOREM VI. *If $F(S)$ is positive definite and positive quasi-regular on A, then it is lower semi-continuous on the class of all surfaces of type C interior to A.*

We now proceed to the proof of

THEOREM VII. *Let the function $f(x, X)$ be defined and satisfy conditions* (DI 2.1) *on a closed point set A of (x^1, x^2, x^3)-space. Let $x^i = x_0^i(u)$, u on B_0, be a typical representation of a surface S_0 of type C interior to A. Let $f(x, X)$ satisfy the conditions:*

(14.2) $f(x, X) \geqq 0$ *for every x in a neighborhood of the points of S_0, and for every X;*

(14.3) $E(x_0(u), X_0(u), {}_uX) > 0$ *for almost every point u of B_0 at which $X_0(u)$ is defined and $\|X_0(u)\| \neq 0$, and for every ${}_uX \neq {}_uX_0(u)$. Then $F(S)$ is lower semi-continuous at S_0 with respect to the class of all surfaces of type C interior to A.*

As before, we need only prove that for every square Q in B_0 and every sequence $\{x_n(u)\}$ of functions quasi-linear on Q and converging uniformly to $x_0(u)$ we have

$$\liminf \int_Q f(x_n, X_n)\, du \geqq \int_Q f(x_0, X_0)\, du;$$

and as before we can consider that Q is the unit square $0 \leqq u^i \leqq 1$. Let now ε be an arbitrary positive number less than 1. We define E_N as the set of all points u of Q at which $X_0(u)$ is defined, inequality (14.3) holds, and $\|X_0(u)\| < N$. Since $\|X_0(u)\|$ is summable, the measure of $Q - E_N$ tends to zero with $1/N$; hence if N be chosen sufficiently large we have

$$\left| F(S_0) - \int_{E_N} f(x_0, X_0)\, du \right| \leqq \frac{\varepsilon}{16}. \tag{14.4}$$

We now denote by E_1 the subset of E_N on which $f(x_0(u), X_0(u)) \geqq \varepsilon/16$; then by (14.2) and (14.4) we have

$$\begin{aligned} \int_{E_1} f(x_0, X_0)\, du \leqq F(S_0) &= \int_{E_1} + \int_{E_N - E_1} + \int_{Q - E_N} f(x_0, X_0)\, du \\ &\leqq \int_{E_1} f(x_0, X_0)\, du + \frac{\varepsilon}{8}. \end{aligned} \tag{14.5}$$

Denote by $\alpha(u)$ the angle between the vectors $X_0(u)$ and $f_{X^\alpha}(x_0(u), X_0(u))$; on E_1 we have

$$\begin{aligned} \frac{\varepsilon}{16} \leqq f(x_0, X_0) = X_0^\alpha f_{X^\alpha}(x_0, X_0) &= \|X_0\| \cdot \|f_X(x_0, X_0)\| \cdot \cos\alpha(u) \\ &\leqq N \cos\alpha(u) \cdot \|f_X(x_0, X_0)\|, \end{aligned} \tag{14.6}$$

in which $\|f_X(x_0, X_0)\|$ denotes $[f_{X^\alpha}(x_0, X_0) \cdot f_{X^\alpha}(x_0, X_0)]^{1/2}$. On the one hand $\|f_X(x_0, X_0)\|$ is bounded,[23] hence there exists a constant $\theta > 0$ such that on E_1

$$0 \leqq \alpha(u) \leqq \frac{\pi}{2} - 2\theta; \tag{14.7}$$

on the other hand, $\|f_X(x_0, X_0)\| \geqq L = \varepsilon/16N$ for all points u in E_1.

We now turn to DI, and follow the proof there given, beginning just below equation (7.7). We justify inequalities (7.10) and (7.11) as in § 13. But we define H, not as an upper bound for $f(x_0, X)$ (which might not exist) but as $1 + F(S_0)$. To inequality (DI 7.15) we adjoin the inequality

$$\begin{aligned} F(S_0) &\geqq \sum_{j=1}^{h} \int_{q_j} f(x_0(u), X_0(u))\, du \\ &\geqq \sum_{j=1}^{h} \int_{q_j} f(x_0(u_j), X_0(u_j))\, du - \frac{\varepsilon}{8}, \end{aligned} \tag{14.8}$$

which follows from (14.2) and (DI 7.10).

Neglecting the last stage in inequality (DI 7.21), we have

$$\int_{T_{1,j,n}} \|X_n\|\, du \leqq \left\{\frac{2}{L \sin\theta}\right\} \left\{\frac{\varepsilon\, m(q_j)}{8} + \int_{q_j} f(x_0(u_j), X_0(u_j))\, du\right\}. \tag{14.9}$$

Summing for $j = 1, \cdots, h$ we have by (14.8)

$$\begin{aligned} \sum_{j=1}^{h} \int_{T_{1,j,n}} \|X_n\|\, du &\leqq \left\{\frac{2}{L \sin\theta}\right\} \left\{\frac{1}{8} + \sum_{j=1}^{h} \int_{q_j} f(x_0(u_j), X_0(u_j))\, du\right\} \\ &\leqq [F(S_0) + 1] \cdot \frac{2}{L \sin\theta} = \frac{2H}{L \sin\theta}. \end{aligned} \tag{14.10}$$

[23] See inequality (9.6), or (DI 5.10).

Inequality (14.10) enables us to argue in identity (DI 7.23) that the sum for $j = 1, \cdots, h$ of the integrals called i_1 is not less than $-\varepsilon/8$. The remarks concerning $i_2, \cdots, i_6$ are unchanged, and the theorem is thus demonstrated.

As an immediate corollary we have

THEOREM VIII. *If for every x in A, every unit vector ${}_uX$, and every unit vector ${}_uX^* \neq {}_uX$ we have*

$$f(x, {}_uX) \geqq 0, \qquad E(x, {}_uX, {}_uX^*) > 0,$$

then $F(S)$ is lower semi-continuous on the class of all surfaces of type C interior to A.

15. Concerning §§ 8, 9 of DI, we shall only remark that they remain valid if we replace everywhere the words "rectifiable surface" by "surface of type C".

PART IV. Addenda.

16. The surfaces considered in the preceding theorems have all been of type C, and the representations $x^i = x^i(u)$, u on B, have been assumed to satisfy conditions (2.1). It is therefore important to have some simple conditions on the functions $x^i(u)$ which will enable us to state that conditions (2.1) are satisfied. We can derive such conditions from the following lemma:

If $g(u)$ is a summable function defined in a rectangle r: $\alpha^i \leqq u^i \leqq \beta^i$, and $f(u)$ is a summable function defined in the rectangle R: $\alpha^i - \varepsilon^i \leqq u^i \leqq \beta^i + \varepsilon^i$ $(\varepsilon^1 > 0, \varepsilon^2 > 0)$, then the integral $\int_r f(u+h)\, g(u)\, du$ exists for each h such that $|h^i| < \varepsilon^i$ and the equation

$$\lim_{h \to 0} \int_r |f(u+h) - f(u)| \cdot |g(u)| \cdot du = 0 \tag{16.1}$$

is valid, provided that f and g satisfy any one of the three following sets of conditions:

(16.2)[24] *g is bounded in r;*

(16.3)[24] *$|f(u)|^p$ is summable over R and $|g(u)|^q$ is summable over r for some positive numbers p, q such that $1/p + 1/q = 1$;*

(16.4)[25] *the functions $f(u)$ and $g(u)$ have associated summabilities in the sense of Young.*

---

[24] Hobson, *Theory of functions of a real variable*, vol. II, p. 331. The introduction of absolute values in (16.1) causes no difficulty.

[25] W. H. Young, *A new set of conditions for a formula for an area*, Proc. London Math. Soc., vol. 21 (1922), p. 78. We here consider $g(u)$ to be defined not merely on r, but on all of R; the absolute values in (16.1) cause no trouble.

Let us therefore suppose that the functions $x^i(u)$ are defined on a Jordan region B and satisfy conditions (2.1a) and (2.1b), and moreover that the partial derivatives $\partial x^i/\partial u^j$ are all summable over B. Suppose further that each[26] of the six pairs of functions $(\partial x^i/\partial u^1, \partial x^j/\partial u^2)$, $i \neq j$, satisfies one of the three conditions

(16.5) one function in the pair is bounded; or

(16.6) one function of the pair belongs to the class[27] L_p and the other to the class L_q, where $p>0$, $q>0$, $1/p+1/q=1$; or

(16.7) the functions of the pair possess associated summabilities.

Then the functions $x^i(u)$ satisfy conditions (2.1).

To prove this we observe that under the given hypotheses each product $(\partial x^i/\partial u^1)\cdot(\partial x^j/\partial u^2)$, $i \neq j$, is summable, so that the jacobians are summable. Hence (2.1c) is fulfilled. To establish (2.1d) we change notation, replacing $x^1, x^2, x^3, X^1, X^2, X^3, u^1, u^2$ by x, y, z, X, Y, Z, u, v respectively. Then for every rectangle r interior to B we have

$$
\begin{aligned}
&\int_r |Z^{(x)}(u,v;h,k) - Z(u,v)|\,du\,dv \\
&\quad = \int_r |[x_u(u+h, v+k) - x_u(u,v)]\,y_v(u,v) \\
&\qquad - [x_v(u+h, v+k) - x_v(u,v)]\,y_u(u,v)|\,du\,dv \\
&\quad \leqq \int_r |x_u(u+h, v+k) - x_u(u,v)| \cdot |y_v(u,v)| \cdot du\,dv \\
&\qquad + \int_r |x_v(u+h, v+k) - x_v(u,v)| \cdot |y_u(u,v)| \cdot du\,dv.
\end{aligned} \tag{16.8}
$$

But by the lemma of this section the right member of (16.8) tends to zero with h and k, so that

$$
\lim_{h\to 0} \frac{1}{4h^2} \int_{-h}^{h}\int_{-h}^{h}\int\int_r |Z^{(x)}(u,v;h,k) - Z(u,v)|\,du\,dv\,dh\,dk = 0. \tag{16.9}
$$

The five other equations analogous to (16.9) can be established in exactly the same manner, and we therefore see that condition (2.1d) is satisfied.

17. As a special case of the surfaces considered in § 16, we may consider the class of surfaces satisfying the following conditions:

(17.1) *The functions $x(u,v)$, $y(u,v)$ satisfy a Lipschitz condition;*

(17.2) *The function $z(u,v)$ satisfies conditions* (2.1a) *and* (2.1b), *and the derivatives z_u and z_v are summable.*

For such surfaces, each pair of derivatives (x_u, y_v), etc., satisfies condition (16.5), and the surface is therefore of type C. In particular, if $z(u,v)$ also satisfies a Lipschitz condition, then the surface is of type C; hence all

[26] It is not necessary that all of the pairs satisfy the same condition.

[27] A function f is said to belong to the class L_p if $|f|^p$ is summable.

rectifiable surfaces are of type C. Specializing in a different direction, let us suppose that $x(u, v) = u$, $y(u, v) = v$; the surface is then representable in the form $z = z(x, y)$, and condition (17.2) is exactly equivalent to Tonelli's[28] "absolute continuity". By Theorem II, we know that for all such surfaces the area is given by the classical double integral,

$$(17.3) \qquad L(S) = \int\int [X^2 + Y^2 + Z^2]\, du\, dv = \int\int [1 + p^2 + q^2]\, du\, dv.$$

But it has already been shown by Tonelli[29] that for surfaces $z = z(x, y)$ the condition of absolute continuity (i. e., condition (17.2)) is not merely a sufficient but also a necessary condition for equation (17.3) to hold. We may therefore reasonably suspect that conditions (2.1a) and (2.1b) are indispensable for our purposes.

18. It might be of some interest to point out that under much weaker conditions than (2.1) we can show that the classical double integral does not exceed the area of the surface. Let the continuous surface S have the representation $x^i = x^i(u)$, u on Q, where Q is the unit square. Then *in order that the inequality*

$$(18.1) \qquad \int_Q \| X \|\, du \leqq L(S)$$

shall hold ($L(S)$ *denoting the Lebesgue area of the surface* S) *it is sufficient that*

(18.2a) *the jacobians* X^i *are defined almost everywhere in* Q, *and are summable;*

(18.2b) *on an everywhere dense set* U^1 *of ordinates the functions* $x^i(u^1, u^2)$ *are absolutely continuous functions of* u^2; *and on an everywhere dense set* U^2 *of abscissas the functions* $x^i(u^1, u^2)$ *are absolutely continuous functions of* u^1;

(18.2c) *for every rectangle* r *bounded by ordinates and abscissas belonging respectively to* U^1 *and* U^2, *the formula*

$$(18.3) \qquad \int_r X^1\, du = \frac{1}{2} \int_0^A (x^2\, x^{3\prime} - x^3\, x^{2\prime})\, dt$$

is valid, together with its analogues for X^2 *and* X^3.

The statement (18.2b) is obviously weaker than (2.1a, b). Moreover, the statement (18.2c) is in a sense "almost" a consequence of (2.1a, b). For it has been shown[30] that if (2.1a, b) hold, then for every rectangle r in Q the formula

[28] L. Tonelli, *Sulla quadratura delle superficie,* Lincei Rendiconti, vol. III, ser. VI, p. 633.

[29] Loc. cit. [28].

[30] W. H. Young, loc. cit. [25], p. 92. Young's hypotheses are equivalent to (2.1a, b) with the word "*almost*" removed; but this difference is easily seen to be unimportant.

$$\int_r \left[\frac{\partial\,(x^2(u^1+\alpha^1, u^2+\alpha^2), x^3(u^1, u^2))}{\partial\,(u^1, u^2)}\right] du^1\,du^2 \tag{18.4}$$
$$= -\int x^3 \cdot x^{2\prime}(u^1+\alpha^1, u^2+\alpha^2)\,dt$$

is valid for almost all values of (α^1, α^2). It cannot be immediately concluded, however, that (18.3) holds for the particular values $\alpha^1 = \alpha^2 = 0$ which concern us.

In order to prove that the inequality (18.1) follows from hypotheses (18.2) we let $\{S_n\}$ be a sequence of polyhedra tending to S for which $\lim L(S_n) = L(S)$. For these polyhedra we choose (as is always possible) typical representations $x^i = x_n^i(u)$, u on Q, such that $\lim x_n^i(u) = x^i(u)$, uniformly on Q. Denoting $\|X\|$ by $f(x, X)$, we can follow the proof of Lemma 8 with only the trivial change that the squares q_j (with centers u_j) occurring in the proof are replaced by rectangles r_j (containing u_j) whose sides belong to U^1 and U^2 and whose shorter side is at least half the longer side. We thus arrive, as in Lemma 8, at the inequality

$$L(S) = \liminf \int_Q \|X_n\|\,du \geqq \int_Q \|X\|\,du.$$

Moreover, let us suppose that $f(x, X)$ is a function satisfying the usual conditions of continuity and homogeneity (DI 2.1). As we have seen,[31] for a sufficiently large number H both of the functions $H \cdot \|X\| \pm f(x, X)$ are positive quasi-regular and positive definite. Hence, by the proof of Lemma 8 as modified above, we know that for every surface S: $x^i = x^i(u)$, u on Q, which satisfies (18.2) and every sequence $\{S_n\}$ of polyhedra $x^i = x_n^i(u)$, u on Q such that $\lim x_n^i(u) = x^i(u)$ uniformly and $\lim L(S_n) = L(S)$, the inequalities

$$H \lim L(S_n) + \liminf \int_Q f(x_n, X_n)\,du \geqq H\int_Q \|X\|\,du + \int_Q f(x, X)\,du, \tag{18.5}$$

$$H \lim L(S_n) - \liminf \int_Q f(x_n, X_n)\,du \geqq H\int_Q \|X\|\,du - \int_Q f(x, X)\,du. \tag{18.6}$$

If now we add to conditions (18.2) the further hypothesis

(18.7) *the area $L(S)$ of the surface S is given by the double integral* $\int_Q \|X\|\,du$,

then inequalities (18.5) and (18.6) imply

$$\lim \int_Q f(x_n, X_n)\,du = \int_Q f(x, X)\,du.$$

Therefore it follows that if the representation $x^i = x^i(u)$, u on Q of the surface S satisfies conditions (18.2) and (18.7), then for every $\varepsilon > 0$ there

[31] DI, p. 475.

exists a polyhedron S_n, typically represented in the form $x^i = x_n^i(u)$, u on Q, such that

$$\|S, S_n\| < \varepsilon, \quad \left| \int_Q f(x, X)\, du - \int_Q f(x_n, X_n)\, du \right| < \varepsilon.$$

The proof of Lemma 8 is therefore applicable to the class of all representations of continuous surfaces satisfying conditions (18.2) and (18.7); we thus find that if a surface S has two such representations $x^i = x^i(u)$, u on Q and $x^i = \bar{x}^i(u)$, u on Q, then

$$\int_Q f(\bar{x}, \bar{X})\, du = \int_Q f(x, X)\, du,$$

allowing us to represent the integral by the symbol $F(S)$; and we also find that $F(S)$ is lower semi-continuous on the class of all surfaces of this type of area $\leqq N$ (N a constant), provided that $F(S)$ is positive semi-definite and positive quasi-regular.

It is easy to see that the rectangularity of the regions r_j is unimportant; we could have replaced conditions (18.2 b, c) by the less stringent condition that

(18.8a) *to almost every point P of the square Q there corresponds a sequence $r_j(P)$ of regions bounded by rectifiable Jordan curves, with diameters tending to zero, and such that $m(r_j(P))/m(W_j(P)) \geqq \alpha(P) > 0$, where $W_j(P)$ is the smallest square with center at P containing $r_j(P)$, and $\alpha(P)$ depends on P alone;*

(18.8b) *on the boundary of each region $r_j(P)$ the functions $x^i(u)$ are absolutely continuous, and the formulas* (18.3) *hold.*

Moreover, the range of definition of the functions $x^i(u)$ need not be the square Q, but may be any Jordan region.

University of Chicago.

A SET OF POSTULATES FOR THE FOUNDATION OF LOGIC.[1]

(SECOND PAPER.)

By Alonzo Church.

1. **Revision of the list of formal postulates.** In a recent paper[2] the author proposed a set of postulates which, it was believed, would lead to a system of mathematical logic free of some of the complications entailed by Bertrand Russell's theory of types, and would at the same time avoid the well known paradoxes, in particular the Russell paradox, by weakening the classical principle of reductio ad absurdum. But, in the course of working with these postulates, it has since become clear that the set as originally given requires some modification in order to render it free from contradiction.

Postulate 19 makes it possible to prove $\sim\Sigma(\{\mathbf{F}\}(\mathbf{U}))$ if there can be found some consequence of $\Sigma(\{\mathbf{F}\}(y))$ which is false of $\mathbf{U}$, subject only to the restriction that $\{\mathbf{F}\}(\mathbf{U}, x)$ shall be known to be false for some x. This is not far from assuming the principle of reductio ad absurdum in its full strength as applied to propositions of the form $\Sigma(\mathbf{A})$. But, given a proposition which is not of the form $\Sigma(\mathbf{A})$, it is ordinarily possible in various ways, to write another proposition of much the same import, which is of the form $\Sigma(\mathbf{A})$. And in this way we find that Postulate 19 contains enough of the principle of reductio ad absurdum to lead to a modified form of the Russell paradox, in outline as follows.

We take $\mathfrak{Y}$ to be $\lambda u \lambda v E(u) E(v)$ and prove, in a straightforward manner, $\sim\sim\{\mathfrak{Y}\}\left(\Lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi)), \{\mathfrak{Y}\}\left(\{\mathfrak{Y}\}(\Lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi)))\right)\right)$, where Λ is defined as in § 5 below. Hence, by Rule III, $\sim\{\lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi))\}\left(\{\mathfrak{Y}\}(\Lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi)))\right)$. Hence, by Theorem 8 below, $\sim\{\Lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi))\}\left(\{\mathfrak{Y}\}(\Lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi)))\right)$. And hence we prove $\Sigma y \Sigma x \,.\sim x(y(x)) \,.\, x = \Lambda\varphi \,.\sim\varphi(y(\varphi))$. But, assuming $\Sigma x \,.\sim x(y(x)) \,.\, x = \Lambda\varphi \,.\sim\varphi(y(\varphi))$, we can prove $\Sigma r \,.\, r(r) \,.\sim r(y(r))$, because $\sim\{\Lambda\varphi \,.\sim\varphi(y(\varphi))\}\left(y(\Lambda\varphi \,.\sim\varphi(y(\varphi)))\right)$ is convertible into $\{\lambda\varphi \,.\sim\varphi(y(\varphi))\}(\Lambda\varphi \,.\sim\varphi(y(\varphi)))$, which, by Theorem 6 below, yields in turn $\{\Lambda\varphi \,.\sim\varphi(y(\varphi))\}(\Lambda\varphi \,.\sim\varphi(y(\varphi)))$. Therefore, by Theorem I, $\Sigma(\{\mathfrak{F}\}(y)) \supset_y \{\mathfrak{G}\}(y)$, where $\{\mathfrak{F}\}(y)$ stands for $\lambda x \,.\sim x(y(x)) \,.\, x = \Lambda\varphi \,.\sim\varphi(y(\varphi))$ and $\{\mathfrak{G}\}(y)$ stands for $\Sigma r \,.\, r(r) \,.\sim r(y(r))$. Now we

[1] Received January 3, 1933.

[2] These Annals, vol. 33 (1932), pp. 346–366.

exists a polyhedron S_n, typically represented in the form $x^i = x_n^i(u)$, u on Q, such that

$$\|S, S_n\| < \varepsilon, \quad \left| \int_Q f(x, X)\, du - \int_Q f(x_n, X_n)\, du \right| < \varepsilon.$$

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1. **Revision of the list of formal postulates.** In a recent paper[2] the author proposed a set of postulates which, it was believed, would lead to a system of mathematical logic free of some of the complications entailed by Bertrand Russell's theory of types, and would at the same time avoid the well known paradoxes, in particular the Russell paradox, by weakening the classical principle of reductio ad absurdum. But, in the course of working with these postulates, it has since become clear that the set as originally given requires some modification in order to render it free from contradiction.

Postulate 19 makes it possible to prove $\sim\Sigma(\{\mathbf{F}\}(\mathbf{U}))$ if there can be found some consequence of $\Sigma(\{\mathbf{F}\}(y))$ which is false of $\mathbf{U}$, subject only to the restriction that $\{\mathbf{F}\}(\mathbf{U}, x)$ shall be known to be false for some x. This is not far from assuming the principle of reductio ad absurdum in its full strength as applied to propositions of the form $\Sigma(\mathbf{A})$. But, given a proposition which is not of the form $\Sigma(\mathbf{A})$, it is ordinarily possible in various ways, to write another proposition of much the same import, which is of the form $\Sigma(\mathbf{A})$. And in this way we find that Postulate 19 contains enough of the principle of reductio ad absurdum to lead to a modified form of the Russell paradox, in outline as follows.

We take $\mathfrak{Y}$ to be $\lambda u \lambda v E(u) E(v)$ and prove, in a straightforward manner, $\sim\sim\{\mathfrak{Y}\}\left(\Lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi)), \{\mathfrak{Y}\}\left(\{\mathfrak{Y}\}(\Lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi)))\right)\right)$, where Λ is defined as in § 5 below. Hence, by Rule III, $\sim\{\lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi))\}\left(\{\mathfrak{Y}\}(\Lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi)))\right)$. Hence, by Theorem 8 below, $\sim\{\Lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi))\}\left(\{\mathfrak{Y}\}(\Lambda\varphi \,.\sim\varphi(\{\mathfrak{Y}\}(\varphi)))\right)$. And hence we prove $\Sigma y \Sigma x \,.\sim x(y(x)) \,.\, x = \Lambda\varphi \,.\sim\varphi(y(\varphi))$. But, assuming $\Sigma x \,.\sim x(y(x)) \,.\, x = \Lambda\varphi \,.\sim\varphi(y(\varphi))$, we can prove $\Sigma r \,.\, r(r) \,.\sim r(y(r))$, because $\sim\{\Lambda\varphi \,.\sim\varphi(y(\varphi))\}\left(y(\Lambda\varphi \,.\sim\varphi(y(\varphi)))\right)$ is convertible into $\{\lambda\varphi \,.\sim\varphi(y(\varphi))\}(\Lambda\varphi \,.\sim\varphi(y(\varphi)))$, which, by Theorem 6 below, yields in turn $\{\Lambda\varphi \,.\sim\varphi(y(\varphi))\}(\Lambda\varphi \,.\sim\varphi(y(\varphi)))$. Therefore, by Theorem I, $\Sigma(\{\mathfrak{F}\}(y)) \supset_y \{\mathfrak{G}\}(y)$, where $\{\mathfrak{F}\}(y)$ stands for $\lambda x \,.\sim x(y(x)) \,.\, x = \Lambda\varphi \,.\sim\varphi(y(\varphi))$ and $\{\mathfrak{G}\}(y)$ stands for $\Sigma r \,.\, r(r) \,.\sim r(y(r))$. Now we

[1] Received January 3, 1933.

[2] These Annals, vol. 33 (1932), pp. 346–366.

take $\mathfrak{U}$ to be λzz and, with the aid of Postulate 19, we prove $\sim\{\mathfrak{G}\}(\mathfrak{U})$. Therefore, by Postulate 19, $\sim\Sigma(\{\mathfrak{F}\}(\mathfrak{U}))$, that is, $\sim\Sigma x.\sim x(x).x=\Lambda\varphi.\sim\varphi(\varphi)$. Then, by means of Theorem 1, we prove $\sim\varphi(\varphi)\supset_{\varphi}\Sigma x.\sim x(x).x=\varphi$. Therefore, by Theorem 9 below, $\sim\{\Lambda\varphi.\sim\varphi(\varphi)\}(\Lambda\varphi.\sim\varphi(\varphi))$. Therefore, by Rule III, $\{\lambda\varphi.\sim\varphi(\varphi)\}(\Lambda\varphi.\sim\varphi(\varphi))$. Therefore, by Theorem 6 below, $\{\Lambda\varphi.\sim\varphi(\varphi)\}(\Lambda\varphi.\sim\varphi(\varphi))$, a contradiction.

For this reason we shall omit Postulate 19 from our list. And with it we shall omit Postulates 20, 28, and 29, because these three postulates entirely lose their point after the omission of Postulate 19.

The chief effect of the omission of Postulate 19 is that we are no longer able to prove formulas of the form $\sim\Sigma(\mathbf{A})$.[3] We believe, however, that this restriction will not render our system inadequate, because the formula $\sim\{\Lambda y\Sigma(\{\mathbf{F}\}(y))\}(\mathbf{U})$ can often be proved and used instead of $\sim\Sigma(\{\mathbf{F}\}(\mathbf{U}))$.

Let us call a formula *vacuous* when it has, or is convertible into, the form $\Pi(\mathbf{F},\mathbf{G})$, and there is no formula $\mathbf{M}$ such that $\{\mathbf{F}\}(\mathbf{M})$ is provable. The omission of Postulates 28 and 29 does away with our chief means of proving vacuous formulas. It remains possible to prove vacuous formulas of certain particular kinds by means of Postulate 25 or of Postulate 34, but this property of these two postulates is largely accidental, and we propose to eliminate it by means of appropriate slight changes in them. It then becomes impossible, we think, to prove any vacuous formula. And we recognize this property of our system by adding to it (tentatively) the postulate, $\Sigma\psi\Pi(\varphi,\psi)\supset_{\varphi}\Sigma(\varphi)$. This postulate is, probably, not indispensable, but it is sometimes convenient. Its addition to the list renders Postulate 32 non-independent, and the latter postulate may, therefore, be omitted.

And it is now also necessary to modify Postulates 17 and 18 to read, respectively, $\Sigma x[\varphi(x).\sim\psi(x)]\supset_{\varphi\psi}\sim\Pi(\varphi,\psi)$ and $\sim\Pi(\varphi,\psi)\supset_{\varphi\psi}\Sigma x.\varphi(x).\sim\psi(x)$. For otherwise it would be impossible to prove any formula $\sim\Pi(\mathbf{F},\mathbf{G})$ unless $\Sigma(\mathbf{G})$ were provable.

And finally, Postulate 2 fails to be independent, and may, therefore, be omitted from the list. For the proofs of cases 9 and 12 under Theorem I can be modified so that Postulate 7 takes the place of Postulate 2, and Theorem I can therefore be proved from Postulates 1, 3–11.[4]

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[3] It is to be observed, however, that another symbol, meaning, "There exists", can be defined as follows:

$$\mathfrak{I}\rightarrow\lambda\varphi\lambda\psi\sim.\varphi(x)\supset_{x}\sim\psi(x).$$

It will then be found that $\mathfrak{I}(\mathbf{A},\mathbf{B})$ is provable whenever $\Sigma x.\{\mathbf{A}\}(x)\{\mathbf{B}\}(x)$ is provable and conversely. And, for particular formulas $\mathbf{A}$ and $\mathbf{B}$, it frequently is possible to prove $\sim\mathfrak{I}(\mathbf{A},\mathbf{B})$.

The expression $\mathfrak{I}(\mathbf{A},\mathbf{B})$ is to be read, "Some $\mathbf{A}$ is $\mathbf{B}$".

[4] This observation is due to Mr. J. B. Rosser.

The question whether our other postulates are all independent remains at present open to doubt, but there appears to be no reason why this question cannot be investigated by the usual method of constructing independence examples.

Our revised list of formal postulates is now as follows:

1. $\Sigma(\varphi) \supset_{\varphi} \Pi(\varphi, \varphi)$.
3. $\Sigma(\sigma) \supset_{\sigma} . [\sigma(x) \supset_{x} \varphi(x)] \supset_{\varphi} . \Pi(\varphi, \psi) \supset_{\psi} . \sigma(x) \supset_{x} \psi(x)$.
4. $\Sigma(\varrho) \supset_{\varrho} . \Sigma y [\varrho(x) \supset_{x} \varphi(x, y)] \supset_{\varphi} . [\varrho(x) \supset_{x} \Pi(\varphi(x), \psi(x))] \supset_{\psi} .$
$[\varrho(x) \supset_{x} \varphi(x, y)] \supset_{y} . \varrho(x) \supset_{x} \psi(x, y)$.
5. $\Sigma(\varphi) \supset_{\varphi} . \Pi(\varphi, \psi) \supset_{\psi} . \varphi(f(x)) \supset_{fx} \psi(f(x))$.
6. $'x . \varphi(x) \supset_{\varphi} . \Pi(\varphi, \psi(x)) \supset_{\psi} \psi(x, x)$.
7. $\varphi(x, f(x)) \supset_{\varphi fx} . \Pi(\varphi(x), \psi(x)) \supset_{\psi} \psi(x, f(x))$.
8. $\Sigma(\varrho) \supset_{\varrho} . \Sigma y [\varrho(x) \supset_{x} \varphi(x, y)] \supset_{\varphi} . [\varrho(x) \supset_{x} \Pi(\varphi(x), \psi)] \supset_{\psi} .$
$[\varrho(x) \supset_{x} \varphi(x, y)] \supset_{y} \psi(y)$.
9. $'x . \varphi(x) \supset_{\varphi} \Sigma(\varphi)$.
10. $\Sigma x \varphi(f(x)) \supset_{f\varphi} \Sigma(\varphi)$.
11. $\varphi(x, x) \supset_{\varphi x} \Sigma(\varphi(x))$.
12. $\Sigma(\varphi) \supset_{\varphi} \Sigma x \varphi(x)$.
13. $\Sigma(\varphi) \supset_{\varphi} . [\varphi(x) \supset_{x} \psi(x)] \supset_{\psi} \Pi(\varphi, \psi)$.
14. $p \supset_{p} . q \supset_{q} p q$.
15. $p q \supset_{qp} p$.
16. $p q \supset_{pq} q$.
17. $\Sigma x [\varphi(x) . \sim \psi(x)] \supset_{\varphi\psi} \sim \Pi(\varphi, \psi)$.
18. $\sim \Pi(\varphi, \psi) \supset_{\varphi\psi} \Sigma x . \varphi(x) . \sim \psi(x)$.
21. $p \supset_{p} . \sim q \supset_{q} \sim . p q$.
22. $\sim p \supset_{p} . q \supset_{q} \sim . p q$.
23. $\sim p \supset_{p} . \sim q \supset_{q} \sim . p q$.
24. $p \supset_{p} . [\sim . p q] \supset_{q} \sim q$.
25. $\big[\sim [\varphi(u) \psi(u)] . [[\varphi(x) . \sim \psi(x)] \supset_{x} \varrho(x)] . [[\sim \varphi(x) . \psi(x)] \supset_{x} \varrho(x)] .$
$[\sim \varphi(x) . \sim \psi(x)] \supset_{x} \varrho(x)\big] \supset_{\varphi\psi\varrho u} \varrho(u)$.
26. $p \supset_{p} \sim \sim p$.
27. $\sim \sim p \supset_{p} p$.
30. $'x . \varphi(x) \supset_{\varphi} . [\theta(x) . \psi(x) \supset_{\psi} \Pi(\theta, \psi)] \supset_{\theta} \varphi(\iota(\theta))$.
31. $\varphi(\iota(\theta)) \supset_{\theta\varphi} \Pi(\theta, \varphi)$.
33. $[\varphi(x, y) \supset_{xy} . \varphi(y, z) \supset_{z} \varphi(x, z)] [\varphi(x, y) \supset_{xy} \varphi(y, x)] \supset_{\varphi} . \varphi(u, v)$
$\supset_{uv} . \psi(A(\varphi, u)) \supset_{\psi} \psi(A(\varphi, v))$.

34. $[\varphi(x, y) \supset_{xy} \cdot \varphi(y, z) \supset_z \varphi(x, z)]\, [\varphi(x, y) \supset_{xy} \varphi(y, x)]\, [\varphi(x, y) \supset_{xy} \theta(x, y)]\, [\sim \theta(u, v)] \supset_{\varphi\theta uv} \sim \cdot \psi(A(\varphi, u)) \supset_\psi \psi(A(\varphi, v))$.

35. $[\psi(A(\varphi, u)) \supset_\psi \psi(A(\varphi, v))] \supset_{\varphi uv} \varphi(u, v)$.

36. $E(A(\varphi)) \supset_\varphi \cdot \varphi(x, y) \supset_{xy} \cdot \varphi(y, z) \supset_z \varphi(x, z)$.

37. $E(A(\varphi)) \supset_\varphi \cdot \varphi(x, y) \supset_{xy} \varphi(y, x)$.

38. $\Sigma\psi\Pi(\varphi, \psi) \supset_\varphi \Sigma(\varphi)$.

2. **Possibility of a proof of freedom from contradiction.** Our present project is to develop the consequences of the foregoing set of postulates, until a contradiction is obtained from them, or until the development has been carried so far consistently as to make it empirically probably that no contradiction can be obtained from them. And in this connection it is to be remembered that just such empirical evidence, although admittedly inconclusive, is the only existing evidence of the freedom from contradiction of any system of mathematical logic which has a claim to adequacy.

It is worth observing, however, that there may be a possibility of proving that there is no formula **A** such that both **A** and $\sim \cdot$ **A** are consequences of our postulates, or, failing this, of proving the same theorem about some related set of postulates. This is conceivable on account of the entirely formal character of the system which makes it possible to abstract from the meaning of the symbols and to regard the proving of theorems (of formal logic) as a game played with marks on paper according to a certain arbitrary set of rules. It is then proposed to use our intuitive logic (some portion of which we have already had to presuppose) to prove about this game, regarded objectively, that if it be played according to the rules it cannot lead to combinations of marks of certain particular kinds. Indeed the whole set of undefined terms of the formal logic, including enumerably many symbols available for use as variables, is enumerable, and hence the whole set of formulas and of possible proofs is also enumerable. And it is therefore even to be hoped that the portion of intuitive logic necessary to the proof of freedom from contradiction should not go beyond the logic of enumerable classes or employ any number system more elaborate than the system of positive integers.[5]

The impossibility of such a proof of freedom from contradiction for the system of *Principia Mathematica* has recently been established by Kurt Gödel.[6]

[5] The conception of a consistency proof of this kind is due to David Hilbert. See *Die logischen Grundlagen der Mathematik,* Math. Annalen, vol. 88 (1923), pp. 151–165, and *Die Grundlagen der Mathematik,* Abh. a. d. Math. Sem. d. Hamb. Univ., vol. 6 (1928), pp. 65–92.

[6] Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I,* Monatshefte für Mathematik und Physik, vol. 38 (1931), pp. 173–198.

His argument, however, makes use of the relation of implication U between propositions in a way which would not be permissible under the system of this paper, and there is no obvious way of modifying the argument so as to make it apply to the system of this paper. It therefore remains, at least for the present, conceivable that there should be found a proof of freedom from contradiction for our system.

3. **Abbreviations of the statement of proofs.** In the proofs of theorems which follow, we indicate after each formula in the proof the means by which it is inferred, that is, we give the number of the rule of procedure and a reference to the premise or premises. Formal postulates are referred to by the number written alone, and theorems are referred to by the abbreviation "Thm" followed by the number.

We use Theorems I, II, and III, and the corollary of Theorem I as if they were rules of procedure. This is justified because our proof of these theorems was constructive in character, so that their use in this way can be regarded as a mere device for abbreviating the statement of a proof. For instance, whenever we have two particular formulas **M** and **N**, each of which contains **x** as a free variable, and a particular proof of $\Sigma \mathbf{x} . \mathbf{M}$, and a particular scheme for proving **N** as a consequence of **M**, then, by the method used in the proof of Theorem I, we can obtain a particular proof of $[\mathbf{M}] \supset_{\mathbf{x}} . \mathbf{N}$.

Moreover we allow uses of Theorems I, II, and III, and the corollary of Theorem I, in the role of rules of procedure, even in the course of proving **N** as a consequence of **M** preparatory to another application of one of the Theorems I, II, III, or Theorem I Corollary. For our proof of these theorems remains valid even after the formula **M** is added to our list as an additional formal postulate, provided only that the variable **x**, which occurs in **M** as a free variable, is not used elsewhere as a bound variable.

In the proof of a theorem we give before each formula a symbol which will be used in referring to it, namely a letter **A**, **C**, or **D**, with a subscript. The formulas marked $\mathbf{A}_i$ have been proved outright. Those marked $\mathbf{D}_i$ are assumed, in order to derive other formulas from them, preparatory to an application of Theorem I, or II, or III, or Theorem I Corollary. And those marked $\mathbf{C}_i$ are proved on the basis of the assumption of certain of the formulas $\mathbf{D}_i$, as indicated in each case.

We abbreviate further by omitting steps which involve only an application of one of the rules of procedure I, II, III, or by condensing several such steps into one, using "conv" to mean, "By conversion", or, "By applications of Rules I, II, and III".

We use IV^n to stand for n successive applications of Rule IV. And we use V^n to stand for n successive applications of Rule V, the result of

each application of Rule V being used in turn as the major premise[7] for the next application of Rule V. After V^n, so used, we give a reference, first to the major premise of the first application of Rule V, and then to the successive minor premises[7] in order. But in passing from $\{\mathbf{F}\}(\mathbf{x}, \mathbf{y})$ $\supset_{\mathbf{xy}} \{\mathbf{G}\}(\mathbf{x}, \mathbf{y})$ and $\{\mathbf{F}\}(\mathbf{A}, \mathbf{B})$ to $\{\mathbf{G}\}(\mathbf{A}, \mathbf{B})$ or in passing from $\{\mathbf{F}\}(\mathbf{x}, \mathbf{y})$ $\supset_{\mathbf{xy}} \{\mathbf{G}\}(\mathbf{y})$ and $\{\mathbf{F}\}(\mathbf{A}, \mathbf{B})$ to $\{\mathbf{G}\}(\mathbf{B})$, we refer to $\{\mathbf{F}\}(\mathbf{A}, \mathbf{B})$ as the only minor premise, omitting reference to the other minor premise $\Sigma \mathbf{y} \{\mathbf{F}\}(\mathbf{A}, \mathbf{y})$. Similarly we shall sometimes refer to a formula $\mathbf{M}$ as minor premise when the true minor premise is not $\mathbf{M}$ but some formula obtained from $\mathbf{M}$ by use of Rule IV or conversion or both. And we shall sometimes refer to two minor premises, $\mathbf{P}$ and $\mathbf{Q}$, when the true minor premise is the logical product $[\mathbf{P}][\mathbf{Q}]$, obtained from the $\mathbf{P}$ and $\mathbf{Q}$ by use of Postulate 14.

4. **Identity.** We adopt the following definitions of identity, or equality,[8] and of distinctness, or inequality:

$$= \longrightarrow \lambda\mu\,\lambda\nu\,.\,\psi(\mu) \supset_{\psi} \psi(\nu).$$

$$\neq \longrightarrow \lambda\mu\,\lambda\nu \sim .\,\psi(\mu) \supset_{\psi} \psi(\nu).$$

We shall abbreviate $\{=\}(\mathbf{A}, \mathbf{B})$ as $[\mathbf{A}] = [\mathbf{B}]$, and $\{\neq\}(\mathbf{A}, \mathbf{B})$ as $[\mathbf{A}] \neq [\mathbf{B}]$.

In terms of these definitions, Postulates 33, 34, 35 can be rewritten as follows:

33′. $[\varphi(x, y) \supset_{xy} \varphi(y, z) \supset_{z} \varphi(x, z)]\,[\varphi(x, y) \supset_{xy} \varphi(y, x)] \supset_{\varphi} \varphi(u, v) \supset_{uv} A(\varphi, u) = A(\varphi, v)$.

34′. $[\varphi(x, y) \supset_{xy} \varphi(y, z) \supset_{z} \varphi(x, z)]\,[\varphi(x, y) \supset_{xy} \varphi(y, x)]\,[\varphi(x, y) \supset_{xy} \theta(x, y)]\,[\sim \theta(u, v)] \supset_{\varphi\theta uv} A(\varphi, u) \neq A(\varphi, v)$.

35′. $[A(\varphi, u) = A(\varphi, v)] \supset_{\varphi uv} \varphi(u, v)$.

It is seen that the propositions 33′, 34′, 35′ are derivable by conversion from Postulates 33, 34, 35, respectively. And Postulate 30 is seen to be essentially the same as Theorem 4 below.

THEOREM 1. $\iota x \,.\, x = x$.

$\mathbf{D}_1$: $E(x)$.

$\mathbf{C}_1$: $\Sigma\psi \,.\, \psi(x)$ —conv, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\psi(x) \supset_{\psi} \psi(x)$ —V, 1, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_3$: $x = x$ —conv, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Sigma\psi \,.\, \psi(\lambda\varphi\, \Pi(\varphi, \varphi))$ —IV, 1.

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[7] In an application of Rule V, we call the formula $\Pi(\mathbf{F}, \mathbf{G})$ the *major premise* and the formula $\{\mathbf{F}\}(\mathbf{A})$ the *minor premise*.

[8] This definition is similar to the one used in *Principia Mathematica*. It is a translation into symbolic notation of the definition originally given by Leibniz. Cf. C. I. Lewis, *A Survey of Symbolic Logic*, 1918, p. 373.

$\mathbf{A}_2$: $E\varphi\Pi(\varphi, \varphi)$ —conv, $\mathbf{A}_1$.
$\mathbf{A}_3$: $\Sigma x E(x)$ —IV, $\mathbf{A}_2$.
$\mathbf{A}_4$: $\iota x \,.\, x = x$ —Thm I, $\mathbf{C}_3$, $\mathbf{A}_3$.

THEOREM 2. $x = y \supset_{xy} . y = x$

$\mathbf{D}_1$: $x = y$.
$\mathbf{C}_1$: $\Sigma\varphi \,.\, \varphi(x)$ —IV, $\mathbf{D}_1$.
$\mathbf{C}_2$: $E(x)$ —III, $\mathbf{C}_1$; $\mathbf{D}_1$.
$\mathbf{C}_3$: $x = x$ —V, Thm 1, $\mathbf{C}_2$; $\mathbf{D}_1$.
$\mathbf{C}_4$: $\{\lambda z \,.\, z = x\}(x)$ —III, $\mathbf{C}_3$; $\mathbf{D}_1$.
$\mathbf{C}_5$: $\{\lambda z \,.\, z = x\}(y)$ —V, $\mathbf{D}_1$, $\mathbf{C}_4$; $\mathbf{D}_1$.
$\mathbf{C}_6$: $y = x$ —II, $\mathbf{C}_5$; $\mathbf{D}_1$.
$\mathbf{A}_1$: $Ex \,.\, x = x$ —IV, Thm 1.
$\mathbf{A}_2$: $[\lambda x \,.\, x = x] = . \lambda x \,.\, x = x$ —V, Thm 1, $\mathbf{A}_1$.
$\mathbf{A}_3$: $\Sigma x \Sigma y \,.\, x = y$ —IV^2, $\mathbf{A}_2$.
$\mathbf{A}_4$: $x = y \supset_{xy} . y = x$ —Thm I Cor, $\mathbf{C}_6$, $\mathbf{A}_3$.

THEOREM 3. $x = y \supset_{xy} . y = z \supset_z . x = z$.

$\mathbf{D}_1$: $x = y$.
$\mathbf{D}_2$: $y = z$.
$\mathbf{C}_1$: $x = z$ —V, $\mathbf{D}_2$, $\mathbf{D}_1$.
$\mathbf{C}_2$: $E(y)$ —IV, $\mathbf{D}_1$.
$\mathbf{C}_3$: $y = y$ —V, Thm 1, $\mathbf{C}_2$; $\mathbf{D}_1$.
$\mathbf{C}_4$: $\Sigma z \,.\, y = z$ —IV, $\mathbf{C}_3$; $\mathbf{D}_1$.
$\mathbf{C}_5$: $y = z \supset_z . x = z$ —Thm I, $\mathbf{C}_1$, $\mathbf{C}_4$; $\mathbf{D}_1$.
$\mathbf{A}_1$: $x = y \supset_{xy} . y = z \supset_z . x = z$ —Thm I Cor, $\mathbf{C}_5$, $\mathbf{A}_3$ under Thm 2.

THEOREM 4. $[\theta(x) \,.\, \theta(y) \supset_y . x = y] \supset_{\theta x} . x = \iota(\theta)$.

$\mathbf{D}_1$: $\theta(x) \,.\, \theta(y) \supset_y . x = y$.
$\mathbf{C}_1$: $\theta(x)$ —V^2, 15, $\mathbf{D}_1$.
$\mathbf{C}_2$: $E(x)$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$.
$\mathbf{C}_3$: $x = x$ —V, Thm 1, $\mathbf{C}_2$; $\mathbf{D}_1$.
$\mathbf{C}_4$: $\theta(y) \supset_y . x = y$ —V^2, 16, $\mathbf{D}_1$.
$\mathbf{D}_2$: $\theta(y)$.
$\mathbf{C}_5$: $x = y$ —V, $\mathbf{C}_4$, $\mathbf{D}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.
$\mathbf{D}_3$: $\psi(x)$.
$\mathbf{C}_6$: $\psi(y)$ —V, $\mathbf{C}_5$, $\mathbf{D}_3$; $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$.

$\mathbf{C}_7$: $\Sigma(\theta)$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_8$: $\pi(\theta, \psi)$ —Thm II, $\mathbf{C}_6$, $\mathbf{C}_7$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_9$: $\Sigma\psi \,.\, \psi(x)$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_{10}$: $\psi(x) \supset_\psi \pi(\theta, \psi)$ —Thm I, $\mathbf{C}_8$, $\mathbf{C}_9$; $\mathbf{D}_1$.

$\mathbf{C}_{11}$: $x = \iota(\theta)$ —V[3], 30, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_1$, $\mathbf{C}_{10}$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Sigma y \,.\, [\lambda x \,.\, x = x] = y$ —IV, $\mathbf{A}_2$ under Thm 2.

$\mathbf{A}_2$: $[[\lambda x \,.\, x = x] = y] \supset_y \,.\, [\lambda x \,.\, x = x] = y$ —V, 1, $\mathbf{A}_1$.

$\mathbf{A}_3$: $[[\lambda x \,.\, x = x] = \,.\, \lambda x \,.\, x = x] \,.\, [[\lambda x \,.\, x = x] = y] \supset_y \,.\, [\lambda x \,.\, x = x] = y$
—V[2], 14, $\mathbf{A}_2$ under Thm 2, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\Sigma\theta \,\Sigma x \,.\, \theta(x) \,.\, \theta(y) \supset_y \,.\, x = y$ —IV[2], $\mathbf{A}_3$.

$\mathbf{A}_5$: $[\theta(x) \,.\, \theta(y) \supset_y \,.\, x = y] \supset_{\theta x} \,.\, x = \iota(\theta)$ —Thm I Cor, $\mathbf{C}_{11}$, $\mathbf{A}_4$.

THEOREM 5. $\text{'}x \,.\, x = \,.\, \iota y \,.\, x = y$.

$\mathbf{D}_1$: $E(x)$.

$\mathbf{C}_1$: $x = x$ —V, Thm 1, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\Sigma y \,.\, x = y$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_3$: $[x = y] \supset_y \,.\, x = y$ —V, 1, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{C}_4$: $x = \,.\, \iota y \,.\, x = y$ —V, Thm 4, $\mathbf{C}_1$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\text{'}x \,.\, x = \,.\, \iota y \,.\, x = y$ —Thm I, $\mathbf{C}_4$, $\mathbf{A}_3$ under Thm 1.

5. **The completion of a propositional function.** We call a propositional function $\mathbf{F}$ *significant* for the value $\mathbf{A}$ of the independent variable if $\{\mathbf{F}\}(\mathbf{A})$ is either a true proposition or a false proposition.

We define the symbol $\mathcal{A}$ as follows:

$$\mathcal{A} \longrightarrow \lambda\mu \,\lambda\pi \,.\, \Pi(\mu, \gamma) \supset_\gamma \gamma(\pi).$$

If $\mathbf{F}$ is any propositional function, then $\mathcal{A}(\mathbf{F})$ is a propositional function which we call the *completion* of $\mathbf{F}$. We shall prove (in Theorems 6 and 7) that if $\mathbf{F}$ is at least once true then $\mathcal{A}(\mathbf{F})$ is equivalent to $\mathbf{F}$, that is, that for every value of the independent variable for which either is true the other is also true. We shall prove (in Theorem 9 and its corollary) that, of all propositional functions which are equivalent to $\mathbf{F}$, $\mathcal{A}(\mathbf{F})$ is significant for the greatest range of values of the independent variable. And we shall prove (in Theorems 14 and 15) that there exist functions $\mathbf{F}$ for which the range of significance of $\mathcal{A}(\mathbf{F})$ is greater than that of $\mathbf{F}$.

We define the symbol G as follows:

$$G \longrightarrow \lambda\nu \,.\, \sim \mathcal{A}(\nu, x) \supset_x \sim \nu(x).$$

And $G(\mathbf{F})$ is to be read, "$\mathbf{F}$ is complete". In Theorem 16 we prove that if $\mathbf{F}$ is at least once true and at least once false, then $\mathcal{A}(\mathbf{F})$ is complete.

And Theorem 20 can be regarded as asserting a certain form of the principle of reductio ad absurdum, as applied to complete propositional functions.

THEOREM 6. $\Sigma(\varphi) \supset_{\varphi} \Pi(\varphi, \Lambda(\varphi))$.

$\mathbf{D}_1$: $\Sigma(\varphi)$.

$\mathbf{C}_1$: $\Pi(\varphi, \varphi)$ —V, 1, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\Sigma\gamma\, \Pi(\varphi, \gamma)$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{D}_2$: $\varphi(x)$.

$\mathbf{D}_3$: $\Pi(\varphi, \gamma)$.

$\mathbf{C}_3$: $\gamma(x)$ —V, $\mathbf{D}_3$, $\mathbf{D}_2$.

$\mathbf{C}_4$: $\Pi(\varphi, \gamma) \supset_{\gamma} \gamma(x)$ — Thm I, $\mathbf{C}_3$, $\mathbf{C}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_5$: $\Lambda(\varphi, x)$ —conv, $\mathbf{C}_4$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_6$: $\Pi(\varphi, \Lambda(\varphi))$ —Thm II, $\mathbf{C}_5$, $\mathbf{D}_1$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Sigma(\Pi(\lambda\varphi\, \Sigma(\varphi)))$ —IV, 1.

$\mathbf{A}_2$: $\Sigma\varphi\, \Sigma(\varphi)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $\Sigma(\varphi) \supset_{\varphi} \Pi(\varphi, \Lambda(\varphi))$ — Thm I, $\mathbf{C}_6$, $\mathbf{A}_2$.

COROLLARY. $\varphi(x) \supset_{\varphi x} \Lambda(\varphi, x)$.

THEOREM 7. $\Sigma(\varphi) \supset_{\varphi} \Pi(\Lambda(\varphi), \varphi)$.

$\mathbf{D}_1$: $\Sigma(\varphi)$.

$\mathbf{C}_1$: $\Pi(\varphi, \varphi)$ —V, 1, $\mathbf{D}_1$.

$\mathbf{D}_2$: $\Lambda(\varphi, x)$.

$\mathbf{C}_2$: $\Pi(\varphi, \gamma) \supset_{\gamma} \gamma(x)$ —conv, $\mathbf{D}_2$.

$\mathbf{C}_3$: $\varphi(x)$ —V, $\mathbf{C}_2$, $\mathbf{C}_1$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_4$: $\Pi(\varphi, \Lambda(\varphi))$ —V, Thm 6, $\mathbf{D}_1$.

$\mathbf{D}_3$: $\varphi(y)$.

$\mathbf{C}_5$: $\Lambda(\varphi, y)$ —V, $\mathbf{C}_4$, $\mathbf{D}_3$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_6$: $\Sigma(\Lambda(\varphi))$ —IV, $\mathbf{C}_5$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_7$: $\Sigma(\Lambda(\varphi))$ —Thm III, $\mathbf{C}_6$, $\mathbf{D}_1$; $\mathbf{D}_1$.

$\mathbf{C}_8$: $\Pi(\Lambda(\varphi), \varphi)$ —Thm II, $\mathbf{C}_3$, $\mathbf{C}_7$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Sigma(\varphi) \supset_{\varphi} \Pi(\Lambda(\varphi), \varphi)$ —Thm I, $\mathbf{C}_8$, $\mathbf{A}_2$ under Thm 6.

COROLLARY. $\Lambda(\varphi, x) \supset_{\varphi x} \varphi(x)$.

Postulate 38 is necessary to the proof of the corollary.

THEOREM 8. $[\Sigma(\varphi) . \sim \varphi(x)] \supset_{\varphi x} \sim \Lambda(\varphi, x)$.

$\mathbf{D}_1$: $\Sigma(\varphi) . \sim \varphi(x)$.

$\mathbf{C}_1$: $\Sigma(\varphi)$ —V², 15, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\Pi(\varphi, \varphi)$ —V, 1, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_3$: $\sim\varphi(x)$ —V^2, 16, $\mathbf{D}_1$.

$\mathbf{C}_4$: $\Pi(\varphi, \varphi) . \sim\varphi(x)$ —V^2, 14, $\mathbf{C}_2$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{C}_5$: $\Sigma\gamma . \Pi(\varphi, \gamma) . \sim\gamma(x)$ —IV, $\mathbf{C}_4$; $\mathbf{D}_1$.

$\mathbf{C}_6$: $\sim\Pi(\varphi, \gamma) \supset_\gamma \gamma(x)$ —V^2, 17, $\mathbf{C}_5$; $\mathbf{D}_1$.

$\mathbf{C}_7$: $\sim A(\varphi, x)$ —conv, $\mathbf{C}_6$.

$\mathbf{A}_1$: $\sim\sim . \Sigma(\varphi) \supset_\varphi \Pi(\varphi, \varphi)$ —V, 26, 1.

$\mathbf{A}_2$: $\Sigma(\sim)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $\Sigma(\sim) . \sim\sim . \Sigma(\varphi) \supset_\varphi \Pi(\varphi, \varphi)$ —V, 14, $\mathbf{A}_2$, $\mathbf{A}_1$.

$\mathbf{A}_4$: $\Sigma\varphi\Sigma x . \Sigma(\varphi) . \sim\varphi(x)$ —IV^2, $\mathbf{A}_3$.

$\mathbf{A}_5$: $[\Sigma(\varphi) . \sim\varphi(x)] \supset_{\varphi x} \sim A(\varphi, x)$ —Thm I Cor, $\mathbf{C}_7$, $\mathbf{A}_4$.

THEOREM 9. $[\Pi(\varphi, \psi) . \sim\psi(x)] \supset_{\varphi\psi x} \sim A(\varphi, x)$.

$\mathbf{D}_1$: $\Pi(\varphi, \psi) . \sim\psi(x)$.

$\mathbf{C}_1$: $\sim . \Pi(\varphi, \gamma) \supset_\gamma \gamma(x)$ —V^2, 17, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\sim A(\varphi, x)$ —conv, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\sim\sim . \Sigma(\varphi) \supset_\varphi \Pi(\varphi, \varphi)$ —V, 26, 1.

$\mathbf{A}_2$: $\Sigma(\sim)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $\Pi(\sim, \sim)$ —V, 1, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\Pi(\sim, \sim) . \sim\sim . \Sigma(\varphi) \supset_\varphi \Pi(\varphi, \varphi)$ —V^2, 14, $\mathbf{A}_3$, $\mathbf{A}_1$.

$\mathbf{A}_5$: $\Sigma\varphi\Sigma\psi\Sigma x . \Pi(\varphi, \psi) . \sim\psi(x)$ —IV^3, $\mathbf{A}_4$.

$\mathbf{A}_6$: $[\Pi(\varphi, \psi) . \sim\psi(x)] \supset_{\varphi\psi x} \sim A(\varphi, x)$ —Thm I Cor, $\mathbf{C}_2$, $\mathbf{A}_5$.

COROLLARY. $[Q(\varphi, \psi) . \sim\psi(x)] \supset_{\varphi\psi x} \sim A(\varphi, x)$.

THEOREM 10. $\sim\Pi(E, \sim)$.

$\mathbf{A}_1$: $\sim\sim . p \supset_p \sim\sim p$ —V, 26, 26.

$\mathbf{A}_2$: $E(p \supset_p \sim\sim p)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $E(p \supset_p \sim\sim p) . \sim\sim . p \supset_p \sim\sim p$ —V^2, 14, $\mathbf{A}_2$, $\mathbf{A}_1$.

$\mathbf{A}_4$: $\sim\Pi(E, \sim)$ —V^2, 17, $\mathbf{A}_3$.

THEOREM 11. $\sim\Pi(E, \lambda z z)$.

$\mathbf{A}_1$: $\sim\sim . p \supset_p \sim\sim p$ —V, 26, 26.

$\mathbf{A}_2$: $E(\sim . p \supset_p \sim\sim p)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $E(\sim . p \supset_p \sim\sim p) . \sim \{\lambda z z\} (\sim . p \supset_p \sim\sim p)$ —V^2, 14, $\mathbf{A}_2$, $\mathbf{A}_1$.

$\mathbf{A}_4$: $\sim\Pi(E, \lambda z z)$ —V^2, 17, $\mathbf{A}_3$.

THEOREM 12. $\Sigma(\iota)$.

$\mathbf{A}_1$: $\text{'}x . x = . \iota y . x = y$ —Thm 5.

$\mathbf{A}_2$: $E(\text{'}x . x = . \iota y . x = y)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $[\iota x \,.\, x = \,.\, \iota y \,.\, x = y] = \,.\, \iota y \,.\, [\iota x \,.\, x = \,.\, \iota y \,.\, x = y] = y$ —V, $\mathbf{A}_1$, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\iota y \,.\, [\iota x \,.\, x = \,.\, \iota y \,.\, x = y] = y$ —V, $\mathbf{A}_3$, $\mathbf{A}_1$.

$\mathbf{A}_5$: $\Sigma(\iota)$ —IV, $\mathbf{A}_4$.

THEOREM 13. $\Sigma x \sim \iota(x)$.

$\mathbf{A}_1$: $\sim\sim \,.\, p \supset_p \sim\sim p$ —V, 26, 26.

$\mathbf{A}_2$: $E(\sim \,.\, p \supset_p \sim\sim p)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $[\sim \,.\, p \supset_p \sim\sim p] = \,.\, \iota y \,.\, [\sim \,.\, p \supset_p \sim\sim p] = y$ —V, Thm 5, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\sim \iota y \,.\, [\sim \,.\, p \supset_p \sim\sim p] = y$ —V, $\mathbf{A}_3$, $\mathbf{A}_1$.

$\mathbf{A}_5$: $\Sigma x \sim \iota(x)$ —IV, $\mathbf{A}_4$.

THEOREM 14. $\sim \Lambda(\iota, E)$.

$\mathbf{D}_1$: $\iota(x)$.

$\mathbf{C}_1$: $\Pi(x, \lambda z z)$ —V^2, 31, $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Pi(\iota, \lambda x \Pi(x, \lambda z z))$ —Thm II, $\mathbf{C}_1$, Thm 12.

$\mathbf{A}_2$: $\Pi(\iota, \lambda x \Pi(x, \lambda z z)) \,.\, \sim \Pi(E, \lambda z z)$ —V^2, 14, $\mathbf{A}_1$, Thm 11.

$\mathbf{A}_3$: $\sim \,.\, \Pi(\iota, \gamma) \supset_\gamma \gamma(E)$ —V^2, 17, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\sim \Lambda(\iota, E)$ —conv, $\mathbf{A}_3$.

THEOREM 15. $\sim \{\Lambda x \sim \iota(x)\}(E)$.

$\mathbf{D}_1$: $\sim \iota(x)$.

$\mathbf{C}_1$: $\Pi(x, \sim)$ —V^2, 31, $\mathbf{D}_1$.

$\mathbf{A}_1$: $\sim \iota(x) \supset_x \Pi(x, \sim)$ —Thm I, $\mathbf{C}_1$, Thm 13.

$\mathbf{A}_2$: $[\sim \iota(x) \supset_x \Pi(x, \sim)] \sim \Pi(E, \sim)$ —V^2, 14, $\mathbf{A}_1$, Thm 10.

$\mathbf{A}_3$: $\sim \,.\, \Pi(\lambda x \sim \iota x, \gamma) \supset_\gamma \gamma(E)$ —V^2, 17, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\sim \{\Lambda x \sim \iota(x)\}(E)$ —conv, $\mathbf{A}_3$.

THEOREM 16. $\Sigma x \Sigma y [\varphi(x) \,.\, \sim \varphi(y)] \supset_\varphi G(\Lambda(\varphi))$.

$\mathbf{D}_1$: $\varphi(z)$.

$\mathbf{C}_1$: $\Lambda(\varphi, z)$ —V^2, Thm 6 Cor, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\Lambda(\Lambda(\varphi), z)$ —V^2, Thm 6 Cor, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{D}_2$: $\varphi(x) \,.\, \sim \varphi(y)$.

$\mathbf{C}_3$: $\varphi(x)$ —V^2, 15, $\mathbf{D}_2$.

$\mathbf{C}_4$: $\Sigma(\varphi)$ —IV, $\mathbf{C}_3$; $\mathbf{D}_2$.

$\mathbf{C}_5$: $\Pi(\varphi, \Lambda(\Lambda(\varphi)))$ —Thm II, $\mathbf{C}_2$, $\mathbf{C}_4$; $\mathbf{D}_2$.

$\mathbf{D}_3$: $\sim \Lambda(\Lambda(\varphi), t)$.

$\mathbf{C}_6$: $\sim \Lambda(\varphi, t)$ —V^3, Thm 9, $\mathbf{C}_5$, $\mathbf{D}_3$; $\mathbf{D}_2$, $\mathbf{D}_3$.

$\mathbf{C}_7$: $\Lambda(\varphi, x)$ —V^2, Thm 6 Cor, $\mathbf{C}_3$; $\mathbf{D}_2$.

$\mathbf{C}_8$: $\Sigma(\mathcal{A}(\varphi))$ —IV, $\mathbf{C}_7$; $\mathbf{D}_2$.

$\mathbf{C}_9$: $\sim\varphi(y)$ —V^2, 16, $\mathbf{D}_2$.

$\mathbf{C}_{10}$: $\sim\mathcal{A}(\varphi, y)$ —V^2, Thm 8, $\mathbf{C}_4$, $\mathbf{C}_9$; $\mathbf{D}_2$.

$\mathbf{C}_{11}$: $\sim\mathcal{A}(\mathcal{A}(\varphi), y)$ —V^2, Thm 8, $\mathbf{C}_8$, $\mathbf{C}_{10}$; $\mathbf{D}_2$.

$\mathbf{C}_{12}$: $\Sigma t \sim\mathcal{A}(\mathcal{A}(\varphi), t)$ —IV, $\mathbf{C}_{11}$; $\mathbf{D}_2$.

$\mathbf{C}_{13}$: $\sim\mathcal{A}(\mathcal{A}(\varphi), t) \supset_t \sim\mathcal{A}(\varphi, t)$ —Thm I, $\mathbf{C}_6$, $\mathbf{C}_{12}$; $\mathbf{D}_2$.

$\mathbf{C}_{14}$: $G(\mathcal{A}(\varphi))$ —conv, $\mathbf{C}_{13}$; $\mathbf{D}_2$.

$\mathbf{D}_4$: $\Sigma x\, \Sigma y \,.\, \varphi(x) \,.\, \sim\varphi(y)$.

$\mathbf{C}_{15}$: $G(\mathcal{A}(\varphi))$ —Thm I Cor, $\mathbf{C}_{14}$, $\mathbf{D}_4$; $\mathbf{D}_4$.

$\mathbf{A}_1$: $E(\Pi(E, \sim))$ —IV, Thm 10.

$\mathbf{A}_2$: $\Sigma(E)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $\Pi(E, E)$ —V, 1, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\Pi(E, E) \,.\, \sim\Pi(E, \sim)$ —V^2, 14, $\mathbf{A}_3$, Thm 10.

$\mathbf{A}_5$: $\Sigma\varphi\, \Sigma x\, \Sigma y \,.\, \varphi(x) \,.\, \sim\varphi(y)$ —IV^3, $\mathbf{A}_4$.

$\mathbf{A}_6$: $\Sigma x\, \Sigma y[\varphi(x) \,.\, \sim\varphi(y)] \supset_\varphi G(\mathcal{A}(\varphi))$ —Thm I, $\mathbf{C}_{15}$, $\mathbf{A}_5$.

COROLLARY. $\Sigma x\, \Sigma y[\varphi(x) \,.\, \sim\mathcal{A}(\varphi, y)] \supset_\varphi G(\mathcal{A}(\varphi))$.

THEOREM 17. $\Sigma(G)$.

$\mathbf{A}_1$: $\Sigma x\, \Sigma y \,.\, \Pi(E, x) \,.\, \sim\Pi(E, y)$ —IV^2, $\mathbf{A}_4$ under Thm 16.

$\mathbf{A}_2$: $G(\mathcal{A}(\Pi(E)))$ —V, Thm 16, $\mathbf{A}_1$.

$\mathbf{A}_3$: $\Sigma(G)$ —IV, $\mathbf{A}_2$.

COROLLARY. $\Sigma\varphi\, G(\varphi)$.

THEOREM 18. $G(\varphi) \supset_\varphi \Sigma(\varphi)$.

$\mathbf{D}_1$: $G(\varphi)$.

$\mathbf{C}_1$: $\sim\mathcal{A}(\varphi, x) \supset_x \sim\varphi(x)$ —II, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\Sigma x \sim\mathcal{A}(\varphi, x)$ —V, 38, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{D}_2$: $\sim\mathcal{A}(\varphi, x)$.

$\mathbf{C}_3$: $\sim \,.\, \Pi(\varphi, \gamma) \supset_\gamma \gamma(x)$ —conv, $\mathbf{D}_2$.

$\mathbf{C}_4$: $\Sigma\gamma \,.\, \Pi(\varphi, \gamma) \,.\, \sim\gamma(x)$ —V^2, 18, $\mathbf{C}_3$; $\mathbf{D}_2$.

$\mathbf{D}_3$: $\Pi(\varphi, \gamma) \,.\, \sim\gamma(x)$.

$\mathbf{C}_5$: $\Pi(\varphi, \gamma)$ —V^2, 15, $\mathbf{D}_3$.

$\mathbf{C}_6$: $\Sigma(\varphi)$ —V, 38, $\mathbf{C}_5$; $\mathbf{D}_3$.

$\mathbf{C}_7$: $\Sigma(\varphi)$ —Thm I, $\mathbf{C}_6$, $\mathbf{C}_4$; $\mathbf{D}_2$.

$\mathbf{C}_8$: $\Sigma(\varphi)$ —Thm I, $\mathbf{C}_7$, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $G(\varphi) \supset_\varphi \Sigma(\varphi)$ —Thm I, $\mathbf{C}_8$, Thm 17 Cor.

THEOREM 19. $G(\varphi) \supset_{\varphi} \Sigma x \sim \varphi(x)$.

$\mathbf{D}_1$: $G(\varphi)$.

$\mathbf{C}_1$: $\sim A(\varphi, x) \supset_x \sim \varphi(x)$ —II, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\Sigma x \sim A(\varphi, x)$ —V, 38, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{D}_2$: $\sim A(\varphi, x)$.

$\mathbf{C}_3$: $\sim \varphi(x)$ —V, $\mathbf{C}_1$, $\mathbf{D}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_4$: $\Sigma x \sim \varphi(x)$ —IV, $\mathbf{C}_3$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_5$: $\Sigma x \sim \varphi(x)$ —Thm I, $\mathbf{C}_4$, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $G(\varphi) \supset_{\varphi} \Sigma x \sim \varphi(x)$ —Thm I, $\mathbf{C}_5$, Thm 17 Cor.

THEOREM 20. $G(\varphi) \supset_{\varphi} . [\Pi(\varphi, \psi) . \sim \psi(x)] \supset_{\psi x} \sim \varphi(x)$.

$\mathbf{D}_1$: $\Pi(\varphi, \psi) . \sim \psi(x)$.

$\mathbf{C}_1$: $\sim A(\varphi, x)$ —V, Thm 9, $\mathbf{D}_1$.

$\mathbf{D}_2$: $G(\varphi)$.

$\mathbf{C}_2$: $\sim A(\varphi, x) \supset_x \sim \varphi(x)$ —II, $\mathbf{D}_2$.

$\mathbf{C}_3$: $\sim \varphi(x)$ —V, $\mathbf{C}_2$, $\mathbf{C}_1$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_4$: $\Sigma(\varphi)$ —V, Thm 18, $\mathbf{D}_2$.

$\mathbf{C}_5$: $\Pi(\varphi, \varphi)$ —V, 1, $\mathbf{C}_4$; $\mathbf{D}_2$.

$\mathbf{C}_6$: $\Sigma y \sim \varphi(y)$ —V, Thm 19, $\mathbf{D}_2$.

$\mathbf{D}_3$: $\sim \varphi(y)$.

$\mathbf{C}_7$: $\Pi(\varphi, \varphi) . \sim \varphi(y)$ —V^2, 14, $\mathbf{C}_5$, $\mathbf{D}_3$; $\mathbf{D}_2$, $\mathbf{D}_3$.

$\mathbf{C}_8$: $\Sigma \psi \Sigma x . \Pi(\varphi, \psi) . \sim \psi(x)$ —IV^2, $\mathbf{C}_7$; $\mathbf{D}_2$, $\mathbf{D}_3$.

$\mathbf{C}_9$: $\Sigma \psi \Sigma x . \Pi(\varphi, \psi) . \sim \psi(x)$ —Thm I, $\mathbf{C}_8$, $\mathbf{C}_6$; $\mathbf{D}_2$.

$\mathbf{C}_{10}$: $[\Pi(\varphi, \psi) . \sim \psi(x)] \supset_{\psi x} \sim \varphi(x)$ —Thm I Cor, $\mathbf{C}_3$, $\mathbf{C}_9$; $\mathbf{D}_2$.

$\mathbf{A}_1$: $G(\varphi) \supset_{\varphi} . [\Pi(\varphi, \psi) . \sim \psi(x)] \supset_{\psi x} \sim \varphi(x)$ —Thm I, $\mathbf{C}_{10}$, Thm 17 Cor.

THEOREM 21. $G(\varphi) \supset_{\varphi} Gx \sim \{A y \sim \varphi(y)\}(x)$.

$\mathbf{D}_1$: $G(\varphi)$.

$\mathbf{C}_1$: $\Sigma y \sim \varphi(y)$ —V, Thm 19, $\mathbf{D}_1$.

$\mathbf{D}_2$: $\varphi(u)$.

$\mathbf{C}_2$: $\sim \sim \varphi(u)$ —V, 26, $\mathbf{D}_2$.

$\mathbf{C}_3$: $\sim \{A y \sim \varphi(y)\}(u)$ —V^2, Thm 8, $\mathbf{C}_1$, $\mathbf{C}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{D}_3$: $\Pi(\lambda x \sim \{A y \sim \varphi(y)\}(x), \gamma) . \sim \gamma(z)$.

$\mathbf{C}_4$: $\Pi(\lambda x \sim \{A y \sim \varphi(y)\}(x), \gamma)$ —V^2, 15, $\mathbf{D}_3$.

$\mathbf{C}_5$: $\gamma(u)$ —V, $\mathbf{C}_4$, $\mathbf{C}_3$; $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$.

$\mathbf{C}_6$: $\Sigma(\varphi)$ —V, Thm 18, $\mathbf{D}_1$.

$\mathbf{C}_7$: $\Pi(\varphi, \gamma)$ —Thm II, $\mathbf{C}_5$, $\mathbf{C}_6$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_8$: $\sim\gamma(z)$ —V, 16, $\mathbf{D}_3$.

$\mathbf{C}_9$: $\Pi(\varphi, \gamma) . \sim\gamma(z)$ —V^2, 14, $\mathbf{C}_7$, $\mathbf{C}_8$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_{10}$: $\Sigma\gamma . \Pi(\varphi, \gamma) . \sim\gamma(z)$ —IV, $\mathbf{C}_9$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_{11}$: $\sim A(\varphi, z)$ —V^2, 17, $\mathbf{C}_{10}$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{D}_4$: $\sim\{Ax \sim \{Ay \sim \varphi(y)\}(x)\}(z)$.

$\mathbf{C}_{12}$: $\Sigma\gamma . \Pi(\lambda x \sim \{Ay \sim \varphi(y)\}(x), \gamma) . \sim\gamma(z)$ —V^2, 18, $\mathbf{D}_4$.

$\mathbf{C}_{13}$: $\sim A(\varphi, z)$ —Thm I, $\mathbf{C}_{11}$, $\mathbf{C}_{12}$; $\mathbf{D}_1$, $\mathbf{D}_4$.

$\mathbf{C}_{14}$: $\sim A(\varphi, x) \supset_x \sim\varphi(x)$ —II, $\mathbf{D}_1$.

$\mathbf{C}_{15}$: $\sim\varphi(z)$ —V, $\mathbf{C}_{14}$, $\mathbf{C}_{13}$; $\mathbf{D}_1$, $\mathbf{D}_4$.

$\mathbf{C}_{16}$: $\{Ay \sim \varphi(y)\}(z)$ —V^2, Thm 6 Cor, $\mathbf{C}_{15}$; $\mathbf{D}_1$, $\mathbf{D}_4$.

$\mathbf{C}_{17}$: $\sim\sim\{Ay \sim \varphi(y)\}(z)$ —V, 26, $\mathbf{C}_{16}$; $\mathbf{D}_1$, $\mathbf{D}_4$.

$\mathbf{D}_5$: $\sim\varphi(v)$.

$\mathbf{C}_{18}$: $\{Ay \sim \varphi(y)\}(v)$ —V^2, Thm 6 Cor, $\mathbf{D}_5$.

$\mathbf{C}_{19}$: $\sim\sim\{Ay \sim \varphi(y)\}(v)$ —V, 26, $\mathbf{C}_{18}$; $\mathbf{D}_5$.

$\mathbf{C}_{20}$: $\Sigma x \sim \{Ay \sim \varphi(y)\}(x)$ —IV, $\mathbf{C}_3$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_{21}$: $\Sigma x \sim \{Ay \sim \varphi(y)\}(x)$ —Thm III, $\mathbf{C}_{20}$, $\mathbf{C}_6$; $\mathbf{D}_1$.

$\mathbf{C}_{22}$: $\sim\{Ax \sim \{Ay \sim \varphi(y)\}(x)\}(v)$ —V^2, Thm 8, $\mathbf{C}_{21}$, $\mathbf{C}_{19}$; $\mathbf{D}_1$, $\mathbf{D}_5$.

$\mathbf{C}_{23}$: $\Sigma z \sim\{Ax \sim \{Ay \sim \varphi(y)\}(x)\}(z)$ —IV, $\mathbf{C}_{22}$; $\mathbf{D}_1$, $\mathbf{D}_5$.

$\mathbf{C}_{24}$: $\Sigma z \sim\{Ax \sim \{Ay \sim \varphi(y)\}(x)\}(z)$ —Thm I, $\mathbf{C}_{23}$, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_{25}$: $\sim\{Ax \sim \{Ay \sim \varphi(y)\}(x)\}(z) \supset_z \sim\sim\{Ay \sim \varphi(y)\}(z)$ —Thm I, $\mathbf{C}_{17}$, $\mathbf{C}_{24}$; $\mathbf{D}_1$.

$\mathbf{C}_{26}$: $Gx \sim \{Ay \sim \varphi(y)\}(x)$ —conv, $\mathbf{C}_{25}$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $G(\varphi) \supset_\varphi Gx \sim \{Ay \sim \varphi(y)\}(x)$ —Thm I, $\mathbf{C}_{26}$, Thm 17 Cor.

COROLLARY. $Gx \sim \varphi(x) \supset_\varphi Gx \sim A(\varphi, x)$.

THEOREM 22. $[\Sigma z . x \neq z] \supset_x Gy . x = y$.

$\mathbf{D}_1$: $\Sigma z . x \neq z$.

$\mathbf{C}_1$: $E(x)$ —IV, $\mathbf{D}_1$.

$\mathbf{C}_2$: $x = x$ —V, Thm 1, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_3$: $\{\lambda u . x = u\}(x)$ —III, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{C}_4$: $\{Au . x = u\}(x)$ —V^2, Thm 6 Cor, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{D}_2$: $\sim\{Au . x = u\}(y)$.

$\mathbf{C}_5$: $\{Au . x = u\}(x) . \sim\{Au . x = u\}(y)$ —V^2, 14, $\mathbf{C}_4$, $\mathbf{D}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_6$: $\Sigma\varrho . \varrho(x) . \sim\varrho(y)$ —IV, $\mathbf{C}_5$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_7$: $x \neq y$ —V^2, 17, $\mathbf{C}_6$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_8$: $\Sigma u . x = u$ —IV, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{D}_3$: $x \ddagger z$.

$\mathbf{C}_9$: $\sim\{\lambda u \,.\, x = u\}(z)$ —III, $\mathbf{D}_3$.

$\mathbf{C}_{10}$: $\sim\{\Lambda u \,.\, x = u\}(z)$ —V^2, Thm 8, $\mathbf{C}_8$, $\mathbf{C}_9$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_{11}$: $\Sigma y \sim\{\Lambda u \,.\, x = u\}(y)$ —IV, $\mathbf{C}_{10}$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_{12}$: $\Sigma y \sim\{\Lambda u \,.\, x = u\}(y)$ —Thm I, $\mathbf{C}_{11}$, $\mathbf{D}_1$; $\mathbf{D}_1$.

$\mathbf{C}_{13}$: $\sim\{\Lambda u \,.\, x = u\}(y) \supset_y x \ddagger y$ —Thm I, $\mathbf{C}_7$, $\mathbf{C}_{12}$; $\mathbf{D}_1$.

$\mathbf{C}_{14}$: $G y \,.\, x = y$ —conv, $\mathbf{C}_{13}$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Sigma\varrho \,.\, \varrho(E) \,.\, \sim\varrho(\sim)$ —IV, $\mathbf{A}_4$ under Thm 16.

$\mathbf{A}_2$: $E \ddagger \sim$ —V^2, 17, $\mathbf{A}_1$.

$\mathbf{A}_3$: $\Sigma x \Sigma z \,.\, x \ddagger z$ —IV^2, $\mathbf{A}_2$.

$\mathbf{A}_4$: $[\Sigma z \,.\, x \ddagger z] \supset_x G y \,.\, x = y$ —Thm I, $\mathbf{C}_{14}$, $\mathbf{A}_3$.

6. **Classes.** We recall the following definitions:

$$Q \longrightarrow \lambda\mu\,\lambda\nu \,.\, \Pi(\mu, \nu) \,.\, \Pi(\nu, \mu).$$
$$K \longrightarrow A(Q).$$

And to these we add two new definitions:

$$\varepsilon \longrightarrow \lambda\alpha\,\lambda\beta \,.\, [\beta = K(\varphi)] \supset_\varphi \varphi(\alpha).$$
$$O \longrightarrow \lambda\gamma\, K\pi \sim \{\varepsilon\}(\pi, \gamma).$$

We shall abbreviate $\{\varepsilon\}(\mathbf{A}, \mathbf{B})$ as $[\mathbf{A}]\,\varepsilon\,[\mathbf{B}]$, and hence we may rewrite the definition of O as follows:

$$O \longrightarrow \lambda\gamma\, K\pi \sim . \,\pi\varepsilon\gamma.$$

If $\mathbf{F}$ and $\mathbf{G}$ are propositional functions, $Q(\mathbf{F}, \mathbf{G})$ is to be read, "$\mathbf{F}$ and $\mathbf{G}$ are equivalent", and $K(\mathbf{F})$ is to be read, "The class of x's such that $\{\mathbf{F}\}(x)$ is true". And $[\mathbf{A}]\,\varepsilon\,[\mathbf{B}]$ is to be read, "$\mathbf{A}$ is a member of the class $\mathbf{B}$". And $O(\mathbf{B})$ is to be read, "The class of things which are not members of the class $\mathbf{B}$", or, "The complement of the class $\mathbf{B}$".

THEOREM 23. $Q(x, y) \supset_{xy} . Q(y, z) \supset_z Q(x, z)$.

$\mathbf{D}_1$: $Q(x, y)$.

$\mathbf{C}_1$: $\Pi(x, y)$ —V^2, 15, $\mathbf{D}_1$.

$\mathbf{D}_2$: $x(r)$.

$\mathbf{C}_2$: $y(r)$ —V, $\mathbf{C}_1$, $\mathbf{D}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{D}_3$: $Q(y, z)$.

$\mathbf{C}_3$: $\Pi(y, z)$ —V^2, 15, $\mathbf{D}_3$.

$\mathbf{C}_4$: $z(r)$ —V, $\mathbf{C}_3$, $\mathbf{C}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$.

$\mathbf{C}_5$: $\Sigma(x)$ —V, 38, $\mathbf{C}_1$; $\mathbf{D}_1$.
$\mathbf{C}_6$: $\Pi(x, z)$ —Thm II, $\mathbf{C}_4$, $\mathbf{C}_5$; $\mathbf{D}_1$, $\mathbf{D}_3$.
$\mathbf{C}_7$: $\Pi(z, y)$ —V, 16, $\mathbf{D}_3$.
$\mathbf{D}_4$: $z(s)$.
$\mathbf{C}_8$: $y(s)$ —V, $\mathbf{C}_7$, $\mathbf{D}_4$; $\mathbf{D}_3$, $\mathbf{D}_4$.
$\mathbf{C}_9$: $\Pi(y, x)$ —V, 16, $\mathbf{D}_1$.
$\mathbf{C}_{10}$: $x(s)$ —V, $\mathbf{C}_9$, $\mathbf{C}_8$; $\mathbf{D}_1$, $\mathbf{D}_3$, $\mathbf{D}_4$.
$\mathbf{C}_{11}$: $\Sigma(z)$ —V, 38, $\mathbf{C}_7$; $\mathbf{D}_3$.
$\mathbf{C}_{12}$: $\Pi(z, x)$ —Thm II, $\mathbf{C}_{10}$, $\mathbf{C}_{11}$; $\mathbf{D}_1$, $\mathbf{D}_3$.
$\mathbf{C}_{13}$: $Q(x, z)$ —V^2, 14, $\mathbf{C}_6$, $\mathbf{C}_{12}$; $\mathbf{D}_1$, $\mathbf{D}_3$.
$\mathbf{C}_{14}$: $\Sigma(y)$ —V, 38, $\mathbf{C}_9$; $\mathbf{D}_1$.
$\mathbf{C}_{15}$: $\Pi(y, y)$ —V, 1, $\mathbf{C}_{14}$; $\mathbf{D}_1$.
$\mathbf{C}_{16}$: $Q(y, y)$ —V^2, 14, $\mathbf{C}_{15}$, $\mathbf{C}_{15}$; $\mathbf{D}_1$.
$\mathbf{C}_{17}$: $\Sigma z\, Q(y, z)$ —IV, $\mathbf{C}_{16}$; $\mathbf{D}_1$.
$\mathbf{C}_{18}$: $Q(y, z) \supset_z Q(x, z)$ —Thm I, $\mathbf{C}_{13}$, $\mathbf{C}_{17}$; $\mathbf{D}_1$.
$\mathbf{A}_1$: $Q(E, E)$ —V^2, 14, $\mathbf{A}_3$ under Thm 16, $\mathbf{A}_3$ under Thm 16.
$\mathbf{A}_2$: $\Sigma x\, \Sigma y\, Q(x, y)$ —IV^2, $\mathbf{A}_1$.
$\mathbf{A}_3$: $Q(x, y) \supset_{xy} . Q(y, z) \supset_z Q(x, z)$ —Thm I Cor, $\mathbf{C}_{18}$, $\mathbf{A}_2$.

THEOREM 24. $Q(x, y) \supset_{xy} Q(y, x)$.
$\mathbf{D}_1$: $Q(x, y)$.
$\mathbf{C}_1$: $\Pi(x, y)$ —V^2, 15, $\mathbf{D}_1$.
$\mathbf{C}_2$: $\Pi(y, x)$ —V^2, 16, $\mathbf{D}_1$.
$\mathbf{C}_3$: $Q(y, x)$ —V^2, 14, $\mathbf{C}_2$, $\mathbf{C}_1$; $\mathbf{D}_1$.
$\mathbf{A}_1$: $Q(x, y) \supset_{xy} Q(y, x)$ —Thm I Cor, $\mathbf{C}_3$, $\mathbf{A}_2$ under Thm 23.

THEOREM 25. $Q(\varphi, \psi) \supset_{\varphi\psi} . K(\varphi) = K(\psi)$ — V, 33, Thm 23, Thm 24.

THEOREM 26. $[K(\varphi) = K(\psi)] \supset_{\varphi\psi} Q(\varphi, \psi)$.
$\mathbf{A}_1$: $K(E) = K(E)$ —V^2, Thm 25, $\mathbf{A}_1$ under Thm 23.
$\mathbf{A}_2$: $\Sigma u\, \Sigma v . K(u) = K(v)$ —IV^2, $\mathbf{A}_1$.
$\mathbf{A}_3$: $[K(u) = K(v)] \supset_{uv} Q(u, v)$ —V, 35, $\mathbf{A}_2$.
$\mathbf{A}_4$: $[K(\varphi) = K(\psi)] \supset_{\varphi\psi} Q(\varphi, \psi)$ —conv, $\mathbf{A}_3$.

THEOREM 27. $\sim Q(\varphi, \psi) \supset_{\varphi\psi} . K(\varphi) \neq K(\psi)$.
$\mathbf{D}_1$: $\Sigma y\, Q(x, y)$.
$\mathbf{C}_1$: $Q(x, y) \supset_y Q(x, y)$ —V, 1, $\mathbf{D}_1$.

$\mathbf{A}_1$: $Q(x, y) \supset_{xy} Q(x, y)$ —Thm I, $\mathbf{C}_1$, $\mathbf{A}_2$ under Thm 23.

$\mathbf{D}_2$: $\sim Q(\varphi, \psi)$.

$\mathbf{C}_2$: $K(\varphi) \ddagger K(\psi)$ —V⁴, 34, Thm 23, Thm 24, $\mathbf{A}_1$, $\mathbf{D}_2$.

$\mathbf{D}_3$: $\sim p$.

$\mathbf{C}_3$: $E(p)$ —IV, $\mathbf{D}_3$.

$\mathbf{A}_2$: $\Pi(\sim, E)$ —Thm II, $\mathbf{C}_3$, $\mathbf{A}_2$ under Thm 9.

$\mathbf{A}_3$: $\sim Q(E, \sim)$ —V², 22, Thm 10, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\Sigma\varphi\,\Sigma\psi \sim Q(\varphi, \psi)$ —IV², $\mathbf{A}_3$.

$\mathbf{A}_5$: $\sim Q(\varphi, \psi) \supset_{\varphi\psi} . K(\varphi) \ddagger K(\psi)$ —Thm I Cor, $\mathbf{C}_2$, $\mathbf{A}_4$.

THEOREM 28. $\Sigma x\,\Sigma y\,[\varphi(x) . \sim \varphi(y)] \supset_{\varphi} G\psi . K(\varphi) = K(\psi)$.

$\mathbf{D}_1$: $Q(x, y)$.

$\mathbf{C}_1$: $K(x) = K(y)$ —V², Thm 25, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\{\Lambda\psi . K(x) = K(\psi)\}(y)$ —V², Thm 6 Cor, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $Q(x, y) \supset_{xy} \{\Lambda\psi . K(x) = K(\psi)\}(y)$ —Thm I Cor, $\mathbf{C}_2$, $\mathbf{A}_2$ under Thm 23.

$\mathbf{D}_2$: $\sim \{\Lambda\psi . K(u) = K(\psi)\}(v)$.

$\mathbf{C}_3$: $K(u) \ddagger K(v)$ —V⁴, 34, Thm 23, Thm 24, $\mathbf{A}_1$, $\mathbf{D}_2$.

$\mathbf{D}_3$: $u(x) . \sim u(y)$.

$\mathbf{C}_4$: $u(x)$ —V², 15, $\mathbf{D}_3$.

$\mathbf{C}_5$: $\sim \sim u(x)$ —V, 26, $\mathbf{C}_4$; $\mathbf{D}_3$.

$\mathbf{C}_6$: $\sim \Pi(u, \lambda x \sim u(x))$ —V², 17, $\mathbf{C}_4$, $\mathbf{C}_5$; $\mathbf{D}_3$.

$\mathbf{C}_7$: $\sim u(y)$ —V², 16, $\mathbf{D}_3$.

$\mathbf{C}_8$: $\sim \Pi(\lambda x \sim u(x), u)$ —V², 17, $\mathbf{C}_7$, $\mathbf{C}_7$; $\mathbf{D}_3$.

$\mathbf{C}_9$: $\sim Q(u, \lambda x \sim u(x))$ —V², 23, $\mathbf{C}_6$, $\mathbf{C}_8$; $\mathbf{D}_3$.

$\mathbf{C}_{10}$: $K(u) \ddagger Kx \sim u(x)$ —V², Thm 27, $\mathbf{C}_9$; $\mathbf{D}_3$.

$\mathbf{C}_{11}$: $\Sigma(u)$ —IV, $\mathbf{C}_4$; $\mathbf{D}_3$.

$\mathbf{C}_{12}$: $\Pi(u, u)$ —V, 1, $\mathbf{C}_{11}$; $\mathbf{D}_3$.

$\mathbf{C}_{13}$: $Q(u, u)$ —V², 14, $\mathbf{C}_{12}$, $\mathbf{C}_{12}$; $\mathbf{D}_3$.

$\mathbf{C}_{14}$: $K(u) = K(u)$ —V², Thm 25, $\mathbf{C}_{13}$; $\mathbf{D}_3$.

$\mathbf{C}_{15}$: $\Sigma\psi . K(u) = K(\psi)$ —IV, $\mathbf{C}_{14}$; $\mathbf{D}_3$.

$\mathbf{C}_{16}$: $\sim \{\Lambda\psi . K(u) = K(\psi)\}(\lambda x \sim u(x))$ —V², Thm 8, $\mathbf{C}_{15}$, $\mathbf{C}_{10}$; $\mathbf{D}_3$.

$\mathbf{C}_{17}$: $\Sigma v \sim \{\Lambda\psi . K(u) = K(\psi)\}(v)$ —IV, $\mathbf{C}_{16}$; $\mathbf{D}_3$.

$\mathbf{C}_{18}$: $\sim \{\Lambda\psi . K(u) = K(\psi)\}(v) \supset_{v} . K(u) \ddagger K(v)$ —Thm I, $\mathbf{C}_3$, $\mathbf{C}_{17}$; $\mathbf{D}_3$.

$\mathbf{C}_{19}$: $G\psi . K(u) = K(\psi)$ —conv, $\mathbf{C}_{18}$; $\mathbf{D}_3$.

$\mathbf{D}_4$: $\Sigma x\,\Sigma y . u(x) . \sim u(y)$.

$\mathbf{C}_{20}$: $G\psi . K(u) = K(\psi)$ —Thm I Cor, $\mathbf{C}_{19}$, $\mathbf{D}_4$; $\mathbf{D}_4$.

$\mathbf{A}_2$: $\Sigma u \Sigma x \Sigma y \,.\, u(x) \,.\, \sim u(y)$ —I, $\mathbf{A}_5$ under Thm 16.

$\mathbf{A}_3$: $\Sigma x \Sigma y [u(x) \,.\, \sim u(y)] \supset_u G\psi \,.\, K(u) = K(\psi)$ —Thm I, $\mathbf{C}_{20}$, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\Sigma x \Sigma y [\varphi(x) \,.\, \sim \varphi(y)] \supset_\varphi G\psi \,.\, K(\varphi) = K(\psi)$ —I, $\mathbf{A}_3$.

THEOREM 29. $\varphi(x) \supset_{\varphi x} \,.\, x \,\varepsilon\, K(\varphi)$.

$\mathbf{D}_1$: $K(\psi) = K(\varphi)$.

$\mathbf{C}_1$: $Q(\psi, \varphi)$ —V^2, Thm 26, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\Pi(\psi, \varphi)$ —V^2, 15, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{D}_2$: $\psi(x)$.

$\mathbf{C}_3$: $\varphi(x)$ —V, $\mathbf{C}_2$, $\mathbf{D}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_4$: $\Sigma(\psi)$ —IV, $\mathbf{D}_2$.

$\mathbf{C}_5$: $\Pi(\psi, \psi)$ —V, 1, $\mathbf{C}_4$; $\mathbf{D}_2$.

$\mathbf{C}_6$: $Q(\psi, \psi)$ —V^2, 14, $\mathbf{C}_3$, $\mathbf{C}_5$; $\mathbf{D}_2$.

$\mathbf{C}_7$: $K(\psi) = K(\psi)$ —V^2, Thm 25, $\mathbf{C}_6$; $\mathbf{D}_2$.

$\mathbf{C}_8$: $\Sigma\varphi \,.\, K(\psi) = K(\varphi)$ —IV, $\mathbf{C}_7$; $\mathbf{D}_2$.

$\mathbf{C}_9$: $[K(\psi) = K(\varphi)] \supset_\varphi \varphi(x)$ —Thm I, $\mathbf{C}_3$, $\mathbf{C}_8$; $\mathbf{D}_2$.

$\mathbf{C}_{10}$: $x \,\varepsilon\, K(\psi)$ —conv, $\mathbf{C}_9$; $\mathbf{D}_2$.

$\mathbf{A}_1$: $\Sigma\psi \Sigma x \,.\, \psi(x)$ —IV^2, 1.

$\mathbf{A}_2$: $\psi(x) \supset_{\psi x} \,.\, x \,\varepsilon\, K(\psi)$ —Thm I Cor, $\mathbf{C}_{10}$, $\mathbf{A}_1$.

$\mathbf{A}_3$: $\varphi(x) \supset_{\varphi x} \,.\, x \,\varepsilon\, K(\varphi)$ —I, $\mathbf{A}_2$.

THEOREM 30. $[x \,\varepsilon\, K(\varphi)] \supset_{\varphi x} \varphi(x)$.

$\mathbf{D}_1$: $x \,\varepsilon\, K(\psi)$.

$\mathbf{C}_1$: $E(K(\psi))$ —IV, $\mathbf{D}_1$.

$\mathbf{C}_2$: $K(\psi) = K(\psi)$ —V, Thm 1, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_3$: $[K(\psi) = K(\varphi)] \supset_\varphi \varphi(x)$ —conv, $\mathbf{D}_1$.

$\mathbf{C}_4$: $\psi(x)$ —V, $\mathbf{C}_3$, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $[\lambda\varphi \Pi(\varphi, \varphi)] \,\varepsilon\, K(\Pi(\lambda\varphi \,\Sigma(\varphi)))$ —V^2, Thm 29, 1.

$\mathbf{A}_2$: $\Sigma\psi \Sigma x \,.\, x \,\varepsilon\, K(\psi)$ —IV^2, $\mathbf{A}_1$.

$\mathbf{A}_3$: $[x \,\varepsilon\, K(\psi)] \supset_{\psi x} \psi(x)$ —Thm I Cor, $\mathbf{C}_4$, $\mathbf{A}_2$.

$\mathbf{A}_4$: $[x \,\varepsilon\, K(\varphi)] \supset_{\varphi x} \varphi(x)$ —I, $\mathbf{A}_3$.

COROLLARY 1. $\Sigma(\varphi) \supset_\varphi Q(\varphi, \lambda x \,.\, x \,\varepsilon\, K(\varphi))$.

COROLLARY 2. $\Sigma(\varphi) \supset_\varphi \,.\, K(\varphi) = Kx \,.\, x \,\varepsilon\, K(\varphi)$.

THEOREM 31. $E(K(\varphi)) \supset_\varphi \Sigma(\varphi)$.

$\mathbf{D}_1$: $E(K(\varphi))$.

$\mathbf{C}_1$: $K(\varphi) = K(\varphi)$ —V, Thm 1, $\mathbf{D}_1$.

$\mathbf{C}_2$: $Q(\varphi, \varphi)$ —V^2, Thm 26, $\mathbf{C}_1$; $\mathbf{D}_1$.
$\mathbf{C}_3$: $\Pi(\varphi, \varphi)$ —V^2, 15, $\mathbf{C}_2$; $\mathbf{D}_1$.
$\mathbf{C}_4$: $\Sigma(\varphi)$ —V, 38, $\mathbf{C}_3$; $\mathbf{D}_1$.
$\mathbf{A}_1$: $E\left(K(\Pi(\lambda\varphi\,\Sigma(\varphi)))\right)$ —IV, $\mathbf{A}_1$ under Thm 30.
$\mathbf{A}_2$: $\Sigma\varphi\, E(K(\varphi))$ —IV, $\mathbf{A}_1$.
$\mathbf{A}_3$: $E(K(\varphi)) \supset_\varphi \Sigma(\varphi)$ —Thm I, $\mathbf{C}_4$, $\mathbf{A}_2$.

COROLLARY 1. $E(K(\varphi)) \supset_\varphi \Sigma x \,.\, x \varepsilon K(\varphi)$.

COROLLARY 2. $\Sigma x[\sim . x \varepsilon \alpha] \supset_\alpha \Sigma x \,.\, x \varepsilon \alpha$.

THEOREM 32. $\Sigma x[x \varepsilon \alpha] \supset_\alpha . \alpha = Kx \,.\, x \varepsilon \alpha$.

$\mathbf{D}_1$: $\alpha = K(\varphi)$.
$\mathbf{C}_1$: $E(K(\varphi))$ —IV, $\mathbf{D}_1$.
$\mathbf{C}_2$: $\Sigma(\varphi)$ —V, Thm 31, $\mathbf{C}_1$; $\mathbf{D}_1$.
$\mathbf{C}_3$: $K(\varphi) = Kx \,.\, x \varepsilon K(\varphi)$ —V, Thm 30 Cor 2, $\mathbf{C}_2$; $\mathbf{D}_1$.
$\mathbf{C}_4$: $\alpha = Kx \,.\, x \varepsilon K(\varphi)$ —V^3, Thm 3, $\mathbf{D}_1$, $\mathbf{C}_3$; $\mathbf{D}_1$.
$\mathbf{C}_5$: $K(\varphi) = \alpha$ —V^2, Thm 2, $\mathbf{D}_1$.
$\mathbf{C}_6$: $\alpha = Kx \,.\, x \varepsilon \alpha$ —V, $\mathbf{C}_5$, $\mathbf{C}_4$; $\mathbf{D}_1$.
$\mathbf{D}_2$: $x \varepsilon \alpha$.
$\mathbf{C}_7$: $[\alpha = K(\varphi)] \supset_\varphi \varphi(x)$ —conv, $\mathbf{D}_2$.
$\mathbf{C}_8$: $\Sigma\varphi \,.\, \alpha = K(\varphi)$ —V, 38, $\mathbf{C}_7$; $\mathbf{D}_2$.
$\mathbf{C}_9$: $\alpha = Kx \,.\, x \varepsilon \alpha$ —Thm I, $\mathbf{C}_6$, $\mathbf{C}_8$; $\mathbf{D}_2$.
$\mathbf{D}_3$: $\Sigma x \,.\, x \varepsilon \alpha$.
$\mathbf{C}_{10}$: $\alpha = Kx \,.\, x \varepsilon \alpha$ —Thm I, $\mathbf{C}_9$, $\mathbf{D}_3$; $\mathbf{D}_3$.
$\mathbf{A}_1$: $\Sigma\alpha\, \Sigma x \,.\, x \varepsilon \alpha$ —IV^2, $\mathbf{A}_1$ under Thm 30.
$\mathbf{A}_2$: $\Sigma x[x \varepsilon \alpha] \supset_\alpha . \alpha = Kx \,.\, x \varepsilon \alpha$ —Thm I, $\mathbf{C}_{10}$, $\mathbf{A}_1$.

COROLLARY. $[[x \varepsilon \alpha] \equiv_x . x \varepsilon \beta] \supset_{\alpha\beta} . \alpha = \beta$.

THEOREM 33. $[\Sigma(\varphi) \,.\, \sim \varphi(x)] \supset_{\varphi x} \sim . x \varepsilon K(\varphi)$.

$\mathbf{D}_1$: $\psi(y)$.
$\mathbf{C}_1$: $y \varepsilon K(\psi)$ —V^2, Thm 29, $\mathbf{D}_1$.
$\mathbf{C}_2$: $E(K(\psi))$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$.
$\mathbf{C}_3$: $K(\psi) = K(\psi)$ —V, Thm 1, $\mathbf{C}_2$; $\mathbf{D}_1$.
$\mathbf{D}_2$: $\Sigma(\psi) \,.\, \sim \psi(x)$.
$\mathbf{C}_4$: $\sim \psi(x)$ —V^2, 16, $\mathbf{D}_2$.
$\mathbf{C}_5$: $K(\psi) = K(\psi) \,.\, \sim \psi(x)$ —V^2, 14, $\mathbf{C}_3$, $\mathbf{C}_4$; $\mathbf{D}_1$, $\mathbf{D}_2$.
$\mathbf{C}_6$: $\Sigma\varphi \,.\, K(\psi) = K(\varphi) \,.\, \sim \varphi(x)$ —IV, $\mathbf{C}_5$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_7$: $\sim . [K(\psi) = K(\varphi)] \supset_{\varphi} \varphi(x)$ —V^2, 17, $\mathbf{C}_6$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_8$: $\sim . x \varepsilon K(\psi)$ —conv, $\mathbf{C}_7$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_9$: $\Sigma(\psi)$ —V^2, 15, $\mathbf{D}_2$.

$\mathbf{C}_{10}$: $\sim . x \varepsilon K(\psi)$ —Thm III, $\mathbf{C}_8$, $\mathbf{C}_9$; $\mathbf{D}_2$.

$\mathbf{A}_1$: $\Sigma\psi \Sigma x . \Sigma(\psi) . \sim \psi(x)$ —I, $\mathbf{A}_4$ under Thm 8.

$\mathbf{A}_2$: $[\Sigma(\psi) . \sim \psi(x)] \supset_{\psi x} \sim . x \varepsilon K(\psi)$ —Thm I Cor, $\mathbf{C}_{10}$, $\mathbf{A}_1$.

$\mathbf{A}_3$: $[\Sigma(\varphi) . \sim \varphi(x)] \supset_{\varphi x} \sim . x \varepsilon K(\varphi)$ —I, $\mathbf{A}_2$.

THEOREM 34. $\Sigma y [\sim . y \varepsilon \alpha] \supset_{\alpha} G x . x \varepsilon \alpha$.

$\mathbf{D}_1$: $x \varepsilon \alpha$.

$\mathbf{C}_1$: $\Sigma x . x \varepsilon \alpha$ —IV, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\Pi(\Lambda x . x \varepsilon \alpha, \lambda x . x \varepsilon \alpha)$ —V, Thm 7, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_3$: $\Pi(\lambda x . x \varepsilon \alpha, \Lambda x . x \varepsilon \alpha)$ —V, Thm 6, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_4$: $Q(\Lambda x . x \varepsilon \alpha, \lambda x . x \varepsilon \alpha)$ —V^2, 14, $\mathbf{C}_2$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{C}_5$: $K(\Lambda x . x \varepsilon \alpha) = K x . x \varepsilon \alpha$ —V^2, Thm 25, $\mathbf{C}_4$; $\mathbf{D}_1$.

$\mathbf{C}_6$: $\alpha = K x . x \varepsilon \alpha$ —V, Thm 32, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{C}_7$: $K x [x \varepsilon \alpha] = \alpha$ —V^2, Thm 2, $\mathbf{C}_6$; $\mathbf{D}_1$.

$\mathbf{C}_8$: $K(\Lambda x . x \varepsilon \alpha) = \alpha$ —V^3, Thm 3, $\mathbf{C}_5$, $\mathbf{C}_7$; $\mathbf{D}_1$.

$\mathbf{C}_9$: $\{\Lambda x . x \varepsilon \alpha\}(x)$ —V, $\mathbf{C}_3$, $\mathbf{D}_1$; $\mathbf{D}_1$.

$\mathbf{C}_{10}$: $\Sigma(\Lambda x . x \varepsilon \alpha)$ —IV, $\mathbf{C}_9$; $\mathbf{D}_1$.

$\mathbf{D}_2$: $\sim \{\Lambda x . x \varepsilon \alpha\}(z)$.

$\mathbf{C}_{11}$: $\sim . z \varepsilon K(\Lambda x . x \varepsilon \alpha)$ —V^2, Thm 33, $\mathbf{C}_{10}$, $\mathbf{D}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_{12}$: $\sim . z \varepsilon \alpha$ —V, $\mathbf{C}_8$, $\mathbf{C}_{11}$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{D}_3$: $\sim . y \varepsilon \alpha$.

$\mathbf{C}_{13}$: $\sim \{\Lambda x . x \varepsilon \alpha\}(y)$ —V^2, Thm 8, $\mathbf{C}_1$, $\mathbf{D}_3$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_{14}$: $\Sigma z \sim \{\Lambda x . x \varepsilon \alpha\}(z)$ —IV, $\mathbf{C}_{13}$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_{15}$: $\sim \{\Lambda x . x \varepsilon \alpha\}(z) \supset_z \sim . z \varepsilon \alpha$ —Thm I, $\mathbf{C}_{12}$, $\mathbf{C}_{14}$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_{16}$: $G x . x \varepsilon \alpha$ —conv, $\mathbf{C}_{15}$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{D}_4$: $\Sigma y \sim . y \varepsilon \alpha$.

$\mathbf{C}_{17}$: $\Sigma x . x \varepsilon \alpha$ —V, Thm 31 Cor 2, $\mathbf{D}_4$.

$\mathbf{C}_{18}$: $G x . x \varepsilon \alpha$ —Thm I, $\mathbf{C}_{16}$, $\mathbf{C}_{17}$; $\mathbf{D}_3$, $\mathbf{D}_4$.

$\mathbf{C}_{19}$: $G x . x \varepsilon \alpha$ —Thm I, $\mathbf{C}_{18}$, $\mathbf{D}_4$; $\mathbf{D}_4$.

$\mathbf{A}_1$: $\Sigma(\Pi(E))$ —IV, $\mathbf{A}_3$ under Thm 16.

$\mathbf{A}_2$: $\sim . [\sim] \varepsilon K(\Pi(E))$ —V^2, Thm 33, $\mathbf{A}_1$, Thm 10.

$\mathbf{A}_3$: $\Sigma\alpha \Sigma y \sim . y \varepsilon \alpha$ —IV^2, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\Sigma y [\sim . y \varepsilon \alpha] \supset_{\alpha} G x . x \varepsilon \alpha$ —Thm I, $\mathbf{C}_{19}$, $\mathbf{A}_3$.

THEOREM 35. $[\sim . x \varepsilon \alpha] \supset_{x\alpha} . x \varepsilon O(\alpha)$.

$\mathbf{D}_1$: $\sim . x \varepsilon \alpha$.

$\mathbf{C}_1$: $x \varepsilon K\pi \sim . \pi \varepsilon \alpha$ —V^2, Thm 29, $\mathbf{D}_1$.

$\mathbf{C}_2$: $x \varepsilon O(\alpha)$ —III, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Sigma x \Sigma \alpha \sim . x \varepsilon \alpha$ —IV^2, $\mathbf{A}_2$ under Thm 34.

$\mathbf{A}_2$: $[\sim . x \varepsilon \alpha] \supset_{x\alpha} . x \varepsilon O(\alpha)$ —Thm I Cor, $\mathbf{C}_2$, $\mathbf{A}_1$.

THEOREM 36. $[x \varepsilon O(\alpha)] \supset_{x\alpha} \sim . x \varepsilon \alpha$.

$\mathbf{D}_1$: $x \varepsilon O(\alpha)$.

$\mathbf{C}_1$: $x \varepsilon K\pi \sim . \pi \varepsilon \alpha$ —II, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\sim . x \varepsilon \alpha$ —V^2, Thm 30, $\mathbf{C}_1$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $[\sim] \varepsilon O(K(\Pi(E)))$ —V^2, Thm 35, $\mathbf{A}_2$ under Thm 34.

$\mathbf{A}_2$: $\Sigma x \Sigma \alpha . x \varepsilon O(\alpha)$ —IV^2, $\mathbf{A}_1$.

$\mathbf{A}_3$: $[x \varepsilon O(\alpha)] \supset_{x\alpha} \sim . x \varepsilon \alpha$ —Thm I Cor, $\mathbf{C}_2$, $\mathbf{A}_2$.

THEOREM 37. $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha . [x \varepsilon \alpha] \supset_x \sim . x \varepsilon O(\alpha)$.

$\mathbf{D}_1$: $x \varepsilon \alpha$.

$\mathbf{C}_1$: $\sim \sim . x \varepsilon \alpha$ —V, 26, $\mathbf{D}_1$.

$\mathbf{D}_2$: $\Sigma y \sim . y \varepsilon \alpha$.

$\mathbf{C}_2$: $\sim . x \varepsilon Ky \sim . y \varepsilon \alpha$ —V^2, Thm 33, $\mathbf{D}_2$, $\mathbf{C}_1$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_3$: $\sim . x \varepsilon O(\alpha)$ —conv, $\mathbf{C}_2$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_4$: $\Sigma x . x \varepsilon \alpha$ —V, Thm 31 Cor 2, $\mathbf{D}_2$.

$\mathbf{C}_5$: $[x \varepsilon \alpha] \supset_x \sim . x \varepsilon O(\alpha)$ —Thm I, $\mathbf{C}_3$, $\mathbf{C}_4$; $\mathbf{D}_2$.

$\mathbf{A}_1$: $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha . [x \varepsilon \alpha] \supset_x \sim . x \varepsilon O(\alpha)$ —Thm I, $\mathbf{C}_5$, $\mathbf{A}_3$ under Thm 34.

COROLLARY 1. $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha . [x \varepsilon \alpha] \supset_x x \varepsilon O(O(\alpha))$.

COROLLARY 2. $[\sim . x \varepsilon \alpha] \supset_{x\alpha} \sim . x \varepsilon O(O(\alpha))$.

COROLLARY 3. $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha . [\sim . x \varepsilon O(O(\alpha))] \supset_x \sim . x \varepsilon \alpha$.

COROLLARY 4. $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha . [\sim . x \varepsilon O(O(\alpha))] \supset_x . x \varepsilon O(\alpha)$.

THEOREM 38. $\Sigma y [\sim . y \varepsilon \alpha] \supset_\alpha Gx \sim . x \varepsilon O(\alpha)$.

$\mathbf{D}_1$: $\Sigma y \sim . y \varepsilon \alpha$.

$\mathbf{D}_2$: $x \varepsilon \alpha$.

$\mathbf{C}_1$: $\sim . x \varepsilon O(\alpha)$ —V^2, Thm 37, $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_2$: $\{Ay \sim . y \varepsilon O(\alpha)\}(x)$ —V^2, Thm 6 Cor, $\mathbf{C}_1$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_3$: $\Sigma y . y \varepsilon \alpha$ —V, Thm 31 Cor 2, $\mathbf{D}_1$.

$\mathbf{C}_4$: $\Pi(\lambda y . y \varepsilon \alpha, Ay \sim . y \varepsilon O(\alpha))$ —Thm II, $\mathbf{C}_2$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{C}_5$: $Gy . y \varepsilon \alpha$ —V, Thm 34, $\mathbf{D}_1$.

$\mathbf{D}_3$: $\sim\{\Lambda y \sim . y \varepsilon O(\alpha)\}(z)$.

$\mathbf{C}_6$: $\sim . z \varepsilon \alpha$ —V^3, Thm 20, $\mathbf{C}_5$, $\mathbf{C}_4$, $\mathbf{D}_3$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_7$: $z \varepsilon O(\alpha)$ —V^2, Thm 35, $\mathbf{C}_6$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{C}_8$: $\sim\sim . z \varepsilon O(\alpha)$ —V, 26, $\mathbf{C}_7$; $\mathbf{D}_1$, $\mathbf{D}_3$.

$\mathbf{D}_4$: $\sim . u \varepsilon \alpha$.

$\mathbf{C}_9$: $u \varepsilon O(\alpha)$ —V^2, Thm 35, $\mathbf{D}_4$.

$\mathbf{C}_{10}$: $\sim\sim . u \varepsilon O(\alpha)$ —V, 26, $\mathbf{C}_9$; $\mathbf{D}_4$.

$\mathbf{C}_{11}$: $\Sigma y \sim . y \varepsilon O(\alpha)$ —IV, $\mathbf{C}_1$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_{12}$: $\sim\{\Lambda y \sim . y \varepsilon O(\alpha)\}(u)$ —V^2, Thm 8, $\mathbf{C}_{11}$, $\mathbf{C}_{10}$; $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_4$.

$\mathbf{C}_{13}$: $\Sigma z \sim\{\Lambda y \sim . y \varepsilon O(\alpha)\}(z)$ —IV, $\mathbf{C}_{12}$; $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_4$.

$\mathbf{C}_{14}$: $\Sigma z \sim\{\Lambda y \sim . y \varepsilon O(\alpha)\}(z)$ —Thm I, $\mathbf{C}_{13}$, $\mathbf{D}_1$; $\mathbf{D}_1$, $\mathbf{D}_2$.

$\mathbf{C}_{15}$: $\Sigma z \sim\{\Lambda y \sim . y \varepsilon O(\alpha)\}(z)$ —Thm I, $\mathbf{C}_{14}$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{C}_{16}$: $\sim\{\Lambda y \sim . y \varepsilon O(\alpha)\}(z) \supset_z \sim\sim . z \varepsilon O(\alpha)$ —Thm I, $\mathbf{C}_8$, $\mathbf{C}_{15}$; $\mathbf{D}_1$.

$\mathbf{C}_{17}$: $Gx \sim . x \varepsilon O(\alpha)$ —conv, $\mathbf{C}_{16}$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\Sigma y[\sim . y \varepsilon \alpha] \supset_\alpha Gx \sim . x \varepsilon O(\alpha)$ —Thm I, $\mathbf{C}_{17}$, $\mathbf{A}_3$ under Thm 34.

7. **The Russell paradox.** The existence of the propositional function $\lambda\varphi . \sim \varphi(\varphi)$ follows readily from our assumptions. In fact, it can be shown that the value of this function is a true proposition for the value $\lambda x . \sim E(x)$ of the independent variable (among others), and that the value of the function is a false proposition for each of the values E and Σ of the independent variable (among others). If, however, we inquire whether the value of the function $\lambda\varphi . \sim \varphi(\varphi)$ is a true proposition or a false proposition for the value $\lambda\varphi . \sim \varphi(\varphi)$ of the independent variable, we are led at once into difficulty. For let $\mathfrak{P}$ stand for the formula $\{\lambda\varphi . \sim \varphi(\varphi)\}(\lambda\varphi . \sim \varphi(\varphi))$. Then $\mathfrak{P}$ is convertible into $\sim . \mathfrak{P}$, and $\sim . \mathfrak{P}$ is convertible into $\mathfrak{P}$. And hence if $\mathfrak{P}$ be a true proposition it must also be a false proposition, and if $\mathfrak{P}$ be a false proposition it must also be a true proposition.

We are not, however, forced to the conclusion that our system is self-contradictory. For no way appears by which we can prove either of the formulas $\mathfrak{P}$ or $\sim . \mathfrak{P}$, and in the absence of such a proof there is no obvious method of obtaining a contradiction out of the situation just described. There is, in fact, no reason to suppose that the propositional function $\lambda\varphi . \sim \varphi(\varphi)$ is significant for all values of the independent variable, or that the formula $\mathfrak{P}$ is anything but a meaningless aggregate of symbols.

Of course, if there were any way of constructing a propositional function $\mathbf{F}$, equivalent to $\lambda\varphi . \sim \varphi(\varphi)$ and significant for all values of the independent variable, then a contradiction could be obtained by considering the

formula $\{\mathsf{F}\}(\mathsf{F})$. But there is no reason to suppose that such a function F can be constructed. In particular, it does not appear that $\hat{\varphi} . \sim \varphi(\varphi)$ could be taken as F.

The property of the formula $\mathfrak{P}$, that its falsehood is provable as a consequence of its truth, and its truth as a consequence of its falsehood, is shared by many other formulas. We cite as examples the formulas $\{\lambda\varphi\, \lambda x . \sim \varphi(x, \varphi)\}$ $(\lambda\varphi\, \lambda x . \sim \varphi(x, \varphi), \lambda\varphi\, \lambda x . \sim \varphi(x, \varphi))$, and $\{\hat{\varphi} . \sim \varphi(\varphi)\}$ $(\hat{\varphi} . \sim \varphi(\varphi))$, and $Kx[\sim . x \in x] \in Kx \sim . x \in x$.

Let Ω stand for the formula $Kx[\sim . x \in x] \in Kx \sim . x \in x$. Then if Ω were true, Theorem 30would enable us to prove $\sim . \Omega$. And if $\sim . \Omega$ were true, Theorem 29 would enable us to prove Ω. This means that if we assume Ω we can deduce the contradiction $\Omega . \sim . \Omega$, and hence, under the classical principle of reductio ad absurdum, we should be justified in inferring that Ω is false, that is, $\sim . \Omega$. And once we had $\sim . \Omega$, we could infer Ω by Theorem 29.

In this way we see that the classical principle of reductio ad absurdum, and the assumption that for every propositional function there is an equivalent function significant for all values of the independent variable, are from a certain point of view the same.

And our omission of the principle of reductio ad absurdum from our list of assumptions is justified by reference to cases, like that just discussed, to which the principle appears to be inapplicable.

8. **Properties of the expression $\lambda\varphi . \sim \varphi(x(\varphi))$.** We now proceed to the proof of a number of theorems which bear a close relation to the Russell paradox.

THEOREM 39. $\Sigma x \{\lambda\varphi . \sim \varphi(x(\varphi))\} \left(\lambda\varphi . \sim \varphi(x(\varphi))\right)$.

$\mathbf{A}_1$: $Ep . q \supset_q pq$ —IV, 14.

$\mathbf{A}_2$: $E(E)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $E(E)\, E(E)$ —V^2, 14, $\mathbf{A}_2$, $\mathbf{A}_2$.

$\mathbf{A}_4$: $Ev . E(E)\, E(v)$ —IV, $\mathbf{A}_3$.

$\mathbf{A}_5$: $\sim \sim Ev . E(E)\, E(v)$ —V, 26, $\mathbf{A}_4$.

$\mathbf{A}_6$: $E\varphi . \sim \varphi(\lambda v . E(\varphi)\, E(v))$ —IV, $\mathbf{A}_5$.

$\mathbf{A}_7$: $E\varphi[\sim \varphi(\lambda v . E(\varphi)\, E(v))]\, E(E)$ —V^2, 14, $\mathbf{A}_6$, $\mathbf{A}_2$.

$\mathbf{A}_8$: $Ew . E\varphi[\sim \varphi(\lambda v . E(\varphi)\, E(v))]\, E(w)$ —IV, $\mathbf{A}_7$.

$\mathbf{A}_9$: $Ew[E\varphi[\sim \varphi(\lambda v . E(\varphi)\, E(v))]\, E(w)]\, E(E)$ —V^2, 14, $\mathbf{A}_8$, $\mathbf{A}_2$.

$\mathbf{A}_{10}$: $Ey . Ew[E\varphi[\sim \varphi(\lambda v . E(\varphi)\, E(v))]\, E(w)]\, E(y)$ —IV, $\mathbf{A}_9$.

$\mathbf{A}_{11}$: $E\varphi[\sim \varphi(\lambda v . E(\varphi)\, E(v))] . Ey . Ew[E\varphi[\sim \varphi(\lambda v . E(\varphi)\, E(v))]\, E(w)]\, E(y)$
—V^2, 14, $\mathbf{A}_6$, $\mathbf{A}_{10}$.

$\mathbf{A}_{12}$: $\sim\sim . E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi)\,E(v))] . Ey . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi)\,E(v))]\,E(w)]\,E(y)$ —V, 26, $\mathbf{A}_{11}$.

$\mathbf{A}_{13}$: $\{\lambda\varphi . \sim\varphi\,(\lambda v . E(\varphi)\,E(v))\}\,(\lambda\varphi . \sim\varphi\,(\lambda v . E(\varphi)\,E(v)))$ —conv, $\mathbf{A}_{12}$.

$\mathbf{A}_{14}$: $\Sigma x\,\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))$ —IV, $\mathbf{A}_{13}$.

THEOREM 40. $\Sigma x \sim . \{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))$.

$\mathbf{A}_1$: $Ep . \sim q \supset_q \sim . pq$ —IV, 21.

$\mathbf{A}_2$: $E(E)$ —IV, $\mathbf{A}_1$.

$\mathbf{A}_3$: $\sim\sim E(E)$ —IV, 26, $\mathbf{A}_2$.

$\mathbf{A}_4$: $\sim . E(E) . \sim E(E)$ —V^2, 21, $\mathbf{A}_2$, $\mathbf{A}_3$.

$\mathbf{A}_5$: $Ev . E(E) . \sim E(v)$ —IV, $\mathbf{A}_4$.

$\mathbf{A}_6$: $\sim\sim Ev . E(E) . \sim E(v)$ —V, 26, $\mathbf{A}_5$.

$\mathbf{A}_7$: $E\varphi . \sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))$ —IV, $\mathbf{A}_6$.

$\mathbf{A}_8$: $\sim . E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(E)$ —V^2, 21, $\mathbf{A}_7$, $\mathbf{A}_3$.

$\mathbf{A}_9$: $Ew . E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(w)$ —IV, $\mathbf{A}_8$.

$\mathbf{A}_{10}$: $\sim . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(w)] . \sim E(E)$ —V^2, 21, $\mathbf{A}_9$, $\mathbf{A}_3$.

$\mathbf{A}_{11}$: $Ey . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(w)] . \sim E(y)$ —IV, $\mathbf{A}_9$.

$\mathbf{A}_{12}$: $\sim\sim Ey . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(w)] . \sim E(y)$ —V, 26, $\mathbf{A}_{11}$.

$\mathbf{A}_{13}$: $\sim . E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim Ey . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(w)] . \sim E(y)$ —V^2, 21, $\mathbf{A}_7$, $\mathbf{A}_{12}$.

$\mathbf{A}_{14}$: $\sim\sim\sim . E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim Ey . Ew\,[E\varphi\,[\sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))] . \sim E(w)] . \sim E(y)$ —V, 26, $\mathbf{A}_{13}$.

$\mathbf{A}_{15}$: $\sim . \{\lambda\varphi . \sim\varphi\,(\lambda v . E(\varphi) . \sim E(v))\}\,(\lambda\varphi . \sim\varphi\,(\lambda v . E(\varphi) . \sim E(v)))$ —conv, $\mathbf{A}_{14}$.

$\mathbf{A}_{16}$: $\Sigma x \sim . \{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))$ —IV, $\mathbf{A}_{15}$.

THEOREM 41. $\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi))) \supset_x . x \neq \lambda z z$.

$\mathbf{D}_1$: $\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))$.

$\mathbf{C}_1$: $\sim\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,\big(x(\lambda\varphi . \sim\varphi\,(x(\varphi)))\big)$ —II, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\sim\sim\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,(\lambda\varphi . \sim(x(\varphi)))$ —V, 26, $\mathbf{D}_1$.

$\mathbf{C}_3$: $\sim\sim\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,\big(\{\lambda z z\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))\big)$ —III, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{C}_4$: $\sim\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,\big(x(\lambda\varphi . \sim\varphi\,(x(\varphi)))\big) . \sim\sim\{\lambda\varphi . \sim\varphi\,(x(\varphi))\}\,\big(\{\lambda z z\}\,(\lambda\varphi . \sim\varphi\,(x(\varphi)))\big)$ —V^2, 14, $\mathbf{C}_1$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{C}_5$: $\Sigma y . y(x) . \sim y(\lambda z z)$ —IV, $\mathbf{C}_4$; $\mathbf{D}_1$.

$\mathbf{C}_6$: $x \ddagger \lambda z z$ —V², 17, $\mathbf{C}_5$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\{\lambda\varphi . \sim\varphi(x(\varphi))\}(\lambda\varphi . \sim\varphi(x(\varphi))) \supset_x . x \ddagger \lambda z z$ —Thm I, $\mathbf{C}_6$, Thm 39.

COROLLARY. $\sim . \{\Lambda x \{\lambda\varphi . \sim\varphi(x(\varphi))\}(\lambda\varphi . \sim\varphi(x(\varphi)))\}(\lambda z z)$.

THEOREM 42. $\sim\{\lambda\varphi . \sim\varphi(x(\varphi))\}(\lambda\varphi . \sim\varphi(x(\varphi))) \supset_x . x \ddagger \lambda z z$.

$\mathbf{D}_1$: $\sim\{\lambda\varphi . \sim\varphi(x(\varphi))\}(\lambda\varphi . \sim\varphi(x(\varphi)))$.

$\mathbf{C}_1$: $\sim\sim\{\lambda\varphi . \sim\varphi(x(\varphi))\}\left(x(\lambda\varphi . \sim\varphi(x(\varphi)))\right)$ —II, $\mathbf{D}_1$.

$\mathbf{C}_2$: $\sim\sim\sim\{\lambda\varphi . \sim\varphi(x(\varphi))\}(\lambda\varphi . \sim\varphi(x(\varphi)))$ —V, 26, $\mathbf{D}_1$.

$\mathbf{C}_3$: $\sim\sim\sim\{\lambda\varphi . \sim\varphi(x(\varphi))\}\left(\{\lambda z z\}(\lambda\varphi . \sim\varphi(x(\varphi)))\right)$ —III, $\mathbf{C}_2$; $\mathbf{D}_1$.

$\mathbf{C}_4$: $\sim\sim\{\lambda\varphi . \sim\varphi(x(\varphi))\}\left(x(\lambda\varphi . \sim\varphi(x(\varphi)))\right) . \sim\sim\sim\{\lambda\varphi . \sim\varphi(x(\varphi))\}$
$\left(\{\lambda z z\}(\lambda\varphi . \sim\varphi(x(x)))\right)$ —V², 14, $\mathbf{C}_1$, $\mathbf{C}_3$; $\mathbf{D}_1$.

$\mathbf{C}_5$: $\Sigma y . y(x) . \sim y(\lambda z z)$ —IV, $\mathbf{C}_4$; $\mathbf{D}_1$.

$\mathbf{C}_6$: $x \ddagger \lambda z z$ —V², 17, $\mathbf{C}_5$; $\mathbf{D}_1$.

$\mathbf{A}_1$: $\sim\{\lambda\varphi . \sim\varphi(x(\varphi))\}(\lambda\varphi . \sim\varphi(x(\varphi))) \supset_x . x \ddagger \lambda z z$ —Thm I, $\mathbf{C}_6$, Thm 40.

COROLLARY. $\sim . \{\Lambda x \sim\{\lambda\varphi . \sim\varphi(x(\varphi))\}(\lambda\varphi . \sim\varphi(x(\varphi)))\}(\lambda z z)$.

9. **Positive integers.** We define:

$$1 \longrightarrow \lambda f \lambda x . f(x).$$
$$S \longrightarrow \lambda\varrho\lambda f\lambda x . f(\varrho(f, x)).$$
$$N \longrightarrow \lambda y . [\varphi(1) . \varphi(x) \supset_x \varphi(S(x))] \supset_\varphi \varphi(y).$$

The symbol 1 is to be read, "One", and $S(\mathbf{A})$ is to be read, "The successor of $\mathbf{A}$", and $N(\mathbf{A})$ is to be read, "$\mathbf{A}$ is a positive integer".

If we define $2 \longrightarrow S(1)$, and $3 \longrightarrow S(2)$, and so on, we find that 2 is convertible into $\lambda f\lambda x . f(f(x))$, and 3 is convertible into $\lambda f\lambda x . f(f(f(x)))$, and so on. The form of these expressions provides a convenient method of writing definitions by induction, as we may illustrate in the case of the definition of the sum, $m+n$, of two positive integers m and n. The equivalent of the recursion formulas, $m+1 = S(m)$, and $m+(k+1) = S(m+k)$, is, in fact, obtained by defining:

$$+ \longrightarrow \lambda m\lambda n . n(S, m).$$

And the operations of subtraction and multiplication may then be defined as follows:

$$- \longrightarrow \lambda r\lambda s\iota x . \{+\}(x, s) = r.$$
$$\times \longrightarrow \lambda m\lambda n . \{-\}(m(n(S), 1), 1).$$

Peano's axioms for the positive integers[9] may be expressed in our notation as follows:

[9] Rivista di Mathematica, vol. 1 (1891), pp. 87–102.

1. $N(1)$.
2. $N(x) \supset_x N(S(x))$.
3. $[N(x) \cdot N(y) \cdot S(x) = S(y)] \supset_{xy} \cdot x = y$.
4. $N(x) \supset_x \cdot S(x) \neq 1$.
5. $[\varphi(1) \cdot \varphi(x) \supset_x \varphi(S(x))] \supset_\varphi \cdot N(y) \supset_y \varphi(y)$.

It is believed that each of these five propositions will turn out to be a theorem which can be proved as a consequence of our postulates. And indeed proofs of 1, 2, and 5 are immediately evident.

Our program is to develop the theory of positive integers on the basis which we have just been describing, and then, by known methods or appropriate modifications of them, to proceed to a theory of rational numbers and a theory of real numbers.

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A SYSTEM ARISING FROM A WEAKENED SET OF GROUP POSTULATES.

By A. H. Clifford.[1]

Let G be a class of elements $a, b, c, \cdots$ satisfying the three postulates:

I. *To every ordered pair of elements (a, b) of G there corresponds a unique element c of G, which we indicate by writing $ab = c$.*

II. *For every three elements a, b, c of G it is true that*

$$(ab)c = a(bc).$$

III. *There exists at least one element e in G such that*

$$ea = a$$

for every element a of G. (Such an element will be called a *left identity.*)

Consider the following possibilities for a fourth postulate:

IV L. *For every element a of G, and for every left identity e, there exists at least one element b of G such that*

$$ba = e.$$

(b will be called a *left inverse of a with respect to the left identity e,* and a will be called a *right inverse of b with respect to e.*)

IV R. *For every element a of G, and for every left identity e, there exists at least one element b of G such that*

$$ab = e.$$

V L. *For every element a of G there exist at least one left identity e and at least one element b such that*

$$ba = e.$$

(This differs from IV L in that a need not have a left inverse with respect to *every* left identity, but only with respect to *at least one.*)

V R. For every element a of G there exist at least one left identity e and at least one element b of G such that

$$ab = e.$$

It is well known that the system (I, II, III, IV L) defines a group.[2] The systems (I, II, III, IV R), (I, II, III, V L), and (I, II, III, V R), how-

[1] Received June 7, 1932.

[2] L. E. Dickson. "Definitions of a group and a field by independent postulates." (Trans. Amer. Math. Soc. vol. 6, 1905, pp. 198–204).

ever, do not.[3] In Theorem 1 it is shown that these three are equivalent, and in Theorem 10 a complete determination of the structure of the system thus defined is given.

I wish here to thank Professor E. T. Bell for the interest he has shown in the paper, and in particular for his assistance in constructing the numerical example given below.

THEOREM 1. *The systems* (I, II, III, IV R), (I, II, III, V L), *and* (I, II, III, V R) *are equivalent.*

Proof: It is obvious that (I, II, III, IV R) implies (I, II, III, V R). Hence we have to show only that
(i) (I, II, III, VR) implies (I, II, III, V L), and
(ii) (I, II, III, V L) implies (I, II, III, IV R).

Proof of (i): Let G be a system (I, II, III, V R). Let a be any element of G, and let b be any right inverse of a, the existence of which is postulated in V R. That is, a left identity e exists such that

$$ab = e. \tag{1}$$

Let

$$ba = c. \tag{2}$$

Then

$$baba = c^2.$$

By (1),

$$bea = c^2.$$

$$ba = c^2.$$

By (2),

$$c = c^2. \tag{3}$$

Let d be any right inverse of c. Then there exists an identity e' (not necessarily the same as e) such that

$$cd = e'. \tag{4}$$

From (3),

$$cd = c^2 d = c(cd).$$

Using (4),

$$ce' = e'. \tag{5}$$

Now let x be any element of G. Then from (5)

$$cx = ce'x = e'x = x.$$

[3] The present investigation was suggested by an ambiguous statement of the fourth postulate in van der Waerden's "Moderne Algebra" (Springer, Berlin, 1930, p. 15) which can be interpreted as either IV L or V L. He evidently intended the former, judging from subsequent deductions; nevertheless I thought it would be of interest to see what the weaker postulate would lead to.

Hence by definition c is a left identity, and (2) shows that postulate VL is satisfied, q. e. d.

Proof of (ii): Let G be a system (I, II, III, VL). Let a be any element of G, and let e be any left identity. To establish IVR we have to exhibit a right inverse of a with respect to the arbitrarily chosen identity e.

Let b be any left inverse of a, the existence of which is postulated in VL. That is, a left identity e' exists such that

$$ba = e'. \tag{1}$$

Let c in turn be a left inverse of b, so that a left identity e'' exists such that

$$cb = e''. \tag{2}$$

From (1),

$$bab = e'b = b,$$

so that

$$cbab = cb.$$

From (2),

$$e''ab = e'',$$

whence

$$ab = e''.$$

From this it follows that the desired right inverse of a with respect to e is be. For

$$a(be) = (ab)e = e''e = e, \text{ q. e. d.}$$

This completes the proof of Theorem 1.

The system defined by any one of the three equivalent sets of postulates will be called a *multiple group*. We proceed now to investigate the properties of multiple groups, the investigation culminating in Theorem 10, by which the structure thereof is completely determined. To save space we shall omit the proofs of the intervening theorems. The latter are, however, arranged in order of logical development.

We remark here that if a multiple group G contains a right identity it becomes a group. Hence in general G will not contain a right identity. We can of course replace postulate III by one postulating the existence of a right identity, and the four subsequent postulates by four others in which left identities are replaced by right identities, in which case G contains only right identities, but evidently nothing new is gained thereby. We therefore tacitly understand in the following that G is of the left-hand type.

THEOREM 2. *For any three elements a, b, c of a multiple group G, $ca = cb$ always implies that $a = b$.*

THEOREM 3. *If $ab = e$, where e is any left identity, then there exists a left identity e' such that $ba = e'$.*

THEOREM 4. *For every element a of a multiple group G, and for every left identity e in G, there exists a unique right inverse of a, namely be, where b is any left inverse of a.*

THEOREM 5. *For each element a of a multiple group G there exists one and only one left identity e in G with respect to which a has left inverses. Furthermore a is permutable with e, that is, $ae = ea = a$, and with no other left identity in G.*

Let us designate by α the number of left identities in a multiple group G, and by e_i the left identities of G, where the index i ranges over some class I of cardinal number α. We shall call α the *index* or *multiplicity* of G. A multiple group of index 1 is evidently a group.

In accordance with Theorem 4 we shall designate by a_i^{-1} the right inverse of an element a of G with respect to the left identity e_i, so that

$$a a_i^{-1} = e_i \qquad \text{(for each } i \text{ in } I\text{)}.$$

Let e_j be the left identity with respect to which a has its left inverses, in accordance with Theorem 5. Applying Theorem 3 we deduce that

$$a_i^{-1} a = e_j \qquad \text{(for each } i \text{ in } I\text{)}.$$

We see then that each element a of G has precisely α inverses a_i^{-1}, each being both a right and a left inverse, the former with respect to e_i and the latter with respect to the fixed left identity e_j. From Theorem 4 we have also the relations

$$a_k^{-1} e_i = a_i^{-1} \qquad \text{(for all } i, k \text{ in } I\text{)}.$$

If we designate by K_i the class of those elements of G which have their left inverses with respect to the left identity e_i, for each i in I, we get a dissection of G into α categories K_i, which are non-overlapping and completely cover G.

THEOREM 6. *The elements of each category K_i of a multiple group constitute a group.*

We shall call the categories K_i of a multiple group G the *groups of composition* of G.

Let K_i and K_j be any two groups of composition of a multiple group G. Let a_i be any element in K_i, and consider the element b defined as follows:

$$b = a_i e_j.$$

Let a_i^{-1} be the inverse of a_i in the group K_i, so that

$$a_i^{-1} a_i = e_i.$$

Then

$$a_i^{-1} b = a_i^{-1} a_i e_j = e_i e_j = e_j,$$

so that b is in K_j. When we observe that

$$be_i = a_i e_j e_i = a_i e_i = a_i,$$

it appears that we can pair off the elements of K_i and K_j in a one-to-one manner. We shall designate by a_j the element b defined above, so that we have the reciprocal relations

$$\text{(A)} \qquad \begin{cases} a_i e_j = a_j, \\ a_j e_i = a_i. \end{cases}$$

Two such elements will be called *conjugate*. Conjugacy as thus defined is an equivalence relation, that is, it is reflexive, symmetric, and transitive, and hence leads to a dissection of G into categories $J(a)$, where $J(a)$ is defined to be the class of elements of G conjugate to the element a. This dissection is "orthogonal" to that afforded by the groups of composition of G, that is, the intersection of any J with any K contains precisely one element of G.

Using the one-to-one correspondence between two groups of composition K_i aud K_j afforded by pairing off conjugate elements, we can show the following:

THEOREM 7. *The groups of composition of a multiple group are simply isomorphic.*

The abstract group isomorphic to all the groups of composition of a multiple group G will be called the *composition group* of G.

THEOREM 8. *The categories J separating the elements of a multiple group G into classes of conjugate elements constitute a group under the operation of multiplication after the fashion of residue classes. This group is simply isomorphic to the composition group of G.*

Any multiple group consisting solely of identity elements will be called a *unitary multiple group*. The set of all identity elements of a multiple group G is evidently a multiple group, which we shall call the *unitary multiple subgroup* of G. Any two unitary multiple groups having the same number of elements are simply isomorphic, any one-to-one correspondence between them being an isomorphism. Hence a unitary multiple group is completely characterized by its order, that is, the number of its elements, which in this case is evidently the same as its index.

THEOREM 9. *The groups of composition of a multiple group G themselves constitute a multiple group under the operation of multiplication after the fashion of residue classes. This multiple group is simply isomorphic with the unitary multiple subgroup of G.*

THEOREM 10. *If H is any group, and α any cardinal number, then we can construct a multiple group G with index α having H as composition*

group. Conversely, any multiple group G is completely determined by its index α and its composition group H.

Proof: Let H be a group with elements $\{e, a, b, c, \cdots\}$, e being the identity. Let $K_i = \{e_i, a_i, b_i, c_i, \cdots\}$ be a set of distinct groups isomorphic with H, that is,

$$a_i b_i = (ab)_i,$$

where the index i ranges over a class I of cardinal number α.

Let G be the sum of all the K_i. We define multiplication in G as follows:

$$a_i b_j = (ab)_j.$$

This very definition implies the closure property I. To show the associativity II we have that

$$(a_i b_j) c_k = (ab)_j c_k = (abc)_k,$$
$$a_i (b_j c_k) = a_i (bc)_k = (abc)_k.$$

From

$$e_i a_j = (ea)_j = a_j$$

we see that postulate III is satisfied. From

$$(a^{-1})_i a_i = (a^{-1} a)_i = e_i$$

we see that postulate VL is satisfied.

Hence G is a multiple group. Since it has α identities, it is of index α. Furthermore the groups K_i must be its groups of composition, for a necessary and sufficient condition that an element of G be permutable with e_i is that it be in K_i, as follows from

$$a_j e_i = a_i.$$

This completes the proof of the first part of the theorem.

Now let G be any multiple group of index α and composition group $H = \{e, a, b, c, \cdots\}$. Let us denote the identities of G by e_i, where i ranges over a class I of cardinal number α. Let $K_i = \{e_i, a_i, b_i, c_i, \cdots\}$ be the group of composition of G corresponding to the left identity e_i, a_i being the element corresponding to a in the isomorphism between K_i and H. Then G is generated by the symbol x_i as x ranges over H and i over I. As in Theorem 7 we set up the isomorphism so that a_i and a_j are conjugate (i, j in I). Then by (A)

$$a_i e_j = a_j,$$

whence

$$a_i b_j = a_j b_j.$$

But by the isomorphism,

$$a_j b_j = (a b)_j.$$

Hence

$$a_i b_j = (a b)_j.$$

Thus G has precisely the same structure as the multiple group previously constructed out of H and α, that is, is simply isomorphic to it. This shows that the multiple group (H, α) is unique to within isomorphisms, q. e. d.

As an illustration we give here the multiplication table of the abstract multiple group of index 2 having the cyclic group of order 3 as composition group:

	e_1	a_1	b_1	e_2	a_2	b_2
e_1	e_1	a_1	b_1	e_2	a_2	b_2
a_1	a_1	b_1	e_1	a_2	b_2	e_2
b_1	b_1	e_1	a_1	b_2	e_2	a_2
e_2	e_1	a_1	b_1	e_2	a_2	b_2
a_2	a_1	b_1	e_1	a_2	b_2	e_2
b_2	b_1	e_1	a_1	b_2	e_2	a_2

We include one further theorem:

THEOREM 11. *If a and b are any two elements of a multiple group G, then the equation $ax = b$ has a unique solution in G.*

It is of interest to note that postulates I and II and Theorem 11 can be taken as a new definition of a multiple group.

A Numerical Example of a Multiple Group:

Let G be the class of all complex numbers $\neq 0$. The product (a, b) of any two elements a, b of G is defined as follows:

$$(a, b) = |a| b.$$

The associativity is easily verified. The left identities are $e^{i\varphi}$, where φ is real. $\frac{1}{a}$ is seen to be an inverse of a. The index of G is the cardinal number of the continuum, and the composition group of G is isomorphic to the multiplicative group of the field of real numbers.

CALIFORNIA INSTITUTE OF TECHNOLOGY.

NOTE ON THE INVARIANTS OF THE CLASS GROUP OF A CYCLIC FIELD.[1]

By Claiborne G. Latimer.

1. **Introduction.** Let F be an algebraic field of odd degree e, which is cyclic with respect to the rational field. The classes of ideals in F are the elements of a finite Abelian group $\mathfrak{G}$. By the fundamental theorem on such groups, $\mathfrak{G}$ has a basis $B_1, B_2, \cdots, B_m$ and the degree of every B_i is a power of a prime. These m prime powers are the invariants of $\mathfrak{G}$.

Let $K_1, K_2, \cdots, K_\mu$ be the distinct fields defined by the various eth roots of unity $\neq 1$, the order being arbitrarily chosen. The purpose of this note is to prove the

Theorem. *Let p be a rational prime and k a positive integer. If the prime ideal divisors of p in $K_1, K_2, \cdots, K_\mu$ are of degree $g_1, g_2, \cdots, g_\mu$ respectively, then the number of invariants of $\mathfrak{G}$ equal to p^k may be written in the form $g_1x_1 + g_2x_2 + \cdots + g_\mu x_\mu$, where each x is an integer $\geqq 0$.*

It will be shown in the last paragraph that this theorem and its proof lead to results closely related to certain theorems previously obtained.

2. **Certain lemmas on congruences.** Let A be a square matrix of order s, the elements of which are rational integers, let $f(x) = |A - xI|$ be the characteristic determinant of A, I being the identity matrix of order s, and let p be a rational prime. Since $f(A) = 0$, A is a root of a congruence $\varphi(x) \equiv 0 \pmod{p}$, of lowest degree, where φ is a polynomial with rational integral coefficients, the leading coefficient being unity and the remaining coefficients being uniquely determined, modulo p. $\varphi(x) \equiv 0 \pmod{p}$ will be called the minimum congruence, modulo p, of A.

The following lemmas on congruences are analogues of known results on equations[2] and proofs may be made similar to those in the reference cited. It will be understood that A, φ, p are as above.

Lemma 1. *If $\lambda(x)$ is a polynomial with rational integral coefficients such that $\lambda(A) \equiv 0 \pmod{p}$, then $\lambda(x)$ is divisible, modulo p, by $\varphi(x)$.*

Lemma 2. *$\varphi(x)$ is the quotient, modulo p, of the characteristic determinant $|A - xI|$ of A by the g. c. d., modulo p, of its $(s-1)$-rowed minors.*

Lemma 3. *Every irreducible factor, modulo p, of $|A - xI|$ is a factor, modulo p, of $\varphi(x)$.*

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[1] Received December 3, 1932.

[2] Dickson, *Algebras and Their Arithmetics*, pp. 109, 10; *Algebren und ihre Zahlentheorie*, pp. 20, 1.

3. **Proof of theorem.** Let p be a rational prime divisor of the class number of F and let $B_1, B_2, \cdots, B_s$ be those basal classes of $\mathfrak{G}$ whose degrees are powers of p. Let the distinct powers of p which are invariants of $\mathfrak{G}$ be $p^{k_1}, p^{k_2}, \cdots, p^{k_t}$. If $t>1$, the notation may be so chosen that $k_h<k_{h+1}$ $(h=1,2,\cdots,t-1)$ and $B_1, B_2, \cdots, B_{s_1}$ are of degree p^{k_1}, $B_{s_1+1}, B_{s_1+2}, \cdots, B_{s_1+s_2}$ are of degree p^{k_2}, etc., $s_1+s_2+\cdots+s_t=s$.

Every substitution of the Galois group of F is a power of one of them, S. If B is a class of ideals in F, it will be understood that $B'=S(B)$, $B''=S^2(B), \cdots, B^{(n)}=S^n(B)$, $n=e-1$. If the degree of B is a power of p, it is expressible as a product of powers of $B_1, B_2, \cdots, B_s$. Hence

$$(1)\qquad \begin{aligned} B_i' &= B_1^{\alpha_{i1}} \cdot B_2^{\alpha_{i2}} \cdot \cdots \cdot B_s^{\alpha_{is}}, \\ B_i'' &= B_1^{\alpha_{i1}^{(2)}} \cdot B_2^{\alpha_{i2}^{(2)}} \cdot \cdots \cdot B_s^{\alpha_{is}^{(2)}}, \\ &\vdots \quad\ \vdots \quad\ \vdots \quad\ \vdots \\ B_i^{(n)} &= B_1^{\alpha_{i1}^{(n)}} \cdot B_2^{\alpha_{i2}^{(n)}} \cdot \cdots \cdot B_s^{\alpha_{is}^{(n)}}, \end{aligned} \qquad (i=1,2,\cdots,s),$$

where the α_{ij} are integers and if the matrix $(\alpha_{ij})=A$, then $(\alpha_{ij}^{(k)})=A^k$ $(i,j=1,2,\cdots,s;\ k=2,3,\cdots,n)$. But every $B_i \cdot B_i' \cdot B_i'' \cdot \cdots \cdot B_i^{(n)}$ $(i=1,2,\cdots,s)$ is the principal class. Therefore by (1),

$$(2)\qquad \lambda(A) = A^n + A^{n-1} + \cdots + A + I \equiv 0 \pmod{p}.$$

If $h \leqq s_1$, $B_h'^{p^{k_1}}$ is principal and if $h>s_1$, $B_h^{p^{k_1}}$ is non-principal. It follows from (1) that if $i \leqq s_1$, $j>s_1$, then $\alpha_{ij} \equiv 0 \pmod{p}$. Similarly, if $i \leqq s_1+s_2$, $j>s_1+s_2$, $\alpha_{ij} \equiv 0 \pmod{p}$. By repetition of this argument, we find that

$$A \equiv \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_{t1} & A_{t2} & \cdots & A_{tt} \end{pmatrix} \pmod{p},$$

where every 0 stands for a rectangular block of zeroes and A_{ij} is a matrix with s_i rows and s_j columns. From (2) and the form of the matrix on the right above, it follows that

$$(3)\qquad \lambda(A_{hh}) \equiv 0 \pmod{p}, \qquad (h=1,2,\cdots,t).$$

After proper choice of notation, we have $\lambda(x)=\lambda_1(x)\cdot\lambda_2(x)\cdot\cdots\cdot\lambda_\mu(x)$, where each field K_i is defined by a root ζ_i of $\lambda_i(x)=0$ and each λ_i is a polynomial, irreducible in the rational field, with rational integral coefficients, the leading coefficient being unity. Each ζ_i is a primitive e_ith root of unity, where e_i is a divisor >1 of e and λ_i is of degree $E_i=\varphi(e_i)$.

The prime ideal divisors of p in a field K_i are all of the same degree g_i. Therefore

$$\lambda_i(x) \equiv \lambda_i^{(1)}(x) \cdot \lambda_i^{(2)}(x) \cdot \cdots \cdot \lambda_i^{(f_i)}(x) \pmod{p}, \qquad (i = 1, 2, \cdots, \mu),$$

where $f_i = E_i/g_i$ and every $\lambda_i^{(j)}$ $(j = 1, 2, \cdots, f_i)$ is of degree g_i and is irreducible, modulo p.[3] Hence we have

$$\lambda(x) \equiv \prod_{i=1}^{\mu} \prod_{j=1}^{f_i} \lambda_i^{(j)}(x) \pmod{p}. \tag{4}$$

If $\varphi(x) \equiv 0 \pmod{p}$ is the minimum congruence of A_{hh}, by (3) and Lemma 1, $\varphi(x)$ divides $\lambda(x)$, modulo p. Hence by (4), we may assume that $\varphi(x)$ is the product of certain of the $\lambda_i^{(j)}$. Let I_h be the identity matrix of order s_h and $f_h(x) = |A_{hh} - xI_h|$, $(h = 1, 2, \cdots, t)$. By Lemma 3, $f_h(x)$ is a product of powers of the irreducible factors, modulo p, of $\varphi(x)$ and therefore the degree of f_h may be written in the form $g_1x_1 + g_2x_2 + \cdots + g_\mu x_\mu$, where each $x_i \geqq 0$. But the degree of f_h is s_h. The theorem follows.

4. **Certain consequences of the theorem and its proof.** If exactly one invariant of $\mathfrak{G}$ is equal to a given prime power, p^k, where p is not a divisor of e, then the prime ideal divisors of p in one of the fields $K_1, K_2, \cdots, K_\mu$ are of the first degree. Hence, by a known result, $p \equiv 1 \pmod{e_i}$,[4] where e_i is a divisor >1 of e.[5]

For a fixed h, from (3) there is a least positive integer e', a divisor of e, such that $A_{hh}^{e'} \equiv I_h \pmod{p}$. Hence if there are exactly s invariants of $\mathfrak{G}$ equal to p^k, there is a corresponding cyclic linear homogeneous substitution group, modulo p, on s variables which is simply isomorphic with a divisor of the Galois group of F.[6] By (3), the order e', of this group is >1 if p is prime to e.

By the theorem, the product of the invariants of $\mathfrak{G}$, i. e. the class number of F, is equal to the product of the norms of μ ideals, one in each of the fields K_i. For the case where e is a prime, this result was obtained by the writer in an entirely different manner.[7]

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[3] Fricke, Lehrbuch der Algebra, III, pp. 57, 198.

[4] Fricke, loc. cit., p. 198.

[5] Cf. Tschebotarow, *Zur Gruppentheorie des Klassenkörpers,* Journal für Mathematik, vol. 161 (1929), p. 188, Theorem 11.

[6] Cf. Tschebotarow, loc. cit., Theorem 10.

[7] *On the Class Number of a Cyclic Field,* Transactions of the American Mathematical Society, vol. 35 (1933).

THE UNIVERSITY OF KENTUCKY.

ON UNIVERSAL SETS OF POSITIVE TERNARY QUADRATIC FORMS.[1]

By A. Adrian Albert.

1. **Introduction.** One of the most interesting topics in the Theory of Numbers is the theory of *universal forms*, that is forms representing all positive integers. No positive ternary quadratic form represents all positive integers, so that there can be no theory of universal positive ternaries. But this theory may be replaced by a theory of universal *sets of a finite number of forms*. It is my purpose, in the present article, to introduce such a new theory and to solve perhaps the first problem.

2. **Preliminary concepts.** Let Σ be a set of positive ternary quadratic forms of which only a finite number are non-equivalent. Then we shall say that a positive integer a *is represented by* Σ if there exists a form $f(x, y, z)$ in Σ and integers α, β, γ for which $a = f(\alpha, \beta, \gamma)$, that is if $f(x, y, z)$ represents a. In particular Σ will be called a *universal set* if it represents all positive integers.

Suppose now that $\Sigma_1, \Sigma_2, \cdots$ are a certain system of sets Σ_i of forms such that every form of Σ_i has the same determinant[2] d_i. Then we shall call d_i the determinant of Σ_i. Let $\Sigma = (\Sigma_1, \cdots, \Sigma_n)$ be a finite sub-system of the above system of forms so that Σ is a set of forms representing the aggregate of all the integers represented by its components Σ_i. Write

$$\Sigma - \Sigma_i = (\Sigma_1, \cdots, \Sigma_{i-1}, \Sigma_{i+1}, \cdots, \Sigma_n) \tag{1}$$

and assume that $\Sigma - \Sigma_i$ is *not* universal for every i but that Σ is a universal set. Then we shall say that Σ is a *chain of sets* Σ_i and call n the length of Σ. In particular if the sets Σ_i consist each of a single form (in the sense of equivalence) then we shall call Σ a *chain of forms*. It is evident that of the universal sets of forms it is the chains of forms which are of greatest importance and it is the problem of the existence of such chains and of the nature of the forms contained therein which I propose for the consideration of my readers and which I hope to study more in future research.

We may immediately write a necessary condition that a set $\Sigma = (\Sigma_1, \cdots, \Sigma_n)$ shall be universal. Let $\Sigma_i^* = \Sigma(d_i)$ be the set of all positive ternaries of determinant d_i. Define

$$\Sigma^* = \Sigma(d_1, \cdots, d_n) \equiv (\Sigma_1^*, \cdots, \Sigma_n^*). \tag{2}$$

[1] Received May 22, 1933.—Presented to the American Mathematical Society April 15, 1933.

[2] and hence there are only a finite number of non-equivalent forms in each Σ_i.

Then Σ is contained in Σ^* and we have

THEOREM 1. *A necessary condition that a set* $\Sigma = (\Sigma_1, \cdots, \Sigma_n)$, Σ_i *of determinant* d_i, *shall be universal is that* $\Sigma^* = \Sigma(d_1, \cdots, d_n)$ *shall be universal.*

Thus as a first problem in our new theory we have the important problem of finding necessary and sufficient conditions that a set $\Sigma(d_1, \cdots, d_n)$ shall be universal. It is this problem that will be solved in the next section. We shall also determine all chains of sets $\Sigma(d_i)$.

3. Universal sets $\Sigma(d_1, \cdots, d_n)$. I have proved

THEOREM 2. *Let* $\Sigma(d)$ *be the set of all positive ternary quadratic forms of determinant* $d = \gamma^2\delta$, δ *with no square factor. Then an integer* $a = \lambda^2\mu$, μ *with no square factor, is represented by* $\Sigma(d)$ *if and only if* a *has not the form*

$$\mu = \alpha\delta, \qquad \alpha \equiv 7 \pmod 8, \tag{3}$$

such that α *is prime to* d *and the Jacobi symbol*

$$\left(\frac{p}{\alpha}\right) = 1 \tag{4}$$

for every odd prime factor p *of* d.

We shall apply the above theorem[3] then to our sets $\Sigma(d_1, \cdots, d_n)$. We write $d_i = \gamma_i^2\delta_i$ and write δ for the least common multiple of the δ_i. Then

$$\delta = \delta_i\varepsilon_i \tag{5}$$

has no square factor and every prime factor of ε_i divides some δ_j. It is obvious that if a does not have the property $\mu \equiv 0 \bmod \delta_i$ then a is represented by $\Sigma(d_i)$. Hence the only integers which may possibly be not represented by $\Sigma(d_1, \cdots, d_n)$ are the integers a for which μ is divisible by all the δ_i and hence by their least common multiple δ. Hence we may write

$$\mu = \delta\mu_0 = \delta_i\alpha_i, \qquad \alpha_i = \varepsilon_i\mu_0 \tag{6}$$

for all integers considered.

Moreover we must have $\alpha_i \equiv 7 \pmod 8$ so that it must be so that

$$\varepsilon_i\mu_0 \equiv 7 \pmod 8 \qquad (i = 1, \cdots, n). \tag{7}$$

If μ_0 is even then (7) is not satisfied and a is represented by $\Sigma(d_1, \cdots, d_n)$. Let then μ_0 be odd. Then (7) becomes

$$\mu_0 \equiv 7\varepsilon_i \pmod 8, \tag{8}$$

from which $7\varepsilon_i \equiv 7\varepsilon_j \pmod 8$, $\varepsilon_i \equiv \varepsilon_j \pmod 8$. Hence we have

[3] See my paper *The integers represented by sets of positive ternary quadratic forms*, American Journal of Mathematics, vol. LV (1933), pp. 274–292.

LEMMA 1. *A necessary condition that a set* $\Sigma(d_1, \cdots, d_n)$ *shall not be universal is that* $\varepsilon_i \equiv \varepsilon_j \pmod 8$.

Let now (8) be satisfied so that $\varepsilon_i \equiv \varepsilon_j \pmod 8$ for all i and j. By (6) we have $\alpha_i = \varepsilon_i \mu_0$ so that if α_i is prime to γ_i then ε_i must be prime to γ_i. But if there exists a factor p_i of ε_i which divides γ_i then this is impossible and $\Sigma(d_1, \cdots, d_n)$ represents all positive integers. Since p_i is prime to δ_i and divides some δ_j it must be so that in fact $\Sigma(d_i, d_j)$ represents all positive integers. For $\Sigma(d_j)$ represents all a's for which μ is not divisible by p_i, a factor of δ_j, while $\Sigma(d_i)$ represents all a's for which μ is divisible by p_i so that α_i has a factor in common with γ_i. We have therefore proved

THEOREM 3. *Let* $d_i = \gamma_i^2 \delta_i$ $(i = 1, \cdots, n)$ *where* δ_i *has no square factor and let*

$$\delta = \delta_i \varepsilon_i \tag{9}$$

be the least common multiple of the δ_i. *Then if there exist a pair of integers* (i, j) *for which either*

$$\varepsilon_i \not\equiv \varepsilon_j \pmod 8, \tag{10}$$

or

$$\varepsilon_i \equiv \varepsilon_j \pmod 8, \quad \gamma_i \equiv \delta_j \equiv 0 \pmod p, \quad \delta_i \not\equiv 0 \pmod p \tag{11}$$

for a prime p, *the set* $\Sigma(d_1, \cdots, d_n)$ *is universal and contains a chain* $\Sigma(d_i, d_j)$, *with respect to sets* $\Sigma(d_i)$, *of length two.*

It is of course evident that in applying the above sufficient condition to a pair of sets $\Sigma(d_1)$ and $\Sigma(d_2)$ that we may replace δ by the least common multiple of δ_1 and δ_2 (instead of all the δ_i) and apply the above with the new ε_1 and ε_2 so obtained.

We shall assume that the above sufficient conditions are not satisfied. Then if $\alpha_i = \mu_0 \varepsilon_i$ the integer α_i is prime to d_i if and only if μ_0 is prime to d_i. Then a is not represented by $\Sigma(d_i)$ if and only if $\left(\frac{p_i}{\alpha_i}\right) = 1$ for every odd prime factor p_i of d_i. But

$$\left(\frac{p_i}{\alpha_i}\right) = (-1)^{\frac{p_i-1}{2} \cdot \frac{\alpha_i-1}{2}} \left(\frac{\alpha_i}{p_i}\right) = (-1)^{\frac{p_i-1}{2}} \left(\frac{\mu_0}{p_i}\right) \left(\frac{\varepsilon_i}{p_i}\right),$$

since $\alpha_i \equiv 7 \pmod 8$, $\frac{1}{2}(\alpha_i - 1) \equiv 3 \pmod 4$. Hence $\left(\frac{p_i}{\alpha_i}\right) = 1$ if and only if

$$\left(\frac{\mu_0}{p_i}\right) = (-1)^{\frac{p_i-1}{2}} \left(\frac{\varepsilon_i}{p_i}\right) = \left(\frac{-\varepsilon_i}{p_i}\right). \tag{12}$$

Hence if the sufficient conditions of Theorem 3 are not satisfied an integer a is represented by the set $\Sigma(d_i)$ if and only if $\mu_0 \equiv 7\varepsilon_1 \pmod 8$, μ_0 is prime

to $d = \prod_i d_i$, and if (12) is satisfied for every odd prime factor p_i of d_i. If η_i is an integer chosen to have the same quadratic character $\left(\frac{\eta_i}{p_i}\right)$ as $-\varepsilon_i$ and hence prime to p_i, then the congruences

$$\mu_0 \equiv 7\,\varepsilon_1 \pmod 8, \quad \mu_0 \equiv \eta_i \pmod{p_i} \tag{13}$$

have relatively prime moduli and hence have solutions μ_0. But then $\Sigma(d_1, \cdots, d_n)$ does not represent such integers $a = \lambda^2(\mu_0\,\delta)$ and is not universal. Also if the η_i are permitted to range over the least integers having the desired quadratic characters the resulting congruences (13) determine arithmetic progressions of all integers not represented by $\Sigma(d_1, \cdots, d_n)$ and we say that $\Sigma(d_1, \cdots, d_n)$ is regular. We have therefore proved

THEOREM 4. *Let $\alpha_i = \gamma_i^2\,\delta_i (i = 1, \cdots, n)$, δ_i with no square factor and let δ and ε_i be defined as in Theorem* 3. *Then the sufficient conditions of Theorem* 3 *that a set $\Sigma(d_1, \cdots, d_n)$ be universal are also necessary. Also every set $\Sigma(d_1, \cdots, d_n)$ which is not universal represents all integers not of the form $a = \lambda^2\mu$, $\mu = \mu_0\,\delta$ where μ_0 is prime to $d = d_1 \cdots d_n$,*

$$\mu_0 \equiv 7\,\varepsilon_1 \pmod 8, \quad \left(\frac{\mu_0}{p}\right) = \left(\frac{-\varepsilon}{p}\right) \tag{14}$$

for every odd prime factor p of d_i, and is regular.

As an immediate corollary of Theorem 4 we have

THEOREM 5. *Every chain of sets $\Sigma(d)$ has length two and is one of the chains determined in Theorem* 3.

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AND

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